Around Kudla's Green function for SO$(3,2)$

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Abstract

One of Kudla's conjectures about deep relations between arithmetic intersection theory, Eisenstein series and their derivatives comes down to a relation between certain Green function integrals and the special value of the derivative of a corresponding Eisenstein series. Though this is in the mean time a well treated field, this shall be discussed in a pedestrian way for the homogenous space belonging to the orthogonal group of signature (3,2). On the way, some (2,2)- and (1,2)-material is also collected.

Contents

Preface

In 1997, Kudla presented in [Ku0] and [Ku2] conjectures about deep relations between arithmetic intersection theory, Eisenstein series and their derivatives, and special values of Rankin L-series. Almost 20 years ago, Ulf Kühn made me acquainted with some of his former work concerning the Arakelov theory and the arithmetic of the world round about the orthogonal and unitary groups in special cases. Based on Kühn's old unpublished draft on his attempt to prove Kudla's conjectures for the case of the product of two modular curves two joint articles discussing Kudla's Green function concerning the orthogonal group $\mathrm{SO}(2,2)$ appeared in the arxiv in 2012 [BeKI] and [BeKII].[1] There, in part one it is proved that the generating series of certain modified arithmetic special cycles is as predicted by Kudlas conjectures a modular form with values in the first arithmetic Chow group. In part two this generating series is paired with the square of the first arithmetic Chern class of the line bundle of modular forms. Using part one and previously known results like the Faltings heights of Hecke correspondences this calculation boils down to determine the integrals of the Green functions $\Xi(m)$ over the associated homogenous space X. The resulting arithmetic intersection numbers turn out to be as predicted by Kudla to be strongly related to the Fourier coefficients of the derivative of the classical real analytic Eisenstein series $E_2(\tau, s)$.
In the following years, an attempt to do all this for the group $\mathrm{SO}(3,2)$ using the in [2003] written articles [BK] by Bruinier and Kühn and [Ku1] by Kudla and though helped by remarks and hints by Jan Bruinier and Jens Funke got lost in time and details as Kühn mainly was taken over by other tasks. And there was much more different and more general work in several directions on orthogonal and unitary groups by a lot of authors (Kudla, Rapoport, Bruinier, Funke, Yang, ...). Recently, encouraged by Kühn and with his help, I revised the material we had covered and assembled it. Though there is in principle no result not known in the meantime, one may hope that our pedestrian way to complete our approach to calculate Kudla's Green function integral is still interesting to some readers.

[1] In the meantime both articles are joined in the paper 'Kudla's conjecture for $X(1) \times X(1)$' which will appear in the volume dedicated to Kudla's 70-th birthday

Introduction

Kudla's program, presented e.g. in his 2002-ICM-talk 'Derivatives of Eisenstein series and arithmetic geometry' [Ku], with sources in [Ku0] and [Ku2], proposes two ways to relate the generating series for subspaces of the arithmetic space $\mathcal{M}$ belonging to the homogenous space $X = \Gamma\backslash\mathbb{D}$ of the orthogonal group $G = \mathrm{SO}(p,2)$ to the appropriate Eisenstein series $\mathcal{E}(\tau, s)$ of weight $p/2+1$ (e.g. the introduction of Kudla, Rapoport, and Yang [KRY]) where $\tau = u + iv \in \mathbb{H}$. The first one is via the degree series $\phi_{\mathrm{deg}}(\tau)$ and the second one via the height series $\hat{\phi}_{\mathrm{height}}(\tau)$: For $m \in \mathbb{Z}$, let be $\hat{Z}(m) = (Z(m), \widetilde{\Xi}(m,v)) \in \widehat{\mathrm{CH}}^1(\mathcal{M})$ where $Z(m)$ is a special cycle and $\widetilde{\Xi}$ is a possibly modified version (caused by the influence of the compactification) of Kudla's Green function Ξ. Then one has

$$\phi_{\mathrm{deg}}(\tau) = \sum_m \deg(\hat{Z}(m,v))q^m = \mathcal{E}(\tau, 0),$$

where $\deg(\hat{Z}(m,v))$ is given by

$$\int_X dd^c\widetilde{\Xi}(m,v) \wedge c_1(\mathcal{L})^{p-1} = \mathrm{vol}(Z(m)).$$

On the way in the following, this relation is proven again for $p = 1, 2$ and 3. For the height series Kudla conjectures

$$\hat{\phi}_{\mathrm{height}}(\tau) := \sum \hat{Z}(m)q^m \cdot \hat{c}_1(\mathcal{L})^p = \frac{d}{ds}\mathbb{E}(\tau, s)|_{s=0}$$

where $\mathbb{E}$ is a certain normalized Eisenstein series and the coefficients of the height series are essentially given by

$$\hat{Z}(m,v) \cdot c_1(\mathcal{L})^p = \mathrm{ht}_{\mathcal{M}}(Z(m)) + \int_X \widetilde{\Xi}(m,v)c_1(\mathcal{L})^p.$$

This is already established for $p = 1$ (see Yang [Ya] and Kudla-Rapoport-Yang [KRY]) and $p = 2$ in a particular cases (as mentioned in the Preface in [BeKI] and [BeKII] and the thesis of Buck [Bu]) and will be briefly recorded below.

For $p = 3$, as in (2.1.1), we take the vector valued Eisenstein series from [BK] (3.1) with the two components for $\beta \equiv 0$ or 1 from [BK] (3.4)

$$E_\beta(\tau, s) := (1/2) \sum_{(M,\phi)\in\Gamma_\infty\backslash\mathrm{Mp}_2(\mathbb{Z})} (\mathbf{e}_\beta v^s)|_\ell^*(M,\phi). = \sum_{\gamma\in L'/L}\sum_{m\in\mathbb{Z}-q(\gamma)} c_\beta(\gamma, m, s', v)\mathbf{e}_\gamma(mu).$$

As in [BK], we specialize to $\beta = 0$ and compare these coefficients $c_0(\gamma, m, s', v)$ and their derivatives $c_0'(\gamma, m, s', v) := \frac{d}{ds'}c_0(\gamma, m, s', v)$ for $s' = 0$ to the integrals

$$I(\gamma, m, v) = \int_X \Xi(\gamma, m, v)d\mu$$

of Kudla's Green function $\Xi(\gamma, m, v)$ displayed in section 4 of this text and $I^{BK}(\gamma, m, v) = \int_X G_{\gamma,m} d\mu$ of the Green function $G_{\gamma,m}, m < 0$ from Bruinier-Kühn [BK] Definition 4.5 and Theorem 4.10. We see that for general m we get into some elementary algebraic number theory, As in [BK] (3.23), for $m \in \mathbb{Z}$ we put $4m = D_0 f^2$ and for $m - 1/4 \in \mathbb{Z}\, 16m = D_0 f^2$ where D_0 is a fundamental discriminant and $f \in \mathbb{N}$, and χ_{D_0} the associated Dirichlet character (e.g.,[Za] p.38). With $a = 4\pi m v$, and $\sigma_{\gamma,m}$ generalized divisor sums from [BK] (3.23) (here see (12.22)) and integrals $J_\pm$ (12.14.2), we end up with the following.

0.1. Proposition. For the (3,2) case, one has

$$c_0(\gamma, m, 0, v) = \begin{cases} C(\gamma, m, 0)e^{-a/2} \text{ for } m > 0, \\ 0, \quad \text{for } m < 0, \end{cases}$$

$$c_0'(\gamma, m, 0, v) = \begin{cases} C(\gamma, m, 0)e^{-a/2}(J_+(3/2, a) + \frac{C'(\gamma,m,0)}{C(\gamma,m,0)}), \text{ for } m > 0, \\ C(\gamma, m, 0)e^{-|a|/2} \cdot J_-(3/2, a), \text{ for } m < 0, \end{cases}$$

(0.1.1) where $$C(\gamma, m, 0) := -2^6 \cdot 3 \cdot 5 \cdot \pi^{-2} |m|^{3/2} L(\chi_{D_0}, 2)\sigma_{\gamma,m}(5/2) \text{ for } m \neq 0,$$

and the

0.2. Theorem. For the (3,2) case, one has

$$(4/B) \cdot I(\gamma, m, v) = \begin{cases} C(\gamma, m, 0)J_+(3/2, a), \text{ for } m > 0, \\ C(\gamma, m, 0)J_-(3/2, a)e^{-|a|}, \text{ for } m < 0. \end{cases}$$

(0.2.1)

$$(4/B) \cdot I^{BK}(\gamma, -m, v) = \begin{cases} -C(\gamma, -m, 0)\big(\frac{C'(\gamma,-m,0)}{C(\gamma,-m,0)} + \log(4\pi) - \Gamma'(1)\big), \text{ for } m > 0, \\ 0, \text{ for } m < 0. \end{cases}$$

The second equation is simply [BK] Theorem 4.10 and the first one is an immediate consequence of our main result:

0.3. Proposition (Green Integral). We have the integrals of Kudla's Green function for the case of the $\mathrm{SO}(2, 3)$

$$\begin{aligned} I(\gamma, v, m) &= \int_X \Xi(\gamma, m, v) d\mu \\ &= -2^6 \cdot 3 \cdot 5 \cdot \pi^{-2} |m|^{3/2} L(\chi_{D_0}, 2)\sigma_{\gamma,m}(5/2)J_+(v, m), \text{ for } m > 0, \\ &= -2^6 \cdot 3 \cdot 5 \cdot \pi^{-2} |m|^{3/2} L(\chi_{D_0}, 2)\sigma_{\gamma,m}(5/2)J_-(v, m)e^{-|a|}, \text{ for } m < 0. \end{aligned} \tag{0.3.1}$$

0.4. Corollary. We have

$$c_0'(\gamma, m, 0, v) = e^{-a/2}((4/B) \cdot (I(\gamma, m, v) - I^{BK}(\gamma, -m, v)) + * \, c_0(\gamma, m, 0, v)) \tag{0.4.1}$$

Hence, Kudla's Green function and the Green function from Bruinier-Kühn sum up to create a modular form.

While the work at the elementary details of the paper was interrupted, a lot of other work on this and similar topics was done. As example, we only mention two items:

0.5. In 2018, an article by Ehlen and Sankaran [ES] appeared which also treated the two ways to define Green functions, there with the notation $Gr_0^K(m,v)$ and $Gr_0^B(m)$. In [ES] Theorem 3.3 they recognize $Gr_0^K(m,v)$ as a regularized theta lift and, among others, they prove the Theorem 3.6 identifying the differences of these Green functions as Fourier coefficients of a modular form.

0.6. In the paper [GS], Garcia and Sankaran treat a very general situation which does not include our special example. There they get as their result a very similar looking relation. **Theorem 1.2.** Suppose that $\mathbb{V}$ is anisotropic and, in the unitary case, that $q = 1$. Then for any T , there is an explicit constant $\kappa(T, \Phi_f)$, given by Definition 5.7, such that

$$\frac{(-1^r\kappa_0}{2\mathrm{Vol}(X_{\mathbb{V},K}, \Omega_{\mathcal{E}})} \int_{[\mathcal{X}_K(\mathbb{C})]} \mathfrak{g}(T, \mathbf{y}, \varphi_f) \wedge \Omega_{\mathcal{E}}^{p+1-r} q^T = E_T'(\boldsymbol{\tau}, \Phi_f, s_0) - \kappa(T, \Phi_f) q^T.$$

Here $q^T = e^{2\pi i \mathrm{tr}(T\tau)}$, and $\kappa_0 = 1$ if $s_0 > 0$ and $\kappa_0 = 2$ if $s_0 = 0$.

In the special case that T is non-degenerate, one has a factorization

$$E_T(\boldsymbol{\tau}, \Phi_f, s) = W_{T,\infty}(\boldsymbol{\tau}, \Phi_\infty^l, s) \cdot W_{T,f}(e, \Phi_f, s)$$

where the factors on the right are the products of the archimedean and nonarchimedean local Whittaker functionals, respectively. Let

$$E_T'(\boldsymbol{\tau}, \Phi_f, s_0)_\infty = W_{T,\infty}'(\boldsymbol{\tau}, \Phi_\infty^l, s_0) \cdot W_{T,f}(e, \Phi_f, s_0)$$

denote the archimedean contribution to the special derivative. Then, if T is totally positive definite, Theorem 1.2 specializes to the identity

$$\begin{aligned}\frac{(-1)^r\kappa_0}{2\mathrm{Vol}(X_{\mathbb{V},K}, \Omega_{\mathcal{E}})} \int_{[\mathcal{X}_K(\mathbb{C})]} \mathfrak{g}(T, \mathbf{y}, \varphi_f) \wedge \Omega_{\mathcal{E}}^{p+1-r} q^T &= E_T'(\boldsymbol{\tau}, \Phi_f, s_0)_\infty \\ &- E_T(\boldsymbol{\tau}, \Phi_f, s_0)((\iota d)/2(r\log\pi - \frac{\Gamma'(\iota m/2)}{\Gamma(\iota m/2)} + \frac{\iota}{2}\log N_{F/\mathbb{Q}} \det T),\end{aligned}$$

here $\iota = 1$(resp. $\iota = 2$) in the orthogonal (resp. unitary) case. If T is not totally positive definite, the last summand is zero.

Organization of the text

The whole time, we follow the principle to give a lot of details which a reader familiar with the subject easily can and will skip. Besides the main topic of the signature (3,2), often we also look into (2,2) and (1,2) cases.

The main tool in our proofs comes from the special *exceptional* homomorphisms between the groups

$$\begin{aligned} \mathrm{SL}(2,\mathbb{R}) &\to \mathrm{SO}(2,1) \\ \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) &\to \mathrm{SO}(2,2) \\ \mathrm{Sp}(2,\mathbb{R}) &\to \mathrm{SO}(3,2) \end{aligned} \tag{0.6.1}$$

As the symmetric space associated with our orthogonal group $G = \mathrm{SO}(p,2)$ may be identified with the set

$$\mathbb{D} = \{ \text{ oriented negative 2-planes } X \subset V = \mathbb{R}^n \}, \tag{0.6.2}$$

i.e. $X =< v_1, v_2 >$, $v_j \in V$, $q(v_j) < 0$, $(v_1, v_2) = 0$. It is well-known that, for $p = 3$, $\mathbb{D}$ has two connected components $\mathbb{D}^+$ and $\mathbb{D}^-$, and $\mathbb{D}^+$ is isomorphic to the Siegel half plane $\mathbb{H}_2$ of genus 2, for $p = 2$ one has $\mathbb{D}^+ \simeq \mathbb{H} \times \mathbb{H}$, and $\mathbb{D}^+ \simeq \mathbb{H}$ for $p = 1$. Hence, in the first section, we display this, in the second, we gather from [BK], what we need for the coefficients of the Eisenstein series for the case $p = 3$. In sections 3 and 4, we introduce Kudla's Green function $\Xi(\beta, m, v)$ belonging to a divisor, rep., to a lattice $L_m \subset V$, resp. an unit group $\Gamma = \Gamma(L) = \mathrm{SO}(L)$ which is a discrete subgroup preserving the lattice L. Section 5 contains notions from elementary number theory needed for the description of the unit groups and the volumes of their fundamental domains in the sections 6 to 11. Helped by material from Siegel, we use the *exceptional* homomorphisms to translate the well known SL_2- and Sp_2-results for the cases later needed in the sections 12, 13, and 14 to determine the integrals of Kudla's Green function over $X = \Gamma\backslash\mathbb{D}$ for the cases $p = 3, 2, 1$. In these sections, we follow Kudla's approaches and, concerning the treatment of certain measures, plunge into papers by Flensted-Jensen, Bruinier and Yang and others which at the end are further spread in the appendix-section 16. In epilogue-section 15, we give an overview and comparison of the obtained results.

1 Notation and Coordinates

1.1. We take a real quadratic space $(V, (.,.))$ with signature $(p,q), p + q = n$ and, for $x, y \in V$, write

$$q(x) = (1/2)(x,x), (x,y) = {}^t x Q y \quad (x,y) := q(x+y) - q(x) - q(y),\ Q \in \mathrm{Sym}_n(\mathbb{R}).$$

Following Siegel (e.g.[S3]), sometime, we also write $Q[x] = {}^t x Q x$. In particular, for

$$E_{pq} := \begin{pmatrix} E_p & \\ & -E_q \end{pmatrix} \tag{1.1.1}$$

and for $x, y \in \mathbb{R}^n$, we have

$$(1.1.2) \qquad 2q_0(x) := \sum_{\alpha=1}^{p} x_\alpha^2 - \sum_{\mu=p+1}^{n} x_\mu^2.$$

For the identity component of the corresponding special orthogonal group, we write

$$(1.1.3) \qquad G := \mathrm{SO}_0(p, q).$$

Here we are mainly interested in the case $p = 3, q = 2$ but we also look at some other low dimensional cases and start by some generalities for the cases with arbitrary (small) p and $q = 2$.
For a real symmetric $n \times n-$matrix Q, we write $\mathrm{SO}(Q) = \{A \in \mathrm{SL}_n(\mathbb{R}); {}^tAQA = Q\}$

Our Symmetric Space

1.2. Realization 1. As usual (e.g. [BF], [Ku1]), for $q = 2$, the symmetric space

$$(1.2.1) \qquad \mathbb{D} = \mathrm{SO}(p, q)/(\mathrm{SO}(p) \times \mathrm{SO}(2))$$

may be identified with the set of oriented negative 2-planes in V, i.e.,

$$(1.2.2) \qquad \mathbb{D} \simeq \{ \text{ oriented negative 2-planes } X \subset V = \mathbb{R}^n \},$$

with $X = \langle v_1, v_2 \rangle$, $v_j \in V$, $q(v_j) < 0$, $(v_1, v_2) = 0$. It is well-known that $\mathbb{D}$ has two connected components $\mathbb{D}^+$ and $\mathbb{D}^-$. For $p = 3$, $\mathbb{D}^+$ is isomorphic to the Siegel half plane $\mathbb{H}_2$ of genus 2, for $p = 2$ one has $\mathbb{D}^+ \simeq \mathbb{H} \times \mathbb{H}$, and for $p = 1$ $\mathbb{D}^+ \simeq \mathbb{H}$. There are several ways to realize these isomorphisms and to fix coordinates. These depend on the special situation where certain notation has become customary but unfortunatily with slight deviations in different papers.

1.3. Realization 2. $\mathbb{D}$ is isomorphic to the subset

$$(1.3.1) \qquad \mathbb{D}_Q = \{w \in V(\mathbb{C}); (w, w) = 0, (w, \bar{w}) < 0\}/\mathbb{C}^\times \subset \mathbb{P}(V(\mathbb{C})).$$

The isomorphism is given by

$$(1.3.2) \qquad X = \langle v_1, v_2 \rangle \mapsto w = v_1 + iv_2.$$

1.4. Realization 3. Moreover, there is the realization of $\mathbb{D}$ as a tube domain: Take a Witt decomposition

$$(1.4.1) \qquad V(\mathbb{R}) = a\mathbb{R} + V_0 + c\mathbb{R},$$

where $a, c \in V$ with $(a, a) = (c, c) = 0, (a, c) = 1$ span a hyperbolic plane with orthogonal complement V_0, and let

$$C = \{v \in V_0; (v, v) < 0\}$$

be the negative cone in V_0. Then $\mathbb{D} \simeq \mathbb{D}_Q$ is isomorphic to

$$\mathbb{D}_T = \{z \in V_0(\mathbb{C}); y = \text{ Im } z \in C\} \tag{1.4.2}$$

via the map

$$\mathbb{D}_T \to V(\mathbb{C}),\ z \mapsto w(z) := v = z + a - q(z)c \tag{1.4.3}$$

composed with the projection to $\mathbb{D}_Q$.

1.5. Example (1,2). We take

$$Q = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}, \quad \tilde{Q} = -\begin{pmatrix} 2 & & \\ & & 1 \\ & 1 & \end{pmatrix}. \tag{1.5.1}$$

and

$$\tilde{V} = \{M = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, a, b, c \in \mathbb{R}\} \simeq \mathbb{R}^3 \tag{1.5.2}$$

with $a = (1/\sqrt{2})x_3, b = (1/\sqrt{2})(x_2 - x_1), c = (1/\sqrt{2})(x_2 + x_1), \mathbf{a} := {}^t(a, b, c)$ and, hence,

$$\det M = -a^2 - bc = (1/2)^t\mathbf{a}\,\tilde{Q}\,\mathbf{a} = (1/2)^t xQx = (1/2)(x_1^2 - x_2^2 - x_3^2) \tag{1.5.3}$$

$\mathrm{SL}(2, \mathbb{R})$ acts on $\tilde{V}$ via $M \mapsto g \cdot M = gMg^{-1} =: M'$ and one has a map

$$\rho : \mathrm{SL}(2, \mathbb{R}) \to \tilde{G} := \mathrm{SO}(\tilde{Q}), \quad g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \rho(g) = \begin{pmatrix} \alpha\delta + \beta\gamma & -\alpha\gamma & \beta\delta \\ -2\alpha\beta & \alpha^2 & -\beta 2 \\ 2\gamma\delta & -\gamma^2 & \delta^2 \end{pmatrix} \tag{1.5.4}$$

where $\rho(g)$ is defined by: $\mathbf{a}'$ belonging to M' is given by $\mathbf{a}' = \rho(g)\mathbf{a}$. For

$$g_z := \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix} \tag{1.5.5}$$

we get

$$\rho(g_z) := \begin{pmatrix} 1 & 0 & x/y \\ -2x & y & -x^2/y \\ 0 & 0 & 1/y \end{pmatrix}, \quad (\rho(g_z))^{-1} := \begin{pmatrix} 1 & 0 & -x \\ 2x/y & 1/y & -x^2/y \\ 0 & 0 & y \end{pmatrix}. \tag{1.5.6}$$

From (1.5.3) we take

$$\mathbf{a} = Cx, \quad C := (1/\sqrt{2})\begin{pmatrix} 1 & & \\ & 1 & 1 \\ & 1 & -1 \end{pmatrix}, \quad C^{-1} := (1/\sqrt{2})\begin{pmatrix} 2 & & \\ & 1 & 1 \\ & 1 & -1 \end{pmatrix}, \tag{1.5.7}$$

and get

$$\nu : \mathrm{SO}(\tilde{Q}) \to \mathrm{SO}(Q), \quad A \mapsto C^{-1}AC, \tag{1.5.8}$$

i.e.,

(1.5.9)

$$\nu \cdot \rho : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SO}(Q),$$
$$g \mapsto \nu(\rho(g)) = (1/2)\begin{pmatrix} 2(\alpha\delta+\beta\gamma) & 2(-\alpha\gamma+\beta\delta) & 2(-\alpha\gamma-\beta\delta) \\ 2(-\alpha\beta+\gamma\delta) & \alpha^2-\beta^2-\gamma^2+\delta^2 & \alpha^2+\beta^2-\gamma^2-\delta^2 \\ 2(-\alpha\beta-\gamma\delta) & \alpha^2-\beta^2+\gamma^2-\delta^2 & \alpha^2+\beta^2+\gamma^2+\delta^2 \end{pmatrix}$$

and

$$\nu(\rho(g_z)) = (1/(2y))\begin{pmatrix} 2y & 2x & -2x \\ -2x^2 & 1-x^2+y^2 & -1+x^2+y^2 \\ -2x^2 & -1-x^2+y^2 & 1+x^2+y^2 \end{pmatrix}. \tag{1.5.10}$$

For the Realization 1, as a base point for $\mathbb{D}^+$, we take the plane

$$X_i := \langle M_1, M_2\rangle, \quad M_1 = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}, M_2 = \begin{pmatrix} & -1 \\ -1 & \end{pmatrix}. \tag{1.5.11}$$

Application of $A(g_z)$ transforms to

$$M_1' = (1/y)\begin{pmatrix} -y & 2xy \\ & y \end{pmatrix}, M_2' = (1/y)\begin{pmatrix} -x & x^2-y^2 \\ -1 & x \end{pmatrix}, \tag{1.5.12}$$

hence, g_z transforms X_i to

$$X_z = \langle \mathrm{Im}\ Z, \mathrm{Re}\ Z\rangle, \quad \bar{Z} = \begin{pmatrix} -z & z^2 \\ -1 & z \end{pmatrix}, \tag{1.5.13}$$

and we have a (bijective) map

$$\iota : \mathbb{H} \to \mathbb{D}^+, \quad z = x+iy \mapsto X_z. \tag{1.5.14}$$

The Realization 3 comes out as follows. We have

$$V(\mathbb{R}) = \sum_{j=1}^{3} e_j\mathbb{R} = \sum_{j=1}^{3} e_j'\mathbb{R}, e_1' = (e_1+e_3)/\sqrt{2}, e_3' = (e_1-e_3)/\sqrt{2}, e_2' = e_2 \tag{1.5.15}$$

with

$$(e_1')^2 = 0, (e_3')^2 = 0, (e_1', e_3') = 1$$

and

$$C = \{v \in V_0 = e_2'\mathbb{R}; (v,v) < 0\}, \text{ i.e., } v = ae_2', a > 0.$$

Hence

$$\mathbb{D}_T^+ = \{z \in V_0(\mathbb{C}), y = \text{ Im } z \in C\} \simeq \{z \in \mathbb{C}; \text{ Im } z > 0\} = \mathbb{H}. \tag{1.5.16}$$

1.6. Example (1,2)bis. Though the action treated above seems to be customary, as well we can take

$$\tilde{Q} = \begin{pmatrix} & & 1/2 \\ & -1 & \\ 1/2 & & \end{pmatrix}, \tag{1.6.1}$$

and

$$\tilde{V} = \{M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, a, b, c \in \mathbb{R}\} \simeq \mathbb{R}^3 \tag{1.6.2}$$

with $\mathbf{a} := {}^t(a,b,c)$ and, hence,

$$(\mathbf{a}, \mathbf{a}) = \det M = ac - b^2 = {}^t\mathbf{a}\,\tilde{Q}\,\mathbf{a} \tag{1.6.3}$$

$\mathrm{SL}(2,\mathbb{R})$ acts on $\tilde{V}$ via $M \mapsto g \cdot M = gM{}^tg =: M'$ and one has a map

$$\rho : \mathrm{SL}(2,\mathbb{R}) \to \tilde{G} := \mathrm{SO}(\tilde{Q}), \quad g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \rho(g) = \begin{pmatrix} \alpha^2 & 2\alpha\beta & \beta^2 \\ \alpha\gamma & \alpha\delta + \beta\gamma & \beta\delta \\ \gamma^2 & 2\gamma\delta & \delta^2 \end{pmatrix} \tag{1.6.4}$$

where $\rho(g)$ is defined by: $\mathbf{a}'$ belonging to M' is given by $\mathbf{a}' = \rho(g)\mathbf{a}$. For

$$g_z := \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix} \tag{1.6.5}$$

we get

$$\rho(g_z) := \begin{pmatrix} v & 2x & x^2/y \\ & 1 & x/y \\ 0 & 0 & 1/y \end{pmatrix}, \quad (\rho(g_z))^{-1} := \begin{pmatrix} 1/y & -2x/y & x^2/y \\ & 1 & -x \\ 0 & 0 & y \end{pmatrix}. \tag{1.6.6}$$

1.7. Example (2,2). We take

$$Q = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad \tilde{Q} = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}. \tag{1.7.1}$$

and

(1.7.2) $$\tilde V = \{M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a,b,c,d \in \mathbb{R}\} \simeq \mathbb{R}^4$$

with

(1.7.3) $$M \equiv \mathbf{a} := {}^t(a,b,c,d) = (1/\sqrt{2})^t(x_1+x_4, -x_2-x_3, x_2-x_3, x_1-x_4)$$

and hence

(1.7.4)
$$\tilde q(\mathbf{a}) = \det M = ad - bc = (1/2)^t\mathbf{a}\,\tilde Q\,\mathbf{a} = (1/2)^t xQx = (1/2)(x_1^2+x_2^2-x_3^2-x_4^2) = q(x)$$

Action 1. $\bar G := \mathrm{SL}(2,\mathbb{R})^2$ acts on $\tilde V$ via $M \mapsto g\cdot M = M^g := g_1 M g_2^{-1} =: M'$ and one has a homomorphism of $\bar G$ to $\mathrm{O}_0(\tilde Q)$ given by $g = (g_1, g_2) \mapsto A'(g)$ with

$$g_1 = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix}, g_2 = \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix}$$

and

(1.7.5) $$g := (g_1, g_2) \longmapsto A'(g) = \begin{pmatrix} \alpha_1\delta_2 & -\alpha_1\gamma_2 & \beta_1\delta_2 & -\beta_1\gamma_2 \\ -\alpha_1\beta_2 & \alpha_1\alpha_2 & -\beta_1\beta_2 & \beta_1\alpha_2 \\ \gamma_1\delta_2 & -\gamma_1\gamma_2 & \delta_1\delta_2 & -\delta_1\gamma_2 \\ -\gamma_1\beta_2 & \gamma_1\alpha_2 & -\delta_1\beta_2 & \delta_1\alpha_2 \end{pmatrix}.$$

Now, as above, fixing as base point of $\mathbb{D}$ the negative 2-plane spanned by $M_1 := \left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right)$ and $M_2 := \left(\begin{smallmatrix} & -1 \\ -1 & \end{smallmatrix}\right)$ we get

$$g_{z_1} M_1 g_{z_1}^{-1} = (y_1y_2)^{-1/2} \begin{pmatrix} y_1 & -x_1y_2 - x_2y_1 \\ 0 & -y_2 \end{pmatrix} = -(y_1y_2)^{-1/2}\,\mathrm{Re}\,\tilde Z$$

$$g_{z_1} M_2 g_{z_2}^{-1} = (y_1y_2)^{-1/2} \begin{pmatrix} -x_1 & x_1x_2 - y_1y_2 - x_2 \\ -1 & x_2 \end{pmatrix} = -(y_1y_2)^{-1/2}\,\mathrm{Im}\,\tilde Z$$

with

$$\tilde Z = \begin{pmatrix} -\bar z_1 & \bar z_1 \bar z_2 \\ -1 & \bar z_2 \end{pmatrix}.$$

This explains a Realization 1 isomorphism

(1.7.6) $$\begin{aligned} &\mathbb{H}^2 \to \mathbb{D}^+ \\ &z = (z_1, z_2) \mapsto X_z := \left\langle g_{z_1}\left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right) g_{z_2}^{-1}, g_{z_1}\left(\begin{smallmatrix} & -1 \\ -1 & \end{smallmatrix}\right) g_{z_2}^{-1}\right\rangle = <\mathrm{Re}\,\tilde Z, \mathrm{Im}\,\tilde Z> . \end{aligned}$$

We observe the relations

$$\begin{aligned} -y_1y_2 &= (\mathrm{Re}\,\tilde Z, \mathrm{Re}\,\tilde Z) = (\mathrm{Im}\,\tilde Z, \mathrm{Im}\,\tilde Z) \\ 0 &= (\mathrm{Re}\,\tilde Z, \mathrm{Im}\,\tilde Z) \end{aligned}$$

Action 2. There is also an alternative action of $\bar G$ on $\tilde V$ which is pursued in [BeKI]. $\bar G = \mathrm{SL}(2,\mathbb{R})^2$ acts on $\tilde V$ via $M \mapsto g\cdot M = M^g := g_1 M\, {}^t g_2 =: M'$, resp. $\mathbf{a}\mapsto \mathbf{a}' = A(g)\mathbf{a}$, and one has a map

$$(1.7.7)\qquad \mathrm{SL}(2,\mathbb{R})^2 \to \tilde G := \mathrm{SO}_0(\tilde Q),\quad g\mapsto A'(g) = \begin{pmatrix} \alpha_1\alpha_2 & \alpha_1\beta_2 & \beta_1\alpha_2 & \beta_1\beta_2\\ \alpha_1\gamma_2 & \alpha_1\delta_2 & \beta_1\gamma_2 & \beta_1\delta_2\\ \gamma_1\alpha_2 & \gamma_1\beta_2 & \delta_1\alpha_2 & \delta_1\beta_2\\ \gamma_1\gamma_2 & \beta_1\delta_2 & \delta_1\gamma_2 & \delta_1\delta_2\end{pmatrix}.$$

In particular we get

$$(1.7.8)\qquad A(z) := A(g_{z_1}, g_{z_2}) = \begin{pmatrix} \sqrt{y_1y_2} & \sqrt{y_1/y_2}\,x_2 & \sqrt{y_2/y_1}\,x_1 & \sqrt{y_1y_2}^{-1}x_1x_2\\ 0 & \sqrt{y_1/y_2} & 0 & \sqrt{y_1y_2}^{-1}x_1\\ 0 & 0 & \sqrt{y_2/y_1} & \sqrt{y_1y_2}^{-1}x_2\\ 0 & 0 & 0 & \sqrt{y_1y_2}^{-1}\end{pmatrix}.$$

And, fixing again as base point of $\mathbb{D}$ the negative 2-plane spanned by $M_1 := \left(\begin{smallmatrix}1 & \\ & -1\end{smallmatrix}\right)$ and $M_2 = \left(\begin{smallmatrix} & -1\\ -1 & \end{smallmatrix}\right)$ we get

$$g_{z_1} M_1 {}^t g_{z_2} = (y_1y_2)^{-1/2}\begin{pmatrix} y_1y_2 - x_1x_2 & -x_1\\ -x_2 & -1\end{pmatrix} = -(y_1y_2)^{-1/2}\,\mathrm{Re}\, Z$$

$$g_{z_1} M_2 {}^t g_{z_2} = (y_1y_2)^{-1/2}\begin{pmatrix} -x_1y_2 - x_2y_1 & -y_1\\ -y_2 & 0\end{pmatrix} = -(y_1y_2)^{-1/2}\,\mathrm{Im}\, Z$$

where Z is given by $Z = \left(\begin{smallmatrix} z_1z_2 & z_1\\ z_2 & 1\end{smallmatrix}\right)$. This explains the formula for the isomorphism

$$(1.7.9)\qquad \begin{aligned}&\mathbb{H}^2 \longrightarrow \mathbb{D}^+\\ &z = (z_1,z_2)\mapsto X_z := \left\langle g_{z_1}\left(\begin{smallmatrix}1 & \\ & -1\end{smallmatrix}\right){}^t g_{z_2},\, g_{z_1}\left(\begin{smallmatrix} & -1\\ -1 & \end{smallmatrix}\right){}^t g_{z_2}\right\rangle = \langle\,\mathrm{Re}\, Z,\, \mathrm{Im}\, Z\rangle.\end{aligned}$$

The coordinates in this description are related to those in the first one by

$$(1.7.10)\qquad z_1\mapsto \bar z_1,\ \bar z_2\mapsto -1/\bar z_2.$$

The Realization 3 comes out as follows. We have

$$(1.7.11)\quad V(\mathbb{R}) = \sum_{j=1}^4 e_j\mathbb{R} = \sum_{j=1}^4 e'_j\mathbb{R},$$
$$e'_1 = (e_1+e_4)/\sqrt2, e'_2 = (e_2+e_3)/\sqrt2, e'_3 = (e_2-e_3)/\sqrt2, e'_4 = (e_1-e_4)/\sqrt2,$$

with

$$(e'_1)^2 = (e'_4)^2 = 0, (e'_1, e'_4) = 1$$

and

$$C = \{v\in V_0 = e'_2\mathbb{R} + e'_3\mathbb{R}; (v,v) < 0\},\text{ i.e., } v = ae'_2 + be'_3, (v,v) = ab < 0.$$

Hence

$$\mathbb{D}_T = \{z \in V_0(\mathbb{C}), y \in \text{ Im } z \in C\} \simeq \{z = (z_1, z_2) \in \mathbb{C}^2; y_1 y_2 > 0\} = \mathbb{H}^2 \cup \overline{\mathbb{H}}^2. \tag{1.7.12}$$

1.8. Example (3,2). We put

$$Q_0 = \begin{pmatrix} 1 &&&& \\ & 1 &&& \\ && 1 && \\ &&& -1 & \\ &&&& -1 \end{pmatrix}, \quad \tilde{Q} = \begin{pmatrix} &&&& -1 \\ &&& -1 & \\ && 2 && \\ & -1 &&& \\ -1 &&&& \end{pmatrix}. \tag{1.8.1}$$

and with

$$u = {}^t(u_1, u_2, u_3, u_4, u_5) = (1/\sqrt{2})^t(x_1 + x_5, x_2 + x_4, x_3, x_4 - x_1, x_5 - x_1) \tag{1.8.2}$$

we have

(1.8.3)

$$2\tilde{q}(u) = 2(u_3^2 - u_2 u_4 - u_1 u_5) = {}^t u \tilde{Q} u = {}^t x Q x = (x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5^2) = 2q_0(x).$$

Hence, we have $u = Cx$ and $x = C^{-1}u$ with

$$C = (1/\sqrt{2}) \begin{pmatrix} 1 &&&& 1 \\ & 1 && 1 & \\ && 1 && \\ & -1 && 1 & \\ -1 &&&& 1 \end{pmatrix}, \quad C^{-1} = (1/\sqrt{2}) \begin{pmatrix} 1 &&&& -1 \\ & 1 && -1 & \\ && 2 && \\ & 1 && 1 & \\ 1 &&&& 1 \end{pmatrix}, \tag{1.8.4}$$

and $Q_0 = {}^t C \tilde{Q} C$, such that

$$\tilde{G} = \text{SO}(\tilde{Q}) = \{\tilde{A}; {}^t\tilde{A}\tilde{Q}\tilde{A} = \tilde{Q}\} = CG_0C^{-1}, \quad G_0 = \text{SO}(Q_0). \tag{1.8.5}$$

Here (clearly going back to Siegel and as in [GN]), we realize $V = \mathbb{R}^5$ as the space $\mathcal{V}$ of skew-symmetric matrices

$$M = M(u) = \begin{pmatrix} u_1 J & XJ \\ JX & -u_5 J \end{pmatrix} \in \text{M}_4(\mathbb{R}) \tag{1.8.6}$$

with

$$X = \begin{pmatrix} u_2 & u_3 \\ u_3 & u_4 \end{pmatrix}, J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \text{M}_2(\mathbb{R}) \tag{1.8.7}$$

The quadratic form $\tilde{q}(u) = (1/2)^t u \tilde{Q} u = u_3^2 - u_2 u_4 - u_1 u_5$ comes in as one has

$${}^t M(u) \begin{pmatrix} & E \\ -E & \end{pmatrix} M(u) = \tilde{q}(u) \begin{pmatrix} & E \\ -E & \end{pmatrix}, E = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \tag{1.8.8}$$

and

$$\det M(u) = (u_3^2 - u_2u_4 - u_1u_5)^2. \tag{1.8.9}$$

The symplectic group $\check{G} = \mathrm{Sp}(2,\mathbb{R})$ acts (transitively) on $\mathcal{V}$ via

$$(g, M(u)) \longmapsto gM(u)^t g =: M(A(g)u) =: M(u') \tag{1.8.10}$$

preserving the quadratic form $\tilde{q}$. As usual, this leads to a homomorphism $\check{G} \longrightarrow \tilde{G}$ where $g \in \check{G}$ is mapped to the matrix $A(g)$ with $u' = A(g)u$. Some calculation shows that one has

$$\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \longmapsto \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{1.8.11}$$

for $B = \begin{pmatrix} b_1 & b_2 \\ b_2 & b_3 \end{pmatrix}$

$$\begin{pmatrix} E & B \\ 0 & E \end{pmatrix} \longmapsto \begin{pmatrix} 1 & b_3 & -2b_2 & b_1 & b_2^2 - b_1b_3 \\ 0 & 1 & 0 & 0 & -b_1 \\ 0 & 0 & 1 & 0 & -b_2 \\ 0 & 0 & 0 & 1 & -b_3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} E & \\ B & E \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ b_3 & 1 & 0 & 0 & 0 \\ -b_2 & 0 & 1 & 0 & 0 \\ b_1 & 0 & 0 & 1 & 0 \\ -(b_1b_3 - b_2^2) & -b_1 & -2b_2 & -b_4 & 1 \end{pmatrix}, \tag{1.8.12}$$

and for $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\eta = \det U$

$$\begin{pmatrix} U & 0 \\ 0 & {}^tU^{-1} \end{pmatrix} \longmapsto (1/\eta) \begin{pmatrix} \eta^2 & & & & \\ & a^2 & 2ab & b^2 & \\ & ac & ad+bc & bd & \\ & c^2 & 2cd & d^2 & \\ & & & & 1 \end{pmatrix}. \tag{1.8.13}$$

We choose

$$g_z = \begin{pmatrix} E & B \\ & E \end{pmatrix} \begin{pmatrix} U & \\ & {}^tU^{-1} \end{pmatrix}, \tag{1.8.14}$$

such that $g_z < iE >= z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \in \mathbb{H}_2$, i.e.,

$$B = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}, \quad U = \begin{pmatrix} a & b \\ & d \end{pmatrix},$$

(1.8.15) $$d = \sqrt{y_3}, b = y_2/\sqrt{y_3}, a = \eta/\sqrt{y_3}, \eta = ad = \sqrt{y_1 y_3 - y_2^2}, \zeta^2 = x_1 x_3 - x_2^2.$$

Hence, one has

(1.8.16)
$$A(g_z) = \begin{pmatrix} \eta & \eta x_3/y_3 & 2(x_3 y_2 - x_2 y_3)/y_3 & (x_3 y_2^2 - 2x_2 y_2 y_3 + x_1 y_3^2)/(\eta y_3) & -\zeta^2/\eta \\ & \eta/y_3 & 2y_2/y_3 & y_2^2/(\eta y_3) & -x_1/\eta \\ & & 1 & y_2/\eta & -x_2/\eta \\ & & & y_3/\eta & -x_3/\eta \\ & & & & 1/\eta \end{pmatrix}$$

and

(1.8.17) $$A(g_z)^{-1} = \begin{pmatrix} 1/\eta & -x_3/\eta & 2x_2/\eta & -x_1/\eta & -\zeta^2/\eta \\ & y_3/\eta & -2y_2/\eta & y_2^2/(y_3\eta) & (x_3 y_2^2 - 2x_2 y_2 y_3 + x_1 y_3^2)/(\eta y_3) \\ & & 1 & -y_2/y_3 & (x_2 y_3 - y_2 x_3)/y_3 \\ & & & \eta/y_3 & \eta x_3/y_3 \\ & & & & \eta \end{pmatrix}.$$

1.9. Remark: We take as a base point of $\mathbb{D}$ the plane

(1.9.1) $$X_{iE_2} = < u^{(1)}, u^{(2)} >, \ u^{(1)} = {}^t(1,0,0,0,1), u^{(2)} = {}^t(0,-1,0,-1,0)$$

and then we get

(1.9.2) $$\begin{aligned} A(g_z)u^{(1)} &= (1/\eta)^t(\eta^2 - \zeta^2, -x_1, -x_2, -x_3, 1), \\ A(g_z)u^{(2)} &= (1/\eta)^t(2x_2 y_y - x_1 y_3 - y_1 x_3, -y_1, -y_2, -y_3, 0), \end{aligned}$$

and with

(1.9.3) $$u_1(z) = z_2^2 - z_1 z_3, u_2(z) = -z_1, u_3(z) = -z_2, u_4(z) = -z_3, u_5(z) = 1$$

(1.9.4) $$X_z = \langle \operatorname{Re} u(z), \operatorname{Im} u(z) \rangle = A(g_z) X_{iE_2}.$$

We observe that one has

(1.9.5) $$(\operatorname{Re} u(z), \operatorname{Re} u(z)) = (\operatorname{Im} u(z), \operatorname{Im} u(z)) = -\eta^2 < 0.$$

In this case, the Realization 3 comes out as follows. We have

(1.9.6)

$$V(\mathbb{R}) = \sum_{j=1}^{5} e_j\mathbb{R} = \sum_{j=1}^{5} e'_j\mathbb{R},$$

$$e'_1 = (e_1+e_5)/\sqrt{2}, e'_2 = -(e_2+e_4)/\sqrt{2}, e'_3 = e_3\sqrt{2}, e'_4 = (e_2-e_4)/\sqrt{2}, e'_5 = (e_1-e_5)/\sqrt{2},$$

with

$$(e'_1)^2 = (e'_5)^2 = 0, (e'_1, e'_5) = 1$$

and

$$C = \{v \in V_0 = e'_2\mathbb{R} + e'_3\mathbb{R} + e'_4\mathbb{R}; (v,v) < 0\}$$

i.e.,

$$v = ae'_2 + be'_3 + ce'_4, (v,v) = -2ac + 2b^2 < 0.$$

Hence, a connected component of $\mathbb{D}_T = \{z \in V_0(\mathbb{C}), y \in \text{ Im } z \in C\}$ can be identified with the Siegel upper half plane $\{z = (z_1, z_2, z_3) \in \mathbb{C}^3; y_1y_3 - y_2^2 > 0\} = \mathbb{H}_2$.

1.10. For the special cases with p or $q = 1$, one can take the Grassmanian of the positive lines, i.e., we have as **Realization 4.**

(1.10.1) $$\mathbb{D}_1 = \{\langle v\rangle; v \in V, q(v) > 0\}.$$

1.11. Example (1,1). We take

(1.11.1) $$Q = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad \tilde{Q} = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}.$$

and

(1.11.2) $$V = \{x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, x_1, x_2 \in \mathbb{R}\} \simeq \mathbb{R}^2$$

Here we have

(1.11.3) $$G = \text{SO}(1,1) \ni g(t) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix},$$

acting on $V = \{x = {}^t(x_1, x_2)\}$ with $q(x) = x_1^2 - x_2^2$ and a map

(1.11.4) $$A : \tilde{G} = \mathbb{R} \to G = \text{SO}(1,1), \ t \mapsto g(t).$$

If we take $x^0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as base point and put $x(t) := g(t)x^0$, we get the coordinization of $\mathbb{D}_1$

$$\mathbb{R} \ni t \mapsto \langle x(t) \rangle \in \mathbb{D}_1. \tag{1.11.5}$$

There is a slightly different approach: For $\tilde{Q} = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ one has

$$G = \mathrm{SO}(1,1) \simeq \tilde{G} = \mathrm{SO}(\tilde{Q}) = \{g(\alpha) = \begin{pmatrix} \alpha & \\ & (1/\alpha) \end{pmatrix}, \alpha \neq 0\}$$

and

$$q(x) = x_1^2 - x_2^2 = 2y_1y_2 = {}^t y\tilde{Q}y = \tilde{q}(y) \tag{1.11.6}$$

i.e., $y_1 = (1/\sqrt{2})(x_1 + x_2)$, $y_2 = (1/\sqrt{2})(x_1 - x_2)$. As ${}^t y' := {}^t(g(\alpha)^{-1}y) = {}^t((1/\alpha)y_1, \alpha y_2)$, we have another coordinization

$$\mathbb{R}^* \ni \alpha \mapsto \langle g(\alpha)^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle = \langle \begin{pmatrix} \alpha^{-1} \\ \alpha \end{pmatrix} \rangle \in \mathbb{D}_1. \tag{1.11.7}$$

1.12. Example (1,2). The (1,2)-case discussed above using 2-planes in V as well can be treated using positive lines. Already above in (1.5.1) we fixed

$$Q = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}, \quad \tilde{Q} = -\begin{pmatrix} 2 & & \\ & & 1 \\ & 1 & \end{pmatrix}.$$

and

$$\tilde{V} = \{M = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad a, b, c \in \mathbb{R}\} \simeq \mathbb{R}^3$$

with $a = (1/\sqrt{2})x_3, b = (1/\sqrt{2})(x_2 - x_1), c = (1/\sqrt{2})(x_2 + x_1), \mathbf{a} := {}^t(a, b, c)$ and, hence,

$$\det M = -a^2 - bc = (1/2)^t\mathbf{a}\,\tilde{Q}\,\mathbf{a} = q(x) = (1/2)(x_1^2 - x_2^2 - x_3^2)$$

$\mathrm{SL}(2,\mathbb{R})$ acts on $\tilde{V}$ via $M \mapsto g \cdot M = gMg^{-1} =: M'$ and one has a map

$$\mathrm{SL}(2,\mathbb{R}) \to \tilde{G} := \mathrm{O}_0(\tilde{Q}), \quad g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto A(g) = \begin{pmatrix} \alpha\delta + \beta\gamma & -\alpha\gamma & \beta\delta \\ -2\alpha\beta & \alpha^2 & -\beta^2 \\ 2\gamma\delta & -\gamma^2 & \delta^2 \end{pmatrix}$$

where $A(g)$ is defined by: $\mathbf{a}'$ belonging to M' is given by $\mathbf{a}' = A(g)\mathbf{a}$. For

$$g_z := \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix} \tag{1.12.1}$$

we get

$$(1.12.2)\qquad A(g_z) := \begin{pmatrix} 1 & 0 & x/y \\ -2x & y & -x^2/y \\ 0 & 0 & 1/y \end{pmatrix}, \quad (A(g_z))^{-1} := \begin{pmatrix} 1 & 0 & -x \\ 2x/y & 1/y & -x^2/y \\ 0 & 0 & y \end{pmatrix}.$$

Again, one has

$$\mathbb{D}_1 = \{\langle v\rangle; v \in \tilde{V}, q(v) > 0\}.$$

If we take $M^0 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$, resp. $\mathbf{a}^0 := {}^t(0,1,-1)$ as base point, we get the coordinization

$$(1.12.3)\qquad \mathbb{H} \to \mathbb{D}_1, \quad z \mapsto \langle (1/y) \begin{pmatrix} -x \\ x^2+y^2 \\ -1 \end{pmatrix} \rangle,$$

resp. the line fixed by the matrix $X(z) := (1/y) \begin{pmatrix} -x & |z|^2 \\ -1 & x \end{pmatrix}$.

1.13. Example (1,2). Sometimes, it is useful to have **hyperbolic coordinates**: We take $\tilde{V}$ as above with $\tilde{q}(M) = -a^2 - bc$ and $\mathbb{D}_1 = \{\langle v\rangle; v \in \tilde{V}, q(v) > 0\}$ with

$$\mathbb{H} \to \mathbb{D}_1, \quad z = x + yi \mapsto \langle (1/y) \begin{pmatrix} x \\ x^2+y^2 \\ -1 \end{pmatrix} \rangle.$$

Now, we introduce

$$(1.13.1)\qquad \mathbb{D}_2 = \{\mathbf{z} = (z_1, z_2, z_3) \in \mathbb{R}^3, q(\mathbf{z}) = z_1^2 - z_2^2 - z_3^2 = 1, z_1 > 0\}.$$

and via

$$x/y = z_2, (x^2+y^2)/y = z_1 + z_3, -1/y = z_3 - z_1$$

have $\mathbb{H} \simeq \mathbb{D}_2$. Let $S^1 = \{(\alpha,\beta); \alpha^2+\beta^2 = 1\}$ and

$$(1.13.2)\qquad (0,\infty) \times S^1 \to \mathbb{D}_2, \quad (r, w = (\alpha,\beta)) \mapsto (\operatorname{ch} r, \operatorname{sh} r \cdot w), \quad \alpha = \cos\vartheta, \beta = \sin\vartheta.$$

Hence, one has

$$(1.13.3)\qquad x = \operatorname{sh} r \cdot \alpha/N =: s\alpha/N, y = 1/N, \quad N = \operatorname{ch} r - \operatorname{sh} r \cdot \beta =: c - s\beta.$$

And, using $c^2 - s^2 = 1$,

$$(1.13.4)\qquad dx = \frac{c\alpha dr - s\beta d\vartheta}{N} - \frac{s\alpha dN}{N^2}, dy = -\frac{dN}{N^2}, dN = (s - c\beta)dr - s\alpha d\vartheta,$$

$$\frac{dx \wedge dy}{y^2} = \operatorname{sh}\vartheta\, dr \wedge d\vartheta.$$

$$z = x + yi = (s\alpha + i)/N$$

$$1 + z^2 = 2(s^2 - cs\beta + s\alpha i)/N^2$$

$$\frac{|1+z^2|^2}{y^2} = 4s^2 = 4(\operatorname{sh}\vartheta)^2.$$

1.14. Example (1,3). In this case, we have

$$Q = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{1.14.1}$$

and

$$\tilde{Q} = \begin{pmatrix} & & & 1/2 \\ & -1 & & \\ & & -1 & \\ 1/2 & & & \end{pmatrix}. \tag{1.14.2}$$

with the forms

$${}^t x Q x = x_1^2 - x_2^2 - x_3^2 - x_4^2 = y_1 y_4 - y_2^2 - y_3^2 = {}^t y \tilde{Q} y. \tag{1.14.3}$$

The associated homogeneous space is the hyperbolic three-space. We report on some material from the first section of the book by Elstrodt, Grunewald and Mennicke [EGM].

1.15. There are several models for the 3-dimensional hyperbolic space:
1. The **upper half space model** $\mathbb{H}^+ = \mathbb{C} \times \mathbb{R}_{>0}$. Points are written as

$$P = (z, r) = (x, y, r) = z + rj \tag{1.15.1}$$

where $\mathbb{H}^+$ may be treated as subset of the Hamilton quaternions $\mathcal{H} = \mathbb{R} + i\mathbb{R} + j\mathbb{R} + k\mathbb{R}$ and has as it boundary $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$. The group $\mathrm{SL}(2, \mathbb{C})$ acts transitively on $\mathbb{H}^+$

(1.15.2)
$$P = (z, r) \mapsto M(P) = (z', r'),\ z' = \frac{(az + b)(\bar{c}\bar{z} + \bar{d}) + a\bar{c}r^2}{|cz + d|^2 + |c|^2 r^2},\ r' = \frac{r}{|cz + d|^2 + |c|^2 r^2}$$

and the stabilizer of j is $\mathrm{SU}(2)$. The element

$$g_P = \begin{pmatrix} 1/\sqrt{r} & -z/\sqrt{r} \\ 0 & \sqrt{r} \end{pmatrix} \tag{1.15.3}$$

maps P unto j. One has invariant line and volume elements

$$ds^2 = \frac{dx^2 + dy^2 + dr^2}{r^2},\ dv = \frac{dx \wedge dy \wedge dr}{r^3}. \tag{1.15.4}$$

2. The **unit ball model** $\mathbb{B} = \{u = u_0 + u_1 i + u_2 j \in \mathcal{H}; \|u\|^2 < 1\}$. One has an isometry $\eta_0 : \mathbb{H}^+ \to \mathbb{B}$ given by

$$u_0 = \frac{2x}{x^2 + x^2 + (r+1)^2}, u_1 = \frac{2y}{x^2 + x^2 + (r+1)^2}, u_2 = \frac{x^2 + y^2 + r^2 - 1}{x^2 + x^2 + (r+1)^2}. \tag{1.15.5}$$

3. The **hyperboloid model** $\mathbb{S} = \{y = y_0 f_0 + y_1 f_1 + y_2 f_2 + y_3 f_3 \in E_1; y_0 > 0, q_1(y) = 1\}$. Here E_1 is a 4-dimensional $\mathbb{R}-$vector space with basis $f_0, \dots, f_3$ and quadratic form

$$q_1(y) = y_0^2 - y_1^2 - y_2^2 - y_3^2. \tag{1.15.6}$$

One has an isometry $\pi_0 : \mathbb{H}^+ \to \mathbb{S}$ given by

$$\pi_0(P) = \frac{1}{2r}((1 + P\bar{P})f_0 + (1 - P\bar{P})f_1 - 2xf_2 - 2yf_3). \tag{1.15.7}$$

4. The **Kleinian model** $\mathbb{K} = \{[y] \in \mathbb{P}(E_1); q_1(y) > 0\}$. One has an isometry $\psi_0 : \mathbb{H}^+ \to \mathbb{K}$ given by

$$\psi_0(P) = [(1 + P\bar{P})f_0 + (1 - P\bar{P})f_1 - 2xf_2 - 2yf_3]. \tag{1.15.8}$$

For later use, we here change the basis $f_0 =: e_1 + e_4, f_1 =: e_1 - e_4, f_2 =: e_2, f_3 =: e_3$ and get the coordinization of the space of positive lines given by

$$\psi_1(P) = [e_1 - xe_2 - ye_3 + P\bar{P}e_4]. \tag{1.15.9}$$

where, now, the form is $q(y) = y_1y_4 - y_2^2 - y_3^2$.

1.16. Remark. Proposition 1.4.2 in [EGM] states that π_0 is equivariant with respect to the homomorphism $\Psi : \mathrm{SL}(2, \mathbb{C}) \to \mathrm{SO}_4^+(\mathbb{R}, q_1)$ which is described in [EGM] Section 1.3. We try a slightly different approach and take

$$\mathcal{V} = \{X = X(y) = \begin{pmatrix} y_1 & w \\ \bar{w} & y_4 \end{pmatrix}; y_1, y_4 \in \mathbb{R}; w = y_2 + iy_3 \in \mathbb{C})\}. \tag{1.16.1}$$

$\mathrm{SL}(2, \mathbb{C})$ acts on $\mathcal{V}$ and for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C})$ we have

$$X(y) \mapsto AX(y)^t\bar{A} = X(y'), \text{ i.e., } y' = \rho(A)y \tag{1.16.2}$$

with $\rho(A) \in \mathrm{SO}(\tilde{Q})$ for ${}^t y\tilde{Q}y = y_1y_4 - y_2^2 - y_3^2$ and

(1.16.3)
$$\rho(A) = \begin{pmatrix} |a|^2 & a\bar{b} + \bar{a}b & (a\bar{b} - \bar{a}b)i & |b|^2 \\ (a\bar{c} + \bar{a}c)/2 & (a\bar{d} + \bar{a}d + b\bar{c} + \bar{c}b)/2 & (a\bar{d} - \bar{a}d + c\bar{b} - \bar{c}b)i/2 & (b\bar{d} + \bar{b}d)/2 \\ (a\bar{c} - \bar{a}c)/(2i) & (a\bar{d} - \bar{a}d + b\bar{c} - \bar{c}b)/(2i) & (a\bar{d} + \bar{a}d - c\bar{b} - \bar{c}b)/2 & (b\bar{d} - \bar{b}d)/(2i) \\ |c|^2 & c\bar{d} + \bar{d}c & (c\bar{d} - \bar{c}d)i & |d|^2 \end{pmatrix}.$$

1.17. Remark. For square free $m < 0$, this map ρ induces a homomorphism

$$\rho_m : \mathrm{SL}(2, \mathbb{Z}[\sqrt{m}]) \to \mathrm{SO}(\mathbb{Z}, q_m), q_m(u) = u_1u_4 - u_2^2 + mu_3^2 \tag{1.17.1}$$

(in $\rho(A)$ replace i by $j := \sqrt{m}$).

From (1.15.3) we have $g_P = \begin{pmatrix} 1/\sqrt{r} & -z/\sqrt{r} \\ 0 & \sqrt{r} \end{pmatrix}$, and hence

$$\rho(A_P^{-1}) = \rho(g_P^{-1}) = \begin{pmatrix} 1/r & -2x/r & -2y/r & |z|^2/r \\ & 1 & 0 & -x \\ & & 1 & -y \\ & & & r \end{pmatrix}. \tag{1.17.2}$$

and

$$\rho(A_P^{-1})y = \begin{pmatrix} y_1/r - 2xy_2/r - 2yy_3/r + |z|^2 y_4/r \\ y_2 - xy_4 \\ y_3 - yy_4 \\ ry_4 \end{pmatrix}. \tag{1.17.3}$$

In particular, for $g = g_P$, we have

$$\rho(g_P) = \begin{pmatrix} r & 2x & 2y & |z|^2/r \\ & 1 & & x/r \\ & & 1 & y/r \\ & & & 1/r \end{pmatrix} =: \bar{A}_P. \tag{1.17.4}$$

1.18. The Cayley-Klein and the Poincaré slice model. There is another way to treat the Minkowski-case, i.e., signature with only one negative term. As example, we treat the case (1,2), that is $q(\mathbf{x}) = x_1^2 - x_2^2 - x_3^2 = {}^t x E_{1,2} x$. We can take as homogeneous space $\mathbb{D}$ the space of positive lines in $V(\mathbb{R}) = \sum_{i=1}^3 x_i \mathbb{R}$ given by

$$\mathbb{D} = \{{}^t(x_1, x_2, x_3) \in \mathbb{R}^3; x_1^2 - x_2^2 - x_3^2 = 1\}. \tag{1.18.1}$$

i) We take the parametrization

$$\gamma : U = \{(u,v) \in \mathbb{R}; u^2 + v^2 < 1\} \to \mathbb{D}, \quad (u,v) \mapsto \frac{1}{\sqrt{1 - u^2 - v^2}} \begin{pmatrix} 1 \\ u \\ v \end{pmatrix}, \tag{1.18.2}$$

with $(x,y) = {}^t x E_{1,2} y$, have the metric tensor

$$(g_{i,j}(u,v)) = \begin{pmatrix} (({}^t\gamma_u, \gamma_u) & ({}^t\gamma_u, \gamma_v) \\ ({}^t\gamma_v, \gamma_u) & ({}^t\gamma_v, \gamma_v) \end{pmatrix} = -(1 - u^2 - v^2)^{-2} \begin{pmatrix} 1 - v^2 & uv \\ uv & 1 - u^2 \end{pmatrix} \tag{1.18.3}$$

and the volume form

$$dv_{CK} = \sqrt{\det(g_{i,j})} dudv = (1 - u^2 - v^2)^{-3/2} dudv. \tag{1.18.4}$$

ii) We take the parametrization

$$(1.18.5)\quad \gamma : U = \{(u,v) \in \mathbb{R}; u^2+v^2<1\} \to \mathbb{D}, \quad (u,v) \mapsto \frac{1}{1-u^2-v^2}\begin{pmatrix} 1+u^2+v^2 \\ 2u \\ 2v \end{pmatrix},$$

with $(x,y) = {}^t x E_{1,2} y$, have the metric tensor

$$(1.18.6)\qquad (g_{i,j}(u,v)) = \begin{pmatrix} (({}^t\gamma_u,\gamma_u) & ({}^t\gamma_u,\gamma_v) \\ ({}^t\gamma_v,\gamma_u) & ({}^t\gamma_v,\gamma_v) \end{pmatrix} = -(1-u^2-v^2)^{-2} 4 \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$

and the volume form

$$(1.18.7)\qquad dv_P = \sqrt{\det(g_{i,j})}dudv = 4(1-u^2-v^2)^{-2}dudv.$$

iii) For $m \in \mathbb{N}$, another example is given by $q_m(x) = 4mx_1x_2 - x_3^2$ resp. $q_m(u) = u_1^2 - u_2^2 - (1/4m)u_3^2$. We take the parametrization

(1.18.8)

$$\gamma : U = \{(u,v) \in \mathbb{R}; u^2+v^2<1\} \to \mathbb{D}, \quad (u,v) \mapsto \frac{1}{1-u^2-(1/4m)v^2}\begin{pmatrix} 1+u^2+(1/4m)v^2 \\ 2u \\ 2v \end{pmatrix},$$

with $(x,y) = x_1y_1 - x_2y_2 - (1/4m)x_3y_3$, have the metric tensor

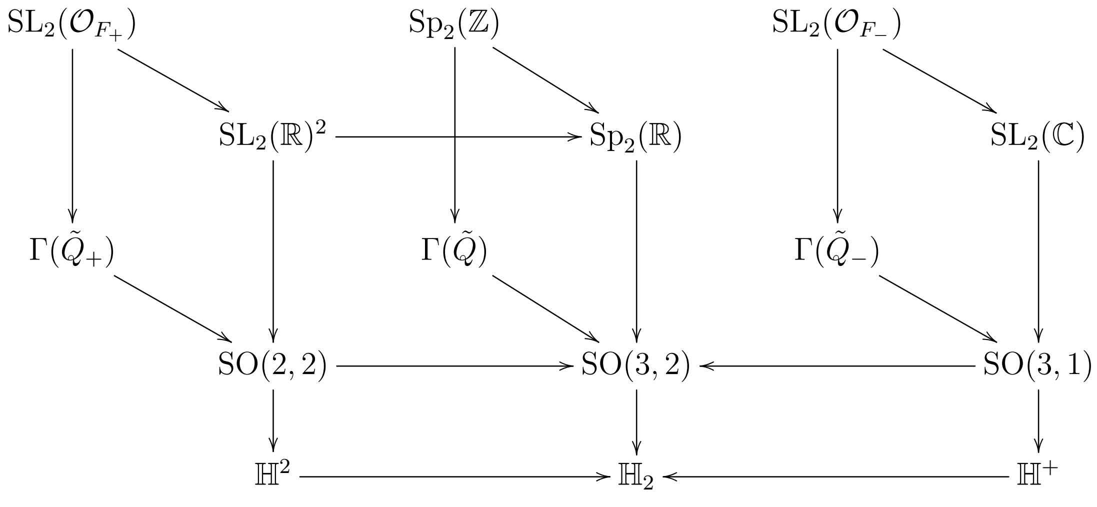
$$(1.18.9)\quad (g_{i,j}(u,v)) = \begin{pmatrix} (({}^t\gamma_u,\gamma_u) & ({}^t\gamma_u,\gamma_v) \\ ({}^t\gamma_v,\gamma_u) & ({}^t\gamma_v,\gamma_v) \end{pmatrix} = (1-u^2-(1/4m)v^2)^{-2} 4 \begin{pmatrix} -4 & \\ & -1/m \end{pmatrix}$$

and the volume form

$$(1.18.10)\qquad dv_P = \sqrt{\det(g_{i,j})}dudv = (2/\mu)(1-u^2-(1/4m)v^2)^{-2}dudv, \quad \mu^2 = m.$$

1.19. An overview over part of all this and objects appearing and to be explained later is given as follows.

Our orthogonal world.



Our Lattice
In the following, primarily, we look at $V = \mathbb{R}^5$ and the lattice $L = \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_5 \simeq \mathbb{Z}^5$ with quadratic form

$$\tilde{q}(u) = u_3^2 - u_2u_4 - u_1u_5 = (1/2)^t u\tilde{Q}u, \tag{1.19.1}$$

$$\tilde{Q} = \begin{pmatrix} &&&&-1\\ &&&-1&\\ &&2&&\\ &-1&&&\\ -1&&&& \end{pmatrix},$$

with orthogonal group and its unit group

$$\tilde{G} = \mathrm{SO}(\tilde{Q}) = \{g \in \mathrm{SL}(5,\mathbb{R}); {}^t g\tilde{Q}g = \tilde{Q}\} \simeq \mathrm{SO}(3,2), \tag{1.19.2}$$
$$\Gamma(\tilde{Q}) = \{W \in \mathrm{SL}(5,\mathbb{Z}); {}^t W\tilde{Q}W = \tilde{Q}\}.$$

We have the dual lattice $L' = L \cup ((1/2)e_3 + L)$ with $L'/L \simeq \mathbb{Z}/2\mathbb{Z}$ and quadratic form

$$q'(v) = v_3^2/4 - v_2v_4 - v_1v_5 = (1/2)^t vQ'v, \tag{1.19.3}$$

$$Q' = \begin{pmatrix} &&&&-1\\ &&&-1&\\ &&1/2&&\\ &-1&&&\\ -1&&&& \end{pmatrix} = \tilde{Q}^{-1}.$$

with groups

$$G' = \mathrm{SO}(Q') = \{g \in \mathrm{SL}(5,\mathbb{R}); {}^t gQ'g = Q'\} \simeq \mathrm{SO}(3,2), \tag{1.19.4}$$
$$\Gamma(Q') = \{W \in \mathrm{SL}(5,\mathbb{Z}); {}^t WQ'W = Q'\}.$$

1.20. Remark. As $Q' = \tilde{Q}^{-1}$, for $g \in \tilde{G}$, one has $({}^t g\tilde{Q}g)^{-1} = g^{-1}Q'{}^t g^{-1} = \tilde{Q}^{-1} = Q'$, hence

$$\tilde{G} \simeq G', \quad g \mapsto g' = {}^t g^{-1}. \tag{1.20.1}$$

2 Eisenstein series of weight 5/2

2.1. In [BK], Bruinier and Kühn study classical real analytic vector valued Eisenstein series for Mp_2 transforming with the Weil representation ρ_L. We want to take over their results. Hence, we have to look at the following specialization of their situation
- a real quadratic space (V, q) of signature (2,3) and rank $r = 5$,
- $(\cdot,\cdot)$ the bilinear form corresponding to q with $q(x) = (1/2)(x,x) = x_1x_2 + x_3x_4 - x_5^2$ (i.e.,

the negative of our form $\tilde{q}$ above),
- the lattice $L = \mathbb{Z}^5$ in V with form q and dual L',
- $(\mathbf{e}_\gamma)_{\gamma \in L'/L}$ the standard basis of the group ring $\mathbb{C}[L'/L]$,
- ρ_L the representation of $\mathrm{Mp}_2(\mathbb{Z})$ in $\mathbb{C}[L'/L]$ as in [BK] (2.3), and
- $\Gamma(L)$ the kernel of the natural homomorphism from $O(L)$ to $O(L'/L)$.

For $\kappa \in (1/2)\mathbb{Z}$, here $\kappa = 5/2$, the Eisenstein series of weight κ is defined by

$$E_\beta(\tau, s) := (1/2) \sum_{(M,\phi) \in \Gamma_\infty \backslash \mathrm{Mp}_2(\mathbb{Z})} (\mathbf{e}_\beta v^s)|^*_\kappa (M, \phi). \tag{2.1.1}$$

E_β has the Fourier expansion

$$E_\beta(\tau, s') = \sum_{\gamma \in L'/L} \sum_{m \in \mathbb{Z} - q(\gamma)} c_\beta(\gamma, m, s', v) \mathbf{e}_\gamma(mu). \tag{2.1.2}$$

Kudla's conjecture relates

$$E'_\beta(\tau, 0) = \frac{\partial}{\partial s} E_\beta(\tau, s)|_{s=0}, \text{ i.e., } c'_\beta(\gamma, m, 0, v) = \frac{\partial}{\partial s} c_\beta(\gamma, m, s, v)|_{s=0} \tag{2.1.3}$$

to appropriate Green function integrals to be defined later. Here, we follow [BK] in their determination of $c_\beta(\gamma, m, s, v)$.

2.2. Proposition 3.1 in [BK] says that E_β has the Fourier expansion (here we change $\kappa =: \ell$)

$$\begin{aligned} E_\beta(\tau, s') &= \sum_{\gamma \in L'/L} \sum_{m \in \mathbb{Z} - q(\gamma)} c_\beta(\gamma, m, s', v) \mathbf{e}_\gamma(mu), \\ c_\beta(\gamma, m, s', v) &= (\delta_{\beta,\gamma} + \delta_{-\beta,\gamma}) v^{s'} + 2\pi v^{1-\ell-s'} \frac{\Gamma(\ell + 2s' - 1)}{\Gamma(\ell + s')\Gamma(s')} \mathcal{H}(\beta, \gamma, 0, s') \text{ for } m = 0, \\ &= \frac{2^\ell \pi^{s'+\ell} |m|^{s'+\ell-1}}{\Gamma(\ell + s')} \mathcal{W}_{s'}(4\pi m v) \mathcal{H}(\beta, \gamma, m, s') \text{ for } m > 0, \\ &= \frac{2^\ell \pi^{s'+\ell} |m|^{s'+\ell-1}}{\Gamma(s')} \mathcal{W}_{s'}(4\pi |m| v) \mathcal{H}(\beta, \gamma, m, s') \text{ for } m < 0. \end{aligned} \tag{2.2.1}$$

I.e., in our case, we have a two component series where each component has coefficients $c(\bar{0}, m, s', v)$ indexed by integers $m \in \mathbb{Z}$ and the other coefficients $c(\bar{1}, m, s', v)$ indexed by $m - 1/4, m \in \mathbb{Z}$. Here, using a generalized Kloosterman sum H^*_c as in [BK] (3.6),

$$\begin{aligned} \mathcal{H}(\beta, \gamma, m, s) &= \sum_{c \in \mathbb{Z} - \{0\}} |2c|^{1-\ell-2s} H^*_c(\beta, 0, \gamma, m) \text{ for } m = 0, \\ &= \sum_{c \in \mathbb{Z} - \{0\}} |c|^{1-\ell-2s} H^*_c(\beta, 0, \gamma, m) \text{ for } m \neq 0, \end{aligned} \tag{2.2.2}$$

and the Whittaker term

$$\begin{aligned}
\mathcal{W}_{s'}(a) &= |a|^{-\ell/2} W_{\operatorname{sgn}(a)\ell/2,(1-\ell)/2-s'}(|a|) \\
&= \frac{e^{-|a/2|}|a|^{1-\ell-s'}}{\Gamma(1-s'-\ell)} \int_0^\infty e^{-|a|t} t^{-s'-\ell}(1+t)^{-s'} dt \quad \text{for } a = 4\pi mv > 0, \\
&= \frac{e^{-|a/2|}|a|^{1-\ell-s'}}{\Gamma(1-s')} \int_0^\infty e^{-|a|t} t^{-s'}(1+t)^{-s'-\ell} dt \quad \text{for } a = 4\pi mv < 0, \qquad (2.2.3)
\end{aligned}$$

where (as in [AS] p.190)

$$W_{\nu,\mu}(z) := \frac{e^{-z/2} z^{\mu+1/2}}{\Gamma(\mu-\nu+1/2)} \int_0^\infty e^{-tz} t^{\mu-\nu-1/2}(1+t)^{\mu+\nu-1/2} dt. \qquad (2.2.4)$$

One has the special cases

$$\begin{aligned}
\mathcal{W}_0(a) &= e^{-a/2} \quad \text{for } a = 4\pi mv > 0, \\
&= e^{-a/2}\Gamma(1-\ell, |a|) \quad \text{for } a = 4\pi mv < 0, \qquad (2.2.5)
\end{aligned}$$

where $\Gamma(s,x) = \int_x^\infty e^{-t} t^{s-1} dt$.

2.3. For later use, we want to put the Whittaker term into another form and take over expressions and relations from an adelic treatment in [KRY] (15.2-6) and material on the confluent hypergeometric function of the second kind as in e.g.[Le] p.324-326. For a short time, we change to some new notation. We write s' for the variable s from [BK] to distinguish from the variable s in the usual adelic version and have

$$s' = (1/2)(s+1-\ell) \qquad (2.3.1)$$

and the special value $s = \ell - 1$ there corresponds to $s' = s = 0$ in [BK]. We have $\ell := \kappa$ and

$$\alpha := (1/2)(s+1+\ell), \beta := (1/2)(s+1-\ell) = s'$$

and the confluent hypergeometric function of the second kind

$$\Psi(a,b;z) := \frac{1}{\Gamma(a)} \int_0^\infty e^{-zr}(r+1)^{b-a-1} r^{a-1} dr, \qquad (2.3.2)$$

where $a > 0, z > 0, b \in \mathbb{R}$ with the functional equation

$$\Psi(a,b;z) = z^{1-b}\Psi(1+a-b, 2-b; z). \qquad (2.3.3)$$

As in [KRY] (15.4), we define

$$\Psi(0,b;z) := \lim_{a\to 0^+} \Psi(a,b;z) = 1 \qquad (2.3.4)$$

and as in [KRY] (15.5), for any number n the function

$$\Psi_n(s,a) := \Psi((1/2)(1+n+s), s+1; a) \tag{2.3.5}$$

which has the functional equation

$$\Psi_n(s,a) = a^{-s}\Psi_n(-s,a). \tag{2.3.6}$$

Now, we take

$$\begin{aligned}
\Psi_\ell(s,a) &:= \Psi((1/2)(1+\ell+s), s+1; a) \\
&= \frac{1}{\Gamma((1/2)(1+\ell+s))}\int_0^\infty e^{-ar}(r+1)^{(1/2)(s-1-\ell)}r^{(1/2)(s+\ell-1)}dr \\
&= \frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-ar}(r+1)^{\beta-1}r^{\alpha-1}dr \\
&= a^{-s}\Psi_\ell(-s,a) \\
&= \frac{a^{-s}}{\Gamma((1/2)(1+\ell-s))}\int_0^\infty e^{-ar}(r+1)^{-(1/2)(s+1+\ell)}r^{(1/2)(\ell-s-1)}dr \\
&= \frac{a^{-s}}{\Gamma(1-\beta)}\int_0^\infty e^{-ar}(r+1)^{-\alpha}r^{-\beta}dr,
\end{aligned} \tag{2.3.7}$$

resp.

$$\begin{aligned}
\Psi_{-\ell}(s,a) &= \frac{1}{\Gamma(\beta)}\int_0^\infty e^{-ar}(r+1)^{\alpha-1}r^{\beta-1}dr \\
&= a^{-\beta} + \frac{1}{\Gamma(\beta)}\int_0^\infty e^{-ar}((r+1)^{s-\beta}-1)r^{\beta-1}dr \\
&= a^{-s}\Psi_\ell(-s,a) \\
&= \frac{a^{-s}}{\Gamma(1-\alpha)}\int_0^\infty e^{-ar}(r+1)^{-\beta}r^{-\alpha}dr.
\end{aligned} \tag{2.3.8}$$

As they will become very important later, we introduce the abbreviation of the integrals

$$\begin{aligned}
J_+(s,a) &:= \int_0^\infty e^{-aw}((w+1)^s-1)dw/w, \\
J_-(s,a) &:= \int_0^\infty e^{-aw}w^s dw/(w+1).
\end{aligned} \tag{2.3.9}$$

Using these, we also have

$$\begin{aligned}
\Psi_{-\ell}(s,a) &= a^{-\beta} + \frac{1}{\Gamma(\beta)}J_+(s-\beta,a), \\
\Psi_{+\ell}(\ell-1,a) &= \frac{1}{\Gamma(\ell)}J_-(\ell-1,a).
\end{aligned} \tag{2.3.10}$$

Hence, remembering $s = 2s' + \ell - 1$, from (2.2.3) we get

$$\mathcal{W}_{s'}(a) = |a|^{\beta} e^{-|a|/2} \Psi_{\mp\ell}(s, |a|) \quad \text{for } a = 4\pi m v > 0, \text{ resp. } < 0. \tag{2.3.11}$$

2.4. Remark. With $\Gamma(s, x) = \int_x^{\infty} e^{-t} t^{s-1} dt$ and (2.2.5), finally, we have

$$\begin{aligned} \mathcal{W}_0(a) &= e^{-a/2} \quad \text{for } a = 4\pi m v > 0, \\ &= e^{-a/2} \Gamma(1 - \ell, |a|) \quad \text{for } a = 4\pi m v < 0 \\ &= e^{-|a|/2} \frac{1}{\Gamma(\ell)} J_{-}(\ell - 1, |a|), \\ &= |a|^{1-\ell} e^{|a|/2} \int_1^{\infty} e^{-|a|t} dt/t^{\ell}. \end{aligned} \tag{2.4.1}$$

2.5. Restricting to E_0, for $m \neq 0$, in [BK] (3.22), one defines coefficients $C(\gamma, m, s)$ by

$$c_0(\gamma, m, s', v) = C(\gamma, m, s') \mathcal{W}_{s'}(4\pi m v)$$

and in Theorem 3.3, analyzing the Kloosterman sums, for positive m these coefficients in our case come out as

$$\begin{aligned} C(\gamma, m, s') &= 2^{2s'+2\ell-1/2} \pi^{-s'} |m|^{\ell+s'-1} \\ &\times \frac{\sin(\pi(2s' + \ell)) |D_0|^{1-2s'-\ell} \Gamma(2s' + \ell)}{\cos(\pi(s' - (\delta/2)) \sqrt{|L'/L|} \Gamma(s' + \ell)} \\ &\times \frac{L(\chi_{D_0}, (3/2) - 2s' - \ell)}{\zeta(2 - 4s' - 2\ell)} \sigma_{\gamma,m}(2s' + \ell). \end{aligned} \tag{2.5.1}$$

In this formula, the index m of the coefficient and the fundamental discriminant D_0 are linked by the essential relation from [BK] (3.24)

$$D := D_0 f^2 := -2d_{\gamma}^2 m \det(L), d_{\gamma} = \min \{b \in \mathbb{Z}_{>0}; b\gamma \in L\} \tag{2.5.2}$$

and $\delta = 0$ if $D_0 > 0$, $\delta = 1$ if $D_0 < 0$, and $\sigma_{\gamma,m}(s)$ from [BK] (3.28) is the *generalized divisor sum* which will later reappear in (12.23.1) and then will be discussed a more intensely

$$\sigma_{\gamma,m}(s) := \prod_{p|D} \frac{1 - \chi_{D_0}(p) p^{(1/2)-s}}{1 - p^{1-2s}} L_{\gamma,m}^{(p)}(p^{-(3/2)-s}). \tag{2.5.3}$$

For $m < 0$ analogous formulae hold but with $\Gamma(s' + \ell)$ replaced by $\Gamma(s')$.

2.6. Remark. It will be helpful to note a formula from the proof of [BK] Theorem 3.3, namely, for $m > 0$, one also has

(2.6.1)

$$C(\gamma, m, s') = \frac{(-1)^{(2\ell-b^-+b^+)/4} 2^{\ell+1} \pi^{\ell+s'} |m|^{\ell+s'-1}}{\sqrt{|L'/L|} \Gamma(s' + \ell)} \times \frac{L(\chi_{D_0}, 2s' + \ell - 1/2)}{\zeta(4s' + 2\ell - 1)} \sigma_{\gamma,m}(2s' + \ell).$$

Again, for $m < 0$, one has an analogous formula but with $\Gamma(s' + \ell)$ replaced by $\Gamma(s')$.

2.7. The formulae (2.5.1) and (2.6.1) are related by the functional equations of the zeta and $L-$ functions: In [BK] p.1701, for a primitive character χ_{D_0}, the Dirichlet series $L(\chi_{D_0}, s)$ satisfies the functional equation

$$L(\chi_{D_0}, s) = L(\chi_{D_0}, 1-s)\frac{2^{s-1}\pi^s |D_0|^{1/2-s}}{\cos(\pi(s-\delta)/2)\Gamma(s)} \tag{2.7.1}$$

where $\delta = 0$ if $D_0 > 0$ and $\delta = 1$ if $D_0 < 0$. For $s = 2$ and $\delta = 1$, a quotient of singular values shows up on the right hand side.
In particular, the Riemann zeta function has the functional equation

$$\zeta(s) = \zeta(1-s)\frac{2^{s-1}\pi^s}{\cos(\pi s/2)\Gamma(s)}. \tag{2.7.2}$$

And one also needs the duplication formula

$$\Gamma(z)\Gamma(z+1/2) = 2^{1-2z}\sqrt{\pi}\Gamma(2z). \tag{2.7.3}$$

2.8. For later use, we assemble several standard formulae:

$$\begin{aligned} &\zeta(2) = \pi^2/6,\ \zeta(4) = \pi^4/90,\ \zeta(-1) = -1/12,\ \zeta(-3) = 1/120, \\ &\Gamma(1/2) = \sqrt{\pi},\ \Gamma(-1/2) = 2\sqrt{\pi},\ \Gamma(3/2) = (1/2)\sqrt{\pi},\ \Gamma(5/2) = (3/4)\sqrt{\pi}. \\ &1/\Gamma(s) = 0,\ (1/\Gamma(s))' = 1 \text{ for } s = 0. \end{aligned} \tag{2.8.1}$$

From [BK] (4.77) we have for our odd case

$$\begin{aligned} \frac{\Gamma'(\kappa)}{\Gamma(\kappa)} &= \Gamma'(1) - 2\log(2) + \sum_{j=1}^{\kappa-1/2}(j-(1/2))^{-1} \\ \frac{\Gamma'(5/2)}{\Gamma(5/2)} &= \Gamma'(1) - 2\log(2) + 8/3. \end{aligned} \tag{2.8.2}$$

2.9. Summary 1. Specialized to our case, up to now, we have

$$\begin{aligned} c_0(\gamma, m, s', v) &= C(\gamma, m, s')\mathcal{W}_{s'}(a) \\ &= -\frac{2^3\pi^{s'+5/2}|m|^{s'+3/2}L(\chi_{D_0}, 2s'+2)}{\Gamma(s'+5/2)\zeta(4s'+4)}\sigma_{\gamma,m}(2s'+5/2)\cdot\mathcal{W}_{s'}(a) \text{ for } m > 0 \\ &= -\frac{2^3\pi^{s'+5/2}|m|^{s'+3/2}L(\chi_{D_0}, 2s'+2)}{\Gamma(s')\zeta(4s'+4)}\sigma_{\gamma,m}(2s'+5/2)\cdot\mathcal{W}_{s'}(a) \text{ for } m < 0 \\ c_0(\gamma, m, 0, v) &= -2^6\cdot 3\cdot 5\cdot \pi^{-2}|m|^{3/2}L(\chi_{D_0}, 2)\sigma_{\gamma,m}(5/2)\cdot e^{-a/2} \text{ for } m > 0 \\ &= 0 \text{ for } m < 0. \end{aligned} \tag{2.9.1}$$

For $m > 0$, by (2.5.1), we have as an alternative

$$C(\gamma, m, s') = 2^{2s'+2\ell-1/2}\pi^{-s}|m|^{\ell+s'-1} \times \frac{\sin(\pi(2s'+\ell))|D_0|^{1-2s'-\ell}\Gamma(2s'+\ell)}{\cos(\pi s')\sqrt{|L'/L|}\Gamma(s'+\ell)} \times \frac{L(\chi_{D_0}, (3/2)-2s'-\ell)}{\zeta(2-4s'-2\ell)}\sigma_{\gamma,m}(2s'+\ell), \tag{2.9.2}$$

i.e.,

$$c_0(\gamma, m, 0, v) = -2^7 \cdot 3 \cdot 5 \cdot \pi^{-2}|m/D_0|^{3/2}L(\chi_{D_0}, -1)\sigma_{\gamma,m}(3/2) \cdot e^{-a/2}. \tag{2.9.3}$$

Derivatives

2.10. We hope not to produce too much confusion as we still have the variable s' and use the $'$ for the s'−derivative. We have

$$c_0'(\gamma, m, s', v) = C'(\gamma, m, s')\mathcal{W}_{s'}(a) + C(\gamma, m, s')\mathcal{W}_{s'}'(a) \tag{2.10.1}$$

and want to evaluate this for $s' = 0$. For $m < 0$, one has $C(\gamma, m, 0) = 0$. Moreover, from (2.3.11)

$$\mathcal{W}_{s'}(a) = |a|^{s'}e^{-|a|/2}\Psi_{\mp\ell}(s, |a|) \quad \text{for } a = 4\pi mv > 0, \text{ resp. } < 0,$$

and from (2.4.1)

$$\begin{aligned} \mathcal{W}_0(a) &= e^{-a/2} \quad \text{for } a = 4\pi mv > 0, \\ &= e^{-a/2}\Gamma(-3/2, |a|) \quad \text{for } a = 4\pi mv < 0 \\ &= e^{-|a|/2}\frac{1}{\Gamma(5/2)}J_-(3/2, |a|) \\ &= e^{-|a|/2}|a|^{-3/2}\int_1^\infty e^{|a|t}dt/t^{5/2}. \end{aligned}$$

For $m > 0$, we get

$$\frac{d}{ds'}\mathcal{W}_{s'}(a) = \log a \cdot a^{s'}e^{-|a|/2}\Psi_{-\ell}(s, a) + |a|^{s'}e^{-|a|/2}\frac{d}{ds'}\Psi_{-\ell}(s, a). \tag{2.10.2}$$

Using [KRY] (15.9), one has

$$\Psi_{-\ell}(\ell-1, a) = 1, \quad \Psi'_{-\ell}(\ell-1, a) = -(1/2)(\log a - J(\ell-1, a)) \tag{2.10.3}$$

Here the derivative ' is with respect to s. For $F(s) = F(2s'+\ell-1)$, one has $(d/ds')F(s) = 2F'(s)$. Hence, for $m > 0$,

$$\begin{aligned} (d/ds')\mathcal{W}_0(a) &= \log a\, e^{-a/2} - 2 \cdot (1/2)(\log a - J_+(\ell-1, a))e^{-a/2} \\ &= J_+(3/2, a)e^{-a/2} = J_+(3/2, a)\mathcal{W}_0(a) \end{aligned} \tag{2.10.4}$$

For $m > 0$, by direct computation or [BK] (4.75), from (2.5.1) we get

$$C'(\gamma.m,0) = C(\gamma.m,0)\big(4\frac{\zeta'(-3)}{\zeta(-3)} - 2\frac{L'(\chi_{D_0},-1)}{L(\chi_{D_0},-1)} + 2\frac{\sigma'_{\gamma,m}(5/2)}{\sigma_{\gamma,m}(5/2)} - \log(\pi/4) + \log|m/D_0^2| + \frac{\Gamma'(5/2)}{\Gamma(5/2)}\big). \tag{2.10.5}$$

and from (2.6.1)

$$C'(\gamma,m,0) = 2^6\cdot 3\cdot 5\cdot \pi^{-2}|m|^{3/2}L(\chi_{D_0},2)\sigma_{\gamma,m}(5/2) \times \Big(\log\pi + \log|m| + 2\frac{L'(\chi_{D_0},2)}{L(\chi_{D_0},2)} - 4\frac{\zeta'(4)}{\zeta(4)} - \frac{\Gamma'(5/2)}{\Gamma(5/2)} + \frac{\sigma'_{\gamma,m}(5/2)}{\sigma_{\gamma,m}(5/2)}\Big). \tag{2.10.6}$$

2.11. Summary 2. For $m > 0$, using the formula above or (2.10.5), we get

$$\begin{aligned} c'_0(\gamma,m,0,v) &= C'(\gamma,m,0)\mathcal{W}_0(a) + C(\gamma,m,0)\mathcal{W}'_0(a) \\ &= C(\gamma,m,0)e^{-a/2}(J_+(3/2,a) + C'(\gamma,m,0)/C(\gamma,m,0)) \\ &= 2^6\cdot 3\cdot 5\cdot \pi^{-2}|m|^{3/2}L(\chi_{D_0},2)\sigma_{\gamma,m}(5/2)\, e^{-a/2}(J_+(3/2,a) + \frac{C'(\gamma,m,0)}{C(\gamma,m,0)}) \\ &= 2^7\cdot 3\cdot 5\,|m/D_0|^{3/2}L(\chi_{D_0},-1)\sigma_{\gamma,m}(5/2)e^{-a/2}\big(4\frac{\zeta'(-3)}{\zeta(-3)} - 2\frac{L'(\chi_{D_0},-1)}{L(\chi_{D_0},-1)} \\ &\quad + 2\frac{\sigma'_{\beta,m}(5/2)}{\sigma_{\beta,m}(5/2)} - \log(\pi/4) + \log|m/D_0^2| + \frac{\Gamma'(5/2)}{\Gamma(5/2)} + J_+(3/2,a)\big). \end{aligned} \tag{2.11.1}$$

And for $m < 0$, using (2.9.1) and (2.4.1),

$$\begin{aligned} c'_0(\gamma,m,0,v) &= C'(\gamma,m,0)\mathcal{W}_0(a) \\ &= \frac{2^3\pi^{5/2}|m|^{3/2}L(\chi_{D_0},2)}{\zeta(4)}\sigma_{\gamma,m}(5/2)e^{-|a|/2}\Psi_{5/2}(3/2,|a|) \\ &= 2^4\cdot 3^2\cdot 5\cdot \pi^{-3/2}|m|^{3/2}L(\chi_{D_0},2)\sigma_{\gamma,m}(5/2)e^{|a|/2}\Gamma(-3/2,|a|) \\ &= 2^6\cdot 3\cdot 5\cdot \pi^{-2}|m|^{3/2}L(\chi_{D_0},2)\sigma_{\gamma,m}(5/2)e^{-|a|/2}J_-(3/2,|a|). \end{aligned} \tag{2.11.2}$$

In our case, we have $q(x) = x_3^2 - x_2x_4 - x_1x_5$, $\det L = 2, a = 4\pi mv$ and $L'/L \simeq \mathbb{Z}/2\mathbb{Z}$, hence γ has just the two values $\bar{0} + L$ and $\bar{1} + L$. These correspond to two cases, Case A and B, where $m \in \mathbb{Z}$ resp.$m - 1/4 \in \mathbb{Z}$ which later show up again. In the formulae above, the index m is related to the fundamental discriminant D_0 via (2.5.2), namely by

$$D_0f^2 = 4m \quad \text{for } m \in \mathbb{Z} \quad \textbf{Case A} \tag{2.11.3}$$

$$= 4^2m \quad \text{for } m - 1/4 \in \mathbb{Z} \quad \textbf{Case B}. \tag{2.11.4}$$

The formulae for the coefficients of the Eisenstein series we took over from [BK], are obtained for quadratic forms of signature (2,3) though, following Kudla, we have signature (3,2). But as [BK] have Eisenstein series for a Weil representation dual to the one of Kudla, we can take their formulae (as done for instance in 3.5.1 in Klöcker [Kl]).

Eisenstein coefficients and Geometry

2.12. As well known, the coefficients of our Eisenstein series also have a geometric meaning: Above, in (2.9.1) we got

$$c_0(\gamma, m, 0, v) = -2^6 \cdot 3 \cdot 5 \cdot \pi^{-2} |m|^{3/2} L(\chi_{D_0}, 2) \sigma_{\gamma,m}(5/2) \cdot e^{-a/2} \text{ for } m > 0 \tag{2.12.1}$$

In [BK] (4.3) for $\lambda \in V$ with negatice norm, $\beta \in L'/L$, $m \in \mathbb{Z} + q(\beta)$ a negative number,

$$\mathcal{H}(\beta, m) := \sum_{\lambda \in L+\beta, q(\lambda)=m} \lambda^{\perp} \tag{2.12.2}$$

is a $\Gamma(L)-$invariant divisor on $\mathrm{Gr}(V)$, the *Heegner divisor* of discriminant (β, m). In [BK] $\mathrm{Gr}(V)$ is the Grassmannian of positive definite subspaces $\nu \subset V$ of dimension 2. From [BK] (4.33), one has $\deg(\mathcal{H}(\gamma, -n)) = \int_{\mathcal{H}(\gamma,-n)} \Omega^2$. [BK] Proposition 4.8 (4.52) specialized to our situation, says

$$\begin{aligned} E_0(\tau, 0) &= 2\mathbf{e}_0 - (2/B) \sum_{\gamma \in L'/L} \sum_{n \in \mathbb{Z} - q(\gamma), n>0} \deg(\mathcal{H}(\gamma, -n)) \mathbf{e}_\gamma(n\tau), \\ B &= \zeta(-1)\zeta(-3) = 2^{-5} 3^{-2} 5^{-1}, \end{aligned} \tag{2.12.3}$$

and (4.53)

$$\deg(\mathcal{H}(\gamma, -n))/B = -2^5 \cdot 3 \cdot 5/\pi^2 \cdot n^{3/2} L(\chi_{D_0}, 2) \sigma_{\gamma,n}(5/2). \tag{2.12.4}$$

This is consistent with the usual results concerning the voluminae of Humbert surfaces and we shall come back to this later in the calculation of the Green function integral (see (12.23.8)).

2.13. Summary 3. Via Kloosterman sums [BK] (3.29) leads to

$$c_0(\gamma, m, 0, v) = -2^6 \cdot 3 \cdot 5 \cdot \pi^{-2} |m|^{3/2} L(\chi_{d_F}, 2) \sigma_{\gamma,m}(5/2) e^{-a/2},$$

where one has from [BK] (3.24) $4m = d_F f^2$ for $m \in \mathbb{Z}$ and $16m = d_F f^2$ for $m = M + 1/4, M \in \mathbb{Z}$. As mentioned above, [BK] Proposition 4.8 (4.52) specialized to our situation, says

$$\begin{aligned} E_0(\tau, 0) &= 2\mathbf{e}_0 - (2/B) \sum_{\gamma \in L'/L} \sum_{n \in \mathbb{Z} - q(\gamma), n>0} \deg(\mathcal{H}(\gamma, -n)) \mathbf{e}_\gamma(n\tau), \\ B &= \zeta(-1)\zeta(-3) = 2^{-5} 3^{-2} 5^{-1}, \end{aligned} \tag{2.13.1}$$

and (4.53)

$$\deg(\mathcal{H}(\gamma, -n))/B = -2^5 \cdot 3 \cdot 5/\pi^2 \cdot n^{3/2} L(\chi_{D_0}, 2)\sigma_{\gamma,n}(5/2). \tag{2.13.2}$$

The coefficients in the two cases A and B relate to integrals of appropriate versions of Kudla's Green function which will be introduced using the following object.

3 The Majorant

The majorant of an indefinite quadratic form and its description have been propagated by Siegel while introducing and discussing thetas for indefinite quadratic forms and this is also essential for the development of the Schwartz forms à la Kudla-Millson. There are different approaches. At first we shall follow Siegel's article [S4] and specialize it to our situation.

3.1. We have a vector space $V = \mathbb{R}^n$ with quadratic form

$$q(x) = (1/2)(x, x), (x, y) = {}^t x Q y, \text{ signature } Q = (p, q), p + q = n. \tag{3.1.1}$$

and have $(x, y) = q(x+y) - q(x) - q(y)$. Following Siegel, we also write $Q[x] = (x, x) = {}^t x Q x$. A *majorant of* $Q[x]$ is a positive definite quadratic form $P[x]$ such that $P[x] \geq Q[x]$ for all $x \in \mathbb{R}^n$. On the first pages of [S4] it is shown that $P \in M_n(\mathbb{R})$ defines such a majorant exactly if

$$PQ^{-1}P = Q, \ {}^t P = P > 0. \tag{3.1.2}$$

And, if C is such that for $x = Cy$ one has ${}^t x Q x = {}^t y Q_0 y = y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_n^2$, we get such a

$$P = (C^t C)^{-1}. \tag{3.1.3}$$

The orthogonal group $G = \mathrm{SO}(Q) = \{A; {}^t A Q A = Q\}$ via

$$A \mapsto {}^t A P A$$

acts transitively on the set $\mathcal{P} = \mathcal{P}(Q)$ of these P, which Siegel calls the **representation space.** For $\tau = u + iv \in \mathbb{H}$

$$\theta(\tau, P) := \sum_{x \in \mathbb{Z}^n} e^{2i\pi R[x]}, \ R = uQ + ivP \tag{3.1.4}$$

is a *Siegel theta series* with its well known nice properties of convergence and transformation.

3.2. The following approach (as for instance contained in [Ku1]) is more adequate to Kudla's world. We restrict to the special case of **signature (**p**,2)** and realize the space G/K by the set $\mathbb{D}$ of oriented negative planes $X =< v_1, v_2 >\subset V$, i.e., with $(v_1, v_2) = 0, (v_1, v_1) = (v_2, v_2) = -\eta < 0$. A *minimal majorant* $(x, x)_z$ of (x, x) with respect to X_z is given by

$$
\begin{aligned}
(x,x)_X &= (x,x) && \text{for } x \in X^{\perp} \\
&= -(x,x) && \text{for } x \in X.
\end{aligned}
\tag{3.2.1}
$$

To make this more explicit, we decompose $x \in V$ into its positive and negative part with respect to X_z. For

$$x = x' + \alpha v_1 + \beta v_2,\ \alpha, \beta \in \mathbb{R} \tag{3.2.2}$$

with

$$(x', v_1) = (x', v_2) = 0, \tag{3.2.3}$$

one has

$$\alpha = -(x, v_1)/\eta,\ \beta = -(x, v_2)/\eta \tag{3.2.4}$$

and, hence,

$$(x, x) = (x', x') - (\alpha^2 + \beta^2)\eta = (x', x') - ((x, x(z))^2 + (x, y(z))^2)/\eta. \tag{3.2.5}$$

Now we see that one has the minimal majorant given by

$$
\begin{aligned}
(x,x)_X &= (x',x') + (\alpha^2+\beta^2)\eta = (x',x') + ((x,v_1)^2 + (x,v_2)^2)/\eta \\
&= (x,x) + 2((x,v_1)^2 + (x,v_2)^2)/\eta.
\end{aligned}
\tag{3.2.6}
$$

Here we take over Kudla's notation and write

$$(x, x)_X = (x, x) + 2R(x, X), R(x, X) := ((x, v_1)^2 + (x, v_2)^2)/\eta. \tag{3.2.7}$$

Following Kudla, we also remark $R(x, X) = -(\mathrm{pr}_X x, \mathrm{pr}_X x)$, where $\mathrm{pr}_X : V \to X$ is the projection with kernel $X^{\perp}$, i.e., $x \mapsto \mathrm{pr}_X(x) = ((x, v_1)v_1 + (x, v_2)v_2)/(-\eta)$.
Using (1.4.3), for $z \in \mathbb{D}_T$, we can also write as in [Ku1] (1.16)

$$R(x, X) := |(x, w(z))|^2 |y|^{-2}. \tag{3.2.8}$$

Lacking a better expression, we call R the *kernel* of the majorant.

3.3. Remark 1. With $A \in G = \mathrm{SO}(Q)$, one has the following invariance property

$$R(x, AX) = R(A^{-1}x, X). \tag{3.3.1}$$

Here, AX is the plane spanned by Av_1 and Av_2 and, with $QA = {}^tA^{-1}Q$, we have

$$\begin{aligned} R(x, AX) &= ((x, Av_1)^2 + (x, Av_2)^2)/\eta \\ &= (({}^txQAv_1)^2 + ({}^txQAv_2)^2)/\eta \\ &= (({}^tx{}^tA^{-1}Qv_1)^2 + ({}^tx{}^tA^{-1}Qv_2)^2)/\eta \\ &= (({}^t(A^{-1}x)Qv_1)^2 + ({}^t(A^{-1}x)Qv_2)^2)/\eta \\ &= R(A^{-1}x, X). \end{aligned}$$

Remark 2. As a consequence, one has the following kind of covariance relation. For ${}^tyQ_0y = {}^txQx, x = Cy$, we have ${}^tCQC = Q_0$ and for $A \in \mathrm{SO}(Q), \hat{A} := C^{-1}AC \in \mathrm{SO}(Q_0)$. With $X =< v_1, v_2 >$ and $Y = C^{-1}X =< u_1 := C^{-1}v_1, u_2 := C^{-1}v_2 >$, we get

$$\begin{aligned} R(x, X) &= ((x, v_1)^2 + (x, v_2)^2)/\eta \\ &= (({}^txQv_1)^2 + ({}^txQv_2)^2)/\eta \\ &= (({}^ty{}^tCQCu_1)^2 + ({}^ty{}^tCQCu_2)^2)/\eta \\ &= ((y, u_1)^2 + (y, u_2)^2)/\eta = R(y, Y). \end{aligned}$$

ergo

(3.3.2) $$R(x, X) = R(u, U).$$

3.4. For **signature (1,q)**, one takes as space the space $\mathbb{D}$ of positive lines $X = \langle v\rangle, v \in V, (v, v) = \eta > 0$ and the majorant is now

(3.4.1) $$\begin{aligned} (x, x)_X &= -(x, x) \quad \text{for } x \in X^\perp \\ &= (x, x) \quad \text{for } x \in X. \end{aligned}$$

Writing $x = x' + \alpha v, (x', v) = 0, (v, v) = \eta > 0$, one has $\alpha = (x, v)/\eta$ and

(3.4.2) $$\begin{aligned} (x, x) &= (x', x') + \alpha^2(v, v) = (x', x') + (x, v)^2/\eta \\ (x, x)_X &= -(x', x') + \alpha^2\eta = -(x, x) + 2(x, v)^2/\eta. \end{aligned}$$

For $\mathrm{pr} : V \to X$, the projection with kernel $X^\perp$, we have $R^1(x, X) := (\mathrm{pr}_Xx, \mathrm{pr}_Xx) = (x, v)^2/\eta$. Hence, parallel to the notation from the signature (p,2)-case, here one has

(3.4.3) $$(x, x)_X = -(x, x) + 2(x, v)^2/\eta = -(x, x) + 2R^1(x, X).$$

3.5. Key Relation Using Siegel's approach, the majorant can be determined as follows. Given a matrix Q with the form ${}^txQx$, look for a matrix P_0 with $P_0Q^{-1}P_0 = Q$. For $A \in \mathrm{SO}(Q)$, we have a majorant ${}^tx{}^tAP_0Ax$. In particular, this is helpful if one has an homomorphism $\rho : \tilde{G} \to \mathrm{SO}(Q)$ where $z \in \tilde{G}/\tilde{K} =: \tilde{D}$ is a parameter for the representation space $\mathcal{P}(Q)$. Hence, take $A := A(z)^{-1} = \rho((g_z)^{-1})$, where $g_z \in \tilde{G}$ is such that $g_z(z_0) = z$ for a base point $z_0 \in \tilde{D}$. For $X = \rho((g_z)^{-1})X_0 =: X_z$, one has

(3.5.1) $$(x, x)_z = {}^t(\rho((g_z)^{-1})x)P_0(\rho((g_z)^{-1})x)$$

In the following examples below, this way, we shall get the same value for the majorant as before.

3.6. Remark. It is a triviality but perhaps useful to observe the following: If one has $(x,x)_z = {}^t(\rho((g_z)^{-1})x)P_0(\rho((g_z)^{-1})x)$ and another form

$$(y,y) = {}^t y\hat{Q}y = (x,x) \quad \text{with} \quad x = Cy, \hat{A} = C^{-1}AC, \hat{Q} = {}^tCQC, \tag{3.6.1}$$

as above one has $(x,x)_z = (y,y)_z$. Hence, also

$$\hat{R}(y,z) = R(C^{-1}x,z). \tag{3.6.2}$$

3.7. Summary. For V with $(x,x) = {}^txQx$ and signature (p,2) and $\mathbb{D} \ni X = \langle v_1, v_2\rangle \subset V, (v_1,v_1) = (v_2,v_2) = -\eta < 0$ we have the majorant

$$\begin{aligned}(x,x)_X &= (x,x) + 2((x,v_1)^2 + (x,v_2)^2)/\eta = (x,x) + 2R(x,X),\\ R(x,X) &= -(\mathrm{pr}_X(x), \mathrm{pr}_X(x)) = ((x,v_1)^2 + (x,v_2)^2)/\eta.\end{aligned} \tag{3.7.1}$$

For signature (1,q) and $X = \langle v\rangle \subset V, (v,v) = \eta > 0$ we have the majorant

$$\begin{aligned}(x,x)_X &= -(x,x) + 2((x,v)^2/\eta = -(x,x) + 2R^1(x,X),\\ R^1(x,X) &= (\mathrm{pr}_X(x), \mathrm{pr}_X(x)) = (x,v)^2/\eta.\end{aligned} \tag{3.7.2}$$

if, as in the Key Relation above, one has a homomorphism $\rho : \tilde{G} \to \mathrm{SO}(Q)$ where $z \in \tilde{G}/\tilde{K} =: \tilde{D}$ is a parameter for the representation space $\mathcal{P}(Q)$, i.e., such that $X = X_z$, in the following examples, we get the same value for the majorant $(x,x)_z := (x,x)_X$,

$$(x,x)_z = {}^t(\rho((g_z)^{-1})x)P_0(\rho((g_z)^{-1})x) \tag{3.7.3}$$

Now, we discuss the explicit outcome for some special cases.

3.8. Example (3,2). We have from (1.8.1

$$Q_0 = \begin{pmatrix}1&&&&\\&1&&&\\&&1&&\\&&&-1&\\&&&&-1\end{pmatrix}, \quad Q = \begin{pmatrix}&&&&-1\\&&&-1&\\&&2&&\\&-1&&&\\-1&&&&\end{pmatrix}.$$

and with

$$u = {}^t(u_1,u_2,u_3,u_4,u_5) = {}^t(x_1+x_5, x_2+x_4, x_3, x_4-x_2, x_5-x_1)/\sqrt{2}$$

i.e.,with $u = Cx$ and $x = C^{-1}u$ we have $(u,v) = {}^tuQv$ and

$${}^txQ_0x = x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5^2 = {}^tuQu = 2(u_3^2 - u_2u_4 - u_1u_5), \tag{3.8.1}$$

hence $Q = {}^t C \tilde{Q} C$, such that

$$\tilde{G} = \mathrm{SO}(Q) = \{\tilde{A}; {}^t\tilde{A}Q\tilde{A} = Q\} = CGC^{-1}, \quad G = \mathrm{SO}(Q_0). \tag{3.8.2}$$

For

$$\begin{aligned} v_1^0 &= {}^t(1,0,0,0,1), (v_1^0, v_1^0) = -2, (v_1^0, u) = -u_5 - u_1 \\ v_2^0 &= {}^t(0,-1,0,-1,0), (v_2^0, v_2^0) = -2, (v_1^0, u) = +u_4 + u_2 \\ X_0 &=< v_1^0, v_2^0 > \end{aligned} \tag{3.8.3}$$

we get

$$\begin{aligned} R(u, X_0) &= -(\mathrm{pr}_{X_0} u, \mathrm{pr}_{X_0} u) = -((v_1^0, u)^2 + (v_2^0, u)^2)/(-2) \\ &= (u_1^2 + u_2^2 + u_4^2 + u_5^2 + 2u_1u_5 + 2u_2u_4)/2 \\ (u,u)_{X_0} &= (u,u) + 2R(X_0, u) \\ &= u_1^2 + u_2^2 + 2u_3^2 + u_4^2 + u_5^2 =: {}^t uPu, \end{aligned} \tag{3.8.4}$$

where

$$P = (C^tC)^{-1} = \begin{pmatrix} 1 &&&& \\ & 1 &&& \\ && 2 && \\ &&& 1 & \\ &&&& 1 \end{pmatrix}, \quad \text{as it should, fulfills} \quad PQ^{-1}P = Q. \tag{3.8.5}$$

As in (16.23.3), we have the homomorphism

$$A : \mathrm{Sp}(2, \mathbb{R}) \to \mathrm{SO}(Q), \quad g \mapsto A(g), gM(u)^t g = M(A(g)u)$$

In particular, for $g = g_z$, as in (1.8.14), such that $g_z < iE >= z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \in \mathbb{H}_2$, with

$$\eta = \sqrt{y_1y_3 - y_2^2}, \quad \zeta = \sqrt{x_1x_3 - x_2^2}, \tag{3.8.6}$$

one has from (1.8.16) and (1.8.17)

$$A(g_z) = \begin{pmatrix} \eta & \eta x_3/y_3 & 2(x_3y_2 - x_2y_3)/y_3 & (x_3y_2^2 - 2x_2y_2y_3 + x_1y_3^2)/(\eta y_3) & -\zeta^2/\eta \\ & \eta/y_3 & 2y_2/y_3 & y_2^2/(\eta y_3) & -x_1/\eta \\ && 1 & y_2/\eta & -x_2/\eta \\ &&& y_3/\eta & -x_3/\eta \\ &&&& 1/\eta \end{pmatrix}$$

and

$$A(g_z)^{-1} = \begin{pmatrix} 1/\eta & -x_3/\eta & 2x_2/\eta & -x_1/\eta & -\zeta^2/\eta \\ & y_3/\eta & -2y_2/\eta & y_2^2/(y_3\eta) & (x_3y_2^2 - 2x_2y_2y_3 + x_1y_3^2)/(\eta y_3) \\ && 1 & -y_2/y_3 & (x_2y_3 - y_2x_3)/y_3 \\ &&& \eta/y_3 & \eta x_3/y_3 \\ &&&& \eta \end{pmatrix}.$$

We take as a base point of $\mathbb{D}$ the plane from (1.9.1)

$$X_0 := X_{iE_2} = < u^{(1)}, u^{(2)} >, \; u^{(1)} = {}^t(1,0,0,0,1), u^{(2)} = {}^t(0,-1,0,-1,0)$$

and then we get

$$\begin{aligned} A(g_z)u^{(1)} &= (1/\eta)^t(\eta^2 - \zeta^2, -x_1, -x_2, -x_3, 1), \\ A(g_z)u^{(2)} &= (1/\eta)^t(2x_2y_y - x_1y_3 - y_1x_3, -y_1, -y_2, -y_3, 0), \end{aligned} \tag{3.8.7}$$

and with $u(z) = {}^t(u_1(z), \dots, u_5(z))$ from (1.9.3)

$$u_1(z) = z_2^2 - z_1z_3, u_2(z) = -z_1, u_3(z) = -z_2, u_4(z) = -z_3, u_5(z) = 1$$

we have the negative plane

$$X_z := < (1/\eta)\,\mathrm{Re}\; u(z), (1/\eta)\,\mathrm{Im}\; u(z) > = A(g_z)X_{iE_2}.$$

where

$$(\,\mathrm{Re}\; u(z), \,\mathrm{Re}\; u(z)) = (\,\mathrm{Im}\; u(z), \,\mathrm{Im}\; u(z)) = -\eta^2 < 0. \tag{3.8.8}$$

Now, for our quadratic form we want to determine the majorant $(u,u)_z := (u,u)_{X_z}$ with respect to X_z. From the key relation (3.2.7)

$$(u,u)_{AX_0} = (u,u) + 2R(u, AX_0) = {}^t(A^{-1}u)P(A^{-1}u)$$

and (3.8.4), by some calculation, we come to

$$R(u,z) := R(u, X_z) = \frac{|u_1 - u_2z_3 + 2u_3z_2 - u_4z_1 + u_5(z_2^2 - z_1z_3)|^2}{2(y_1y_3 - y_2^2)} = \frac{|\psi_u(z)|^2}{2\eta^2}. \tag{3.8.9}$$

One observes that for $\check{z} = (i, 0, i)$ one has

$$(u,u)_{\check{z}} = (u_1^2 + u_2^2 + 2u_3^2 + u_4^2 + u_5^2). \tag{3.8.10}$$

3.9. Example (2,2). Slightly changing (1.7.1) and (1.7.2) we take

$$Q_0 = \begin{pmatrix} 1 &&& \\ & 1 && \\ && -1 & \\ &&& -1 \end{pmatrix}, \quad Q = \begin{pmatrix} &&& 1 \\ && -1 & \\ & -1 && \\ 1 &&& \end{pmatrix}, \quad C = (1/\sqrt{2}) \begin{pmatrix} 1 &&& 1 \\ & -1 & 1 & \\ & 1 & 1 & \\ 1 &&& -1 \end{pmatrix},$$

and

$$V = \{M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a,b,c,d \in \mathbb{R}\} \simeq \mathbb{R}^4$$

with

$$M \equiv \mathbf{a} := {}^t(a,b,c,d) = (1/\sqrt{2})^t(x_1+x_4, -x_2-x_3, x_2-x_3, x_1-x_4), \text{ i.e., } \mathbf{a} = Cx$$

and

$$(3.9.1)\qquad \begin{aligned}(M,M') &= (\mathbf{a},\mathbf{a'}) = (ad'+a'd-bc'-b'c) = {}^t\mathbf{a}Q\mathbf{a'},\\ (M,M) &= 2\det M\\ q(\mathbf{a}) &= (1/2)(\mathbf{a},\mathbf{a}) = \det M = x_1^2+x_2^2-x_3^2-x_4^2.\end{aligned}$$

We fix again as base point of the space $\mathbb{D}$ of oriented negative 2-planes $X = \langle v_1, v_2\rangle$ in V the negative 2-plane X^0 spanned by $v_1^0 := M_1 = \begin{pmatrix}1 & \\ & -1\end{pmatrix}$ and $v_2^0 := M_2 = \begin{pmatrix} & -1\\ -1 & \end{pmatrix}$ with $(M_1,M_1) = (M_2,M_2) = -2 = -\eta$.
We stick to the action of $\bar{G} = \mathrm{SL}(2,\mathbb{R})^2$ on V via $M \mapsto g\cdot M = M^g := g_1 M {}^t g_2 =: M'$ and one has a map $\mathbf{a} \mapsto \mathbf{a}' = A(g)\mathbf{a}$ with $A(g)$ as in (1.7.8) and get

$$(3.9.2)\qquad A(z) := A(g_{z_1}, g_{z_2}) = (1/\sqrt{y_1y_2})\begin{pmatrix} y_1y_2 & y_1x_2 & x_1y_2 & x_1x_2\\ & y_1 & & x_1\\ & & y_2 & x_2\\ & & & 1\end{pmatrix}$$

and, hence,

$$g_{z_1}M_1{}^tg_{z_2} = (y_1y_2)^{-1/2}\begin{pmatrix} y_1y_2 - x_1x_2 & -x_1\\ -x_2 & -1\end{pmatrix} = -(y_1y_2)^{-1/2}\,\mathrm{Re}\, Z =: M_1(z)$$

$$g_{z_1}M_2{}^tg_{z_2} = (y_1y_2)^{-1/2}\begin{pmatrix} -x_1y_2 - x_2y_1 & -y_1\\ -y_2 & 0\end{pmatrix} = -(y_1y_2)^{-1/2}\,\mathrm{Im}\, Z =: M_2(z).$$

where Z is given as above by $Z = \begin{pmatrix} z_1z_2 & z_1\\ z_2 & 1\end{pmatrix}$. One has

$$(3.9.3)\qquad \begin{aligned}(M, M_1(z)) &= (\eta/(\sqrt{y_1y_2})(-a - bx_2 - cx_1 - d(x_1x_2 - y_1y_2))\\ (M, M_2(z)) &= (\eta/(\sqrt{y_1y_2})(-by_2 - cy_1 - d(x_1y_2 + y_1x_2))\end{aligned}$$

and by (3.2.6) for the majorant

$$(3.9.4)\qquad \begin{aligned}(M,M)_z &= (M,M) + 2((M,M_1(z))^2 + (M,M_2(z))^2)/(2)\\ &= (1/(y_1y_2))|a - bz_2 - cz_1 + dz_1z_2|^2 + 2(ad-bc).\end{aligned}$$

The same way, we can take

$$(3.9.5)\qquad A(g_{z_1}^{-1}, g_{z_2}^{-1}) = (1/\sqrt{y_1y_2})\begin{pmatrix} 1 & x_2 & -x_1 & x_1x_2\\ & y_2 & & x_1y_2\\ & & y_1 & -y_1x_2\\ & & & y_1y_2\end{pmatrix}$$

and via

$$\mathbf{a}' := A(g_{z_1}^{-1}, g_{z_2}^{-1})\mathbf{a} \tag{3.9.6}$$
$$= (1/\sqrt{y_1y_2}) \begin{pmatrix} a + bx_2 - cx_1 + dx_1x_2 \\ by_2 + dx_1y_2 \\ cy_1 - dy_1x_2 \\ dy_1y_2 \end{pmatrix}$$

with $P_0 = E_4$ get as above

$$^t\mathbf{a}'P_0\mathbf{a}' = (M, M)_z \tag{3.9.7}$$

and

$$(\mathbf{a}, \mathbf{a})_{X_z} = (\mathbf{a}, \mathbf{a}) + 2R(\mathbf{a}, z),$$
$$R(M, z) = R(\mathbf{a}, z) = (1/(2y_1y_2))|a - bz_2 - cz_1 + dz_1z_2|^2 \tag{3.9.8}$$

In particular, one has

$$(\mathbf{a}, \mathbf{a})_{(i,i)} = 2(ad - bc) + ((a-d)^2 + (b-c)^2 = a^2 + b^2 + c^2 + d^2$$
$$= x_1^2 + x_2^2 + x_3^2 + x_4^2. \tag{3.9.9}$$

3.10. Example (1,2). Here we have

$$\tilde{V} = \{M = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}; a, b, c \in \mathbb{R}\},$$

$$\det M = -(a^2 + bc) = (M, M) = {}^t\mathbf{a}\tilde{Q}\mathbf{a}, \tilde{Q} = -\begin{pmatrix} 1 & & \\ & & 1/2 \\ & 1/2 & \end{pmatrix}$$

$$(M, M') = -(aa' + bc'/2 + cb'/2) \tag{3.10.1}$$

and we identify $M = \mathbf{a} = {}^t(a, b, c),\ x = {}^t(x_1, x_2, x_3)$. With $a = x_3, b = x_2 + x_1, c = x_2 - x_1$ one has

$$-a^2 - bc = x_1^2 - x_2^2 - x_3^2.$$

As fixed in (1.12.2), one has a homomorphism

$$G' = \mathrm{SL}(2, \mathbb{R}) \longrightarrow \tilde{G} = \mathrm{SO}(\tilde{Q}),\ g \longmapsto A(g)$$

where, in particular, for $z = x + iy$, $g_z = \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix}$, we get

$$A(g_z) = \begin{pmatrix} 1 & 0 & x/y \\ -2x & y & -x^2/y \\ 0 & 0 & 1/y \end{pmatrix},\ A(g_z^{-1}) = \begin{pmatrix} 1 & 0 & -x \\ 2x/y & 1/y & -x^2/y \\ 0 & 0 & y \end{pmatrix}, \tag{3.10.2}$$

Looking at $\tilde{V}$ of signature $(p,2)$ with $p = 1$, we take $\mathbb{D}$ as space of negative oriented 2-planes $X \subset V$, and as base point the plane

$$X_i := \langle M_1, M_2 \rangle,\;\; M_1 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, M_2 = \begin{pmatrix} & -1 \\ -1 & \end{pmatrix}.$$

Via g_z this plane is transported to

$$X_z :=< M_1', M_2' >,\;\; M_1' = M_1(z) := (1/y)\begin{pmatrix} y & -2xy \\ & -y \end{pmatrix}, M_2' = M_2(z) = (1/y)\begin{pmatrix} -x & x^2 - y^2 \\ -1 & x \end{pmatrix}.$$

i.e., for $Z = (1/y)\begin{pmatrix} -\bar{z} & \bar{z}^2 \\ -1 & \bar{z} \end{pmatrix}$ one has $X_z = \langle \operatorname{Im} Z, \operatorname{Re} Z \rangle$.

To determine the majorant, as usual, we decompose $M \in \tilde{V}$ with respect to the negative 2-plane X_z into its positive and negative parts

$$M = M' + \alpha M_1(z) + \beta M_2(z).$$

We get

$$\begin{aligned} \alpha &= -(1/2y)(M, M_1(z)) = -(1/2y)(2ax + b - c(x^2 - y^2)) \\ \beta &= -(1/2y)(M, M_2(z)) = -(1/2y)(-2ay + 2cxy) \end{aligned}$$

and, hence, the majorant

$$\begin{aligned} (M, M)_z &= (M, M) + (1/(2y^2))((M, M_1(z))^2 + (M, M_2(z))^2) \\ &= -(a^2 + bc) + (1/2y^2)(4a^2(x^2 + y^2) + b^2 + c^2(x^2 + y^2)^2 \\ &\qquad + 4abx - 4acx(x^2 + y^2) - 2bc(x^2 - y^2) \\ &=: (M, M) + 2R(M, z) \end{aligned} \tag{3.10.3}$$

i.e.,

$$R(M, z) = R(M, X_z) = (1/4y^2)|2az + b - cz^2|^2. \tag{3.10.4}$$

In particular, one has, as to be expected,

$$(M, M)_i = (a^2 + b^2/2 + c^2/2) = 2(x_1^2 + x_2^2 + x_3^2) \tag{3.10.5}$$

3.11. Using the signature $(1,q)$ version for the majorant, we take $(M, M) = -(a^2 + bc)$, $v_0 = {}^t(0, 1, -1)$ and have $(v_0, v_0) = \eta = 1 > 0$. With (3.10.2) we get $M(z) := A(g_z)v_0 = (1/y)^t(-x, |z|^2, -1)$ and from (3.3.2)

$$\begin{aligned} (M, M(z)) &= -(1/(2y))(-2ax - b + c|z|^2) \\ (M, M)_z &= 2(M, M(z))^2 - (M, M) \\ &= (1/(2y^2))(a^2(4x^2 + 2y^2) + b^2 + c^2|z|^4 + 4abx - 4acx|z|^2 - 2bcx^2) \\ R^1(M, z) &= (M, M(z))^2 \\ &= (1/2)((M, M)_z + (M, M) = (1/(4y^2))(2ax + b - c|z|^2)^2 \end{aligned} \tag{3.11.1}$$

3.12. Remark. The same formula for the majorant comes out in both cases (emphazising $p = 1$ or $q = 2$) and also using (1.12.2) resp. (3.10.2) and the 'Key Relation' with $P_0 = \begin{pmatrix} 1 & & \\ & 1/2 & \\ & & 1/2 \end{pmatrix}$

$$\begin{aligned} \mathbf{a}' &= A(g_z^{-1})\mathbf{a} = (1/y)\begin{pmatrix} ay - cxy \\ a2x + b - cx^2 \\ cy^2 \end{pmatrix} \\ {}^t\mathbf{a}' P_0 \mathbf{a}' &= (1/2y^2)(a^2(4x^2+2y^2) + b^2 + c^2|z|^4 + 4abx - 4acx|z|^2 - 2bcx^2 \\ &= -(a^2+bc) + (1/2y^2)|2az + b - cz^2|^2 \\ &= (a^2+bc) + (1/2y^2)(2ax + b - c|z|^2)^2 \end{aligned}$$

(3.12.1)

In [BF] one has the same result in a slightly different shape, namely

(3.12.2) $$R(z, M) = (1/2y^2)(c|z|^2 - 2ax - b)^2 + 2(a^2 + bc)$$

3.13. Example (1,2)bis. To follow the same procedure as in the other examples, we use the alternative action from 1.6 $M \mapsto gM^tg$. For

(3.13.1) $$M = \mathbf{a} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, (M, M) = ac - b^2,\ M \mapsto gM^tg = M' = (A(g)\mathbf{a},$$

with (1.6.6), we come to

$$\mathbf{a}' = A(g_z^{-1})\mathbf{a} = (1/y)\begin{pmatrix} a/y - 2bx/y + cx^2/y \\ b - cx \\ cy \end{pmatrix}$$

$$(M, M)_z = (\mathrm{a}, \mathrm{a})_z = {}^t\mathbf{a}' P_0 \mathbf{a}'$$

(3.13.2) $$R(M, z) = ((M, M)_z - (M, M))/2 = (1/4y^2)|a - 2bz + z^2|^2.$$

3.14. Example (1,3). Here we take

$$\mathcal{V} = \{M = M(y) = \begin{pmatrix} y_1 & y_2 + iy_3 \\ y_2 - iy_3 & y_4 \end{pmatrix}; y_1, y_2, y_3, y_4 \in \mathbb{R}\},$$

$$\det M = (M, M) = {}^tyQy = (y_1y_4 - y_2^2 - y_3^2), \quad Q = \begin{pmatrix} & & & 1/2 \\ & -1 & & \\ & & -1 & \\ 1/2 & & & \end{pmatrix}$$

(3.14.1) $(M, M') = (y_1y_4'/2 - y_2y_2' - y_3y_3' + y_4y_1'/2).$

As in (1.15.2) $G' = \mathrm{SL}(2,\mathbb{C})$ acts transitively on $\mathbb{H}^+ = \mathbb{C} \times \mathbb{R}_{>0}$ where $g_P(0,1) = (z,r) = P$ for $g_P = \begin{pmatrix} r^{1/2} & zr^{-1/2} \\ & r^{-1/2} \end{pmatrix}$. As fixed in (1.16.2), one has a homomorphism

$$\rho : G' = \mathrm{SL}(2,\mathbb{C}) \longrightarrow \tilde{G} = \mathrm{SO}(Q),\ g \longmapsto \rho(g)$$

given by $(g, M(y)) \mapsto gM(y)^t\bar{g} =: M(\rho(g)y)$. From (1.17.2) we get

$$\text{(3.14.2)} \qquad \rho(g_P) = \begin{pmatrix} r & 2x & 2y & |z|^2/r \\ & 1 & & x/r \\ & & 1 & y/r \\ & & & 1/r \end{pmatrix},\ \rho(g_P^{-1}) = \begin{pmatrix} 1/r & -2x/r & -2y/r & |z|^2/r \\ & 1 & 0 & -x \\ & & 1 & -y \\ & & & r \end{pmatrix}.$$

and

(3.14.3)

$$\rho(g_P)y = \begin{pmatrix} ry_1 + 2xy_2 + 2yy_3 + |z|^2y_4/r \\ y_2 + xy_4/r \\ y_3 + yy_4/r \\ y_4/r \end{pmatrix},\ \rho(g_P^{-1})y = \begin{pmatrix} y_1/r - 2xy_2/r - 2yy_3/r + |z|^2y_4/r \\ y_2 - xy_4 \\ y_3 - yy_4 \\ ry_4 \end{pmatrix}.$$

For $M_0 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = M(v_0), v_0 := {}^t(1,0,0,1)$ one has $(M_0, M_0) = 1 > 0$ and

$$\text{(3.14.4)} \qquad M(P) := M(\rho(g_P)v_0) = \rho(g_P)v_0 = (1/r)\begin{pmatrix} r^2 + |z|^2 \\ x \\ y \\ 1 \end{pmatrix}$$

$$(M, M(P)) = (1/2r)(y_1 - 2xy_2 - 2yy_3 + (r^2 + |z|^2)y_4).$$

From (3.4.3) one has the majorant

$$\begin{aligned} \text{(3.14.5)} \qquad (M,M)_P &= 2(M, M(P))^2 - (M,M) = 2R^1(M,P) - (M,M) \\ &= (1/(2r^2))(y_1^2 + (4x^2 + 2r^2)y_2^2 + (4y^2 + 2r^2)y_3 + (r^2 + |z|^2)^2y_4^2 \\ &\quad - 4xy_1y_2 - 4yy_1y_3 + 2|z|^2y_1y_4 \\ &\quad + 8xyy_2y_3 - 4x(r^2 + |z|^2)y_2y_4 - 4y(r^2 + |z|^2)y_3y_4). \end{aligned}$$

The same result comes out evaluating the 'Key Relation', i.e., for the matrix $P_0 = \begin{pmatrix} 1/2 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1/2 \end{pmatrix}$ with $P_0Q^{-1}P_0 = Q$ and $y' := \rho(g_P^{-1})y$ from (3.14.3) one has ${}^ty'P_0y' = (M,M)_P$ as in (3.14.5).
From (3.7.1) and (3.7.2) we have

$$\begin{aligned} \text{(3.14.6)} \qquad R^1(M,P) &= (1/2)((M,M)_P + (M,M)) = (M, M(P))^2 \\ &= (1/(4r^2))(y_1 - 2xy_2 - 2yy_3 + (|z|^2 + r^2)y_4)^2. \end{aligned}$$

3.15. Example (1,1). We take

$$V = \mathbb{R}^2 \ni y = {}^t(y_1, y_2), v_0 = {}^t(1,1)$$
$$(y,y) = {}^t y\hat{Q}y = 2y_1y_2, \quad \hat{Q} = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix},$$
$$\mathrm{SO}(\hat{Q}) \ni g(t) = \begin{pmatrix} t & \\ & 1/t \end{pmatrix}, \; g(t)v_0 = {}^t(t, 1/t) =: v_t, \; g(t)y = {}^t(y_1 t, y_2/t) =: y_t,$$

(3.15.1) $\qquad \mathbb{D} = \{\langle v\rangle \subset \mathbb{R}^2; (v,v) > 0\}.$

With $P_0 = E$, the Key relation gives the majorant

$$(y,y)_t = {}^t y_{-t} \cdot y_{-t} = y_1^2/t^2 + y_2^2 t^2,$$

and one has

(3.15.2) $\qquad R^1(y,t) = (1/2)((y,y)_t + (y,y)) = (1/2)(y_1/t + y_2 t)^2.$

For

(3.15.3)

$$V = \mathbb{R}^2 \ni x = {}^t(x_1, x_2), v_0 = {}^t(1,0)$$
$$(x,x) = {}^t xQx = x_1^2 - x_2^2, \quad Q = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix},$$
$$\mathrm{SO}(Q) \ni g(t) = \begin{pmatrix} \operatorname{ch} t & \operatorname{sh} t \\ \operatorname{sh} t & \operatorname{ch} t \end{pmatrix}, \; g(t)v_0 = {}^t(\operatorname{ch} t, \operatorname{sh} t) =: v_t, \; g(t)x = {}^t(x_1 \operatorname{ch} t, x_2 \operatorname{sh} t) =: x_t,$$
$$\mathbb{D} = \{\langle v\rangle \subset \mathbb{R}^2; (v,v) > 0\},$$

with $P_0 = E$, the Key relation gives the majorant

$$(x,x)_t = {}^t x_{-t} \cdot x_{-t} = ((\operatorname{ch} t)^2 + (\operatorname{sh} t)^2)(x_1^2 + x_2^2) - 4\operatorname{ch} t \cdot \operatorname{sh} t \cdot x_1 x_2$$

and one has

(3.15.4) $\qquad R^1(x,t) = (1/2)((x,x)_t + (x,x)) = (x_1 \operatorname{ch} t - x_2 \operatorname{sh} t)^2.$

3.16. Example Summary. (i) From (3.8.9) one has

(3.16.1)

$$(u,u) = {}^t uQu = 2(u_3^2 - u_1u_5 - u_2u_4), \quad \mathbb{D}^+ \simeq \mathbb{H}_2 \ni z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix},$$
$$R(u,z) = (1/2)((u,u)_z - (u,u)) = \frac{|u_1 - u_2z_3 + 2u_3z_2 - u_4z_1 + u_5(z_2^2 - z_1z_3)|^2}{2(y_1y_3 - y_2^2)},$$
$$w(z) = {}^t(1, z_1, z_2, z_3, -z_2^2 + z_1z_3),$$
$$(x,x) = x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5^2,$$
$$R(x,z) = \frac{|x_1(1 + (z_2^2 - z_1z_3)) + x_2(z_1 - z_2) + x_3z_2 + x_4(-z_1 - z_2) + x_5(1 + (z_2^2 - z_1z_3)|^2}{4(y_1y_3 - y_2^2)}$$

(ii) From (3.9.8) one has

$$(M,M) = {}^t\mathbf{a}Q\mathbf{a} = 2(ad-bc), \quad \mathbb{D}^+ \simeq \mathbb{H}^2 \ni z = (z_1, z_2),$$
$$R(M,z) = R(\mathbf{a},z) = (1/(2y_1y_2))|a - bz_2 - cz_1 + dz_1z_2|^2$$
$$w(z) = {}^t(1, z_1, z_2, z_1z_2), \tag{3.16.2}$$

(iii) From (3.13.2) one has

$$(M,M) = {}^t\mathbf{a}Q\mathbf{a} = b^2 - ac, \quad \mathbb{D}^+ \simeq \mathbb{H} \ni z,$$
$$R(M,z) = R(M,X_z) = (1/4y^2)|a - 2bz + cz^2|^2, R^1(M,X_z) = (1/(4y^2))(a - bx - c|z|^2)^2$$
$$w(z) = (1, z, z^2)$$

(iv) From (3.14.6 we get

$$(M,M) = {}^tyQy = (y_1y_4 - y_2^2 - y_3^2), \quad \mathbb{D}^+ \simeq \mathbb{H}^+ \ni P = (x+iy, r),$$
$$R^1(M,P) = (1/2)(M,M)_P + (M,M)(= (1/(4r^2))(y_1 - 2xy_2 - 2yy_3 + (|z|^2 + r^2)y_4)^2$$
$$u(P) = {}^t(1, x, y, r^2 + x^2 + y^2).$$

(v) From (3.15.1) and (3.15.3) we have

$$(y,y) = {}^ty\hat{Q}y = 2y_1y_2, \quad \mathbb{D}^+ \simeq \mathbb{R}^* \ni t,$$
$$R^1(y,t) = (1/2)((y,y)_t + (y,y)) = (1/2)(y_1/t + y_2t)^2,$$
$$(x,x) = {}^txQx = x_1^2 - x_2^2, \qquad \mathbb{D}^+ \simeq \mathbb{R} \ni t,$$
$$R^1(x,t) = (1/2)((x,x)_t + (x,x)) = (x_1 \mathrm{ch}\, t - x_2 \mathrm{sh}\, t)^2.$$

4 Kudla's Green function

4.1. We go back to our original situation in (1.8.3) and (1.8.1) with $V = \mathbb{R}^5$ and the quadratic form $\tilde{q}(u) = (1/2){}^tu\tilde{Q}u = u_3^2 - u_2u_4 - u_1u_5$ The symplectic group $\check{G} = \mathrm{Sp}(2,\mathbb{R})$ acts (transitively) on (16.23.1) $\mathcal{V} \simeq V$ via

$$(g, M(u)) \longmapsto gM(u){}^tg =: M(A(g)u) =: M(u') \tag{4.1.1}$$

preserving the quadratic form $\tilde{q}$. As usual, this leads to a homomorphism $\check{G} \longrightarrow \tilde{G}$ where $g \in \check{G}$ is mapped to the matrix $A(g)$ with $u' = A(g)u$. We coordinatize the homogeneous space $\mathbb{D} := \tilde{G}/\tilde{K} = \{X(z) \text{ oriented negative plane in } V\}$ via $z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \in \mathbb{H}_2$.

4.2. We look at the majorant of the quadratic form $\tilde{q}(u)$. From (3.16.1) we know that the majorant $(u,u)_z = (M,M)_z$ with respect to the negative plane $X(z)$ belonging to the matrix $Z = M(u(z))$ is given by

$$(M,M)_z =: (M,M) + 2R(u,z), \tag{4.2.1}$$

where

(4.2.2)

$$R(u,z) = R(u;z_1,z_2,z_3) = \frac{|u_1 - u_2z_3 + 2u_3z_2 - u_4z_1 + u_5(z_2^2 - z_1z_3)|^2}{2(y_1y_3 - y_2^2)} =: \frac{|\psi_u(z)|^2}{2\eta^2}.$$

4.3. Remark. As usual, for $x \in V$ with positive norm, one has the special divisor on $\mathbb{D}$

(4.3.1)
$$\begin{aligned} Z(x) &= \{z \in \mathbb{D}; z \perp x\} \\ &= \{z \in \mathbb{D}; R(x,z) = 0\}. \end{aligned}$$

This comes out as follows. In our parametrization by $\mathbb{H}_2$, elements of $\mathbb{D}$ are negative planes $X_z = \langle \operatorname{Re} u(z), \operatorname{Im} u(z)\rangle, z \in \mathbb{H}_2$ with

(4.3.2)
$$u_1(z) = z_2^2 - z_1z_3, u_2(z) = -z_1, u_3(z) = -z_2, u_4(z) = -z_3, u_5(z) = 1.$$

Hence,

(4.3.3)
$$\begin{aligned} Z(x) &= \{z \in \mathbb{D}(V); z \perp x\}, \\ &\simeq \{z \in \mathbb{H}_2; {}^t u(z)\tilde{Q}x = 0\}, \\ &\simeq \{z \in \mathbb{H}_2; \psi_{u(z),x} = x_1 - x_2z_3 + 2x_3z_2 - x_4z_1 + x_5(z_2^2 - z_1z_3) = 0\}. \end{aligned}$$

This is a *Humbert equation* for the tuple $(x_1, x_2, 2x_3, x_4, x_5)$, i.e., with discriminant $\Delta = 4x_3^2 - 4x_1x_5 - 4x_2x_4 = 4\tilde{q}(x)$.

4.4. It is well known that $\Gamma = \mathrm{Sp}_2(\mathbb{Z})$ acts on the set of Humbert surfaces $\mathcal{H}(y)$ with equations $H(y) = y_1 - y_2z_3 + y_3z_2 - y_4z_1 + y_5(z_2^2 - z_1z_3) = 0$ with primitive integer quintuples y. $\Gamma-$orbits with discriminants $\Delta(y) = 4m$ are parametrized by $a_m = (1,0,0,0,-m), m \neq 0$, and those with discriminant $\Delta(y) = 4m+1$ by $a'_m = (0,1,1,-m,0)$. From the discussion of the coefficients of the Eisenstein series we have to look at $u \in L'$ with $\tilde{q}(u) = m$ and $m \in \mathbb{Z}$ and $m \in \mathbb{Z} + 1/4$. Hence, for $m \in \mathbb{Z}$ we put

(4.4.1)
$$\mathcal{Z}(0,m) = \sum_{n, n^2|m} \sum_{\gamma \in \Gamma/\Gamma_{a_{m/n^2}}} Z(n\gamma a_{m/n^2}),$$

and for $m = M + 1/4, M \in \mathbb{Z}$

(4.4.2)
$$\mathcal{Z}(1,m) = \sum_{n, n^2|4m} \sum_{\gamma \in \Gamma/\Gamma_{a'_{4m/n^2}}} Z(n\gamma a'_{4m/n^2}),$$

With $d_\gamma = 1$, resp. $= 2$ for $m \in \mathbb{Z}$ or $m = M + 1/4, M \in \mathbb{Z}$, we also put

(4.4.3)
$$L(\gamma, m) = \sum_{n, n^2 | d_\gamma^2 m} n L^*_{\gamma, m/n^2},$$
$$L^*_{\gamma,m} = \{y \in L_{\gamma,m}; \gcd(y) = 1\}$$
$$L_{0,m} = \{y \in \mathbb{Z}^5; \hat{q}(y) = y_3^2 - 4y_1y_5 - 4y_2y_4 = 4m\} \quad \text{for } m \in \mathbb{Z}$$
$$L_{1,m} = \{y \in \mathbb{Z}^5; \hat{q}(y) = y_3^2 - 4y_1y_5 - 4y_2y_4 = 4M + 1 = 4m\} \quad \text{for } m \in \mathbb{Z} + 1/4.$$

4.5. From Kudla's work we know that $\beta(2\pi v R(z, x))$ is a Green function on $\mathbb{D}$ for the cycle $Z(x)$. Hence, we introduce a two component Kudla Green function on $X = \Gamma \backslash \mathbb{D}$

(4.5.1)
$$\Xi(0, m, v, z) = (1/2) \sum_{x \in L_{0,m}} \beta(2\pi v R(x, z)) \quad \text{for } m \in \mathbb{Z},$$
$$\Xi(1, m, v, z) = (1/2) \sum_{x \in L_{1,m}} \beta(2\pi v R(x, z)) \quad \text{for } m \in \mathbb{Z} + 1/4.$$

4.6. Aim. We want to compare the $s-$derivatives of the Fourier coefficients $c(\gamma, m, s, v)$ (in $s = 0$) of the Eisenstein series with the Green function integrals

(4.6.1)
$$I(0, m, v) := \int_X \Xi(0, m, v, z) d\mu_z = (1/2) \sum_{x \in L_{0,m}} \int_X \beta(2\pi v R(x, z)) d\mu_z,$$
$$I(1, m, v) := \int_X \Xi(1, m, v, z) d\mu_z = (1/2) \sum_{x \in L_{1,m}} \int_X \beta(2\pi v R(x, z)) d\mu_z.$$

4.7. Using (4.4.1) and (4.4.2) and unfolding $X = \Gamma \backslash \mathbb{D}$, one has (by the invariance of $R(x, z)$)

(4.7.1)
$$I(0, m, v) = (1/2) \sum_{n, n^2 | m} \int_{\Gamma_{a_{m/n^2}} \backslash \mathbb{D}} \beta(2\pi v R(n a_{m/n^2}, z)) d\mu_z,$$
$$I(1, m, v) = (1/2) \sum_{n, n^2 | 4m} \int_{\Gamma_{a'_{4m/n^2}} \backslash \mathbb{D}} \beta(2\pi v R(n a'_{4m/n^2}, z) d\mu_z.$$

As a first step, we try to describe the unit groups appearing here and their covolume.

5 Notions from Number Theory

To prepare the field, we collect some elementary facts from number theory (see e.g.[Za]):

For square free rational integer $d_0 \neq 0$ the maximal order in $F = \mathbb{Q}(j), j = \sqrt{d_0}$ is given by

$$(5.0.1) \qquad \mathcal{O} = \mathbb{Z} + \omega\,\mathbb{Z}, \quad \omega = j \quad \text{if } d_0 \equiv 2, 3 \bmod 4 \,(\text{Case } 1)$$
$$\omega = \frac{1+j}{2} \text{ if } d_0 \equiv 1 \bmod 4 \quad (\text{Case } 2).$$

The discriminant of F is given by $d_F = 4d_0$ in Case 1 and $d_F = d_0$ in Case 2. Such a discriminant also is called a *fundamental discriminant* and it is either $\equiv 0$ or $\equiv 1$ mod 4.

For an $\mathbb{Z}-$module $M \subset F$, the *complementary module* M^* is given by $M^* = \{x \in F, \mathrm{Tr}(xy) \in \mathbb{Z} \text{ for all } y \in M\}$ and $(M^*)^{-1} = \{x \in F; (xy) \in M \text{ for all } y \in M^*\}$ is the *different.* In particular, the ideal $\mathfrak{D} := (\mathcal{O}^*)^{-1}$ is the *different of* F. Hence, for Case 1 with $\mathcal{O} = \mathbb{Z}[j] = \mathbb{Z} + j\,\mathbb{Z}$, one has

$$(5.0.2) \qquad \mathcal{O}^* = (1/(2j))\mathcal{O} = (1/2)\mathbb{Z} + (1/(2j))\mathbb{Z},$$
$$\mathfrak{D} = (\mathcal{O}^*)^{-1} = 2j\,\mathcal{O} = 2j\,\mathbb{Z} + 2d\mathbb{Z}.$$

Then, in Case 2 with

$$(5.0.3) \qquad \mathcal{O} = \mathbb{Z}[\omega] = \mathbb{Z} + (j+1)/2\,\mathbb{Z},$$
$$= \{x = (2x_1 + x_2)/2 + x_2 j/2;\; x_1, x_2 \in \mathbb{Z}\}$$

one has

$$(5.0.4) \qquad \mathcal{O}^* = (1/j)\mathcal{O}$$
$$\mathfrak{D} = (\mathcal{O}^*)^{-1} = j\,\mathcal{O}$$

And for $M_f = \mathbb{Z} + fj\mathbb{Z} = \{a = a_1 + fja_2;\; a_1, a_2 \in \mathbb{Z}\}$, one has

$$(5.0.5) \qquad M_f^* = (1/(2fj))M_f = \{c = c_2/2 + c_1/(2fd_0)j;\; c_1, c_2 \in \mathbb{Z}\},$$
$$(M_f^*)^{-1} = 2fjM_f = \{b = b_2 2f^2 d_0 + b_1 2fj;\; b_1, b_2 \in \mathbb{Z}\}.$$

We also need the *order with conductor f,* $f \in \mathbb{N}$, defined by

$$\mathcal{O}_f := \mathbb{Z} + f\omega\mathbb{Z}.$$

Hence, in Case 1, we have

$$(5.0.6) \qquad \mathcal{O}_f = \mathbb{Z} + fj\,\mathbb{Z} = \{a = a_1 + a_2 fj;\; a_1, a_2 \in \mathbb{Z}\},$$
$$\mathcal{O}_f^* = (1/(2fj))M_f = \{c = c_2/2 + c_1/(2fd_0)j;\; c_1, c_2 \in \mathbb{Z}\},$$
$$(\mathcal{O}_f^*)^{-1} = (2fj)M_f = \{b = b_2 2f^2 d_0 + b_1 2fj;\; b_1, b_2 \in \mathbb{Z}\}.$$

Here one has

$$(5.0.7) \qquad a\bar{a} = a_1^2 - f^2 d_0 a_2^2,\; b\bar{b} = 4f^4 d^2 b_2^2 - 4f^2 d_0 b_1^2,\; c\bar{c} = c_2^2/4 - c_1^2/(4f^2 d_0)$$

And in Case 2, $d_0 = 1 + 4M, M \in \mathbb{Z}$,

$$
\begin{aligned}
(5.0.8) \qquad \mathcal{O}_f &= \mathbb{Z} + f((1+j)/2)\mathbb{Z} = \{a = (2a_1 + fa_2)/2 + fa_2 j/2;\ a_1, a_2 \in \mathbb{Z}\},\\
\mathcal{O}_f^* &= (1/(fj))\mathcal{O}_f = \{c = c_2/2 + (2c_1 + fc_2)j/(2fd_0);\ c_1, c_2 \in \mathbb{Z}\},\\
\mathfrak{D}_f = (\mathcal{O}_f^*)^{-1} &= (fj)\mathcal{O}_f = \{b = b_2 f^2 d_0/2 + (2b_1 + fb_2)fj/2;\ b_1, b_2 \in \mathbb{Z}\},
\end{aligned}
$$

with

$$
\begin{aligned}
(5.0.9) \qquad a\bar{a} &= a_1^2 + fa_1a_2 - f^2a_2^2M,\\
b\bar{b} &= -(b_1^2 + fb_1b_2 + f^2b_2^2M)f^2d_0,\\
c\bar{c} &= (c_2^2f^2M - c_1^2 - fc_1c_2)/(f^2d_0).
\end{aligned}
$$

6 Some classical Unit Group Covolumes

6.1. Given a symmetric matrix $S \in M_n(\mathbb{Q})$, we associate to it the *unit group*

$$(6.1.1) \qquad \Gamma(S) = \{A \in \mathrm{SL}_n(\mathbb{Z});\ {}^tASA = S\}.$$

As we shall see later while determining the Green function integrals, we are led to treat two special cases:

Case A. We take

$$
\begin{aligned}
2q(u) = 2\hat{q}(u) &= u_3^2 - 4u_2u_4 - 4u_1u_5 = {}^tu\hat{Q}u = (u,u),\\
S = \hat{Q} &= \begin{pmatrix} &&&&-2\\ &&&-2&\\ &&1&&\\ &-2&&&\\ -2&&&& \end{pmatrix},\\
\hat{G} = \mathrm{SO}(\hat{Q}) &= \{g \in \mathrm{SL}(5,\mathbb{R});\ {}^tg\hat{Q}g = \hat{Q}\} \simeq \mathrm{SO}(3,2),
\end{aligned}
$$

and

$$
\begin{aligned}
(6.1.2) \qquad a_m &= {}^t(1,0,0,0,-m),\ m \in \mathbb{Z}\backslash\{0\},\ \text{ i.e. } 2\hat{q}(a_m) = 4m,\\
\hat{G}_{a_m} &= \{g \in \hat{G};\ ga_m = a_m\},\\
\hat{\Gamma}_{a_m} &= \Gamma(\hat{Q}, a_m) = \{W \in \Gamma(\hat{Q});\ Wa_m = a_m\},
\end{aligned}
$$

Case B. We take

$$
\begin{aligned}
(6.1.3) \qquad a'_m &= {}^t(0,-1,1,M,0),\ M \in \mathbb{Z}\backslash\{0\},\ \text{ i.e. } 2\hat{q}(a'_m) = 4m = 4M+1,\\
\hat{G}_{a'_m} &= \{g \in \hat{G};\ ga'_m = a'_m\},\\
\hat{\Gamma}_{a'_m} &= \Gamma(\hat{Q}, a'_m) = \{W \in \Gamma(\hat{Q});\ Wa'_m = a'_m\},
\end{aligned}
$$

and, for $a = a_m$ resp. $a = a'_m$, and $\mathbb{D}_a$ the symmetric space belonging to $\hat{G}_a$, we want to know

$$\text{(6.1.4)} \qquad \kappa_m = \mathrm{vol}(\hat{\Gamma}_a \backslash \mathbb{D}_a).$$

As $\hat{G}_{a_m}$ and $\hat{G}_{a'_m}$ are either isomorphic to $\mathrm{SO}(2,2)$ or to $\mathrm{SO}(3,1)$ we are in the world of Hilbert or Bianchi groups.

6.2. Remark. In the book by Elstrodt, Grunewald and Mennicke [EGM], one has in their Theorem 1.1 in Chapter 7 the following result:
Let K be an imaginary quadratic field of discriminant $d_K < 0$ and let $\mathcal{O}$ be its ring of integers. Then the covolume of the group $\mathbf{PSL}(2,\mathcal{O})$ (in its action on the 3-dimensional hyperbolic space $\mathbb{H}^+$) is

$$\begin{aligned}\text{(6.2.1)} \qquad V_{1,3} := \mathrm{vol}(\mathbf{PSL}(2,\mathcal{O})\backslash \mathbb{H}^+) &= \frac{|d_K|^{3/2}}{4\pi^2}\zeta_K(2) \\ &= \frac{|d_K|^{3/2}}{24} L(2,\chi_K),\ L(s,\chi_K) := \sum_{n>0} (\frac{d_K}{n}) n^{-s}.\end{aligned}$$

This is also called Humbert's formula and goes back to a result of Humbert from 1919. Here, with $\mathbb{H}^+ \ni P = (x,y,r)$, the volume is measured with the volume form

$$\text{(6.2.2)} \qquad dv_{\mathbb{H}^+} = \frac{dxdydr}{r^3}.$$

6.3. Remark. From [HG] p.172 for $m = m_0 f^2 > 0, m_0 = d_K$ a fundamental discriminant, $K = \mathbb{Q}(\sqrt{m_0})$, we have

$$\begin{aligned}\text{(6.3.1)} \qquad \mathrm{vol}(\mathrm{SL}(2,(\mathcal{O}_f,\mathcal{O}_f^*)\backslash \mathbb{H}^2) &= 2f^3 \prod_{p|f}(1 - (\frac{m_0}{p})p^{-2})\zeta_{\mathbb{Q}(\sqrt{m_0})}(-1) \\ &= 2f^3 \prod_{p|f}(1 - (\frac{m_0}{p})p^{-2})|d_K|^{3/2} 2^{-2}\pi^{-4}\zeta_{\mathbb{Q}(\sqrt{m_0})}(2) \\ &= f^3 \prod_{p|f}(1 - (\frac{m_0}{p})p^{-2})|d_K|^{3/2} L(2,\chi_{\mathbb{Q}(\sqrt{m_0})})1/(12\pi^2)\end{aligned}$$

where, with $\mathbb{H}^2 \ni (z_1 = x_1 + iy_1, z_2 = x_2 + iy_2)$, the volume is measured by

$$\text{(6.3.2)} \qquad dv_{\mathrm{HG}} = \frac{dx_1dy_1dx_2dy_2}{(2\pi y_1 y_2)^2},$$

and we used the functional equation

$$\text{(6.3.3)} \qquad \zeta_K(-1) = \zeta_K(2) d_K^{3/2}/(4\pi^4).$$

Here $\mathcal{O}_f$ is an order of conductor f, and

$$(6.3.4) \qquad \mathrm{SL}(2,(\mathcal{O}_f,\mathcal{O}_f^*)) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix}; a,d \in \mathcal{O}_f, c \in \mathcal{O}_f^*, b \in \mathcal{O}_f^{*-1}, ad-bc=1\}$$

with

$$(6.3.5) \qquad \mathcal{O}_f^* = \{x \in K; \mathrm{Tr}(xa) \in \mathbb{Z} \text{ for } \ \text{all}\, a \in \mathcal{O}_f\}$$

6.4. The formula above goes back to a result by Siegel. As a special case of [S2] (19), one can conclude that

$$(6.4.1) \qquad \begin{aligned} \mathrm{vol}(\mathrm{SL}(2,\mathcal{O})\backslash\mathbb{H}^2) &= (2/\pi^2)|d_K|^{3/2}\zeta_K(2) = |d_K|^{3/2}L(\chi_K,2)/3 =: V_{2,2} \\ &= 8\pi^2\zeta_K(-1) \quad \text{for}\, d_K>0 \end{aligned}$$

where, here, the volume is measured by

$$(6.4.2) \qquad dv_{\mathbb{H}^2} := \frac{dx_1dy_1dx_2dy_2}{(y_1y_2)^2}.$$

6.5. In [BK] section 4, Bruinier-Kühn treat the general situation: If $\beta \in L'/L$ and $m \in \mathbb{Z}+q(\beta) > 0$, then

$$\mathcal{H}(\beta,m) = \sum_{x\in L+\beta, q(x)=m} x^\perp$$

is a $\Gamma(L)-$invariant divisor on $\mathrm{Gr}_2(V)$, called the *Heegner divisor* of discriminant $(\beta.m)$. It gets the same name as divisor on X_Γ. For a divisor D on X_Γ with volume form ω, one defines

$$\deg(D) = \int_D \omega.$$

By some highly nontrivial manipulations Bruinier-Kühn get in [BK] Proposition 4.8 the mysterious and wonderful result:

$$(6.5.1) \qquad \frac{2}{B}\deg\left(\mathcal{H}(\gamma,m)\right) = -C_0(\gamma,m,0).$$

For the (2,3) case from the example above, this comes down to

$$(6.5.2) \qquad \deg\left(\mathcal{H}(\gamma,m)\right) = -(B/2)C(\gamma,m,0) = -3^{-1}\cdot\pi^{-2}|m|^{3/2}L(\chi_{d_F},2)\sigma_{\gamma,m}(5/2)$$

with

$$B = \int_{X_L}\Omega^3 = \zeta(-1)\zeta(-3) = (1/12)(1/120) = 2^{-5}\cdot3^{-2}\cdot5^{-1}$$

$$(6.5.3) \qquad dv_{\mathrm{BK}} := \Omega^2 = \frac{1}{8\pi^2}\frac{dx_1dy_1dx_2dy_2}{(y_1y_2)^2}.$$

where $\Omega = \frac{1}{4\pi}(\frac{dx_1dy_1}{y_1^2} + \frac{dx_2dy_2}{y_2^2})$, and the generalized divisor sum [BK] (3.28) $\sigma_{\gamma,m}(5/2)$ which will here reappear later.

6.6. Aim. We try to relate these results to our volumes κ_m resp. κ'_m. Led by [BK] (3.24) resp. (2.11.3) and (2.11.4), for $F = \mathbb{Q}(\sqrt{d})$ with fundamental discriminant d_F, we have to treat the two cases:
Case A

(6.6.1) $$4m = d_F f^2, \text{ for } m \in \mathbb{Z}$$

and Case B

(6.6.2) $$4 \cdot 4m = d_F f^2, \text{ for } m = M + 1/4, M \in \mathbb{Z}$$

At first, we come to the following.

6.7. Proposition. For $a = a_m$ and $a = a'_m$, in both cases, we have

(6.7.1) $$\hat{\Gamma}_a \simeq \mathrm{PSL}(2, (\mathcal{O}_f, \mathcal{O}_f^*)).$$

Our proof again goes back to Siegel:

7 Siegel's Approach

7.1. Siegel in [S3] Section 3, on his way to define a *Darstellungsmass*, discusses the following situation: We are given a rational symmetric matrix S belonging to a quadratic form $S[y] = {}^t ySy = (y, y) =: 2q(y)$ of type $(p, q), p + q = n$ and $a \in \mathbb{Z}^n$ (primitive) with $2q(a) = m$. We have the notion of an associated *unit group*

$$\Gamma(S) := \{U \in \mathrm{GL}(n, \mathbb{Z}), {}^tUSU = S\}$$

and want to determine the isotropy group

$$\Gamma_a := \Gamma(S, a) := \{U \in \Gamma(S), Ua = a\}.$$

Siegel proposes to describe the isotropy group Γ_a as a unit group in one dimension less. Following his procedure, this leads to

(7.1.1) $$\Gamma(S, a) \simeq \{W \in \Gamma(K); {}^tWb \equiv b \bmod m\},$$

where one chooses a matrix B such that the matrix A with first column a, i.e., $A = (a, B)$ is unimodular and has

(7.1.2) $$b := {}^tBSa, \; K := {}^tBSB - b{}^tb/m.$$

7.2. We give some details leading to his result: With $G := \begin{pmatrix} m & {}^t b \\ & E \end{pmatrix}$, one has

(7.2.1)
$$ {}^tASA = \begin{pmatrix} {}^t a \\ {}^t B \end{pmatrix} S\,(a,B) = \begin{pmatrix} {}^t aSa & {}^t aSB \\ {}^t BSa & {}^t BSB \end{pmatrix} = \begin{pmatrix} m & {}^t b \\ b & K + b{}^t b/m \end{pmatrix} = {}^tG \begin{pmatrix} m^{-1} & \\ & K \end{pmatrix} G. $$

As for $Ua = a$, one has $UA = U(a,B) = (a, UB)$, an element $U \in \Gamma(S,a)$ has a form

$$ UA = A \begin{pmatrix} 1 & {}^t c \\ & W \end{pmatrix} $$

with integer $(n-1)-$column c and integer $(n-1)\times(n-1)-$matrix W. Introducing this into the condition ${}^tUSU = S$ leads to

$$ {}^tUSU = {}^tA^{-1} \begin{pmatrix} 1 & \\ c & {}^tW \end{pmatrix} {}^tASA \begin{pmatrix} 1 & {}^t c \\ & W \end{pmatrix} A^{-1} = S $$

i.e.,

$$ \begin{pmatrix} 1 & \\ c & {}^tW \end{pmatrix} {}^tASA \begin{pmatrix} 1 & {}^t c \\ & W \end{pmatrix} = {}^tASA $$

and, with (7.1.2),

$$ \begin{pmatrix} 1 & \\ c & {}^tW \end{pmatrix} {}^tG \begin{pmatrix} m^{-1} & \\ & K \end{pmatrix} G \begin{pmatrix} 1 & {}^t c \\ & W \end{pmatrix} = {}^tG \begin{pmatrix} m^{-1} & \\ & K \end{pmatrix} G $$

and, moreover,

$$ \begin{pmatrix} 1 & \\ c & {}^tW \end{pmatrix} \begin{pmatrix} m & {}^t b \\ b & K + b{}^t b/m \end{pmatrix} \begin{pmatrix} 1 & {}^t c \\ & W \end{pmatrix} = {}^tG \begin{pmatrix} m^{-1} & \\ & K \end{pmatrix} G = \begin{pmatrix} m & {}^t b \\ b & K + b{}^t b/m \end{pmatrix}. $$

Evaluating the left hand side and comparing to the right hand side, asks for the conditions from the remark above

$$ mc + {}^tWb = b,\ {}^tWKW = K. $$

We evaluate this in some cases near to our situation.

7.3. Example. We look again at

$$
\begin{aligned}
\tilde{q}(u) &= (1/2)(u,u) = (1/2)^t u\tilde{Q}u = u_3^2 - u_2u_4 - u_1u_5,\\
S = \tilde{Q} &= \begin{pmatrix} &&&&-1\\ &&&-1&\\ &&2&&\\ &-1&&&\\ -1&&&& \end{pmatrix},\\
\tilde{G} &= \mathrm{SO}(\tilde{Q}) = \{g \in \mathrm{SL}(5,\mathbb{R}); {}^t g\tilde{Q}g = \tilde{Q}\} \simeq \mathrm{SO}(3,2),\\
\Gamma(\tilde{Q}) &= \{W \in \mathrm{SL}(5,\mathbb{Z}); {}^t W\tilde{Q}W = \tilde{Q}\},\\
a &= a_m = {}^t(1,0,0,0,-m),\ m \in \mathbb{Z}\backslash 0,\ \text{ i.e. } q(a) = \tilde{q}(a_m) = m,\\
\check{G}_a &= \{g \in \tilde{G}; ga = a\},\\
\tilde{\Gamma}_a &= \Gamma(S,a) = \Gamma(\tilde{Q},a_m) = \{U \in \Gamma(\tilde{Q}); Ua = a\}
\end{aligned}
$$

We take

$$
A = (a,B) = \begin{pmatrix} 1&&&&\\ &1&&&\\ &&1&&\\ &&&1&\\ -m&&&&1 \end{pmatrix}, \tag{7.3.1}
$$

and, by Siegel's prescription, get

$$
b = {}^t BSa = \begin{pmatrix} 0\\0\\0\\-1 \end{pmatrix},\ K = {}^t BSB - b^t b/(2m) = \begin{pmatrix} &&-1&\\ &2&&\\ -1&&&\\ &&&-1/(2m) \end{pmatrix}. \tag{7.3.2}
$$

Hence, we have

$$
\Gamma(\hat{Q},a_m) \simeq \{W \in \Gamma(K); {}^t Wb \equiv b \bmod 2m\}, \tag{7.3.3}
$$

or, as well,

$$
\Gamma(\hat{Q},a_m) \simeq \{W \in \Gamma(K_m); {}^t Wb \equiv b \bmod 2m\}, \tag{7.3.4}
$$

where

$$
K_m := 2mK = \begin{pmatrix} &&-2m&\\ &4m&&\\ -2m&&&\\ &&&-1 \end{pmatrix},\ b = {}^t(0,0,0,-1), \tag{7.3.5}
$$

i.e., the unit group $\Gamma(K_m)$ belongs to the quadratic form

$$q_m(u) = -4mu_1u_3 + 4mu_2^2 - u_4^2. \tag{7.3.6}$$

7.4. Example. We look at

$$2q(u) = 2\hat{q}(u) = u_3^2 - 4u_2u_4 - 4u_1u_5 = {}^t u\hat{Q}u = (u,u),$$

$$S = \hat{Q} = \begin{pmatrix} &&&&-2\\ &&&-2&\\ &&1&&\\ &-2&&&\\ -2&&&& \end{pmatrix},$$

$$\hat{G} = \mathrm{SO}(\hat{Q}) = \{g \in \mathrm{SL}(5,\mathbb{R}); {}^t g\hat{Q}g = \hat{Q}\} \simeq \mathrm{SO}(3,2),$$

$$\Gamma(\hat{Q}) = \{W \in \mathrm{SL}(5,\mathbb{Z}); {}^t W\tilde{Q}W = \hat{Q}\},$$

$$a = a_m = {}^t(1,0,0,0,-m),\ m \in \mathbb{Z}\backslash 0,\ \text{ i.e. } (a,a) = 2\hat{q}(a_m) = 4m,$$

$$\hat{G}_a = \{g \in \hat{G}; ga = a\},$$

$$\hat{\Gamma}_a = \Gamma(S,a) = \Gamma(\hat{Q},a_m) = \{U \in \Gamma(\hat{Q}); Ua = a\}$$

We take

$$A = (a,B) = \begin{pmatrix} 1&&&&\\ &1&&&\\ &&1&&\\ &&&1&\\ -m&&&&1 \end{pmatrix}, \tag{7.4.1}$$

and, by Siegel's prescription, get

$$b = {}^t BSa = \begin{pmatrix} 0\\0\\0\\-2 \end{pmatrix},\ K = {}^t BSB - b^t b/4m = \begin{pmatrix} &&-2&\\ &1&&\\ -2&&&\\ &&&-1/m \end{pmatrix}. \tag{7.4.2}$$

Hence, we have

$$\Gamma(\hat{Q},a_m) \simeq \{W \in \Gamma(K); {}^t Wb \equiv b \bmod 4m\}, \tag{7.4.3}$$

or, as well,

$$\Gamma(\hat{Q},a_m) \simeq \{W \in \Gamma(K_m); {}^t Wb \equiv b \bmod 4m\}, \tag{7.4.4}$$

where

$$K_m = mK = \begin{pmatrix} &&-2m&\\ &m&&\\ -2m&&&\\ &&&-1 \end{pmatrix},\ b = {}^t(0,0,0,-2), \tag{7.4.5}$$

i.e., the unit group $\Gamma(K_m)$ belongs to the quadratic form

$$\hat{q}_m(u) = -4mu_1u_3 + mu_2^2 - u_4^2. \tag{7.4.6}$$

7.5. Example. We look at

$$2q(u) = 2\hat{q}(u) = u_3^2 - 4u_2u_4 - 4u_1u_5 = {}^t u\hat{Q}u = (u,u),$$

$$S = \hat{Q} = \begin{pmatrix} &&&&-2\\ &&&-2&\\ &&1&&\\ &-2&&&\\ -2&&&& \end{pmatrix},$$

$$\hat{G} = \mathrm{SO}(\tilde{Q}) = \{g \in \mathrm{SL}(5,\mathbb{R}); {}^t g\hat{Q}g = \hat{Q}\} \simeq \mathrm{SO}(3,2),$$

$$\Gamma(\hat{Q}) = \{W \in \mathrm{SL}(5,\mathbb{Z}); {}^t W\hat{Q}W = \hat{Q}\},$$

$$a = a_m = {}^t(0,-1,1,M,0),\ M \in \mathbb{Z}\backslash 0,\ \text{ i.e. } (a,a) = 2\hat{q}(a_m) = 4M+1 = \Delta,$$

$$\hat{G}_a = \{g \in \tilde{G}; ga = a\},$$

$$\hat{\Gamma}_a = \Gamma(S,a) = \Gamma(\hat{Q},a_m) = \{U \in \Gamma(\hat{Q}); Ua = a\}$$

We take

$$A = (a,B) = \begin{pmatrix} 0&1&&&\\ -1&0&&&\\ 1&&1&&\\ M&&&1&\\ 0&&&&1 \end{pmatrix}, \tag{7.5.1}$$

and, by Siegel's prescription, get

$$b = {}^t BSa = \begin{pmatrix}0\\1\\2\\0\end{pmatrix},\ K = {}^t BSB - b{}^t b/\Delta = \begin{pmatrix} 0&&&-2\\ 0&1-1/\Delta&-2/\Delta&0\\ 0&-2/\Delta&-4/\Delta&0\\ -2&&&0 \end{pmatrix}. \tag{7.5.2}$$

Hence, we have

$$\Gamma(\hat{Q},a_m) \simeq \{W \in \Gamma(K); {}^t Wb \equiv b \bmod \Delta\}, \tag{7.5.3}$$

or, as well,

$$\Gamma(\hat{Q},a_m) \simeq \{W \in \Gamma(K_m); {}^t Wb \equiv b \bmod \Delta\}, \tag{7.5.4}$$

where

$$K_m = \Delta\cdot K = \begin{pmatrix} 0&&&-2\Delta\\ 0&\Delta-1&-2&0\\ 0&-2&-4&0\\ -2\Delta&&&0 \end{pmatrix},\ b = {}^t(0,1,2,0), \tag{7.5.5}$$

i.e., the unit group $\Gamma(K_m)$ belongs to the quadratic form

(7.5.6) $$q_m(u) = -4\Delta u_1 u_4 + (\Delta - 1)u_2^2 - 4u_2u_3 - 4u_3^2.$$

7.6. Pseudo-Example. The same way as above, we are tempted to look at

$$2q(x) = 2q_0(x) = x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5^2 = {}^t x Q_0 x,$$

$$S = Q_0 = \begin{pmatrix} 1 &&&& \\ & 1 &&& \\ && 1 && \\ &&& -1 & \\ &&&& -1 \end{pmatrix},$$

$$G = \mathrm{SO}(3,2)$$

$$a = a_m = {}^t(\alpha, 0, 0, 0, \alpha'),\ \alpha = (1+m)/2, \alpha' = (1-m)/2,\ \text{i.e. } 2q_0(a_m) = m,$$

$$\hat{G}_a = \{g \in \tilde{G}; ga = a\},$$

$$\hat{\Gamma}_a = \Gamma(S, a) = \Gamma(Q_0, a_m) = \{U \in \Gamma(Q_0); Ua = a\}$$

But here a is not integer as it should be. Hence, cheating, we take anyway

(7.6.1) $$A = (a, B) = \begin{pmatrix} \alpha &&&& -1 \\ & 1 &&& \\ && 1 && \\ &&& 1 & \\ \alpha' &&&& 1 \end{pmatrix},$$

and, again using Siegel's prescription, get

(7.6.2) $$b = {}^tBSa = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix},\ K = {}^tBSB - b{}^tb/m = \begin{pmatrix} 1 &&& \\ & 1 && \\ && -1 & \\ &&& -1/m \end{pmatrix}.$$

Hence, we get formally

(7.6.3) $$\Gamma^?(Q_0, a_m) \simeq \{W \in \Gamma(K); {}^tWb \equiv b \bmod m\},$$

or, as well,

(7.6.4) $$\Gamma^?(Q_0, a_m) \simeq \{W \in \Gamma(K_m); {}^tWb \equiv b \bmod m\},$$

where

(7.6.5) $$K_m = m \cdot K = \begin{pmatrix} m &&& \\ & m && \\ && -m & \\ &&& -1 \end{pmatrix},\ b = {}^t(0,0,0,-1),$$

i.e., the unit group $\Gamma(K_m)$ belongs to the quadratic form

(7.6.6) $$q_m^0(u) = mu_1^2 + mu_2^2 - mu_3^2 - u_4^2.$$

But we don't really know what we got?

8 Proof of the Proposition 6.7, Case A

8.1. Orthogonal and SL_2-matrices. The matrix W of the examples above can be realized by an SL_2-matrix. We try the following approach. With $j = \sqrt{m'}, m' \in \mathbb{Z}, m' \neq 0$, and $y_1, ..., y_4 \in \mathbb{Q}$, we put

$$X(y) = \begin{pmatrix} y_1 & y_2 + jy_3 \\ y_2 - jy_3 & y_4 \end{pmatrix},$$

where

(8.1.1) $$\det X(y) = y_1y_4 - y_2^2 + m'y_3^2 = {}^t y D_{m'} y, D_{m'} = \begin{pmatrix} & & & 1/2 \\ & -1 & & \\ & & m' & \\ 1/2 & & & \end{pmatrix}.$$

With

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Q}(j))$$

we get a map $\rho : \mathrm{SL}(2, \mathbb{Q}(j)) \to \mathrm{SO}(D_{m'}),\ g \mapsto \rho(g)$ given by

(8.1.2) $$X(y) \mapsto gX(y)^t\bar{g} = X(y'),\ y' = \rho(g)y,$$

$$\rho(g) = \begin{pmatrix} |a|^2 & a\bar{b} + \bar{a}b & (a\bar{b} - \bar{a}b)j & |b|^2 \\ \frac{(a\bar{c}+\bar{a}c)}{2} & \frac{(a\bar{d}+\bar{a}d+b\bar{c}+\bar{c}b)}{2} & \frac{(a\bar{d}-\bar{a}d+c\bar{b}-\bar{c}b)j}{2} & \frac{(b\bar{d}+\bar{b}d)}{2} \\ \frac{a\bar{c}-\bar{a}c}{2j} & \frac{a\bar{d}-\bar{a}d+b\bar{c}-\bar{b}c}{2j} & \frac{a\bar{d}+\bar{a}d-c\bar{b}-\bar{c}b}{2} & \frac{b\bar{d}-\bar{b}d}{2j} \\ |c|^2 & c\bar{d} + \bar{c}d & (c\bar{d} - \bar{c}d)j & |d|^2 \end{pmatrix},$$

where $\rho(g)$ has elements in $\mathbb{Q}$. We want to use this to analyze units of the quadratic forms coming up in the examples by Siegel's method.

Case A1.

8.2. As in (2.5.2), from [BK] (3.24), we are led to treat the relation between the index $m \in \mathbb{Z}$ of the Fourier coefficient of the Eisenstein series and fundamental discriminant d_F of the associated quadratic field $F = \mathbb{Q}(\sqrt{d})$

(8.2.1) $$4m = d_F f^2, f \in \mathbb{Z}.$$

At first we treat the case where $d \equiv 2 \,\mathrm{or}\, 3 \bmod 4$, i.e,.$d_F = 4d$. Hence, with $m = df^2$ we go to Example 7.4 where for our unit group fixing $a_m = {}^t(1, 0, 0, 0, -m)$ we have as in (7.4.4)

$$\Gamma(\hat{Q}, a_m) \simeq \{W \in \Gamma(K_m); {}^tWb \equiv b \bmod 4m\},$$

with

$$K_m = mK = \begin{pmatrix} & & -2m & \\ & m & & \\ -2m & & & \\ & & & -1 \end{pmatrix},\ b = {}^t(0, 0, 0, -2),$$

i.e., W should be a matrix $W = \begin{pmatrix} W_3 & \mathbf{b} \\ {}^t\mathbf{c} & \mathbf{d} \end{pmatrix}$, where the entries are rational integers and $\mathbf{c}$ and $\mathbf{d}-1$ are multiples of $2m$ and the unit group $\Gamma(K_m)$ belongs to the quadratic form

$$q_m(u) = -4mu_1u_3 + mu_2^2 - u_4^2.$$

Now, in (8.1.1), we choose $m' = d$ and compare

(8.2.2)
$$\begin{aligned} y_1y_4 - y_2^2 + dy_3^2 &= -4mu_1u_3 + mu_2^2 - u_4^2, \\ &= -4f^2du_1u_3 - u_4^2 + f^2du_2^2, \end{aligned}$$

and, with $j = \sqrt{d}$, put

(8.2.3)
$$-4f^2du_1 = y_1, fu_2 = y_3, u_3 = y_4, u_4 = y_2,$$

or

(8.2.4)
$$y = Cu, C = \begin{pmatrix} -4df^2 & & & \\ & & & 1 \\ & & f & \\ & & 1 & \end{pmatrix}.$$

Hence, the matrix $\rho(g)$ for the $y-$variables from (8.1.1), via the map ν given by $A \mapsto C^{-1}AC$, changes to a matrix for the $u-$variables

(8.2.5)
$$W_{\text{Sie}} = \rho'_m(g) = (w_{ij})$$
$$= \begin{pmatrix} |a|^2 & -\frac{(a\bar{b}-\bar{a}b)j}{4fm'} & -\frac{|b|^2}{4f^2m'} & -\frac{a\bar{b}+\bar{a}b}{4f^2m'} \\ -\frac{4fm'(a\bar{c}-\bar{a}c)}{2j} & \frac{a\bar{d}+\bar{a}d-b\bar{c}-\bar{c}b}{2} & \frac{b\bar{d}-\bar{b}d}{2fj} & \frac{a\bar{d}-\bar{a}d-c\bar{b}+\bar{c}b}{2fj} \\ -4f^2m'|c|^2 & (c\bar{d}-\bar{d}c)fj & |d|^2 & c\bar{d}+\bar{c}d \\ -2f^2m'(a\bar{c}+\bar{a}c) & \frac{(a\bar{d}-\bar{a}d+c\bar{b}-\bar{c}b)fj}{2} & \frac{b\bar{d}+\bar{b}d}{2} & \frac{a\bar{d}+\bar{a}d+b\bar{c}+\bar{c}b}{2} \end{pmatrix}.$$

8.3. Remark. We get an integer element $U \in \tilde{\Gamma}_{a_m}$ if the components of $\rho'_m(g)$ are integer and $\mathbf{c}$ and $\mathbf{d}-1$ are divisible by $2m$. And this is the case if we take M_f from (5.0.6) and

(8.3.1)
$$g \in \mathrm{SL}(2, (M_f, M_f^\star)), \text{ i.e., } a, d \in M_f,\ b \in (M_f^\star)^{-1}, c \in M_f^\star.$$

This can be seen by some tiny standard verifications. We present some examples. We have $b = 2f^2db_1 + 2fjb_2, b_1, b_2 \in \mathbb{Z}$, and $c = (1/2)c_1 + (1/(2fj))c_2, c_1, c_2 \in \mathbb{Z}$. Hence, we get

(8.3.2)
$$\begin{aligned} w_{13} &= b\bar{b}/(4f^2d) = (4f^4m'^2b_1^2 - 4f^2j^2b_2^2))/(4f^2d) \in \mathbb{Z}, \\ -w_{31} &= 4f^2dc\bar{c} = 4f^2md((1/4)c_1^2 - (1/4f^2j^2)c_2^2) \in \mathbb{Z}, \\ -w_{41} &= 2f^2d(a\bar{c} + \bar{a}c) \\ &= 2f^2d((a_1 + fja_2)((1/2)c_1 - (1/2fj)c_2) + (a_1 - fja_2)((1/2)c_1 + (1/2fj)c_2) \\ &= 2f^2d(a_1c_1 - a_2c_2) \in 2f^2\mathbb{Z}, \\ w_{32} &= (c\bar{d} - \bar{c}d)2fj = ((c_1/2 + c_2j/(2f))(d_1 - fjd_2) + \dots)2fj = 2(c_2d_1 - c_1d_2f^2d) \in \mathbb{Z} \\ w_{24} &= (-a_1d_2 + a_1d_2 + b_1c_2 - b_2c_1)/2. \end{aligned}$$

From $ad - bc = (a_1 + a_2 fj)(d_1 + d_2 fj) - (b_1 2f^2 m' + b_2 2fj)(c_1/2 + c_2 j/(2fd)) = 1$ we get

$$a_1 d_1 - b_2 c_2 + (a_2 d_2 - b_1 c_1) f^2 m' = 1,\ a_1 d_2 + a_2 d_1 - b_2 c_1 - b_1 c_2 = 0. \tag{8.3.3}$$

Hence, one has

$$\begin{aligned} w_{44} = \mathbf{d} &= (a\bar{d} + \bar{a}d + b\bar{c} + \bar{b}c)/2 \\ &= a_1 d_1 - a_2 d_2 f^2 m' + b_1 c_1 f^2 m' - b_2 c_2 \\ &= 1 + (b_1 c_1 - a_2 d_2) 2 f^2 m'. \end{aligned} \tag{8.3.4}$$

8.4. Summary. As remarked above, in Case A1, we have $M_f = \mathcal{O}_f$. The map $\rho' = \nu\rho : \mathrm{SL}(2, \mathbb{Q}(j)) \to \mathrm{SO}(K_m)$ restricts to $\rho' : \mathrm{SL}(2, (\mathcal{O}_f, \mathcal{O}_f^*)) \to \hat{\Gamma}_{a_m}$ and, hence, we have the formula (6.4.1)

$$\hat{\Gamma}_{a_m} \simeq \mathrm{PSL}(2, (\mathcal{O}_f, \mathcal{O}_f^*)) \tag{8.4.1}$$

from Proposition 6.7.

8.5. Remark. Strangely enough, the same procedure doesn't seem to work starting with the example 7.3 for the quadratic form $\tilde{q}(u) = u_3^2 - u_1 u_5 - u_2 u_4$: As in (8.2.2), we would be led to

$$\begin{aligned} y_1 y_4 - y_2^2 - d y_3^2 &= -4m u_1 u_3 + 4m u_2^2 - u_4^2, \\ &= -4f^2 d u_1 u_3 - u_4^2 + 4f^2 d u_2^2, \end{aligned} \tag{8.5.1}$$

and

$$-4f^2 d u_1 = y_1, 2f u_2 = y_3, u_3 = y_4, u_4 = y_2.$$

Hence, using the same way as above, we get an element $\tilde{w}_{24} = \frac{a\bar{d} - \bar{a}d - c\bar{b} + \bar{c}b}{4fj}$ but which is not an integer.

Case A2.

8.6. We take $d = 4N + 1, N \in \mathbb{Z}$, i.e., $d_F = d$ and, this time, have $4m = df^2$. Again we relate (8.1.1)

$$\det X(y) = y_1 y_4 - y_2^2 + m' y_3^2 = {}^t y D_{m'} y, D_{m'} = \begin{pmatrix} & & & 1/2 \\ & -1 & & \\ & & m' & \\ 1/2 & & & \end{pmatrix}.$$

and (7.4.6)

$$q_m(u) = -4m u_1 u_3 + m u_2^2 - u_4^2.$$

We choose $m' = d$, and

(8.6.1) $$df^2u_1 = y_1, u_4 = y_2, fu_2/2 = y_3, u_3 = y_4.$$

Hence, the relation $y' = \rho(g)y = (a_{ij})y$ with

$$\rho(g) = \begin{pmatrix} |a|^2 & a\bar{b}+\bar{a}b & (a\bar{b}-\bar{a}b)j & |b|^2 \\ \frac{(a\bar{c}+\bar{a}c)}{2} & \frac{(a\bar{d}+\bar{a}d+b\bar{c}+\bar{c}b)}{2} & \frac{(a\bar{d}-\bar{a}d+c\bar{b}-\bar{c}b)j}{2} & \frac{(b\bar{d}+\bar{b}d)}{2} \\ \frac{a\bar{c}-\bar{a}c}{2j} & \frac{a\bar{d}-\bar{a}d+b\bar{c}-\bar{b}c}{2j} & \frac{a\bar{d}+\bar{a}d-c\bar{b}-\bar{c}b}{2} & \frac{b\bar{d}-\bar{b}d}{2j} \\ |c|^2 & c\bar{d}+\bar{c}d & (c\bar{d}-\bar{c}d)j & |d|^2 \end{pmatrix} = (a_{ij})$$

from (8.1.2) changes to $u' = \rho'(g)u = (w_{ij})u$ with

(8.6.2) $$\rho'(g) = (w_{ij}) = \begin{pmatrix} a_{11} & -a_{13}/(2df) & -a_{14}/(df^2) & -a_{12}/(df^2) \\ -a_{31}2df & a_{33} & a_{34}2/f & a_{32}2/f \\ -a_{41}df^2 & a_{43}f/2 & a_{44} & a_{42} \\ -a_{21}df^2 & a_{23}f/2 & a_{24} & a_{22} \end{pmatrix}.$$

We take

(8.6.3) $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2; \mathcal{O}_f, \mathcal{O}_f^*)$$

i.e.,

(8.6.4) $$a = (2a_1 + fa_2 + fja_2)/2, b = fj(2b_1 + fb_2 + fjb_2)/2,$$
$$c = (fj)^{-1}(2c_1 + fc_2 + fjc_2)/2, d = (2d_1 + fd_2 + fjd_2)/2,$$

and get

(8.6.5)
$$a\bar{d} = A_1 + A_2fj, A_1 = a_1d_1 + (a_1d_2 + a_2d_1)f/2 - a_2d_2N, A_2 = (a_2d_1 - a_1d_2)/2,$$
$$a\bar{b} = -fj(A_1' + A_2'fj), A_1' = a_1b_1 + (a_1b_2 + a_2b_1)f/2 - a_2b_2N, A_2' = (a_2b_1 - a_1b_2/2,$$
$$a\bar{c} = -(1/fj)(A_1'' + A_2''fj), A_1'' = a_1c_1 + (a_1c_2 + a_2c_1)f/2 - a_2c_2N, A_2'' = (a_2c_1 - a_1c_2)/2,$$
$$b\bar{c} = -((B_1 + B_2fj), B_1 = b_1c_1 + (b_1c_2 + b_2c_1)f/2 - b_2c_2N, B_2 = (b_2c_1 - b_1c_2)/2,$$
$$b\bar{d} = fj(B_1' + B_2'fj), B_1' = b_1d_1 + (b_1d_2 + b_2d_1)f/2 - b_2d_2N, B_2' = (b_2d_1 - b_1d_2)/2,$$
$$c\bar{d} = (1/fj)(C_1 + C_2fj), C_1 = c_1d_1 + (c_1d_2 + c_2d_1)f/2 - c_2d_2N, C_2 = (c_2d_1 - c_1d_2)/2.$$

Hence, with $4m = df^2 = (1+4N)f^2$, by some standard computation, we come to

(8.6.6)
$$\begin{aligned}
w_{11} &= |a|^2 = a_1^2 + a_1a_2f - a_2^2f^2N\\
w_{12} &= -(a\bar{b} - \bar{a}b)j/(2df) = A_1',\\
w_{13} &= -|b|^2/(df^2) = b_1^2 + b_1b_2f - b_2^2N,\\
w_{14} &= -(a\bar{b} + \bar{a}b)/(df^2) = 2A_2',\\
w_{21} &= -2f(a\bar{c} - \bar{a}c)/(2j) = 2A_1'',\\
w_{22} &= (a\bar{d} - b\bar{c} + \bar{a}d - c\bar{b})/2 = A_1 + B_1\\
w_{23} &= (b\bar{d} - b\bar{d})f/j = 2B_2'\\
w_{24} &= (a\bar{d} + b\bar{c} - b\bar{a} + c\bar{b})f/j = 2(A_2 - B_2),\\
w_{31} &= -df^2|c|^2 = c_1^2 + c_1c_2f - c_2^2N,\\
w_{32} &= (c\bar{d} - d\bar{c})jf/2 = C_1,\\
w_{33} &= |d|^2 = d_1^2 + d_1d_2f - d_2^2N,\\
w_{34} &= c\bar{d} + \bar{c}d = 2C_2,\\
w_{41} &= -df^2|(a\bar{c} + c\bar{a})/2 = df^2A"_2,\\
w_{42} &= (a\bar{d} + c\bar{b} - \bar{a}d - \bar{c}b)fj = (A_2 + B_2)df^2/2,\\
w_{43} &= (b\bar{d} + d\bar{b})/2 = df^2B_2',\\
w_{44} &= (a\bar{d} + b\bar{c} + d\bar{a} + c\bar{b})/2 = A_1 - B_1.
\end{aligned}$$

We observe that these expressions are integers as one has $A_1, 2A_2, A_1', 2A_2' \in \mathbb{Z}$ etc. because in $4m = (1+4N)f^2$ f must be even. Concerning the congruence relations, we prepare some identities: From the determinant relation

$$1 = ad - bc = [(2a_1 + fa_2 + a_2fj)(2d_1 + fd_2 + d_2fj) - (2b_1 + fb_2 + b_2fj)(2c_1 + fc_2 + c_2fj)]/4$$

we get for the 'imaginary part'

(8.6.7)
$$0 = a_1d_2 + a_2d_1 + fa_2d_2 - b_1c_2 - b_2c_1 - fb_2c_2$$

and for the 'real part' using (8.6.7)

(8.6.8)
$$\begin{aligned}
1 &= [(2a_1 + a_2f)(2d_1 + fd_2) + a_2d_2df^2 - (2b_1 + fb_2)(2c_1 + fc_2) - b_2c_2df^2]/4,\\
&= a_1d_1 - b_1c_1 + (1/2)(a_1d_2 + a_2d_1 - b_1c_2 - b_2c_1)f + (a_2d_2 - b_2c_2)f^2(1+d)/4,\\
&= a_1d_1 - b_1c_1 + (1/2)(b_2c_2 - a_2d_2)f^2 + (a_2d_2 - b_2c_2)f^2(1+d)/4,\\
&= a_1d_1 - b_1c_1 + (a_2d_2 - b_2c_2)f^2(d-1)/4 = a_1d_1 - b_1c_1 + (a_2d_2 - b_2c_2)f^2N.
\end{aligned}$$

Hence, using (8.6.5), (8.6.6) and $4m = f^2(4N+1)$, we have

(8.6.9)
$$\begin{aligned}
A_1 - B_1 &= a_1d_1 - b_1c_1 + (1/2)(a_2d_1 + a_1d_2 - b_2c_1 - b_1c_2)f + (a_2d_2 - b_2c_2)f^2(1-d)/4\\
&= 1 - (a_2d_2 - b_2c_2)f^2N + (1/2)(b_2c_2 - a_2d_2)f^2 - (a_2d_2 - b_2c_2)f^2N\\
&= 1 - (a_2d_2 - b_2c_2)2m,
\end{aligned}$$

and

$$A_2 + B_2 = (1/2)(a_2d_1 - a_1d_2 + b_2c_1 - b_1c_2) \\ = (1/2)(b_2c_1 - a_1d_2 + (b_2c_2 - a_2d_2)f - a_1d_2 + b_2c_1) \\ = b_2c_1 - a_1d_2 + (1/2)(b_2c_2 - a_2d_2)f. \tag{8.6.10}$$

Finally for Siegel's congruence relations, using (8.6.6), (8.6.10) and f even, we get

$$w_{41} = A_2''df^2 = A_2''4m \equiv 0 \bmod 2m, \\ w_{42} = (A_2 + B_2)2m \equiv 0 \bmod 2m, \\ w_{43} = B_2'df^2 = B_2'4m \equiv 0 \bmod 2m, \\ w_{44} = A_1 - B_1 \equiv 1 \bmod 2m. \tag{8.6.11}$$

We see that we have integral elements w_{ij} and Siegel's congruence conditions are fulfilled (f odd is not possible in $4m = f^2d = f^2(1 + 4N)$).

8.7. Summary. In Case A2, we have the formula (6.4.1)

$$\hat{\Gamma}_{a_m} \simeq \mathrm{PSL}(2, (\mathcal{O}_f, \mathcal{O}_f^*)) \tag{8.7.1}$$

from Proposition 6.7.

8.8. The treatment of the square free case, i.e., $f = 1$ also is contained in the careful considerations of unit groups for quadratic forms in Section 10.2 in [EGM]: For $m' \in \mathbb{N}$, they take

$$Q_m'(x) = -x_1^2 - m'x_2^2 + x_3x_4 \tag{8.8.1}$$

and, via a prescription $g \mapsto \Psi_m'(g)$, in analogy to the one given above, provide an isomorphism

$$\Psi_m' : \mathbf{PSL}(2, \mathcal{R}_m') \to \mathbf{SO}_4^+(\mathbb{Z}, Q_m') \simeq \mathbf{PSO}_4^+(\mathbb{Z}, Q_m') \tag{8.8.2}$$

where the group with entries in $\mathcal{R}_m' = \mathbb{Z}[\sqrt{-m'}]$ has an index in the group with entries in the ring $\mathcal{O}$ of integers in $\mathbb{Q}(\sqrt{-m'})$ given by

$$[\mathbf{PSL}(2, \mathcal{O}) : \mathbf{PSL}(2, \mathcal{R}_m')] = 1 \quad m \equiv 1, 2 \bmod 4, \\ = 6 \quad m \equiv 7 \bmod 8, \\ = 10 \quad m \equiv 3 \bmod 8. \tag{8.8.3}$$

Namely, with $j=\sqrt{-m'}$, here one has

$$\Psi'_m(g)=\begin{pmatrix}\frac{a\bar d+\bar a d+b\bar c+\bar b c}{2} & \frac{(a\bar d-\bar a d+b\bar c-\bar b c)j}{2} & \frac{a\bar c+\bar a c}{2} & \frac{b\bar d+\bar b d}{2}\\ \frac{a\bar d-\bar a d+b\bar c-\bar b c}{2j} & \frac{a\bar d+\bar a d-b\bar c-\bar b c}{2} & \frac{a\bar c-\bar a c}{2j} & \frac{b\bar d-\bar b d}{2j}\\ a\bar b+\bar a b & (a\bar b-\bar a b)j & a\bar a & b\bar b\\ c\bar d+\bar c d & (c\bar d-\bar c d)j & c\bar c & d\bar d\end{pmatrix}. \tag{8.8.4}$$

In (7.4.6)

$$^t u K_m u=-4mu_1u_3+4mu_2^2-u_4^2=q_m(u).$$

we change $-m=m'$ and, to compare to (8.8.2) put

$$x_3=-4mu_1, x_4=u_3, x_2=2u_2, x_1=u_4.$$

Hence, the matrix $\Psi'_m(g)$ changes to a matrix for the $u-$variables which is the same as in (8.2.5) for $f=1, m=m_0$

$$\begin{aligned} W'_{\text{Sie}} &= \rho'_m(g) \\ &= \begin{pmatrix}|a|^2 & -\frac{(a\bar b-\bar a b)j}{2m_0} & -\frac{|b|^2}{4m_0} & -\frac{a\bar b+\bar a b}{4m_0}\\ -\frac{2fm_0(a\bar c-\bar a c)}{2j} & \frac{a\bar d+\bar a d-b\bar c-\bar c b}{2} & \frac{b\bar d-\bar b d}{4j} & \frac{a\bar d-\bar a d-c\bar b+\bar c b}{4j}\\ -4m_0|c|^2 & (c\bar d-\bar d c)2j & |d|^2 & c\bar d+\bar c d\\ -2m_0(a\bar c+\bar a c) & (a\bar d-\bar a d+c\bar b-\bar c b)j & \frac{b\bar d+\bar b d}{2} & \frac{a\bar d+\bar a d+b\bar c+\bar c b}{2}\end{pmatrix}. \end{aligned} \tag{8.8.5}$$

The same way as in Remark 8.3, we have

8.9. Remark. The components of $\Psi'_m(g)$ are in $\mathbb{Z}$, if g is of the type

$$g=\begin{pmatrix}a & b\\ c & d\end{pmatrix}, a,d\in\mathcal{R}_m, b\in(1/j)\mathcal{R}_m, c\in j\mathcal{R}_m. \tag{8.9.1}$$

And if g is of this type, Siegel's congruence condition from Remark (8.3) is fulfilled automatically.

8.10. As well, we find a discussion of the (2,2)-case in Bruinier's text [Br] Section 2.7. For squarefree $d\in\mathbb{N}$ take $F=\mathbb{Q}(\sqrt{d})$ with discriminant $D=d, \mathcal{O}_F=\mathbb{Z}+\frac{1+\sqrt{d}}{2}\mathbb{Z}$ for $d\equiv 1 \bmod 4$ and $D=4d$, $\mathcal{O}_F=\mathbb{Z}+\sqrt{d}\mathbb{Z}$ for $d\equiv 2,3 \bmod 4$ and different $\mathfrak{d}_F=(D)$. For a fractional ideal $\mathfrak{a}$, one denotes

(8.10.1)
$$\Gamma(\mathcal{O}_F \oplus \mathfrak{a}) := \{g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,F),\ a,d \in \mathcal{O}_F, b \in \mathfrak{a}^{-1}, c \in \mathfrak{a}\}$$
$$\Gamma_F := \Gamma(\mathcal{O}_F \oplus \mathcal{O}_F),$$
$$\tilde{V} := \{X = \begin{pmatrix} y_1 & w' \\ w & y_4 \end{pmatrix}\ y_1, y_4 \in \mathbb{Q}, w \in F\},$$
$$\tilde{Q}(X) = -\mathrm{det} X,$$
$$L := \mathbb{Z} \oplus \mathbb{Z} \oplus \mathcal{O}_F = \{\begin{pmatrix} y_1 & w' \\ w & y_4 \end{pmatrix} \in \tilde{V},\ y_1, y_4 \in \mathbb{Z}, w \in \mathcal{O}_F\}$$
$$L^\wedge := \mathbb{Z} \oplus \mathbb{Z} \oplus \mathfrak{d}_F^{-1} = \{\begin{pmatrix} y_1 & w' \\ w & y_4 \end{pmatrix} \in \tilde{V},\ y_1, y_4 \in \mathbb{Z}, w \in \mathfrak{d}_F^{-1}\}$$
$$L(\mathfrak{a}) := \{\begin{pmatrix} y_1 & w' \\ w & Ay_4 \end{pmatrix} \in \tilde{V},\ y_1, y_4 \in \mathbb{Z},\ w \in \mathfrak{a}\}, A = \mathrm{N}(\mathfrak{a}).$$

$\mathrm{Sp}(2,F) \simeq \mathrm{Spin}_V$ acts isometrically on $\tilde{V}$ by

(8.10.2)
$$X \mapsto gX{}^tg'$$

and (Proposition 2.25) one has $\mathrm{Spin}_L = \Gamma_F$ and, (Remark 2.26) $\Gamma(\mathcal{O}_F \oplus \mathfrak{a})$ preserves the lattice $L(\mathfrak{a})$.

9 Proof of the Proposition 6.7, Case B

9.1. Again, we look at the quadratic form

(9.1.1)
$$2\hat{q}(u) = u_3^2 - 4u_2u_4 - 4u_1u_5 = {}^tu\hat{Q}u = (u,u),\ \hat{Q} = \begin{pmatrix} & & & & -2 \\ & & & -2 & \\ & & 1 & & \\ & -2 & & & \\ -2 & & & & \end{pmatrix}.$$

This time, we take

(9.1.2)
$$a' := {}^t(0,-1,1,M,0), \text{ such that } (a',a') = 4m = 4M+1 =: \Delta.$$

We want to determine $\hat{\Gamma}_{a'} := \mathrm{SO}(\hat{Q},\mathbb{Z})_{a'}$.

9.2. From Example 7.5 in Siegel's approach, we have

(9.2.1)
$$K_m := K \cdot \Delta = \begin{pmatrix} 0 & & & -2\Delta \\ 0 & \Delta - 1 & -2 & 0 \\ 0 & -2 & -4 & 0 \\ -2\Delta & & & 0 \end{pmatrix},\ b = {}^t(0,1,2,0)$$

and

$$q_m(x) := {}^t x K_m x = -4\Delta x_1 x_4 + (\Delta - 1)x_2^2 - 4x_2x_3 - 4x_3^2, \tag{9.2.2}$$

and, via Siegel's prescription

$$\Gamma_{a'} \simeq \{W \in \mathrm{SO}(K_m, \mathbb{Z}); {}^t W b \equiv b \bmod \Delta\}. \tag{9.2.3}$$

These matrices W can be realized by SL_2-matrices. From (2.11.4) we have

$$4\Delta = d_F f^2.$$

Case B1.

For $d_F = 4d, d = 2 + 4N$ or $d = 3 + 4N$ one has $\Delta = 4M + 1 = (2 + 4N)f^2$ or $4M + 1 = (3 + 4N)f^2$ which both is not possible. Hence, we only have to look at:

Case B2.

9.3. For

$$F = \mathbb{Q}(j), j^2 = d = d_F,\ 4\Delta = 4(4M + 1) = f^2 j^2 = f^2 d \tag{9.3.1}$$

with $d \equiv 1 \bmod 4$, we put $d = 4N + 1, N \in \mathbb{Z}$. We observe that here f must be even.

For

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, F), \tag{9.3.2}$$

$$X(y) = \begin{pmatrix} y_1 & w \\ \bar{w} & y_4 \end{pmatrix}, w = y_2 + y_3\omega, y_1, \ldots, y_4 \in \mathbb{Q}, \omega := 1 + fj/2, f \in \mathbb{Z}, f \neq 0,$$

we use the standard proceeding

$$X(y) = \begin{pmatrix} y_1 & w \\ \bar{w} & y_4 \end{pmatrix} \mapsto gX(y){}^t\bar{g} =: X(y'), y' = \rho(g)y. \tag{9.3.3}$$

This map preserves the determinant and, for $\omega = 1 + fj/2, f \in \mathbb{Z}, f \neq 0$, one has

$$\begin{aligned} \det X(y) = y_1y_4 - \bar{w}w &= y_1y_4 - y_2^2 - 2y_2y_3 - |\omega|^2 y_3^2 \\ &= y_1y_4 - y_2^2 - 2y_2y_3 - (1 - f^2 d)y_3^2/4 = y_1y_4 - y_2^2 - 2y_2y_3 + 4My_3^2 \\ &= {}^t y D_m y,\ D_m = \begin{pmatrix} 0 & & & 1/2 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & 4M & 0 \\ 1/2 & & & 0 \end{pmatrix}. \end{aligned} \tag{9.3.4}$$

Hence, generalizing [EGM] p.463 for $\Delta < 0$, one has an isomorphism

$$\rho : \ \mathrm{PSL}(2, F) \to \mathrm{SO}^+(D_m, F), \tag{9.3.5}$$

where

$$\rho(g) = \begin{pmatrix} |a|^2 & a\bar{b} + \bar{a}b & \omega a\bar{b} + \bar{\omega}\bar{a}b & |b|^2 \\ \frac{\bar{\omega}a\bar{c} - \omega\bar{a}c}{\bar{\omega}-\omega} & \frac{-\omega(\bar{a}d + \bar{b}c) + \bar{\omega}(b\bar{c} + \bar{d}a)}{\bar{\omega}-\omega} & \frac{-\omega^2 c\bar{b} + \bar{\omega}^2\bar{c}b - |\omega|^2(d\bar{a} - \bar{d}a)}{\bar{\omega}-\omega} & \frac{-\omega d\bar{b} + \bar{\omega}\bar{d}b}{\bar{\omega}-\omega} \\ \frac{a\bar{c} - \bar{a}c}{\omega - \bar{\omega}} & \frac{a\bar{d} - \bar{a}d + b\bar{c} - \bar{c}b}{\omega - \bar{\omega}} & \frac{(a\bar{d} - \bar{b}c)\omega - (a\bar{d} - \bar{c}b)\bar{\omega}}{\omega - \bar{\omega}} & \frac{b\bar{d} - \bar{b}d}{\omega - \bar{\omega}} \\ |c|^2 & c\bar{d} + \bar{c}d & c\bar{d}\omega + \bar{c}d\bar{\omega} & |d|^2 \end{pmatrix} \tag{9.3.6}$$

9.4. Remark. We observe that matrices g with elements in $\mathcal{O}$ are transformed into matrices $\rho(g)$ with components in $\mathbb{Z}$.

9.5. We use this for the description of our isotropy group $\Gamma_{a'}$: One has

$$\begin{gathered} \omega = 1 + fj/2, 4\Delta = 4(4m) = 4(4M+1) = f^2 j^2 = f^2 d = f^2(1+4N), \\ \omega - \bar{\omega} = 2,\ \omega + \bar{\omega} = jf,\ |\omega|^2 = 1 - f^2 d/4 = 1 - \Delta,\ \omega^2 = 1 + \Delta + fj, \\ \mathcal{O}_f = \mathbb{Z} + f(1+j)/2 \cdot \mathbb{Z}, \mathcal{O}_f^* = (jf)^{-1}\mathcal{O}_f, (\mathcal{O}_f^*)^{-1} = jf\mathcal{O}_f. \end{gathered} \tag{9.5.1}$$

From (9.2.2) and (9.3.4), we take

$$\begin{aligned} q_m(x) &= -4\Delta x_1 x_4 + (\Delta - 1)x_2^2 - 4x_2 x_3 - 4x_3^2, \\ \det X(y) &= y_1 y_4 - y_2^2 - (\omega + \bar{\omega})y_2 y_3 - |\omega|^2 y_3^2 \\ &= y_1 y_4 - y_2^2 - 2y_2 y_3 - (1 - \Delta)y_3^2, \end{aligned}$$

and we put

$$-4\Delta x_1 = y_1, x_4 = y_4, 2x_3 = y_2, x_2 = y_3. \tag{9.5.2}$$

From $y' = \rho(g)y = (a_{ij})$ with (9.3.6)

$$\rho(g) = \begin{pmatrix} |a|^2 & a\bar{b} + \bar{a}b & \omega a\bar{b} + \bar{\omega}\bar{a}b & |b|^2 \\ \frac{\bar{\omega}a\bar{c} - \omega\bar{a}c}{\bar{\omega}-\omega} & \frac{-\omega(\bar{a}d + \bar{b}c) + \bar{\omega}(b\bar{c} + \bar{d}a)}{\bar{\omega}-\omega} & \frac{-\omega^2 c\bar{b} + \bar{\omega}^2\bar{c}b - |\omega|^2(d\bar{a} - \bar{d}a)}{\bar{\omega}-\omega} & \frac{-\omega d\bar{b} + \bar{\omega}\bar{d}b}{\bar{\omega}-\omega} \\ \frac{a\bar{c} - \bar{a}c}{\omega - \bar{\omega}} & \frac{a\bar{d} - \bar{a}d + b\bar{c} - \bar{c}b}{\omega - \bar{\omega}} & \frac{(a\bar{d} - \bar{b}c)\omega - (d\bar{a} - \bar{c}d)\bar{\omega}}{\omega - \bar{\omega}} & \frac{b\bar{d} - \bar{b}d}{\omega - \bar{\omega}} \\ |c|^2 & c\bar{d} + \bar{c}d & c\bar{d}\omega + \bar{c}d\bar{\omega} & |d|^2 \end{pmatrix}$$

we get $x' = \rho'(A)x = (w_{ij})x$ with

$$\begin{aligned} x_1' &= a_{11}x_1 - a_{13}/(4\Delta)x_2 - a_{12}/(2\Delta)x_3 - a_{14}/(4\Delta)x_4 \\ x_2' &= -4\Delta a_{31}x_1 + a_{33}x_2 + a_{32}2x_3 + a_{34}x_4 \\ x_3' &= -2\Delta a_{21}x_1 + a_{23}x_2/2 + a_{22}x_3 + a_{24}x_4/2 \\ x_4' &= -4\Delta a_{41}x_1 + a_{43}x_2 + a_{42}2x_3 + a_{44}x_4. \end{aligned} \tag{9.5.3}$$

We take

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2; \mathcal{O}_f, \mathcal{O}_f^*) \tag{9.5.4}$$

i.e., with $a_1, \ldots, d_2 \in \mathbb{Z}$,

$$\begin{aligned} a &= (2a_1 + fa_2 + fja_2)/2, b = fj(2b_1 + fb_2 + fjb_2)/2, \\ c &= (fj)^{-1}(2c_1 + fc_2 + fjc_2)/2, d = (2d_1 + fd_2 + fjd_2)/2, \end{aligned} \tag{9.5.5}$$

and get

(9.5.6)

$$\begin{aligned}
a\bar{d} &= A_1 + A_2 fj, & A_1 &= a_1d_1 + (a_1d_2 + a_2d_1)f/2 - a_2d_2N, A_2 = (a_2d_1 - a_1d_2)/2, \\
a\bar{b} &= -fj(A_1' + A_2' fj) \\
&= -f^2 dA_2' - A_1' fj, & A_1' &= a_1b_1 + (a_1b_2 + a_2b_1)f/2 - a_2b_2N, A_2' = (a_2b_1 - a_1b_2)/2, \\
a\bar{c} &= -(1/fj)(A_1'' + A_2'' fj) \\
&= -A_2'' - A_1''/(f^2 d)fj, & A_1'' &= a_1c_1 + (a_1c_2 + a_2c_1)f/2 - a_2c_2N, A_2'' = (a_2c_1 - a_1c_2)/2, \\
b\bar{c} &= -(B_1 + B_2 fj), & B_1 &= b_1c_1 + (b_1c_2 + b_2c_1)f/2 - b_2c_2N, B_2 = (b_2c_1 - b_1c_2)/2, \\
b\bar{d} &= fj(B_1' + B_2' fj) \\
&= B_2' f^2 d + B_1' fj, & B_1' &= b_1d_1 + (b_1d_2 + b_2d_1)f/2 - b_2d_2N, B_2' = (b_2d_1 - b_1d_2)/2, \\
c\bar{d} &= (1/fj)(C_1 + C_2 fj) \\
&= C_2 + C_1/(f^2 d)fj, & C_1 &= c_1d_1 + (c_1d_2 + c_2d_1)f/2 - c_2d_2N, C_2 = (c_2d_1 - c_1d_2)/2.
\end{aligned}$$

Hence, by some standard computation, one has

(9.5.7)

$$
\begin{aligned}
w_{11} &= |a|^2 = a_1^2 + a_1a_2f - a_2^2f^2N\\
w_{12} &= -(a\bar{b}\omega + \bar{a}b\bar{\omega})/(4\Delta) = (-f^2dA_2' - A_1'fj)(1+fj/2)+\dots)/(4\Delta) = -2A_2' - A_1'\\
w_{13} &= -(a\bar{b}+\bar{a}b)/(2\Delta) = 4A_2',\\
w_{14} &= -|b|^2/(4\Delta) = b_1^2 + b_1b_2f - b_2^2f^2N,\\
w_{21} &= -4\Delta(a\bar{c}-\bar{a}c)/(fj) = 2A_1'',\\
w_{22} &= ((a\bar{d}-c\bar{d})\omega - (\bar{a}d - b\bar{c})\bar{\omega})/(fj)\\
&= (A_1 + A_2fj + B_1 - B_2fj)(1+fj/2)-\dots)/(fj) = 2A_2 - 2B_2 + A_1 + B_1\\
w_{23} &= 2(a\bar{d}+b\bar{c}) - (\bar{a}d + c\bar{b}))/(fj) = 4A_2 - 4B_2\\
w_{24} &= (b\bar{d} - d\bar{b})/(fj) = 2B_1',\\
w_{31} &= -2\Delta(a\bar{c}\omega - \bar{a}c\bar{\omega})/(fj)\\
&= -2\Delta(-A_2'' - A_1''/(f^2d)fj)(1+fj/2)-\dots)/(fj) = 2A_2''\Delta - A_1'',\\
w_{32} &= ((d\bar{a}-a\bar{d})|\omega|^2 + \bar{b}c\omega^2 - b\bar{c}\bar{\omega}^2)/(2fj)\\
&= (-2A_2fj(1-\Delta) + (-B_1 + B_2fj)(1+\Delta+fj)-\dots)/(2fj)\\
&= -A_2(1-\Delta) + B_2(1+\Delta) - B_1\\
&= \Delta(A_2+B_2) + B_2 - A_2 - B_1 = 4M(A_2+B_2) + 2B_2 - B_1,\\
w_{33} &= ((d\bar{a}+c\bar{b})\omega - (\bar{a}d + b\bar{c}\bar{\omega}))/(fj)\\
&= (A_1 - A_2fj - B_1 + B_2fj)(1+fj/2)-\dots)/(fj) = A_1 - B_1 - 2A_2 + 2B_2\\
w_{34} &= ((d\bar{b}\omega - \bar{d}b\bar{\omega})/(2fj) = B_2'2\Delta - B_1',\\
w_{41} &= -4\Delta|c|^2 = -(c_1^2 + c_1c_2f - c_2^2f^2N),\\
w_{42} &= c\bar{d}\omega + d\bar{c}\bar{\omega} = (C_1 + C_2fj)(1+fj/2)+\dots = 2C_2 + C_1,\\
w_{43} &= 2(c\bar{d} + d\bar{c}) = 4C_2,\\
w_{44} &= |d|^2 = d_1^2 + d_1d_2f - d_2^2f^2N.
\end{aligned}
$$

We observe that these expressions are rational integers as one has $2A_2 \in \mathbb{Z}$ and for even f has also $A_1 \in \mathbb{Z}$ etc. Concerning the congruence relations in (7.5.4), we prepare some identities: From the determinant relation

$$1 = ad - bc = [(2a_1 + fa_2 + a_2fj)(2d_1 + fd_2 + d_2fj) - (2b_1 + fb_2 + b_2fj)(2c_1 + fc_2 + c_2fj)]/4$$

we get for the 'imaginary part'

$$0 = a_1d_2 + a_2d_1 + fa_2d_2 - b_1c_2 - b_2c_1 - fb_2c_2 \tag{9.5.8}$$

and for the 'real part' using (9.5.8)

$$\begin{aligned}(9.5.9)\quad 1 &= [(2a_1+a_2f)(2d_1+fd_2)+a_2d_2f^2d-(2b_1+fb_2)(2c_1+fc_2)-b_2c_2f^2d]/4,\\ &= a_1d_1-b_1c_1+(1/2)(a_1d_2+a_2d_1-b_1c_2-b_2c_1)f+(a_2d_2-b_2c_2)f^2(1+d)/4,\\ &= a_1d_1-b_1c_1+(1/2)(b_2c_2-a_2d_2)f^2+(a_2d_2-b_2c_2)f^2(1+d)/4,\\ &= a_1d_1-b_1c_1+(a_2d_2-b_2c_2)f^2(d-1)/4 = a_1d_1-b_1c_1+(a_2d_2-b_2c_2)f^2N.\end{aligned}$$

Hence, using (9.5.9) and $4\Delta = f^2(4N+1)$, we have

(9.5.10)

$$\begin{aligned}A_1-B_1 &= a_1d_1-b_1c_1+(1/2)(a_2d_1+a_1d_2-b_2c_1-b_1c_2)f+(a_2d_2-b_2c_2)f^2(1-d)/4\\ &= 1-(a_2d_2-b_2c_2)f^2N+(1/2)(b_2c_2-a_2d_2)f^2-(a_2d_2-b_2c_2)f^2N\\ &= 1-(a_2d_2-b_2c_2)f^2(2N+1/2) = 1-(a_2d_2-b_2c_2)2\Delta.\end{aligned}$$

Finally for Siegel's congruence relations, using (9.5.10), we get

$$\begin{aligned}(9.5.11)\quad w_{21}+2w_{31} &= -2A_2''\Delta \equiv 0 \bmod \Delta,\\ w_{24}+2w_{34} &= 4B_2'\Delta \equiv 0 \bmod \Delta,\\ w_{22}+2w_{32} &= 2(A_2-B_2)+A_1+B_1-2B_1+2B_2-2A_2+2\Delta(A_2+B_2)\\ &= A_1-B_1+2\Delta(A_2+B_2) \equiv 1 \bmod \Delta\\ w_{23}+2w_{33} &= 4(A_2-B_2)+2A_1-2B_1-4A_2+4B_2 = 2(A_1-B_1) \equiv 2 \bmod \Delta.\end{aligned}$$

We see that we have integral elements w_{ij} and Siegel's congruence conditions are fulfilled.

9.6. Summary. In Case B2, we have the formula (6.4.1)

$$(9.6.1)\qquad \hat{\Gamma}_{a'} \simeq \mathrm{PSL}(2,(\mathcal{O}_f,\mathcal{O}_f^*))$$

from Proposition 6.7.

10 Some subgroup considerations

10.1. One has the following standard facts (Shimura [Sh] p.23)

(10.1.1)
$$\mathrm{SL}(2,\mathbb{Z}/N\mathbb{Z}) = \prod_{i=1}^{r} \mathrm{SL}(2,\mathbb{Z}/p_i^{e_i}\mathbb{Z}),\ N = \prod_{i=1}^{r} p_i^{e_i}$$
$$|\mathrm{SL}(2,\mathbb{Z}/f\mathbb{Z})| = f^3 \prod_{p|f}(1-1/p^2),$$
$$|\mathrm{GL}(2,\mathbb{Z}/p^l\mathbb{Z})| = p^{4(l-1)}(p^2-1)(p^2-p),\ |(\mathbb{Z}/p^l\mathbb{Z})^*| = p^l - p^{l-1}$$
$$|\mathrm{PSL}(2,\mathbb{Z}/f\mathbb{Z})| = f^3/2 \prod_{p|f}(1-1/p^2),\ f > 2$$
$$|\mathrm{PSL}(2,\mathbb{Z}/2\mathbb{Z})| = 6,$$

and with $q = p^e$ (Artin [Ar] p.169)

(10.1.2)
$$|\mathrm{GL}(2,\mathbb{F}_q)| = (q^2-1)(q^2-q),$$
$$|\mathrm{SL}(2,\mathbb{F}_q)| = (q^2-1)q,$$
$$|\mathrm{PSL}(2,\mathbb{F}_q)| = (q^2-1)q/2,\ q\,\text{odd},$$
$$|\mathrm{PSL}(2,\mathbb{F}_q)| = (q^2-1)q,\ q\,\text{even}.$$

10.2. For $m_0 \in \mathbb{Z}$, there is the Legendre symbol

(10.2.1)
$$(\frac{m_0}{p}) = \pm 1$$

if p is prime and does not divide m_0 and one finds x with $x^2 \equiv m_0 \bmod p$ or not. For $p|m_0$ one has $(\frac{m_0}{p}) = 0$. We observe that, for all odd m_0, one has $(\frac{m_0}{2}) = 1$.

Moreover, if D is a fundamental discriminant, i.e. $D \in \mathbb{Z}$ with

(10.2.2)
$$D \equiv 0 \bmod 4, D/4\,\text{square free}, D/4 \equiv 2\,\text{or}\,3 \bmod 4 \quad \text{(case 1)},$$
or
$$D \equiv 1 \bmod 4, D\,\text{square free}, \quad \text{(case 2)},$$

there is [Za] p.38) a function $\chi_D : \mathbb{N} \to \mathbb{Z}$ modulo $|D|$ defined by

(10.2.3)
$$\chi_D(p) = (\frac{D}{p}) \quad p\ \text{an odd prime}$$
$$\chi_D(2) = 0 \quad \text{if} \quad D \equiv 0 \bmod 4,$$
$$= 1 \quad \text{if} \quad D \equiv 1 \bmod 8,$$
$$= -1\,\text{if} \quad D \equiv 5 \bmod 8,$$
$$\chi_D(p_1^{n_1} \dots p_k^{n_k}) = \chi_D(p_1^{n_1}) \dots \chi_D(p_k^{n_k}).$$

And χ_D defines a primitive Dirichlet character modulo $|D|$ (also denoted by χ_D) with

(10.2.4)
$$\begin{aligned}\chi_D(-1) &= 1 \quad \text{if} \quad D > 0,\\ &= -1 \quad \text{if} \quad D < 0.\end{aligned}$$

And [Za] p.40, for D_1, D_2 fundamental discriminants

(10.2.5)
$$\chi_{D_1 D_2} = \chi_{D_1}\chi_{D_2}.$$

10.3. We want use this to verify the formula from [EGM] above cited as (8.8.3): For $j = \sqrt{-m}, m \in \mathbb{N}$, square free, $\mathcal{R}_m = \mathbb{Z}[j]$ has an index in the ring $\mathcal{O}$ of integers in $\mathbb{Q}(j)$ given by

$$\begin{aligned} r_m := [\mathbf{PSL}(2,\mathcal{O}) : \mathbf{PSL}(2,\mathcal{R}_m)] &= 1 \quad m \equiv 1,2 \bmod 4,\\ &= 6 \quad m \equiv 7 \quad \bmod 8,\\ &= 10 \quad m \equiv 3 \quad \bmod 8.\end{aligned}$$

The first relation is trivial as $\mathcal{O} = \mathbb{Z} + j\mathbb{Z}$. For $m = 7 + 8M$ (case a) and $m = 3 + 8M$ (case b) one has $\mathcal{O} = \mathbb{Z} + \omega\mathbb{Z}, \omega = (1+j)/2$, where

$$\begin{aligned}\omega^2 &= \omega - 2 - 2M, \quad \text{case a},\\ \omega^2 &= \omega - 1 - 2M, \quad \text{case b}.\end{aligned}$$

Here, we have

$$\begin{aligned}\mathcal{R}_m = \mathcal{O}_2 &= \mathbb{Z} + 2\omega\mathbb{Z}\\ 2\mathcal{O} &= 2\mathbb{Z} + 2\omega\mathbb{Z}\end{aligned}$$

and, hence,

(10.3.1)
$$\begin{aligned}\mathcal{O}/2\mathcal{O} = (\mathbb{Z}/2\mathbb{Z})[\omega] &\simeq \mathbb{F}_2 \times \mathbb{F}_2, \quad \text{case a},\\ &\simeq \mathbb{F}_4, \qquad \text{case b},\\ \mathcal{O}_2/2\mathcal{O} &= \mathbb{F}_2.\end{aligned}$$

Now, using

(10.3.2)
$$\mathrm{SL}(2, \mathcal{O}/\mathcal{O}_2) = \mathrm{SL}(2, \mathcal{O}/2\mathcal{O})/\mathrm{SL}(2, \mathcal{O}_2/2\mathcal{O})$$

and (10.1.1) and (10.1.2), we get $r_m = 6 \times 6/6 = 6$ in case a and $r_m = 60/6 = 10$ in case b. In both cases, with (10.6.1), one also may write

(10.3.3)
$$r_m = 2^3(1 - \chi_{-m}(2)/2^2).$$

10.4. We look at $F = \mathbb{Q}(j), j = \sqrt{d}, d \in \mathbb{Z}, d \neq 0$ square free, $f \in \mathbb{N}$, and want to determine

$$\psi(d,f) := [\mathbf{PSL}(2,\mathcal{O}) : \mathbf{PSL}(2,\mathcal{O}_f)], \tag{10.4.1}$$

where $\mathcal{O} = \mathbb{Z}+\omega\mathbb{Z}, \mathcal{O}_f = \mathbb{Z}+f\omega\mathbb{Z}$. Already for $m = d_F f^2 > 0$, from van der Geer's formula (6.3.1) one may deduce

$$[\mathbf{SL}(2,\mathcal{O}) : \mathbf{SL}(2,\mathcal{O}_f)] \sim f^3 \prod_{p|f} (1 - (\frac{d_F}{p})p^{-2}) \tag{10.4.2}$$

with the Legendre symbol $(\frac{d}{p})$, but we look into this a bit more closely:

10.5. We have

$$\mathrm{SL}(2,\mathcal{O}/\mathcal{O}_f) = \mathrm{SL}(2,\mathcal{O}/f\mathcal{O})/\mathrm{SL}(2,\mathcal{O}_f/f\mathcal{O}), \tag{10.5.1}$$

where

$$\begin{aligned} &\mathcal{O}_f/f\mathcal{O} \simeq \mathbb{Z}/f\mathbb{Z} \\ &\mathcal{O}/f\mathcal{O} = (\mathbb{Z}+\omega\mathbb{Z})/(f\mathbb{Z}+f\omega\mathbb{Z}) \simeq (\mathbb{Z}/f\mathbb{Z})[\omega]. \end{aligned} \tag{10.5.2}$$

This is a special case of the general classic formula for a number field F of degree $n = \sum_{i=1}^{g} e_i f_i$

$$\mathcal{O}/f\mathcal{O} = \mathcal{O}/(\mathfrak{p}_1^{e_1} \times \cdots \times \mathfrak{p}_g^{e_g}). \tag{10.5.3}$$

10.6. For a rational prime p in $F = \mathbb{Q}(\sqrt{d})$ with discriminant D, one has (e.g. [Za] p.100)

$$\begin{aligned} &\chi_D(p) = 1 \text{ iff } p \text{ is split: } p = \mathfrak{p}_1 \times \mathfrak{p}_2, \mathfrak{p}_1 \neq \mathfrak{p}_2, \text{ hence } \mathcal{O}/p\mathcal{O} \simeq \mathbb{F}_p \times \mathbb{F}_p, \\ &\chi_D(p) = 0 \text{ iff } p \text{ is ramified: } p = \mathfrak{p}^2, \text{ hence } \mathcal{O}/p\mathcal{O} \simeq \mathbb{Z}/p^2\mathbb{Z} \\ &\chi_D(p) = -1 \text{ iff } p \text{ is inert: } p = \mathfrak{p}, \text{ hence } \mathcal{O}/p\mathcal{O} \simeq \mathbb{F}_{p^2}. \end{aligned} \tag{10.6.1}$$

Hence, in the first case we have $\mathcal{O}/p\mathcal{O} \simeq \mathbb{F}_p \times \mathbb{F}_p$ and

$$\begin{aligned} |\mathrm{GL}(2,\mathcal{O}/p\mathcal{O})| &= ((p^2-p)(p^2-1))^2, \\ |\mathrm{SL}(2,\mathcal{O}/p\mathcal{O})| &= p^6(1-(1/p^2))^2, \end{aligned} \tag{10.6.2}$$

in the third case $\mathcal{O}/f\mathcal{O} \simeq \mathbb{F}_{p^2}$ and

$$\begin{aligned} |\mathrm{GL}(2,\mathcal{O}/p\mathcal{O})| &= (p^4-1)(p^4-p^2), \\ |\mathrm{SL}(2,\mathcal{O}/p\mathcal{O})| &= p^6(1-(1/p^4)) = p^6(1-(1/p^2))(1+(1/p^2)) \end{aligned} \tag{10.6.3}$$

and in the second case $\mathcal{O}/p\mathcal{O} \simeq \mathbb{Z}/p^2\mathbb{Z}$ and

$$|\mathrm{SL}(2, \mathcal{O}/p\mathcal{O})| = p^6(1 - (1/p^2)). \tag{10.6.4}$$

With $|\mathrm{SL}(2, \mathcal{O}_p/p\mathcal{O})| = p^3(1-(1/p^2))$, consistent (at least for odd primes) with the formula of van der Geer, we get

$$|\mathrm{SL}(2, \mathcal{O}/\mathcal{O}_f)| = p^3(1 - \chi_D(p)/p^2). \tag{10.6.5}$$

10.7. In the case $f = pp', p \neq p'$ one has $\mathbb{Z}/f\mathbb{Z} = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p'\mathbb{Z}$ and

(10.7.1)
$$\mathcal{O}/f\mathcal{O} = \mathbb{Z} + \omega\mathbb{Z}/(f\mathbb{Z} + f\omega\mathbb{Z}) = \mathbb{F}_p[\omega] \times \mathbb{F}_{p'}[\omega],$$
$$|\mathrm{SL}(2, \mathcal{O}/f\mathcal{O})| = p^6(1 - (1/p^2))(1 - \chi_D(p)/p^2))p'^6(1 - (1/p'^2))(1 - \chi_D(p')/p'^2),$$
$$|\mathrm{SL}(2, \mathcal{O}/f\mathcal{O})|/|\mathrm{SL}(2, \mathbb{Z}/f\mathbb{Z})| = p^3p'^3(1 - \chi_D(p)/p^2))(1 - \chi_D(p')/p'^2),$$

again as in van der Geer's formula (6.3.1).

10.8. For $f = p^l$, we follow Shimura as in his proof of (10.1.1) and look at the exact sequence

$$1 \to X \to \mathrm{GL}(2, \mathcal{O}/p^l\mathcal{O}) \xrightarrow{\rho} \mathrm{GL}(2, \mathcal{O}/p\mathcal{O}) \to 1, \tag{10.8.1}$$

where the homomorphism ρ is given by $\rho(a + p^l\mathcal{O}) = a + p\mathcal{O}, a \in \mathcal{O}$. For $\chi_D(p) = \pm 1$ the elements of $\mathcal{O}/p^l\mathcal{O}$ are of the type (α, β) resp. $\alpha\omega + \beta, \alpha, \beta \in \mathbb{Z}/p^l\mathbb{Z}$. Elements from $\rho^{-1}(0 + p\mathcal{O})$ show up if α and β are represented by

$$\nu p,\ \nu = 0, 1, \ldots (p^{l-1} - 1). \tag{10.8.2}$$

Hence, in both cases, we have $|\rho^{-1}(0 + p\mathcal{O})| = p^{2(l-1)}$ and the kernel X of ρ has $|X| = p^{4(2l-2)}$. Using (10.6.2) and (10.7.1), from

(10.8.3)
$$\begin{aligned}|\mathrm{GL}(2, \mathcal{O}/p^l\mathcal{O})| &= |X||\mathrm{GL}(2, \mathcal{O}/p\mathcal{O})|,\\ &= p^{4(2l-2)}((p^2 - p)(p^2 - 1))^2 = p^{8l}(i - 1/p^2)^2(1 - 1/p)^2 \text{ if } \chi_D(p) = 1,\\ &= p^{4(2l-2)}(p^4 - 1)(p^4 - p^2) = p^{8l}(1 - 1/p^4)(1 - 1/p^2) \text{ if } \chi_D(p) = -1\end{aligned}$$

we deduce

(10.8.4)
$$\begin{aligned}|\mathrm{SL}(2, \mathcal{O}/p^l\mathcal{O})| &= p^{6l}(i - 1/p^2)^2 \text{ if } \chi_D(p) = 1,\\ &= p^{6l}(1 - 1/p^4) \text{ if } \chi_D(p) = -1,\end{aligned}$$

as one has $|(\mathcal{O}/p^l\mathcal{O}))^*| = (p^l - p^{l-1})^2$ resp. $= p^{2l-2}(p^2 - 1)$. Finally, with $\mathrm{SL}(2, \mathcal{O}_f/p^l\mathcal{O}) = p^{3l}(1 - 1/p^2)$ from (10.1.1) we get

$$|\mathrm{SL}(2, \mathcal{O}/\mathcal{O}_{p^l})| = p^{3l}(1 - \chi_D(p)/p^2). \tag{10.8.5}$$

One gets the same formula in the ramified case $\chi_D(p) = 0$ as one has $\mathcal{O}/p^l\mathcal{O} \simeq \mathbb{Z}/p^{2l}\mathbb{Z}$ and, hence, $|\mathrm{SL}(\mathcal{O}/p^l\mathcal{O})| = p^{6l}(1 - 1/p^2)$.
Thus we have proved

10.9. Proposition. For $F = \mathbb{Q}(\sqrt{d}), d$ squarefree, with discriminant d_F, maximal order $\mathcal{O}$, and order $\mathcal{O}_f$ with conductor f one has

$$\psi(d_F, f) := [\mathbf{PSL}(2, \mathcal{O}) : \mathbf{PSL}(2, \mathcal{O}_f)] = f^3 \prod_{p|f}(1 - \chi_{d_F}(p)/p^2). \tag{10.9.1}$$

10.10. Summary. We looked at the following ingredients.

Case A. We take

$$\begin{aligned} a_m &= {}^t(1, 0, 0, 0, -m),\ m \in \mathbb{Z}\backslash 0,\ \text{ i.e. } 2\hat{q}(a_m) = 4m, \\ \hat{G}_{a_m} &= \{g \in \hat{G}; g a_m = a_m\}, \\ \hat{\Gamma}_{a_m} &= \Gamma(\hat{Q}, a_m) = \{U \in \Gamma(\hat{Q}); U a_m = a_m\}. \end{aligned} \tag{10.10.1}$$

Case B. We take

$$\begin{aligned} a'_m &= {}^t(0, -1, 1, M, 0),\ M \in \mathbb{Z}\backslash 0,\ \text{ i.e. } 2\hat{q}(a_m) = 4m = 4M + 1 = \Delta, \\ \hat{G}_{a'_m} &= \{g \in G'; g a'_m = a'_m\}, \\ \hat{\Gamma}_{a'_m} &= \Gamma(\hat{Q}, a'_m) = \{U \in \Gamma(\hat{Q}); U a'_m = a'_m\}, \end{aligned} \tag{10.10.2}$$

and we want to know in both cases

$$\begin{aligned} \kappa_m^0 &= \mathrm{vol}(\hat{\Gamma}_{a_m}\backslash\hat{G}_{a_m}),\ \text{resp.}\ \kappa_m'^0 = \mathrm{vol}(\hat{\Gamma}_{a'_m}\backslash\hat{G}_{a'_m}) \\ \kappa_m &= \mathrm{vol}(\hat{\Gamma}_{a_m}\backslash\mathbb{D}_{a_m}),\ \text{resp.}\ \kappa'_m = \mathrm{vol}(\hat{\Gamma}_{a'_m}\backslash\mathbb{D}'_{a'_m}), \end{aligned} \tag{10.10.3}$$

where $\mathbb{D}_a$ stands for the respective symmetric space. Above, in Proposition 6.7, we already related the unit groups $\hat{\Gamma}_a$ and $\hat{\Gamma}_a$ to discrete groups belonging to the SL-theory. Thus, at first we shall determine the volumes of these groups in their associated homogeneous spaces $\mathbb{H}^+$ resp. $\mathbb{H}^2$.

10.11. For the integers m resp. Δ, we have to distinguish the way they are related to fundamental discriminants d_F of quadratic fields $F = \mathbb{Q}(\sqrt{d_F})$. With d, f non-zero integers, we put

Case 1.

$$F = \mathbb{Q}(j) \supset \mathcal{O} = \mathbb{Z} + j\mathbb{Z} \supset \mathcal{O}_f = \mathbb{Z} + fj\mathbb{Z}, j^2 = d \equiv 2 \,\mathrm{or}\, 3 \bmod 4, d_F = 4d. \tag{10.11.1}$$

Case 2.

(10.11.2)
$F=\mathbb{Q}(j)\supset\mathcal{O}=\mathbb{Z}+((1+j)/2)\mathbb{Z}\supset\mathcal{O}_f=\mathbb{Z}+f((1+j)/2)\mathbb{Z}, j^2=d\equiv 1 \bmod 4, d_F=d.$

Hence, from Proposition 6.7, we have in both cases

$$\hat{\Gamma}_{a_m} \text{ and } \hat{\Gamma}_{a'_m} \simeq \mathrm{PSL}_2(\mathcal{O}_f,\mathcal{O}_f^*). \tag{10.11.3}$$

10.12. Remark. One has

$$\begin{pmatrix}\alpha & \\ & 1\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1/\alpha & \\ & 1\end{pmatrix}=\begin{pmatrix}a & \alpha b\\ c/\alpha & d\end{pmatrix}.$$

Hence, $\mathrm{PSL}_2(\mathcal{O}_f,\mathcal{O}_f^*)$ is conjugate to $\mathrm{PSL}_2(\mathcal{O}_f)$ and their fundamental domains in $\mathbb{H}^2$ resp. $\mathbb{H}^+$ have the same volume. For $F=\mathbb{Q}(\sqrt{d}), d<0$, we have

$$\mathrm{vol}_{\mathbb{H}^+}(\Gamma_a\backslash\mathbb{H}^+)=\mathrm{vol}_{\mathbb{H}^+}(\mathrm{PSL}_2(\mathcal{O})\backslash\mathbb{H}^+)[\mathrm{PSL}_2(\mathcal{O}):\mathrm{PSL}_2(\mathcal{O}_f)] \tag{10.12.1}$$

and a similar formula for $d>0$.

Hence, using Remarks 6.2.1 and 6.3.1 and Proposition 10.9, we get

10.13. Proposition. For $d_F<0$, with $dv_{\mathbb{H}^+}=dxdydr/r^3$, one has

$$\begin{aligned}\mathrm{vol}_{\mathbb{H}^+}(\Gamma_a\backslash\mathbb{H}^+)&=\mathrm{vol}_{\mathbb{H}^+}(\mathrm{PSL}_2(\mathcal{O})\backslash\mathbb{H}^+)[\mathrm{PSL}_2(\mathcal{O}):\mathrm{PSL}_2(\mathcal{O}_f)]\\&=\frac{|d_F|^{3/2}}{4\pi^2}\zeta_F(2)f^3\prod_{p|f}(1-\chi_{d_F}(p)/p^2)\\&=:\frac{|d_F|^{3/2}}{24}L(\chi_{d_F},2)\psi(d_F,f)=:v_-\end{aligned} \tag{10.13.1}$$

and for $d_F>0$, with $dv_{\mathbb{H}^{\not\in}}=dx_1dy_1dx_2dy_2/(y_1y_2)^2$,

$$\begin{aligned}\mathrm{vol}_{\mathbb{H}^{\not\in}}(\Gamma_a\backslash\mathbb{H}^2)&=\frac{|d_F|^{3/2}}{3}L(\chi_{d_F},2)f^3\prod_{p|f}(1-\chi_{d_F}(p)/p^2)\\&=\frac{|d_F|^{3/2}}{3}L(\chi_{d_F},2)\psi(d_F,f)=:v_+.\end{aligned} \tag{10.13.2}$$

11 Siegel's volume of our fundamental domains

11.1. As already said in the Summary 10.10, we want to know

$$\kappa_a^0:=\mathrm{vol}(\hat{\Gamma}_a\backslash\hat{G}_a) \text{ and } \kappa_a:=\mathrm{vol}(\hat{\Gamma}_a\backslash\mathbb{D}_a). \tag{11.1.1}$$

Above, we related the unit groups $\hat{\Gamma}_a$ to discrete subgroups of SL_2 and determined the volumes of their fundamental domains in $\mathbb{H}^+$ resp. $\mathbb{H}^2$. We will use this to calculate κ_a.

11.2. Siegel's Volume Formula. In [S4], Siegel treats the following situation (simplified for our application): One has a quadratic form

$$S[x] = {}^t xSx = \sum_{k,l=1}^{m} s_{kl}x_k x_l, S = {}^tS \in M_m(\mathbb{R}), \mathrm{sign}(S) = (n, m-n). \tag{11.2.1}$$

Let be $x = Cy$ such that $S[Cy] = y_1^2 + \cdots + y_n^2 - y_{n+1}^2 - \cdots - y_m^2$, resp.

$$S[C] = {}^tCSC = S_0,\ S_0 = \begin{pmatrix} E_n & \\ & -E_{m-n} \end{pmatrix}. \tag{11.2.2}$$

Hence, $P = (C^tC)^{-1}$ is a majorant for S and one has

$$PS^{-1t}P = S,\ {}^tP = P. \tag{11.2.3}$$

The set $\mathbb{D}(S)$ of these majorants is a homogenous space for the orthogonal group $\mathrm{SO}(S)$ of S (the *representation space* from p.88 in [S5]) and can be parametrized as follows.
At first look at $S = S_0$. Take a matrix $Y \in M_{n,m-n}(\mathbb{R})$ with $E - Y^tY > 0$. Hence, an element P of $\mathbb{D}(S_0)$ is given by

$$P = \begin{pmatrix} \frac{E+Y^tY}{E-Y^tY} & (E-Y^tY)^{-1}Y \\ {}^tY(E-Y^tY)^{-1} & \frac{E+{}^tYY}{E-{}^tYY} \end{pmatrix}. \tag{11.2.4}$$

For the general case, one has to choose a fixed $C = C_0$ with $S[C_0] = S_0$ and, in the equation above, replaces P by $P[C_0]$.

We put

$$\mathbb{D}_{n,m-n} := \mathbb{D}(S_0) = \{Y \in M_{n,m-n}(\mathbb{R}); E - Y^tY > 0\} \tag{11.2.5}$$

$G = \mathrm{SO}(S_0)$ acts transitively on $\mathbb{D}(S_0)$

$$G \ni g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ maps } Y \mapsto g(Y) = (AY+B)(CY+D)^{-1} \tag{11.2.6}$$

A $G-$invariant volume element is given by ((6) in [S4])

$$dv_{\mathrm{Sie}} = (\det(E - Y^tY))^{-m/2} \prod_{k=1}^{n} \prod_{l=1}^{m-n} dy_{kl}. \tag{11.2.7}$$

If $\Gamma(S)$ is the unit subgroup of G, i.e. $\Gamma(S) = G \cap M_m(\mathbb{Z})$, $\mu(S)$ the measure of a fundamental domain of $\Gamma(S)$ in G (with respect to the form $d\omega$ given by (12) in [S4]) and

$\kappa_S = \mathrm{vol}(\Gamma(S)\backslash\mathbb{D}_{n,m-n})$ the volume of a fundamental domain in the representation space (with respect to (11.2.7)), one has ((13) in [S4], Theorem 7 in [S5], or (1) in [GHS]) **Siegel's volume formula**

$$2\mu(S) = \rho_n\rho_{m-n}|\mathrm{det}S|^{-(m+1)/2}\kappa_S,\ \rho_l = \prod_{k=1}^{l}\frac{\pi^{k/2}}{\Gamma(k/2)}$$

(11.2.8) $$\rho_1 = 1,\ \rho_2 = \pi,\ \rho_3 = 2\pi^2.$$

We rewrite this as

(11.2.9) $$2\,\mathrm{vol}_S(\Gamma(S)\backslash G(S)) = \rho_n\rho_{m-n}|\mathrm{det}S|^{-(m+1)/2}\mathrm{vol}_S(\Gamma(S)\backslash\mathbb{D}_{m,m-n}).$$

For $n = 2, m = 4$ and $S = S_0 = E_{2,2} = \begin{pmatrix} E_2 & \\ & -E_2 \end{pmatrix}$, we get our prototype formula

(11.2.10) $$\mathrm{vol}_S(\Gamma(S_0)\backslash\mathrm{SO}(2,2)) = (\pi^2/2)\,\mathrm{vol}_S(\Gamma(S_0)\backslash\mathbb{D}_{2,2}).$$

11.3. The (2,2) Case. It is well known that in this case the representation space is isomorphic to $\mathbb{H}^2$. We shall need some more details and put

(11.3.1) $$Q_0 = \begin{pmatrix} E_2 & \\ & -E_2 \end{pmatrix},\ \hat{Q} = (1/2)\begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix},\ \hat{C} = \begin{pmatrix} 1 & & & 1 \\ & -1 & 1 & \\ & 1 & 1 & \\ 1 & & & -1 \end{pmatrix}$$

As in Example 2.2 in 1.7, one has the standard way to connect $G_1 := (\mathrm{SL}_2(\mathbb{R}))^2$ with $G = \mathrm{SO}(Q_0) = \mathrm{SO}(2,2)$:

We identify $\mathbf{a} = {}^t(a,b,c,d) \equiv M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. For $g = (g_1, g_2) \in G_1$ the map

$$\mathbf{a} \equiv M \mapsto g_1 M {}^t g_2 = M' \equiv \mathbf{a'} =: A(g)\mathbf{a}$$

preserves the determinant, i.e. $A(g)$ is an element of $\mathrm{SO}(\check{Q})$, the orthogonal group of the quadratic form ${}^t\mathbf{a}\check{Q}\mathbf{a}$. For $z = (z_1, z_2) \in \mathbb{H}^2$ and $g_{z_j} = \begin{pmatrix} {y_j}^{1/2} & x_j{y_j}^{-1/2} \\ & {y_j}^{-1/2} \end{pmatrix}$ one has

(11.3.2) $$A(z) := A(g_{z_1}, g_{z_2}) = \begin{pmatrix} q_1 & x_2q_2 & x_1/q_2 & x_1x_2/q_1 \\ & q_2 & 0 & x_1/q_1 \\ & & 1/q_2 & x_2/q_1 \\ & & & 1/q_1 \end{pmatrix},\ q_1 = \sqrt{y_1y_2}, q_2 = \sqrt{y_1/y_2}.$$

To go over from $A(z) \in \mathrm{SO}(\hat{Q})$ to $G = \mathrm{SO}(Q_0)$, we change to $x = \hat{C}\mathbf{a}$ with ${}^t x Q_0 x = x_1^2 + x_2^2 - x_3^2 - x_4^2$ and have

$$\hat{A}(z) := \hat{C} A(z) \hat{C}^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$$

(11.3.3)

$$= (1/2) \begin{pmatrix} q_1 + 1/q_1 + x_1 x_2/q_1 & -x_2 q_2 + x_1/q_2 & x_2 q_2 + x_1/q_2 & q_1 - 1/q_1 - x_1 x_2/q_1 \\ (-x_1 + x_2)/q_1 & q_2 + 1/q_2 & -q_2 + 1/q_2 & (x_1 - x_2)/q_1 \\ (x_1 + x_2)/q_1 & -q_2 + 1/q_2 & q_2 + 1/q_2 & -(x_1 + x_2)/q_1 \\ q_1 - 1/q_1 + x_1 x_2/q_1 & -x_2 q_2 + x_1/q_2 & x_2 q_2 + x_1/q_2 & q_1 + 1/q_1 - x_1 x_2/q_1 \end{pmatrix}, .$$

$\mathbb{D}_{2,2}$ is a symmetric space belonging to $G = \mathrm{SO}(2,2)$ and from (11.2.6) one has a map

$$G \ni g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto BD^{-1} = Z \in \mathbb{D}_{2,2}. \tag{11.3.4}$$

If we take $g = \hat{A}(z)$ from (11.3.2), with

$$\zeta = \det D = (1/(4y_1y_2))\xi,\ \xi := ((|z_1|^2 + 1)y_2 + (|z_2|^2 + 1)y_1),$$

we get the map $\Psi : \mathbb{H}^2 \to \mathbb{D}_{2,2}$ we wanted

$$\begin{aligned} \mathbb{H}^2 \ni z = (z_1, z_2) \mapsto Z = BD^{-1} & \\ &= (1/\xi) \begin{pmatrix} 2(x_1 y_2 + x_2 y_1) & (|z_1|^2 - 1)y_2 + (|z_2|^2 - 1)y_1 \\ -(|z_1|^2 - 1)y_2 + (|z_2|^2 - 1)y_1 & 2(x_1 y_2 - x_2 y_1) \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{D}_{2,2}. \end{aligned} \tag{11.3.5}$$

With

$$\delta = ad - bc = \frac{-(|z_1|^2 + 1)y_2 - (|z_2|^2 + 1)y_1}{(|z_1|^2 + 1)y_2 + (|z_2|^2 + 1)y_1},$$

a small calculation leads to

$$\begin{aligned} \Delta := \det(E - Z^t Z) &= 1 - a^2 - b^2 - c^2 - d^2 + \delta^2 \\ &= 2^4 y_1^2 y_2^2 / ((|z_1|^2 + 1)y_2 + (|z_2|^2 + 1)y_1)^2 = \zeta^{-2} \end{aligned} \tag{11.3.6}$$

Another, a bit more substantial, calculation leads to

$$da \wedge db \wedge dc \wedge dd = 2^6 y_1^2 y_2^2 / (|z_1|^2 + 1)y_2 + (|z_2|^2 + 1)y_1)^4\, dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2. \tag{11.3.7}$$

Hence, for our case, Siegel's formula (11.2.7) shows up to the factor 1/4 the 'usual' volume element for $\mathbb{H}^2$

$$dv_{\mathrm{Sie}} \cdot \Psi^* = (1/(4y_1^2 y_2^2))\, dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 = (1/4) dv_{\mathbb{H}^2}. \tag{11.3.8}$$

11.4. The (1,3) Case. For $n=1, m=4$ and $S=S_0=E_{1,3}=\begin{pmatrix} E_1 & \\ & -E_3 \end{pmatrix}$, we get the prototype formula

$$\mathrm{vol}_S(\Gamma(S_0)\backslash \mathrm{SO}(1,3)) = \pi^2 \,\mathrm{vol}_S(\Gamma(S_0)\backslash \mathbb{D}_{1,3}). \tag{11.4.1}$$

As in Example 1.3 in 1.14, similarly, one may treat the case $n=1, m=4$ where Siegel's representation space comes out as the hyperbolic 3-plane $\mathbb{H}^+$. For instance from the first pages of [EGM], we know that $\bar{G}=\mathrm{SL}(2,\mathbb{C})$ has $\mathbb{H}^+$ as homogeneous space and, with $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, acts on $\mathbb{H}^+$ by

$$\mathbb{H}^+ \ni P=(z,r)=(x,y,r) \mapsto g(P)=(z',r')$$

$$z'=\frac{(az+b)(\bar{c}\bar{z}+\bar{d})+a\bar{c}r^2}{|cz+d|^2+|c|^2r^2},\; r'=\frac{r}{|cz+d|^2+|c|^2r^2}. \tag{11.4.2}$$

With $g_P=\begin{pmatrix} \sqrt{r} & z/\sqrt{r} \\ & 1/\sqrt{r} \end{pmatrix}$ and $j=(0,1)\in \mathbb{H}^+$, one has

$$g_P(j)=(z,r) \tag{11.4.3}$$

and the $\bar{G}-$invariant metric and volume form

$$ds^2=\frac{dx^2+dy^2+dr^2}{r^2}, \qquad dv_{\mathbb{H}^+}=\frac{dx\wedge dy\wedge dr}{r^3}. \tag{11.4.4}$$

There is the standard procedure to relate $\bar{G}=\mathrm{SL}(2,\mathbb{C})$ to an orthogonal group of signature (1,3): For $y_j\in\mathbb{R}, j=1,\dots,4$, we take a matrix

$$X(y)=\begin{pmatrix} y_1 & w \\ \bar{w} & y_4 \end{pmatrix}, w=y_2+iy_3, \tag{11.4.5}$$

with

$$\det(X(y))=y_1y_4-y_2^2-y_3^2={}^t y\bar{Q}y,\; \bar{Q}=\begin{pmatrix} & & & 1/2 \\ & -1 & & \\ & & -1 & \\ 1/2 & & & \end{pmatrix}, \tag{11.4.6}$$

and, with $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\bar{G}$, via

$$X(y)\mapsto gX(y)^t\bar{g}=: X(y'),\; y'=\rho(g)y \tag{11.4.7}$$

get a surjection $\rho:\bar{G}\to \mathrm{SO}(\bar{Q}),\; g\mapsto \rho(g)$ where

(11.4.8)

$$\rho(g)=\begin{pmatrix} |a|^2 & a\bar{b}+\bar{a}b & (a\bar{b}-\bar{a}b)i & |b|^2 \\ (a\bar{c}+\bar{a}c)/2 & (a\bar{d}+\bar{a}d+b\bar{c}+\bar{c}b)/2 & (a\bar{d}-\bar{a}d+c\bar{b}-\bar{c}b)i/2 & (b\bar{d}+\bar{b}d)/2 \\ (a\bar{c}-\bar{a}c)/(2i) & (a\bar{d}-\bar{a}d+b\bar{c}-\bar{c}b)/(2i) & (a\bar{d}+\bar{a}d-c\bar{b}-\bar{c}b)/2 & (b\bar{d}-\bar{b}d)/(2i) \\ |c|^2 & c\bar{d}+\bar{d}c & (c\bar{d}-\bar{c}d)i & |d|^2 \end{pmatrix}.$$

In particular, for $g = g_P$, we have

$$\rho(g_P) = \begin{pmatrix} r & 2x & 2y & |z|^2/r \\ & 1 & & x/r \\ & & 1 & y/r \\ & & & 1/r \end{pmatrix} =: \bar{A}_P. \tag{11.4.9}$$

We want to transform this to $G = \mathrm{SO}(1,3) = \mathrm{SO}(Q_0)$ and introduce

(11.4.10)
$$Q_0 = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \; C = \begin{pmatrix} 1/2 & & & 1/2 \\ & 1 & & \\ & & 1 & \\ 1/2 & & & -1/2 \end{pmatrix}, \; C^{-1} = \begin{pmatrix} 1 & & & 1 \\ & 1 & & \\ & & 1 & \\ 1 & & & -1 \end{pmatrix}.$$

With $x = Cy$, we have $\bar{q}(y) = {}^t y \bar{Q} y = {}^t x Q_0 x = q(x)$ and get

$$\nu : \mathrm{SO}(\bar{Q}) \to \mathrm{SO}(Q_0),\ \bar{A} \mapsto \nu(\bar{A}) = C\bar{A}C^{-1} =: A. \tag{11.4.11}$$

Hence, we get

$$\nu(\bar{A}_P) = \begin{pmatrix} (|z|^2+1+r^2)/(2r) & x & y & (r^2-|z|^2-1)/(2r) \\ x/r & 1 & & -x/r \\ y/r & & 1 & -y/r \\ (|z|^2-1+r^2)/(2r) & x & y & (r^2-|z|^2+1)/(2r) \end{pmatrix} =: A_P \tag{11.4.12}$$

and we have a map $\tilde{\rho} : \mathbb{H}^+ \to \mathrm{SO}(1,3)$, $P \mapsto A_P$. For $G = \mathrm{SO}(1,3)$, Siegel's representation space is parametrized by the unit ball

$$\mathbb{D}_{(1,3)} = \{X = (x_1, x_2, x_3) \in \mathbb{R}^3; ||X||^2 = x_1^2 + x_2^2 + x_3^2 < 1\} \tag{11.4.13}$$

From (11.2.6) we know that $G = \mathrm{SO}(1,3)$ acts transitively on $\mathbb{D}_{1,3}$ via

$$G \ni g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ maps } X \mapsto g(X) = (AX+B)(CX+D)^{-1} \tag{11.4.14}$$

where, here, A is a scalar, B and C are triples and D is a three by three matrix. We get a map Ψ

$$\mathbb{H}^+ \ni P \mapsto X_P = BD^{-1} \in \mathbb{D}_{1,3} \tag{11.4.15}$$

if we take

$$B = (x, y, (r^2-|z|^2-1)/(2r)),\ D = \begin{pmatrix} 1 & & -x/r \\ & 1 & -y/r \\ x & y & (r^2-|z|^2+1)/(2r) \end{pmatrix}. \tag{11.4.16}$$

We put $\Xi := \det(D) = (1 + r^2 + x^2 + y^2)/(2r)$ and get

$$X_P = BD^{-1} = (x, y, r - \Xi) \cdot \begin{pmatrix} \Xi - x^2/r & -xy/r & x/r \\ -xy/r & \Xi - y^2/r & y/r \\ -x & -y & 1 \end{pmatrix} (1/\Xi) \tag{11.4.17}$$
$$= (1/(r\Xi))(x, y, r\Xi - 1)$$

One has $1 - X_P {}^t X_P = \Xi^{-2}$ and with $r\Xi = (1/2)(x^2 + y^2 + r^2)$

$$d(x/(r\Xi)) \wedge d(y/(r\Xi)) \wedge d(1 - 1/(r\Xi)) = (r\Xi)^{-4} r dx \wedge dy \wedge dr.$$

Hence, for $n = 1, m = 4$ Siegel's volume element (11.2.7) comes out as

$$dv_{\text{Sie}} = (\det(E - X^t X)^{-m/2} \prod_{k=1}^{n} \prod_{l=1}^{m-n} dx_{kl},$$
$$dv_{\text{Sie}} \cdot \Psi^* = \Xi^4 \cdot (r\Xi)^{-4} r dx \wedge dy \wedge dr,$$
$$= \frac{dx \wedge dy \wedge dr}{r^3} = dv_{\mathbb{H}^+} \tag{11.4.18}$$

i.e., exactly the standard volume element for the hyperbolic three-space.

11.5. The (3,1) Case. By conjugation with $C = \begin{pmatrix} & & & 1 \\ & 1 & & \\ & & 1 & \\ 1 & & & \end{pmatrix}$, one has an isomorphism

$$\sigma : \mathrm{SO}(3,1) \to \mathrm{SO}(1,3), \quad A = (a_{ij}) \mapsto \begin{pmatrix} a_{44} & a_{42} & a_{43} & a_{41} \\ a_{24} & a_{22} & a_{23} & a_{21} \\ a_{34} & a_{32} & a_{33} & a_{31} \\ a_{14} & a_{12} & a_{13} & a_{11} \end{pmatrix}, \tag{11.5.1}$$

Hence, similarly, for $A : \mathrm{SL}_2(\mathbb{C}) \to \mathbb{SO}(3,1)$ and the map $\Psi : \mathrm{H}^+ \to \mathbb{D}_{31}$, we have

$$A_P := A(g_P) = (1/2r) \begin{pmatrix} -|z|^2 + 1 + r^2 & 2xr & 2yr & r^2 + |z|^2 - 1 \\ -2x & 2r & & 2x \\ -2y & & 2r & 2y \\ -|z|^2 - 1 + r^2) & 2xr & 2yr & r^2 + |z|^2 + 1 \end{pmatrix} \tag{11.5.2}$$

and

$$X_P = {}^t b \cdot d^{-1} = \begin{pmatrix} r^2 + |z|^2 - 1 \\ 2y \\ 2x \end{pmatrix} (1/\Xi) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \Xi = r^2 + x^2 + y^2 + 1$$

We get

$$dx_1 \wedge dx_2 \wedge dx_3 = (4r/\Xi^4) dx \wedge dy \wedge dr,$$
$$\det(E_3 - X_P {}^t X_P) = 4r^2 \Xi^{-2},$$

hence, for $dv_{\mathrm{Sie}} = (\det(E_3 - X_P{}^tX_P))^{-2}dx_1 \wedge dx_2 \wedge dx_3$ again

$$\Psi^* dv_{\mathrm{Sie}} = \frac{dx \wedge dy \wedge dr}{r^3} = dv_{\mathbb{H}^+}$$

11.6. The (1,2) Case. In the case $n = 1, m = 3$. we get Siegel's prototype formula

$$\mathrm{vol}_S(\Gamma(S_0)\backslash \mathrm{SO}(1,2)) = \pi/2 \, \mathrm{vol}_S(\Gamma(S_0)\backslash \mathbb{D}_{1,2}). \tag{11.6.1}$$

One has the standard way to relate the signature $(1,2)$ groups to $G_1 := \mathrm{SL}(2,\mathbb{R})$: Take

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G_1 \tag{11.6.2}$$

$$M(\mathbf{a}) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \det(M) = -a^2 - bc = {}^t\mathbf{a}\hat{Q}\mathbf{a},\ \hat{Q} = \begin{pmatrix} -1 & & \\ & & -1/2 \\ & -1/2 & \end{pmatrix}.$$

One has a map $\rho : G_1 \to \mathrm{SO}(\tilde{Q}),\ g \mapsto \rho(g)$ where

$$gM(\mathbf{a})g^{-1} = M(\rho(g)\mathbf{a}),\ \rho(g) = \begin{pmatrix} \alpha\delta + \beta\gamma & -\alpha\gamma & \beta\delta \\ -2\alpha\beta & \alpha^2 & -\beta^2 \\ 2\gamma\delta & -\gamma^2 & \delta^2 \end{pmatrix}. \tag{11.6.3}$$

We put $a = x_3, b = x_2 + x_1, c = x_2 - x_1$, i.e.,

$$\mathbf{a} = C\mathbf{x},\ C = \begin{pmatrix} & & 1 \\ 1 & 1 & \\ -1 & 1 & \end{pmatrix},\ \mathbf{x} = C^{-1}\mathbf{a}, C^{-1} = \begin{pmatrix} & 1/2 & -1/2 \\ & 1/2 & 1/2 \\ 1 & & \end{pmatrix} \tag{11.6.4}$$

to get

$${}^t\mathbf{a}\hat{Q}\mathbf{a} = {}^t\mathbf{x}Q_0\mathbf{x} = x_1^2 - x_2^2 - x_3^2,\ Q_0 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$$

and an isomorphism

$$\nu : \mathrm{SO}(\tilde{Q}) \to \mathrm{SO}(Q_0) = \mathrm{SO}(1,2),\ A \mapsto C^{-1}AC.$$

Hence we have the surjection $\rho' = \nu \cdot \rho : G_1 \to \mathrm{SO}(1,2), g \mapsto C^{-1}\rho(g)C =: A(g)$ with

$$A(g) = (1/2) \cdot \begin{pmatrix} \alpha^2 + \beta^2 + \gamma^2 + \delta^2 & \alpha^2 + \beta^2 - \gamma^2 - \delta^2 & -2(\alpha\beta + \gamma\delta) \\ \alpha^2 + \beta^2 - \gamma^2 - \delta^2 & \alpha^2 - \beta^2 - \gamma^2 + \delta^2 & 2(\gamma\delta - \alpha\beta) \\ -2(\alpha\gamma + \beta\delta) & -2\alpha\gamma + 2\beta\delta & 2(\alpha\delta + \beta\gamma) \end{pmatrix}. \tag{11.6.5}$$

In particular, for $g_z = \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ & 1/\sqrt{y} \end{pmatrix}$, we obtain

$$\rho(g_z) = \begin{pmatrix} 1 & 0 & x/y \\ -2x & y & -x^2/y \\ 0 & 0 & 1/y \end{pmatrix}. \tag{11.6.6}$$

and

$$A(g_z) = (1/2y) \cdot \begin{pmatrix} x^2+y^2+1 & -x^2+y^2-1 & -2xy \\ x^2+y^2-1 & -x^2+y^2+1 & -2xy \\ -2x & +2x & 2y \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{11.6.7}$$

where

$$B = (1/(2y))(-x^2+y^2-1, -2xy),\ D = (1/(2y)) \begin{pmatrix} -x^2+y^2+1 & -2xy \\ 2x & 2y \end{pmatrix}. \tag{11.6.8}$$

One has $\det(D) = (1/(2y))(|z|^2+1) =: (1/(2y))\xi$ and

$$\begin{aligned} X = X(z) &= BD^{-1} \\ &= (-x^2+y^2-1, -2xy) \begin{pmatrix} 2y & 2xy \\ -2x & y^2-x^2+1 \end{pmatrix} (1/\xi) \\ &= (1/(|z|^2+1))(|z|^2-1, -2x) =: (1/\xi)(\zeta, -2x) \end{aligned} \tag{11.6.9}$$

with $1 - X^tX = 4y^2/\xi^2$. Hence, we have a map

$$\begin{aligned} \Psi : \mathbb{H} \to \mathbb{D}_{1,2} = \mathbb{D} = \{X = (x_1, x_2) \in \mathbb{R}^2; X^tX < 1\} \\ z \mapsto (1/(|z|^2+1))(|z|^2-1, -2x) = (1/\xi)(\zeta, -2x) = (x_1, x_2) \end{aligned} \tag{11.6.10}$$

and get

$$dx_1 \wedge dx_2 = d(\zeta/\xi) \wedge d(-2x/\xi) = (2^3y/\xi^3)dx \wedge dy. \tag{11.6.11}$$

Thus, in this case, Siegel's volume (11.2.7) comes out as it should

$$\begin{aligned} dv_{Sie} &= (1 - X^tX)^{-3/2} dx_1 \wedge dx_2 \\ dv_{Sie} \cdot \Psi^* &= (4y^2/\xi^2)^{-3/2}(2^3y/\xi^3)dx \wedge dy = \frac{dx \wedge dy}{y^2}. \end{aligned} \tag{11.6.12}$$

11.7. Volumes of fundamental domains in the representation space. We want to use these singular relations between the orthogonal and the SL_2−world to determine the volumes $\mathrm{vol}(\hat{\Gamma}\backslash\hat{G})$ with the formula (11.2.8) where we determine the volume of the fundamental domain $\mathcal{F}$ of $\hat{\Gamma}$ in the representation space of $\hat{G}$ with the information from the SL_2-world we got above. We have the following general situation.

(11.7.1)
$$\begin{array}{ccc}
\Gamma_1 & \longrightarrow & \Gamma \\
\downarrow & & \downarrow \\
G_1 & \xrightarrow{\ \Phi\ } & G \\
\downarrow & & \downarrow \\
\mathbb{D}_1 & \xrightarrow{\ \Psi\ } & \mathbb{D} \\
\downarrow & & \downarrow \\
\mathcal{F}_1 \simeq \Gamma_1\backslash\mathbb{D}_1 & \longrightarrow & \Gamma\backslash\mathbb{D} \simeq \mathcal{F}
\end{array}$$

Hence, one has

(11.7.2)
$$\int_{\mathcal{F}_1} dv_{Sie} \cdot \Psi^* = \int_{\Psi(\mathcal{F}_1)} dv_{Sie}.$$

We have to make this concrete in our different cases.

11.8. For $F = \mathbb{Q}(j), j^2 = d > 0$, we have the situation with the natural maps and the maps explained below

(11.8.1)
$$\begin{array}{ccccccc}
\Gamma_1 = \mathrm{PSL}_2(\mathcal{O}_f, \mathcal{O}_f^*) & \longrightarrow & & & \Gamma^m & \longrightarrow & \Gamma^0 \\
\downarrow & & & & \downarrow & & \downarrow \\
\mathrm{PSL}_2(F) & \xrightarrow{\ \rho_d\ } & \mathrm{SO}^+(D_d, \mathbb{Q}) & \xrightarrow{\ \nu\ } & \mathrm{SO}^+(K_m, \mathbb{Q}) & \xrightarrow{\ \sigma\ } & \mathrm{SO}^+(2,2) \\
\downarrow \iota & & \downarrow \iota_d & & \downarrow \mathrm{id} & & \downarrow \mathrm{id} \\
(\mathrm{PSL}_2(\mathbb{R}))^2 & \xrightarrow{\ \rho_0\ } & \mathrm{SO}^+(D_0, \mathbb{R}) & \xrightarrow{\ \nu_0\ } & \mathrm{SO}^+(K_m, \mathbb{R}) & \xrightarrow{\ \sigma\ } & \mathrm{SO}^+(2,2) \\
\downarrow & & & & \downarrow & & \downarrow \\
\mathbb{H}^2 & & \xrightarrow{\qquad\qquad \Psi \qquad\qquad} & & \mathbb{D}_m & & \mathbb{D}_{2,2} \\
\downarrow & & & & \downarrow & & \downarrow \\
\mathcal{F}_1 \simeq \mathrm{PSL}_2(\mathcal{O}_f, \mathcal{O}_f^*)\backslash\mathbb{H}^2 & & \longrightarrow & & \Gamma^m\backslash\mathbb{D}_m & & \Gamma^0\backslash\mathbb{D}_{2,2} \simeq \mathcal{F}.
\end{array}$$

Here $\Gamma^m = \Gamma_{a_m}$ or $= \Gamma_{a'_m}$ in Case A or B, and $\Gamma^0 = \sigma(\Gamma^m)$,
and in **Case A1**

(11.8.2)
$$D_d = \begin{pmatrix} & & & 1/2 \\ & -1 & & \\ & & d & \\ 1/2 & & & \end{pmatrix}, D_0 = \begin{pmatrix} & & & 1/2 \\ & & -1/2 & \\ & -1/2 & & \\ 1/2 & & & \end{pmatrix}), K_m = \begin{pmatrix} & & -2m & \\ & m & & \\ -2m & & & \\ & & & -1 \end{pmatrix},$$

- ρ_d is given by

$$\text{for } g \in \mathrm{SL}_2(F), X(y) = \begin{pmatrix} y_1 & w \\ \bar{w} & y_4 \end{pmatrix}, w = y_2 + jy_3, \text{ one has } gX(y)^t\bar{g} = X(\rho_d(g)y)$$

- ρ_0 is given by

$$\text{for } g = (g_1, g_2) \in (\mathrm{SL}_2(\mathbb{R}))^2, X(\mathbf{a}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ one has } g_1 X(\mathbf{a})^t g_2 = X(\rho_0(g)\mathbf{a})$$

- for $m = df^2$($i.e.$, **case A1**), ν, ν_0, σ are given by conjugation with (respectively)

$$C = \begin{pmatrix} -4m & & & \\ & & & 1 \\ & f & & \\ & & 1 & \end{pmatrix}, C_0 = \begin{pmatrix} -4m & & & \\ & \mu & & 1 \\ & -\mu & & 1 \\ & & 1 & \end{pmatrix}, C_\mu = \begin{pmatrix} -1/(4m) & & & 1/(4m) \\ & 1/\mu & & \\ 1 & & & 1 \\ & & 1 & \end{pmatrix}, \mu = \sqrt{m},$$

- ι is given by $\iota(g) = (g, \bar{g})$ and ι_d by conjugation with

$$J = \begin{pmatrix} 1 & & & \\ & 1/2 & 1/2 & \\ & 1/(2\mu) & -1/(2\mu) & \\ & & & 1 \end{pmatrix}, \mu = \sqrt{d},$$

- $\mathbb{D}_m$ and $\mathbb{D}_{2,2}$ are Siegel's representation spaces of the groups above, and Ψ is the map (11.3.5).

11.9. Proposition. Applying (11.7.2) to the diagram we have

$$\begin{aligned} \kappa_m = \mathrm{vol}_{\mathrm{Sie}}(\Gamma^0 \backslash \mathbb{D}_{2,2}) &= \mathrm{vol}_{\mathbb{H}^2}(\mathrm{PSL}_2(\mathcal{O}_f, \mathcal{O}_f^*)\backslash \mathbb{H}^2)/4 \\ &= (1/12)|d_F|^{3/2} L(\chi_F, 2) \cdot f^3 \prod_{p|f} (1 - \chi_{d_F}(p)/p^2). \end{aligned} \tag{11.9.1}$$

11.10. To get the volume of the fundamental domain of Γ^0 in the group G we want to apply Siegel's formula (11.2.8), resp. the prototype formula (11.2.10)

$$\mathrm{vol}(\Gamma(S_0)\backslash G(S_0)) = (1/2)\pi^2 \kappa_0,\ \kappa_0 := \mathrm{vol}_{\mathrm{Sie}}(\Gamma(S_0)\backslash \mathbb{D}_{2,2}).$$

for $S_0 = \begin{pmatrix} E_2 & \\ & -E_2 \end{pmatrix}$ and

$$\mathrm{SO(2,2)}^+ =: G(S_0),\ \Gamma^0 \cap \Gamma(S_0) =: \Gamma^0(S_0),\ [\Gamma^0 : \Gamma^0(S_0)] =: \lambda_m,\ [\Gamma(S_0) : \Gamma^0(S_0)] =: \mu_m$$

From (11.8.1) we deduce

$$
(11.10.1)\qquad
\begin{array}{ccccccc}
\Gamma_1 = \mathrm{PSL}_2(\mathcal{O}_f, \mathcal{O}_f^*) & \longrightarrow & \Gamma^0 & \longleftarrow & \Gamma^0(S_0) & \longrightarrow & \Gamma(S_0) \\
\downarrow & & & \searrow & \downarrow & \swarrow & \\
(\mathrm{PSL}_2(\mathbb{R}))^2 & & \longrightarrow & & \mathrm{SO}^+(2,2) & & \\
\downarrow & & & & \downarrow & & \\
\mathbb{H}^2 & & \xrightarrow{\ \Psi\ } & & \mathbb{D}_{2,2} & & \\
\downarrow & & & \swarrow & \downarrow & \searrow & \\
\mathcal{F}_1 \simeq \mathrm{PSL}_2(\mathcal{O}_f, \mathcal{O}_f^*)\backslash\mathbb{H}^2 & \longrightarrow & \mathcal{F} \simeq \Gamma^0\backslash\mathbb{D}_{2,2} & \longrightarrow & \Gamma^0(S_0)\backslash\mathbb{D}_{2,2} & \longrightarrow & \Gamma(S_0)\backslash\mathbb{D}_{2,2}
\end{array}\ .
$$

Hence

$$\mathrm{vol}_{\mathrm{Sie}}(\Gamma^0(S_0)\backslash\mathbb{D}_{2,2}) = \mu_m \mathrm{vol}_{\mathrm{Sie}}(\Gamma(S_0)\backslash\mathbb{D}_{2,2}) = \lambda_m \mathrm{vol}_{Sie}(\Gamma^0\backslash\mathbb{D}_{2,2}),$$

and

$$\mathrm{vol}_{\mathrm{Sie}}(\Gamma^0(S_0)\backslash G(S_0)) = \mu_m \mathrm{vol}_{\mathrm{Sie}}(\Gamma(S_0)\backslash G(S_0)) = \lambda_m \mathrm{vol}_{Sie}(\Gamma^0\backslash G(S_0)).$$

From here, finally

11.11. Corollary. We have

$$
\begin{aligned}
\kappa_m^0 = \mathrm{vol}(\Gamma^0\backslash\mathrm{SO}(2,2)) &= (\mu_m/\lambda_m)\mathrm{vol}(\Gamma(S_0)\backslash G(S_0)) \\
&= (\mu_m/\lambda_m)(\pi^2/2)\mathrm{vol}_{\mathrm{Sie}}(\Gamma(S_0)\backslash\mathbb{D}_{2,2}) = (\pi^2/2)\mathrm{vol}_{Sie}(\Gamma^0\backslash\mathbb{D}_{2,2}) \\
&= (\pi^2/8)\mathrm{vol}_{\mathbb{H}^2}((\mathrm{PSL}_2(\mathcal{O}_f, \mathcal{O}_f^*)\backslash\mathbb{H}^2) \\
&= (\pi^2/24)|d_F|^{3/2}L(\chi_F, 2)\cdot f^3 \prod_{p|f}(1-\chi_{d_F}(p)/p^2).
\end{aligned}
\tag{11.11.1}
$$

11.12. In **Case A2**, we have $4m = df^2$. with $d \equiv 1 \bmod 4$. Everything is essentially as above: only in the description of ν, we have to change C to

$$C' = \begin{pmatrix} -4m & & & \\ & & & 1 \\ & f/2 & & \\ & & 1 & \end{pmatrix},$$

and again we come to the result (11.11.1).

In **Case B2**, we have to look at $4\Delta = 4(4M+1) = df^2, d = d_F = 4N+1, j = \sqrt{d}$, and we have more changes in the interpretation of the diagram above. Here D_d and K_m change to

$$D_\Delta = \begin{pmatrix} & & & 1/2 \\ & -1 & -1 & \\ & -1 & \Delta-1 & \\ 1/2 & & & \end{pmatrix}, K_\Delta = \begin{pmatrix} & & & -2\Delta \\ & \Delta-1 & -2 & \\ & -2 & -4 & \\ -2\Delta & & & \end{pmatrix},$$

and ρ_d changes to the the map $\rho_\Delta : \mathrm{PSL}_2(F) \to \mathrm{SO}^+(D_\Delta)$, where ρ_Δ is given by

for $g \in \mathrm{SL}_2(F), X(y) = \begin{pmatrix} y_1 & w \\ \bar{w} & y_4 \end{pmatrix}, w = y_2 + (1 + fj/2)jy_3$, one has $gX(y)^t\bar{g} = X(\rho_\Delta(g)y)$,

ν changes to ν_Δ, ν_0 to $\nu_{0,\Delta}$, σ to σ_δ and ι_d to ι_Δ given by conjugation with

$$C_\Delta = \begin{pmatrix} -4\Delta & & & \\ & & 2 & \\ & 1 & & \\ & & & 1 \end{pmatrix},$$

$$C_{0,\Delta} = \begin{pmatrix} -1/(4\Delta) & & & \\ & 1/(2\delta) & 1/(2\delta) & \\ & 1/2-1/(2\delta) & 1/2+1/(2\delta) & \\ & & & 1 \end{pmatrix},$$

$$C_{0,\delta} = \begin{pmatrix} -1/(4\Delta) & & & 1/(4\Delta) \\ & 1/\delta & & \\ & -1/(2\delta) & 1/2 & \\ 1 & & & -1 \end{pmatrix}, \delta = \sqrt{\Delta},$$

$$J_\Delta = \begin{pmatrix} 1 & & & \\ & 1-1/(2\delta) & 1+1/(2\delta) & \\ & 1/(2\delta) & -1/(2\delta) & \\ & & & 1 \end{pmatrix},$$

Using all this, the same way as in Case A1, we get (11.11.1).

11.13. For $F = \mathbb{Q}(j), j^2 = d < 0$, we have slight changes in our diagram and come to

(11.13.1)

$$\begin{array}{ccccccc}
\Gamma_1 = \mathrm{PSL}_2(\mathcal{O}_f, \mathcal{O}_f^*) & \longrightarrow & & \longrightarrow & \Gamma^m & \longrightarrow & \Gamma^0 \\
\downarrow & & & & \downarrow & & \downarrow \\
\mathrm{PSL}_2(F) & \xrightarrow{\rho_d} & \mathrm{SO}^+(D_d, \mathbb{Q}) & \xrightarrow{\nu} & \mathrm{SO}^+(K_m, \mathbb{Q}) & \xrightarrow{\sigma} & \mathrm{SO}^+(1,3) \\
\downarrow \iota & & \downarrow \iota_d & & \downarrow \mathrm{id} & & \downarrow \mathrm{id} \\
\mathrm{PSL}_2(\mathbb{C}) & \xrightarrow{\rho_0} & \mathrm{SO}^+(D_0, \mathbb{R}) & \xrightarrow{\nu_0} & \mathrm{SO}^+(K_m, \mathbb{R}) & \xrightarrow{\sigma} & \mathrm{SO}^+(1,3) \\
\downarrow & & & & \downarrow & & \downarrow \\
\mathbb{H}^+ & & & \xrightarrow{\Psi} & \mathbb{D}_m & & \mathbb{D}_{1,3} \\
\downarrow & & & & \downarrow & & \downarrow \\
\mathcal{F}_1 \simeq \mathrm{PSL}_2(\mathcal{O}_f, \mathcal{O}_f^*)\backslash\mathbb{H}^+ & & & \longrightarrow & \Gamma^m\backslash\mathbb{D}_m & & \Gamma^0\backslash\mathbb{D}_{1,3} \simeq \mathcal{F}
\end{array}$$

Here, we have $\Gamma^m = \Gamma_{a_m}$ or $= \Gamma_{a'_m}$ in Case A or B, and $\Gamma^0 = \sigma(\Gamma^m)$,
and in **Case A1**

$$D_d = \begin{pmatrix} & & & 1/2 \\ & -1 & & \\ & & d & \\ 1/2 & & & \end{pmatrix}, D_0 = \begin{pmatrix} & & & 1/2 \\ & -1 & & \\ & & -1 & \\ 1/2 & & & \end{pmatrix}, K_m = \begin{pmatrix} & & -2m & \\ & m & & \\ -2m & & & \\ & & & -1 \end{pmatrix},$$

- ρ_d is given by

$$\text{for } g \in \mathrm{SL}_2(F), X(y) = \begin{pmatrix} y_1 & w \\ \bar{w} & y_4 \end{pmatrix}, w = y_2 + jy_3, \text{ one has } gX(y){}^t\bar{g} = X(\rho_d(g)y)$$

- ρ_0 is given by

$$\text{for } g \in \mathrm{SL}_2(\mathbb{C}), X(v) = \begin{pmatrix} v_1 & v_2 + iv_3 \\ v_2 - iv_3 & v_4 \end{pmatrix}, i^2 = -1, \text{ one has } gX(v){}^t\bar{g} = X(\rho_0(g)v),$$

- for $m = df^2$($i.e.$, **Case A1**), ν, ν_0, σ are given by conjugation with (respectively)

$$C = \begin{pmatrix} -4m & & & \\ & & & 1 \\ & f & & \\ & & 1 & \end{pmatrix}, C_0 = \begin{pmatrix} -4m & & & \\ & & & 1 \\ & \mu & & \\ & & 1 & \end{pmatrix}, C_\mu = \begin{pmatrix} -1/(4m) & & & 1/(4m) \\ & 1/\mu & & \\ 1 & & & 1 \\ & & 1 & \end{pmatrix}, \mu = \sqrt{-m},$$

- ι is given by $\iota(g) = g$ and ι_d by conjugation with

$$J = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1/\mu & \\ & & & 1 \end{pmatrix}, \mu = \sqrt{-d},$$

- $\mathbb{D}_m$ and $\mathbb{D}_{1,3}$ are Siegel's representation spaces of the groups above, and Ψ is the map (11.4.15).

In **Case A2**, hence $4m = df^2 < 0$, we change ν to $\check{\nu}$ with C to $\check{C} = \begin{pmatrix} -4m & & & \\ & & & 1 \\ & f/2 & & \\ & & 1 & \end{pmatrix}$.

In **Case B**, we have $4\Delta = df^2, \Delta = 4M + 1, d = 4N + 1$ and we change D_d and K_m to

$$D_\Delta = \begin{pmatrix} & & & 1/2 \\ & -1 & -1 & \\ & -1 & \Delta-1 & \\ 1/2 & & & \end{pmatrix}, K_\Delta = \begin{pmatrix} & & & -2\Delta \\ & \Delta-1 & -2 & \\ & -2 & -4 & \\ -2\Delta & & & \end{pmatrix},$$

Moreover, we change ν_d, ν_0, σ to $\nu_\Delta, \nu_0', \sigma'$ given by conjugation (respectively) with

$$C_\Delta = \begin{pmatrix} -4\Delta & & & \\ & & 2 & \\ & f & & \\ & & & 1 \end{pmatrix}, C_\Delta' = \begin{pmatrix} -4\Delta & & & \\ & \delta & & \\ & 1 & 2 & \\ & & & 1 \end{pmatrix}, C' = \begin{pmatrix} -(1/(4\Delta)) & & & -(1/(4\Delta)) \\ & 1/\delta & & \\ & -1/(2\delta) & 1/2 & \\ 1 & & & -1 \end{pmatrix}.$$

ι again is the simple injection and ι_m is to be replaced by ι_Δ with $y = \tilde{C}v$ given by conjugation with

$$\tilde{C} = \begin{pmatrix} 1 & & & \\ & -1/\delta & 1 & \\ & 1/\delta & & \\ & & & 1 \end{pmatrix}.$$

From the diagram (11.13.1) in all cases, one can compute the same way as above for $d > 0$. Usig the protoformula (6.2.1) from [EGM] and Proposition 10.9

$$\text{(11.13.2)} \qquad \begin{aligned} \kappa_m = \mathrm{vol}_{\mathrm{Sie}}(\Gamma^0 \backslash \mathbb{D}_{1,3}) &= \mathrm{vol}_{\mathbb{H}^+}(\mathrm{PSL}_2(\mathcal{O}_f, \mathcal{O}_f^*) \backslash \mathbb{H}^+) \\ &= (1/24)|d_F|^{3/2} L(\chi_F, 2) \cdot f^3 \prod_{p|f} (1 - \chi_{d_F}(p)/p^2). \end{aligned}$$

From Siegel's formula (11.2.8), resp. its prototype (11.4.1), we have

$$\kappa(S_1) = \mathrm{vol}(\Gamma(S_1)\backslash G(S_1)) = \pi^2\kappa_1,\ \kappa_1 = \mathrm{vol}_{Sie}(\Gamma(S_1)\backslash\mathbb{D}_{1,3}),\ S_1 = \begin{pmatrix} 1 & \\ & -E_3 \end{pmatrix}.$$

Applying (11.7.2), one has

$$\mathrm{vol}_{\mathbb{H}^+}(\mathrm{PSL}_2(\mathcal{O}_f, \mathcal{O}_f^*)\backslash\mathbb{H}^+) = \mathrm{vol}_{Sie}(\Gamma^0\backslash\mathbb{D}_{1,3}). \tag{11.13.3}$$

With

$$\Gamma^0_{a'} \cap \Gamma(S_1) =: \Gamma^*,\ [\Gamma^0_{a'} : \Gamma^*] =: \lambda,\ [\Gamma(S_0) : \Gamma^*] =: \mu$$

one has

$$\mathrm{vol}_{Sie}(\Gamma^*\backslash\mathbb{D}_{1,3}) = \mu\,\kappa_1 = \lambda\,\mathrm{vol}_{Sie}(\Gamma^0_{a'}\backslash\mathbb{D}_{1,3}),$$

hence

$$\mathrm{vol}_{Sie}(\Gamma^0\backslash\mathbb{D}_{1,3}) = (\mu/\lambda)\kappa_1$$

and finally with (11.13.3)

$$\begin{aligned}\kappa_m^0 = \mathrm{vol}(\Gamma^0\backslash\mathrm{SO}(1,3)) &= (\mu/\lambda)\pi^2\kappa_1 = \pi^2\mathrm{vol}_{Sie}(\Gamma^0_{a'}\backslash\mathbb{D}_{1,3})\\ &= \pi^2\mathrm{vol}_{\mathbb{H}^+}(\mathrm{PSL}_2(\mathcal{O}_f, \mathcal{O}_f^*)\backslash\mathbb{H}^+)\\ &= (\pi^2/24)|d_F|^{3/2}L(\chi_F, 2)\cdot f^3\prod_{p|f}(1-\chi_{d_F}(p)/p^2).\end{aligned} \tag{11.13.4}$$

11.14. Summary. For $m > 0$, we have the voluminae

$$\begin{aligned}\kappa_m &= \mathrm{vol}_{\mathrm{Sie}}(\Gamma^0\backslash\mathbb{D}_{2,2}) = (1/12)|d_F|^{3/2}L(\chi_F, 2)\cdot f^3\prod_{p|f}(1-\chi_{d_F}(p)/p^2),\\ \kappa_m^0 &= \mathrm{vol}(\Gamma^0\backslash\mathrm{SO}(2,2)) = (\pi^2/24)|d_F|^{3/2}L(\chi_F, 2)\cdot f^3\prod_{p|f}(1-\chi_{d_F}(p)/p^2),\end{aligned} \tag{11.14.1}$$

and for $m < 0$

$$\begin{aligned}\kappa_m &= \mathrm{vol}_{\mathrm{Sie}}(\Gamma^0\backslash\mathbb{D}_{1,3}) = (1/24)|d_F|^{3/2}L(\chi_F, 2)\cdot f^3\prod_{p|f}(1-\chi_{d_F}(p)/p^2),\\ \kappa_m^0 &= \mathrm{vol}(\Gamma^0\backslash\mathrm{SO}(1,3)) = (\pi^2/24)|d_F|^{3/2}L(\chi_F, 2)\cdot f^3\prod_{p|f}(1-\chi_{d_F}(p)/p^2).\end{aligned} \tag{11.14.2}$$

Apparently, these voluminae could be determined more elegantly: κ_m may be interpreted as the real Tamagawa (Haar) measure $\alpha_\infty(L_m)$ of the lattice L_m belonging to K_m, and hence, essentially is the inverse of the product of the (finite) local densities $\alpha_p(L_m)$. But here one easily is lost in subtle calculations in particular concerning the prime $p = 2$.

12 The Kudla Green function integral for signature (3,2)

We want to determine the value of Kudla's Green function integral from (4.6.1).

$$I(0,m,v) := \int_X \Xi(0,m,v,z)d\mu_z = (1/2)\sum_{x\in L_{0,m}}\int_X \beta(2\pi vR(x,z))d\mu_z,$$
$$I(1,m,v) := \int_X \Xi(1,m,v,z)d\mu_z = (1/2)\sum_{x\in L_{1,m}}\int_X \beta(2\pi vR(x,z))d\mu_z,$$

resp. from (4.7.1)

$$I(0,m,v) = (1/2)\sum_{n^2|m}\int_{\Gamma_{a_{m/n^2}}\backslash\mathbb{D}} \beta(2\pi vR(na_{m/n^2},z))d\mu_z =: \sum_{n^2|m} I(0,m,n,v),$$
$$I(1,m,v) = (1/2)\sum_{n^2|4m}\int_{\Gamma_{a'_{4m/n^2}}\backslash\mathbb{D}} \beta(2\pi vR(na'_{4m/n^2,z})d\mu_z =: \sum_{n^2|4m} I(1,m,n,v).$$

Kudla's Green-integral formula

12.1. To evaluate the integrals, we try to follow closely Kudla in [Ku1] p.318. There, he looks at the integral

$$\int_{\Gamma_x\backslash\mathbb{D}^+}\beta_{\sigma+1}(2\pi vR(x,z))d\mu_z, \quad \beta_{\sigma+1} := \int_1^\infty e^{-t\nu}\nu^{-\sigma-1}d\nu, \tag{12.1.1}$$

where he treats a more general $\mathrm{SO}(p,2)-$case and $x \in V(\mathbb{R})$ with $Q(x) = m$: For $m > 0$, he chooses a basis $\mathbf{v}$ for $V(\mathbb{R})$ so that the inner product has matrix $I_{p,2}$ and so that the respective special element x is a nonzero multiple of the first basis vector v_1, i.e. $x = 2\alpha v_1$. Then $\mathrm{SO}(V)(\mathbb{R})^+ = \mathrm{SO}^+(p,2) = G$ and the subgroup stabilizing x is isomorphic to $\mathrm{SO}^+(p-1,2) = G_x$. Kudla further proposes $z_0 \in \mathbb{D}$ to be the oriented negative 2-plane spanned by v_{p+1} and v_{p+2} and let $K = \mathrm{SO}(n)\times\mathrm{SO}(2)$ be its stabilizer in G. The plane spanned by v_1 and v_{p+1}, the first negative basis vector, has signature (1,1). The identity component of the special orthogonal group of this plane is a 1-parameter subgroup

$$A = \{a_t; t\in\mathbb{R}\} \simeq \{\left(\begin{smallmatrix}\cosh t & \sinh t\\ \sinh t & \cosh t\end{smallmatrix}\right); t\in\mathbb{R}\} \tag{12.1.2}$$

where $a_t\cdot v_1 = v_1\cosh t + v_{n+1}\sinh t$. Let A_+ the subset of a_t's with $t\geq 0$. Then, from the general theory of semisimple symmetric spaces - with Flensted-Jensen [FlJ] Sect.2 as a reference - Kudla has a 'double set decomposition'

$$G = G_xA_+K \tag{12.1.3}$$

and the integral formula (3.21)

$$\int_G \Phi(g)dg = \int_{G_x}\int_{A_+}\int_K \Phi(g_x a_t k)|\sinh t|(\cosh t)^{p-1} dg_x dt dk. \tag{12.1.4}$$

For $z = g_x a_t \cdot z_0$, one has

$$R(x,z) = 2m\sinh^2 t. \tag{12.1.5}$$

and, from [Ku1] (3.23) with a positive constant C depending on normalization of invariant measures, what we will cite as **Kudla's Green-integral formula**

$$\begin{aligned}&\int_{\Gamma_x\backslash\mathbb{D}^+}\int_1^\infty e^{-2\pi v R(x,z)r} dr/r d\mu(z)\\ &= C\mathrm{vol}(\Gamma_x\setminus G_x)\mathrm{vol}(K)\int_0^\infty\int_1^\infty e^{-4\pi v|m|\sinh^2 tr} dr/r \sinh t(\cosh t)^{p-1}dt.\end{aligned} \tag{12.1.6}$$

We abbreviate

$$I_+^p(v,m) := \int_0^\infty\int_1^\infty e^{-4\pi v|m|\sinh^2 tr} dr/r \sinh t(\cosh t)^{p-1}dt \tag{12.1.7}$$

and will determine this for $p = 1,2,3$ later in Summary 12.22. For $m < 0$, we do alike and choose a basis for $V(\mathbb{R})$ such that $x = 2\alpha v_{p+1}, Q(x) = -2\alpha^2 = m$. Here, we have $G_x \simeq \mathrm{SO}(p,1)$ and, with a similar reasoning as above, with (12.1.3) and

$$R(x,z) = 2|m|\cosh^2 t. \tag{12.1.8}$$

$$\begin{aligned}&\int_{\Gamma_x\backslash\mathbb{D}^+}\int_1^\infty e^{-2\pi v R(x,z)r} dr/r d\mu(z)\\ &= C\mathrm{vol}(\Gamma_x\setminus G_x)\mathrm{vol}(K) I_-^p(v,m)\end{aligned} \tag{12.1.9}$$

with

$$I_-^p(v,m) := \int_0^\infty\int_1^\infty e^{-4\pi v|m|\cosh^2 tr} dr/r (\sinh t)^{p-1}\cosh t dt. \tag{12.1.10}$$

Flensted-Jensen's integral formula

12.2. We add some comments and variants to Kudla's formulae. The background to the integral formula is something like

$$\int_{\Gamma_x\backslash\mathbb{D}} = \int_{\Gamma_x\backslash G/K} = \int_{\Gamma_x\backslash G_x A_+}. \tag{12.2.1}$$

The main point is the formula (12.1.3) $G = G_x A_+ K$, a variant of the Cartan decomposition to be found for instance in Theorem 2.4 in Heckman-Schlichtkrull [HS]. This is the background to Flensted-Jensen's important integration formula (2.14) in his Theorem 2.6 in [FlJ] which Kudla is using above.

To discuss this, for a moment, we go back to the (3,2)-case: Elements of $G = \mathrm{SO}(3,2)$ are described by 10 parameters, elements of $G_a = \mathrm{SO}(2,2) =: H$ by 6 parameters, elements of $K = \mathrm{SO}(3) \times \mathrm{SO}(2)$ by 4 parameters. Hence on the right hand side of (12.1.3), one has 11 parameters. And, moreover, the element

$$\begin{pmatrix} \cosh t & & & & \sinh t \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ \sinh t & & & & \cosh t \end{pmatrix} = \exp(tX_{1,5}) \in G \tag{12.2.2}$$

looks as if it is not to be found on the right hand side. But one easily verifies $\mathrm{Ad}(\ell)\exp X_{1,5} = \exp X_{1,4}$, for $\ell = \{\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 0 & -1 \\ & & & 1 & 0 \end{pmatrix} \in K$. As to be extracted from [FlJ] Sect.2, we get the following: For the corresponding Lie algebras, we have

$$\begin{aligned} \mathfrak{g} &= \mathfrak{k} + \mathfrak{p} \\ &= \{\begin{pmatrix} A & \\ & B \end{pmatrix}, A \in M_3(\mathbb{R})\,\text{skew}, B \in M_2(\mathbb{R})\,\text{skew}\} + \{\begin{pmatrix} & C \\ {}^tC & \end{pmatrix}, C \in M_{3,2}(\mathbb{R})\} \\ &= \langle X_{1,2}, X_{1,3}, X_{2,3}, X_{4,5}\rangle + \langle X_{1,4}, X_{1,5}, X_{2,4}, X_{2,5}, X_{3,4}, X_{3,5}\rangle \\ &= \mathfrak{h} + \mathfrak{q} \\ &= \langle X_{2,3}, X_{4,5}, X_{2,4}, X_{2,5}, X_{3,4}, X_{3,5}\rangle + \langle X_{1,2}, X_{1,3}, X_{1,4}, X_{1,5}\rangle. \end{aligned} \tag{12.2.3}$$

There are the Lie relations

$$\begin{aligned} &[X_{1,5}, X_{1,2}] = X_{2,5}, \quad [X_{1,5}, X_{1,3}] = X_{3,5} \\ &[X_{1,5}, X_{2,5}] = X_{1,2}, \quad [X_{1,5}, X_{3,5}] = X_{1,3} \\ &[X_{1,4}, X_{4,5}] = X_{1,5}, \quad [X_{1,4}, X_{2,5}] = 0 \end{aligned} \tag{12.2.4}$$

and many similar. Here $\mathfrak{k}, \mathfrak{p}$ are the ± 1 eigenspaces of the Cartan involution τ with $\tau X = -{}^tX$ and $\mathfrak{h}, \mathfrak{q}$ are the ± 1 eigenspaces of the involution σ with $\sigma X = E_{1,4} X E_{1,4}$. Corresponding groups are $K = \mathrm{SO}(3) \times \mathrm{SO}(2)$ and $H \simeq \mathrm{SO}(2,2)$. We put $L = K \cap H$, $\mathfrak{l} = \mathrm{Lie}\, L$, choose $\mathfrak{b}$ maximal abelian in $\mathfrak{p} \cap \mathfrak{q}$, and

$$M = Z_L(\mathfrak{b}) = \{\ell \in L; \mathrm{Ad}(\ell)B = B \text{ for all } B \in \mathfrak{b}\}.$$

In our case, we have $\mathfrak{l} = \mathfrak{k} \cap \mathfrak{h} = \langle X_{2,3}, X_{4,5}\rangle$, $\mathfrak{p} \cap \mathfrak{q} = \langle X_{1,4}, X_{1,5}\rangle$, and $\mathfrak{b} = \langle X_{1,4}\rangle$. Hence $M = \{\exp tX_{2,3}; t \in \mathbb{R}\} \simeq \mathrm{SO}(2)$.

The background of Kudla's and Flensted-Jensen's integral formolae is in the following geometric consideration: In [FlJ] p.261, one observes that the map $L/M \times \mathfrak{b} \to \mathfrak{p} \cap \mathfrak{q}$ given by

$$(lM, B) \mapsto \mathrm{Ad}(l)B \tag{12.2.5}$$

is a diffeomorphism onto an open dense set. Therefore, the map

$$\Phi : \mathfrak{p} \cap \mathfrak{h} \times L/M \times \mathfrak{b} \to G/K \tag{12.2.6}$$

given by

$$\Phi(X, lM, B) = \pi(\exp X\, l \exp B),$$

where $\pi : G \to G/K$ is the canonical map, is a diffeomorphism unto an open dense set and also $\Phi' : X \mapsto \exp X L$ is a diffeomorphism of $\mathfrak{p} \cap \mathfrak{h}$ unto H/L.

The Killing form defines Riemannian (i.e., Euclidean) structures on $\mathfrak{p} \cap \mathfrak{h}, \mathfrak{b}^+$, and L/M, and one lets the measure on L/M be $\mathrm{vol}(L/M)^{-1}$ times the volume element. Via Killing form, one has Riemannian structures on G/K and H/L, and by their volume elements also measures. Moreover, following Flensted-Jensen, we take measures on G and H such that

$$\begin{aligned} \int_G f(x)dx &= \int_{G/K}\int_K f(xk)dkdxK, \quad \int_K dk = 1, \quad \text{for } f \in C_c(G) \\ \int_H f(x)dx &= \int_{H/L}\int_L f(xk)dkdxL, \quad \int_L dk = 1, \quad \text{for } f \in C_c(H). \end{aligned} \tag{12.2.7}$$

Taking the Jacobians $J(X, lM, B) = |\det d\Phi_{(X,lM,B)}|$ and $J_1(X) = |\det d\Phi^1_{(X)}|$ with reference to the respective Riemannian structures, one has for $f \in C_c(G)$ and $f_1 \in C_c(H)$

$$\begin{aligned} \int_{G/K} f(x)dx &= \mathrm{vol}\,(L/M)\int_{\mathfrak{p}\cap\mathfrak{h}}\int_{L/M}\int_{\mathfrak{b}^+} f(\Phi(X, lM, B))J(X, lM, B)dBdlMdX \\ &= \mathrm{vol}\,(L/M)\int_{H/M}\int_{\mathfrak{b}^+} f(h\exp B)\delta_1(B)dBdh \end{aligned} \tag{12.2.8}$$

where $\delta_1(B) = |\det d\Phi_{(0,eM,B)}|$, $B \in \mathfrak{b}^+$. From here (his formula (2.9)), Flensted-Jensen comes to the formula (2.14) in his Theorem 2.6

$$\int_G f(g)dg = \mathrm{vol}\,(L/M)\int_K\int_H\int_{\mathfrak{b}^+} f(k\exp H'h)\delta(H')dH'dhdk \quad \text{for } f \in C_c(G) \tag{12.2.9}$$

where δ given by formula (2.12) comes from the δ_1. This also is taken on in a similar way by Kudla-Millson [KMII] (4.35) and (4.37) as

(12.2.10)

$$\int_{\Gamma\backslash G} f(g)dg = \mathrm{vol}\,(L/M)\int_K\int_{\Gamma\backslash H}\int_{\mathfrak{b}^+} f(h\exp X'k)\delta(X)dXdhdk \quad \text{for } f \in C_c(G).$$

In the sequel, we shall use this to interpret Kudla's Green integral. At first, some remarks concerning the function to be integrated:

The integrand

12.3. In the (3,2)-case, one has a two-component Green function and, as above from (4.7.1), the Green function integrals

$$(12.3.1)\quad I(0,m,v) = (1/2)\sum_{n^2|m}\int_{\Gamma_{a_{m/n^2}}\backslash\mathbb{D}} \beta(2\pi v R(na_{m/n^2},z))d\mu_z =: \sum_{n^2|m} I(0,m,n,v),$$

$$I(1,m,v) = (1/2)\sum_{n^2|4m}\int_{\Gamma_{a'_{4m/n^2}}\backslash\mathbb{D}} \beta(2\pi v R(na'_{4m/n^2},z))d\mu_z =: \sum_{n^2|4m} I(1,m,n,v).$$

We want to determine $I(\gamma,v,m,n)$ and may use Flensted-Jensen's resp.Kudla's formulae, as we know from (3.3.1) that our $R(x,z)$ is left G_x- and right $K-$ invariant and depends only on the hyperbolic group A. To do this, we have to assemble and clarify step by step several items. We distinguish between $m>0$, Case I, and $m<0$, Case II. Moreover, as above, Case A for $m\in\mathbb{Z}$ and Case B for $m\in\mathbb{Z}+1/4$.

12.4. In Case I, the formula (12.1.5)

$$R(x,z) = 2m\sinh^2 t$$

can be derived as follows. We remember the 'Key Relation' (3.5)

$$(x,x)_z = (x,x) + 2R(x,z) = {}^t((A(g_z))^{-1}x)P_0((A(g_z))^{-1}x).$$

Here we have $(x,x) = {}^tx2E_{3,2}x, x = {}^t(2\alpha,0,0,0,0)$ and $A(g_z) = g(x)a_t, g(x)\in G_x$, $P_0 = E$, hence

$${}^t((A(g_z))^{-1}x) = (\cosh t,0,0,-\sinh t,0)2\alpha$$

and, as $(x,x) = 2Q(x) = 4\alpha^2 = 2m$, the Key Relation says

$$(x,x) + 2R(x,z) = 4\alpha^2 + 2R(x,z) = (\cosh^2 t + \sinh^2 t)4\alpha^2 = (2\sinh^2 t + 1)4\alpha^2,$$

i.e., $R(x,z) = 2m\sinh^2 t$.

For $m<0$, i.e., Case II, we do alike and choose a basis for $V(\mathbb{R})$ such that $x = 2\alpha v_{p+1}, Q(x) = -2\alpha^2 = m$. Here, we have $G_x \simeq \mathrm{SO}(p,1)$. And for $x = {}^t(0,\dots,0,2\alpha,0)$ and $A(g_z) = g(x)a_t, g(x)\in G_x$, hence

$${}^t((A(g_z))^{-1}x) = (-\sinh t,0,0,\cosh t,0)2\alpha$$

and, as $(x,x) = 2Q(x) = -4\alpha^2 = 2m$, the Key Relation says

$$(x,x) + 2R(x,z) = -4\alpha^2 + 2R(x,z) = (\cosh^2 t + \sinh^2 t)4\alpha^2 = (2\cosh^2 t - 1)4\alpha^2,$$

i.e.,

$$R(x,z) = 2|m|\cosh^2 t. \tag{12.4.1}$$

Hence we have verified the following.

12.5. Remark. From (12.1.5) and (12.4.1) with $z = g_x a_t \cdot z_0$, one has

$$\begin{aligned} R(x,z) &= 2m\sinh^2 t \quad \text{in Case I with} \quad m > 0 \\ &= 2|m|\cosh^2 t \quad \text{in Case II with} \quad m < 0. \end{aligned}$$

Differential forms and measures

12.6. In Kudla's Green function integrals above for the $(p,2)-$case, the measure $d\mu(z)$ on X is given by Ω^3 as in [Ku1] (5.8) or [BK] (4.50) with

$$\Omega^3 = -\frac{3}{16\pi^3}\det(y)^{-3}(\frac{i}{2})^3 dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \wedge dz_3 \wedge d\bar{z}_3. \tag{12.6.1}$$

In the paper [BY] by Bruinier and Yang, there is another formula relating differential forms and measures. From Proposition 3.4 in [BY], we have

$$(d\ell_x)^*(-\Omega)^p = \pm\frac{p!}{(2\pi)^p}\nu_{\mathfrak{p}} \tag{12.6.2}$$

Here, we are in the following situation. We have a vector space V of dimension $n+1$ with a quadratic form Q of signature (p,2), bilinear form (.,.), the associated orthogonal group $H = \mathrm{SO}(V)$, and the homogenous space $\mathcal{D} = H(\mathbb{R})/K_\infty$, and Shimura variety $X_K = H(\mathbb{Q})\backslash(\mathcal{D} \times H(\mathbb{A}_f)/K)$.

One has a map $Q : V^n \to \mathrm{Sym}_n$, $x \mapsto ((x_i, x_j)_{i,j})$. Let α be a gauge form for V^n, i.e., a top level differential of V^n, a generator of $(\wedge^{n(n+1)}V^n)^*$, and similarly β a gauge $n(n+1)/2-$form for Sym_m. In [BY] Section 2.2, it is explained that one can find $\nu \in (\wedge^{n(n+1)/2}V)^*$ on V^n_{reg} with $\alpha = Q^*(\beta) \wedge \nu$. This ν may be identified with a top degree invariant form on SO(V) which again is called ν., and gives a Haar measure $dh = d_\nu h = dh_+ dh_- dh_{\mathfrak{p}}$ on SO(V).

Here, we apply this with $n = 4, p = 3$ and, hence, may identify $\nu_{\mathfrak{p}}$ with dx in (16.16.12). We take this to realize the Flensted-Jensen formula for an integral of a Green function f as above via

$$\begin{aligned} \int_{\Gamma_x\backslash\mathbb{D}} f(z)d\mu(z) &= (p!/(2\pi)^p)\cdot \mathrm{vol}\,(L/M)\int_K\int_{\Gamma_x\backslash G_x}\int_{\mathfrak{b}+} f(h\exp X'k)\delta(X)dXdhdk, \\ &= (p!/(2\pi)^p)\cdot \mathrm{vol}\,(L/M)\cdot \mathrm{vol}(\Gamma_x\setminus G_x)\cdot I^p_\pm(v,m), \end{aligned} \tag{12.6.3}$$

where $d\mu(z) = \Omega^p$ and Γ_x and $I^p_\pm(v,m)$ has to be spezified in each case.

Now, for $G = \mathrm{SO}(3,2)$, Case I, and $H = \mathrm{SO}(2,2)$, with $a_{0,m/n^2} =: x$ (12.2.10), by the usual unfolding we have

(12.6.4)

$$
\begin{aligned}
I(m,v) &= (1/2)\int_{\Gamma\backslash\mathbb{D}}\sum_{x\in L_m}\beta(2\pi mR(x,z))d\mu(z)\\
&= (1/2)\sum_{x\in\Gamma\backslash L_m}\int_{\Gamma_x\backslash\mathbb{D}}\beta(2\pi mR(x,z))d\mu(z).\\
&= \sum_{n^2|m} I(0,m,n,v)\\
\cdot I(0,m,n,v) &= (1/2)\int_{\Gamma_{a_{0,m/n^2}}\backslash\mathbb{H}_2}\int_1^\infty e^{-4\pi vR(na_{0,m/n^2},z)r}dr/rd\mu(z)\\
&= (1/2)\frac{3!}{(2\pi)^3}\int_{\Gamma_a\backslash G/K} f(z)dz\\
&= (1/2)\frac{3!}{(2\pi)^3}\mathrm{vol}\,(\mathrm{SO}(2))\int_{\Gamma_x\backslash H}\int_0^\infty\int_1^\infty e^{-4\pi vm(\sinh t)^2 r}|\sinh(t)|\cosh^2(t)dr/rdtdh\\
&= (1/2)\frac{3!}{(2\pi)^3}\mathrm{vol}\,(\mathrm{SO}(2))\int_{\Gamma_x\backslash H} dh\, I_+^3(v,m)\\
&= (3/(4\pi^2))\int_{\Gamma_x\backslash H} dh\, I_+^3(v,m)\\
I_+^3(v,m) &= \int_0^\infty\int_1^\infty e^{-4\pi vm(\sinh t)^2 r}|\sinh(t)|\cosh^2(t)dr/rdt
\end{aligned}
$$

Similarly, for Case II, $H=\mathrm{SO}(3,1)$, we get

(12.6.5)

$$
\begin{aligned}
2\cdot I(0,m,n,v) &= \int_{\Gamma_{a_{m/n^2}}\backslash\mathbb{H}_2}\int_1^\infty e^{-4\pi vR(na_{m/n^2},z)r}dr/rd\mu(z)\\
&= \frac{3!}{(2\pi)^3}\int_{\Gamma_a\backslash G/K} f(z)dz\\
&= \frac{3!}{(2\pi)^3}\mathrm{vol}\,(\mathrm{SO}(3)/\mathrm{SO}(2))\int_{\Gamma_x\backslash H}\int_0^\infty\int_1^\infty e^{-4\pi vm(\cosh t)^2 r}(\sinh(t))^2\cosh(t)dr/rdtdh\\
&= \frac{3!}{(2\pi)^3}\mathrm{vol}\,(\mathrm{SO}(3)/\mathrm{SO}(2))\int_{\Gamma_x\backslash H} dh\, I_-^3(v,m).\\
&= (3/\pi^2)\int_{\Gamma_x\backslash H} dh\, I_-^3(v,m).\\
I_-^3(v,m) &= \int_0^\infty\int_1^\infty e^{-4\pi vm(\cosh t)^2 r}(\sinh(t))^2\cosh(t)dr/rdt.
\end{aligned}
$$

In (12.22.1) below, we assemble the $I_\pm-$integrals (with $a = 4\pi mv$) and get

$$I^3_+(v,m) = (1/3)\int_0^\infty e^{-\alpha w}((w+1)^{3/2}-1)dw/w = (1/3)J_+(3/2,a)$$
$$I^3_-(v,m) = (1/3)e^{-|a|}\int_1^\infty e^{-|a|r}r^{3/2}dr/(r+1) = (1/3)e^{-|a|}J_-(3/2,|a|)$$

Here we used that, e.g., from [GHS] (3) and (4), one has $\mathrm{vol}\,(\mathrm{SO}(2)) = 2\pi$, $\mathrm{vol}\,(\mathrm{SO}(3)) = 8\pi^2$ and $\mathrm{vol}\,(\mathrm{SO}(3)/\mathrm{SO}(2)) = 4\pi$. Hence, we have to find the concrete meaning of the measure dh and to determine

$$\mathrm{vol}_*(\Gamma_x\backslash H) := \int_{\Gamma_x\backslash H} dh. \tag{12.6.6}$$

As cornerstones we have the classical results (10.13.1) from [EGM] and (10.13.2) from [HG] For $F = \mathbb{Q}(\sqrt{d})$ with discriminant $d_F < 0$, with $dv_{\mathbb{H}^+} = dxdydr/r^3 = dv_{Sie}$, one has

$$\begin{aligned}\mathrm{vol}_{\mathbb{H}^+}(\Gamma_a\backslash\mathbb{H}^+) &= \mathrm{vol}_{\mathbb{H}^+}(\mathrm{PSL}_2(\mathcal{O})\backslash\mathbb{H}^+)[\mathrm{PSL}_2(\mathcal{O}):\mathrm{PSL}_2(\mathcal{O}_f)]\\ &= \frac{|d_F|^{3/2}}{24}L(\chi_{d_F},2)f^3\prod_{p|f}(1-\chi_{d_F}(p)/p^2) =: v_-\end{aligned}$$

and for $d_F > 0$, with

$$dv_{\mathbb{H}^2} = dx_1dy_1dx_2dy_2/(y_1y_2)^2 = 4\pi^2dv_{HG} = 8\pi^2dv_{BK} = 4dv_{Sie},$$

$$\begin{aligned}\mathrm{vol}_{\mathbb{H}^2}(\Gamma_a\backslash\mathbb{H}^2) &= \mathrm{vol}(\mathrm{SL}(2,(\mathcal{O}_f,\mathcal{O}^*_f)\backslash\mathbb{H}^2)\\ &= \frac{|d_F|^{3/2}}{3}L(\chi_{d_F},2)f^3\prod_{p|f}(1-\chi_{d_F}(p)/p^2) =: v_+.\end{aligned}$$

Moreover, there is **Siegel's volume formula** (11.2.9) relating the volumes of fundamental domains in the group to those in the homogeneous spaces. In our situation, one has from (11.2.10)

$$\begin{aligned}\mathrm{vol}_S(\Gamma(S_0)\backslash\mathrm{SO}(2,2)) &= (\pi^2/2)\,\mathrm{vol}_S(\Gamma(S_0)\backslash\mathbb{D}_{2,2}),\\ \mathrm{vol}_S(\Gamma(S_0)\backslash\mathrm{SO}(3,1)) &= \pi^2\,\mathrm{vol}_S(\Gamma(S_0)\backslash\mathbb{D}_{3,1}).\end{aligned}$$

But in Flensted-Jensen and also in [Ku1], one usually works with the normalization $\int_K dk = 1$. Moreover, by the geometric meaning of the Eisenstein series (see (2.13.1)) we are led to measure the volumes as volumes in the representation space, i.e., here $\mathbb{D}_{22} \simeq \mathbb{H}^2$ resp. $\mathbb{D}_{31} \simeq \mathbb{H}^+$. We try to put all this together starting by (12.2.10) (and do NOT use Siegel's formula above!): From (11.14.1) and (11.13.3) we have

$$\begin{aligned} \mathrm{vol}_{\mathrm{Sie}}(\Gamma^0\backslash\mathbb{D}_{2,2}) &= \mathrm{vol}_{\mathbb{H}^2}(\mathrm{PSL}_2(\mathcal{O}_f,\mathcal{O}_f^*)\backslash\mathbb{H}^2)/4 \\ &= (1/12)|d_F|^{3/2}L(\chi_F,2)\cdot f^3\prod_{p|f}(1-\chi_{d_F}(p)/p^2), \end{aligned}$$

(12.6.7)
$$\begin{aligned} \mathrm{vol}_{\mathrm{Sie}}(\Gamma^0\backslash\mathbb{D}_{1,3}) &= \mathrm{vol}_{\mathbb{H}^+}(\mathrm{PSL}_2(\mathcal{O}_f,\mathcal{O}_f^*)\backslash\mathbb{H}^+) \\ &= (1/24)|d_F|^{3/2}L(\chi_F,2)\cdot f^3\prod_{p|f}(1-\chi_{d_F}(p)/p^2), \end{aligned}$$

where Γ^0 stands for the image of Γ_{a_m} resp. $\Gamma_{a'_m}$ in the corresponding group H. (This comes up to interpret dh in (12.6.6) as dv_{Sie}.)

12.7. To introduce these voluminae into the formulae above, we still have to work out the sum of $n^2|d_\gamma^2 m$ and discuss $\sigma_{\gamma,m}(5/2)$ from (2.5.3) . This is helped by the discussion of the geometric content of the Eisenstein coefficients for positive indices m:

Humbert volumes

12.8. From [vdG] p. 211, resp [Ku1] p.338, for $\Delta = D_0 f^2, D_0$ a fundamental discriminant, and H_Δ a Humbert surface in $\Gamma_2\backslash\mathbb{H}_2$, we have

(12.8.1)
$$G_\Delta := \cup_{r^2|\Delta} H_{\Delta/r^2}.$$

and

(12.8.2)
$$\mathrm{vol}_{HG}H_\Delta = (1/12\pi^2)f^3\prod_{p|f}(1-\chi_{D_0}(p)p^{-2})D_0^{3/2}L(\chi_{D_0},2) = 2\mathrm{vol}_{BK}H_\Delta$$

i.e.,

(12.8.3)
$$\mathrm{vol}_{HG}G_\Delta = \sum_{r|f}(1/12\pi^2)\psi(D_0,f/r)D_0^{3/2}L(\chi_{D_0},2) = (1/12\pi^2)\tau(D_0,f)D_0^{3/2}L(\chi_{D_0},2),$$

where, to abbreviate, for $\Delta = D_0 f^2, D_0$ a fundamental discriminant, we introduce

(12.8.4)
$$\begin{aligned} \psi(D_0,f) &:= f^3\prod_{p|f}(1-\chi_{D_0}(p)p^{-2}), \psi(D_0,1)=1, \\ \tau(D_0,f) &:= \sum_{d^2|\Delta}\psi(D_0,f/d) = \sum_{d|f}\psi(D_0,f/d). \end{aligned}$$

From [vdG] p.213, to G_Δ we associate $g_\Delta \in H^2(V,\mathbb{Q})$ and the series $\sum g_\Delta q^\Delta$ can be identified with Cohen's modular form $\sum_{N\equiv 0,1 \bmod 4} H(2,N)q^N$, where

$$H(2,D_0f^2) = L(-1,(\frac{D_0}{\cdot}))\sum_{d|f}\mu(d)(\frac{D_0}{d})d\,\sigma_3(f/d) \tag{12.8.5}$$
$$= -L(2,(\frac{D_0}{\cdot}))D_0^{3/2}/(2\pi^2)\sum_{d|f}\mu(d)(\frac{D_0}{d})d\,\sigma_3(f/d).$$

Again, to abbreviate, we introduce $\xi(D_0,f) := \sum_{d|f}\mu(d)(\frac{D_0}{d})d\,\sigma_3(f/d)$ and verifying for $f=p^k$ immediately, remark

$$\xi(D_0,f) = \tau(D_0,f). \tag{12.8.6}$$

12.9. In Bruinier-Kühn [BK] (4.3) the Heegner divisor of discriminant (β,m) is introduced as a $\Gamma(L)-$invariant divisor on $\mathrm{Gr}(V)$ for $\beta\in L'/L$ and negative $m\in\mathbb{Z}+q(\beta)$

$$\mathcal{H}(\beta,m) = \sum_{\lambda\in L+\beta, q(\lambda)=m}\lambda^\perp \tag{12.9.1}$$

From [BK] p.1721, we take that the Heegner divisor $(1/2)\mathcal{H}(\beta,-D/4)$ (β determined by $D=D_0f^2$) can be identified with the Humbert surface $G(D)$ of discriminant D of [vdG]. And (5.3) says

$$\deg(G(D)) = -(B/4)C(D,0) = (1/2)\zeta_K(-1) = 2^{-3}3^{-1}(1/2\pi^2)D_0^{3/2}L(\chi_{D_0},2). \tag{12.9.2}$$

From [BK] (4.56) and (4.18), for $4d_\gamma^2 m = D_0f^2$, i.e., $m^{3/2} = D_0^{3/2}(f/2d_\gamma)^3$, we have

$$(2/B)\deg(\mathcal{H}(\beta,m)) = -C(\beta,m,0) \tag{12.9.3}$$

where the [BK] formulae reproduced above, e.g. (2.9.1), give

$$C(\beta,m,0) = 2^6\cdot 3^2\cdot 5\cdot(1/3\pi^2)m^{3/2}L(\chi,2)\sigma_{\beta,m}(5/2),$$

i.e.,

$$\mathrm{vol}_{BK}((\mathcal{H}(\beta,m)) = -(1/3\pi^2)m^{3/2}L(\chi,2)\sigma_{\beta,m}(5/2) \tag{12.9.4}$$

(showing that in (12.9.2) resp. [BK] (5.3) D is a fundamental discriminant). With (12.8.3) $\mathrm{vol}_{HG}G_\Delta = 2\mathrm{vol}_{BK}G_\Delta = (1/12\pi^2)\tau(D_0,f)D_0^{3/2}L(\chi_{D_0},2)$ we come to the important relation

$$m^{3/2}\sigma_{\beta,m}(5/2) = (1/8)D_0^{3/2}\tau(D_0,f). \tag{12.9.5}$$

Further down, we will discuss generalized divisor sums following [BrKu], and give hints to a direct proof of this relation even for nonpositive m (see (12.23.8)).

12.10. To compare, we also reproduce Kudla's treatment. In [Ku1] p.337ff, things look like this. We have

$$(12.10.1)\qquad E(\tau,3/2;\varphi_0) = 1+\zeta(-3)^{-1}\sum_{m=1}^{\infty} H(2,4m)q^m = \sum A_0(m,v)q^m$$

$$E(\tau,3/2;\varphi_1) = \zeta(-3)^{-1}\sum_{m-1/4=0}^{\infty} H(2,4m)q^m = \sum A_1(m,v)q^m$$

$$\zeta(-3)^{-1} = 2^3\cdot 3\cdot 5,\quad 4m = D_0f^2, D_0\equiv 0,1 \bmod 4 \quad ([\text{Ku1}](5.18)),$$

$$\begin{aligned} H(2,4m) &= L(-1,\chi_{D_0})\sum_{d|f}\mu(d)\chi_{D_0}(d)d\,\sigma_3(f/d) \\ &= -L(2,\chi_{D_0})D_0^{3/2}/(2\pi^2)\sum_{d|f}\mu(d)\chi_{D_0}(d)d\,\sigma_3(f/d) \\ &= -(1/(2\pi^2))L(2,\chi_{D_0})D_0^{3/2}\xi(D_0,f), \end{aligned}$$

$$\xi(D_0,f) = \sum_{d|f}\mu(d)\chi_{D_0}(d)d\,\sigma_3(f/d).$$

Hence, Kudla's Eisenstein series is half of the series in Bruinier-Kühn and here we have the coefficients

$$(12.10.2)\qquad \begin{aligned} A(m,v) &= 2^3\cdot 3\cdot 5\cdot H(2,4m) \\ &= 2^3\cdot 3\cdot 5\cdot L(2,\chi_{D_0})D_0^{3/2}/(2\pi^2)\sum_{d|f}\mu(d)\chi_{D_0}(d)d\,\sigma_3(f/d), \\ &= 2^6\cdot 3\cdot 5\cdot L(2,\chi_{D_0})m^{3/2}/(2\pi^2)f^{-3}\xi(D_0,f). \end{aligned}$$

From [Ku1] (5.22), one has

$$(12.10.3)\qquad \begin{aligned} \deg(Z(m,\varphi_\mu)) = \deg G_{4m} &= -(1/12)H(2,4m) \\ &= (1/(24\pi^2))\cdot D_0^{3/2}L(\chi_{D_0},2)\xi(D_0,f). \end{aligned}$$

Together with (12.8.3) and $4m = D_0f^2$, again we get

$$(12.10.4)\qquad \sum_{d|f}\psi(D_0,f/d) = \tau(D_0,f) = \xi(D_0,f).$$

12.11. [BrKu] on p.447, for $\beta\in L'/L$ with $q(\beta)\in\mathbb{Z}$ look at the Eisenstein series

$$(12.11.1)\qquad \begin{aligned} E_\beta(\tau) &= (1/2)\sum_{(M,\varphi)\in\tilde\Gamma_\infty\backslash \mathrm{Mp}_2(\mathbb{Z})}\mathfrak{e}_\beta|_k^*(M,\varphi) \\ &= \sum_{\gamma\in L'/L}\sum_{n\in\mathbb{Z}-q(\gamma),n\geqslant 0} q_\beta(\gamma,n)\mathfrak{e}_\gamma(n\tau). \end{aligned}$$

Remark. From [BrKu] Example 10 p.454, for $-4m = \Delta f^2$, one has

(12.11.2)
$$q(\gamma, m) = -\frac{2^3\pi^{5/2}m^{3/2}}{\Gamma(5/2)}\frac{L(2,\chi_\Delta)}{\zeta(4)}\sum_{d|f}\mu(d)\chi_\Delta(d)d^{-2}\,\sigma_{-3}(f/d)$$
$$= -2^6\cdot 3\cdot 5/\pi^2\cdot m^{3/2}L(2,\chi_\Delta)\sum_{d|f}\mu(d)\chi_\Delta(d)d^{-2}\,\sigma_{-3}(f/d)$$
$$= -2^6\cdot 3\cdot 5/\pi^2\cdot m^{3/2}L(2,\chi_\Delta)f^{-3}\sum_{d|f}\mu(d)\chi_{D_0}(d)d\,\sigma_3(f/d),$$

i.e., here we find $q(\gamma, m) = -2A(m, v)$.

Remark. Comparizon of the $A(m, v)$ with the coefficients (2.12.1) from Bruinier-Kühn
$$c_0(\gamma, m, 0, v) = -2^6\cdot 3\cdot 5\cdot \pi^{-2}|m|^{3/2}L(\chi_{D_0}, 2)\sigma_{\gamma,m}(5/2)\cdot e^{-a/2} \text{ for } m > 0$$
again leads to

(12.11.3)
$$\sigma_{\gamma,m}(5/2) = f^{-3}\xi(D_0, f).$$

From [BK] p.1721, with $D = D_0f^2$ we also have

(12.11.4)
$$\sigma_{\beta,D/4}(5/2) = \sum_{d|f}\mu(d)\chi_{D_0}(d)d^{-2}\,\sigma_{-3}(d/f),$$
$$= f^{-3}\sum_{d|f}\mu(d)\chi_{D_0}(d)d\,\sigma_3(d/f) = f^{-3}\xi(D_0, f).$$

12.12. Summary. We have

(12.12.1)
$$\psi(D_0, f) = f^3\prod_{p|f}(1-\chi_{D_0}(p)p^{-2}), \psi(D_0, 1) = 1$$
$$\xi(D_0, f) = \sum_{d|f}\mu(d)\chi_{D_0}(d)d\,\sigma_3(f/d) = f^3\sigma_{\beta,m}(5/2)$$
$$\tau(D_0, f) = \sum_{d|f}\psi(f/d) = \xi(D_0, f)$$

Our final aim

We go back to the determination of the Green function integral (12.3.1) and want to use the preceding to show:

12.13. Proposition. For $m > 0$, with $a = 4\pi m v$, and $J_+(3/2, a) = \int_0^\infty e^{-ar}((r+1)^{3/2} - 1)dr/r$, one has

(12.13.1)
$$(4/B)I(0, m, v) = C(0, m, 0)J_+(3/2, a)$$
$$2^7\cdot 3^2\cdot 5\cdot I(0, m, v) = 2^6\cdot 3\cdot 5/\pi^2\cdot m^{3/2}L(2,\chi_{D_0})\sigma_{0,m}(5/2)J_+(3/2, a)$$
$$I(0, m, v) = (1/6\pi^2)\cdot m^{3/2}L(2,\chi_{D_0})\sigma_{0,m}(5/2)J_+(3/2, a).$$

Proof. In the (3,2)-case, one has a two-component Green function and, as above from (4.7.1), the Green function integrals

$$(12.13.2)\quad I(0,m,v) = (1/2)\sum_{n^2|m}\int_{\Gamma_{a_{m/n^2}}\backslash\mathbb{D}} \beta(2\pi vR(na_{m/n^2},z))d\mu_z =: \sum_{n^2|m} I(0,m,n,v),$$

$$I(1,m,v) = (1/2)\sum_{n^2|4m}\int_{\Gamma_{a'_{4m/n^2}}\backslash\mathbb{D}} \beta(2\pi vR(na'_{4m/n^2},z)d\mu_z =: \sum_{n^2|4m} I(1,m,n,v).$$

For Case I, $m\in\mathbb{N}$, $H=\mathrm{SO}(2,2)$, with $a_{m/n^2} =: x$, we got above with (12.2.10) and (12.17.2)

(12.13.3)

$$I(0,m,n,v) = (1/2)(3/2\pi^2)\int_{\Gamma_x\backslash H} dh\, I^3_+(v,m) = (1/2)(1/2\pi^2)\int_{\Gamma_x\backslash H} dh\,(1/3)J^3_+(3/2,a).$$

As we already assembled

$$dv_{\mathbb{H}^2} = dx_1dy_1dx_2dy_2/(y_1y_2)^2 = 4\pi^2 dv_{HG} = 8\pi^2 dv_{BK} = 4dv_{Sie},$$

and (11.14.1) and (11.13.3), we have

$$(12.13.4)\quad \begin{aligned}\mathrm{vol}_{\mathrm{Sie}}(\Gamma^0\backslash\mathbb{D}_{2,2}) &= \mathrm{vol}_{\mathbb{H}^2}(\mathrm{PSL}_2(\mathcal{O}_f,\mathcal{O}^*_f)\backslash\mathbb{H}^2)/4\\ &= (1/12)|d_F|^{3/2}L(\chi_F,2)\cdot f^3\prod_{p|f}(1-\chi_{d_F}(p)/p^2),\\ \mathrm{vol}_{\mathrm{Sie}}(\Gamma^0\backslash\mathbb{D}_{1,3}) &= \mathrm{vol}_{\mathbb{H}^+}(\mathrm{PSL}_2(\mathcal{O}_f,\mathcal{O}^*_f)\backslash\mathbb{H}^+)\\ &= (1/24)|d_F|^{3/2}L(\chi_F,2)\cdot f^3\prod_{p|f}(1-\chi_{d_F}(p)/p^2).\end{aligned}$$

If we use this, with (12.10.4)

$$\tau(D_0,f) = \sum_{d|f}\psi(D_0,f/d) = \sum_{d|f}\mu(d)\chi_{D_0}(d)d\,\sigma_3(d/f) = \xi(D_0,f)$$

and

$$\sum_{r|f}\psi(D_0,f/r) = \tau(D_0,f) = f^3\sigma_{\beta,m}(5/2)$$

we get

$$(12.13.5)\quad \begin{aligned}I(0,m,v) &= \sum_{n|f} I(0,m,n,v),\\ &= \sum_{n|f}(1/4\pi^2)\frac{|D_0|^{3/2}}{12}L(\chi_{D_0},2)\psi(D_0,f/n)J_+(3/2,a)\\ &= (1/(6\pi^2))|D_0|^{3/2}L(\chi_{D_0},2)(f/2)^3\sigma_{\beta,m}(5/2)J_+(3/2,a).\end{aligned}$$

With $m^{3/2} = D_0^{3/2}(f/2)^3$, this is the formula (12.13.1) we aimed at. □

12.14. For $m \in \mathbb{Z}, m < 0$, with $J_-(3/2, a) = e^{-a} \int_0^\infty e^{-|a|r} r^{3/2}(r+1)^{-1} dr$, one has the corresponding formula

$$(12.14.1) \qquad (4/B)I(0,m,v) = C(0,|m|,0)J_-(3/2,a)e^{-a/2}$$
$$I(0,m,v) = (1/6\pi^2) \cdot |m|^{3/2} L(2, \chi_{D_0}) \sigma_{0,m}(5/2) J_-(3/2,a) e^{-a}.$$

Proof. We have $H = \mathrm{SO}(3,1)$ and from (12.6.5) with $\mathrm{vol}\,((\mathrm{SO}(3)/\mathrm{SO}(2)) = 4\pi$ and (12.13.4)

$$\begin{aligned} I(0,m,n,v) &= (1/2) \int_{\Gamma_{a_{0,m/n^2}} \backslash \mathbb{H}_2} \int_1^\infty e^{-2\pi v R(na_{0,m/n^2}, z)r} dr/r d\mu(z) \\ &= (1/2) \frac{3!}{(2\pi)^3} \mathrm{vol}\,((\mathrm{SO}(3)/\mathrm{SO}(2)) \int_{\Gamma_x \backslash H} dh\, I_-^3(v,m) \\ &= (1/2)(3/\pi^2) \int_{\Gamma_x \backslash H} dh\, I_+^3(v,m) = (1/\pi^2) \int_{\Gamma_x \backslash H} dh\, (1/3) J_-(3/2,a) e^{-|a|} \\ &= (1/2\pi^2) \frac{|D_0|^{3/2}}{24} L(\chi_{D_0}, 2) \psi(D_0, f/n) J_-(3/2,a) e^{-|a|} \end{aligned}$$

Above, the relations in (12.12.1) $\xi(D_0, f) = \sum_{d|f} \mu(d)\chi_{D_0}(d) d\, \sigma_3(f/d) = \tau(D_0, f) = f^3 \sigma_{\beta,m}(5/2)$ came up for positive m but, as to be seen later in (12.22), may be used here too to give

$$\begin{aligned} I(0,m,v) &= \sum_{n|f} I(0,m,n,v), \\ &= \sum_{n|f} (1/2\pi^2) \frac{|D_0|^{3/2}}{24} L(\chi_{d_F}, 2) \psi(D_0, f/n) J_+(3/2,a) \\ &= (1/(6\pi^2)) |D_0|^{3/2} L(\chi_{d_F}, 2)(f/2)^3 \sigma_{0,m}(5/2) J_-(3/2,a). \end{aligned}$$

Here again, for the determination of the integral $J_-(3/2,a)$, we refer to below in (12.22.1). □

With corresponding considerations and results, one can treat the case B with $m - 1/4 = M \in \mathbb{Z}$ (This may appear a bit more subtle and lead to Example 10 in [BrKu]. One has to be careful as in [BK] one has $D = 4 \cdot 4 \cdot m$ and in [Ku1] $D = 4m$ all the way). □

Finally, with the relations already introduced above and to be shown below in ((12.22.1))

$$
\begin{aligned}
I_+(v,m) &= (1/3)J_+(3/2,a),\ a = 4\pi v|m|, \\
I_-(v,m) &= (1/3)e^{-a}J_-(3/2,a), \\
J_+(3/2,a) &= \int_0^\infty e^{-ar}((r+1)^{3/2}-1)dr/r, \\
J_-(3/2,a) &= e^{-a}\int_0^\infty e^{-|a|r}r^{3/2}(r+1)^{-1}dr,
\end{aligned} \tag{12.14.2}
$$

we obtain

12.15. Proposition (Green Integral Summary). We have the Green function integrals

$$
\begin{aligned}
I(\gamma,v,m) &= \int_X \Xi(\gamma,m,v)d\mu \\
&= (1/(6\pi))m^{3/2}L(\chi_{D_0},2)\sigma_{\gamma,m}(5/2)(1/3)J_+(3/2,a),\ \text{for } m>0, \\
&= (1/(6\pi))|m|^{3/2}L(\chi_{D_0},2)\sigma_{\gamma,m}(5/2)(1/3)J_-(3/2,a)e^{-a},\ \text{for } m<0.
\end{aligned} \tag{12.15.1}
$$

If we join this with the results for the coefficients of the Eisenstein series

$$
\begin{aligned}
C(\gamma,m,0) &:= -2^6\cdot 3\cdot 5\cdot \pi^{-2}|m|^{3/2}L(\chi_{d_F},2)\sigma_{\gamma,m}(5/2) \\
c_0(\gamma,m,0,v) &= \begin{cases} C(\gamma,m,0)e^{-a/2} \text{ for } m>0, \\ 0, \quad \text{for } m<0, \end{cases} \\
c_0'(\gamma,m,0,v) &= \begin{cases} C(\gamma,m,0)e^{-a/2}(J_+(3/2,a)+\frac{C'(\gamma,m,0)}{C(\gamma,m,0)}), \text{ for } m>0, \\ C(\gamma,m,0)e^{-|a|/2}\cdot J_-(3/2,a), \text{ for } m<0, \end{cases}
\end{aligned} \tag{12.15.2}
$$

we get our central result.

12.16. Theorem. For the (3,2) case, one has

$$
\begin{aligned}
(4/B)\cdot I(\gamma,m,v) &= \begin{cases} C(\gamma,m,0)J_+(3/2,a), \text{ for } m>0, \\ C(\gamma,m,0)J_-(3/2,a)e^{-|a|}, \text{ for } m<0. \end{cases} \\
(4/B)\cdot I^{BK}(\gamma,-m,v) &= \begin{cases} -C(\gamma,m,0)\big(\frac{C'(\gamma,m,0)}{C(\gamma,m,0)}+\log(4\pi)-\Gamma'(1)\big), \text{ for } m>0, \\ 0, \text{ for } m<0. \end{cases}
\end{aligned} \tag{12.16.1}
$$

Here $I^{BK}(\gamma,m,v) = \int_X G_{\gamma,m}(Z)\Omega^3$ is the integral of the Green function $G_{\gamma,m}(Z)$ from Bruinier-Kühn [BK] Definition 4.5 and Theorem 4.10.

12.17. Corollary. We have

$$
c_0'(\gamma,m,0,v) = e^{-a/2}((4/B)\cdot(I(\gamma,m,v)-I^{BK}(\gamma,-m,v))+*\,c_0(\gamma,m,0,v)). \tag{12.17.1}
$$

Now, we treat the results we used above.

The $I^p_\pm-$Integrals

Here, we determine the integrals $I_\pm(v,m)$ from (12.1.7) and (12.1.10)

$$I^p_+(v,m) = \int_0^\infty \int_1^\infty e^{-4\pi v|m|\sinh^2 tr} dr/r \sinh t (\cosh t)^{p-1} dt, \tag{12.17.2}$$

$$I^p_-(v,m) = \int_0^\infty \int_1^\infty e^{-4\pi v|m|\cosh^2 tr} dr/r (\sinh t)^{p-1} \cosh t dt.$$

We will do this for $p = 3, 2$ and 1:

12.18. To evaluate $I_-(v,m)$ in the (3,2)-case, we try the substitution $\sinh^2 t = w$, i.e., $2\cosh t \sinh t dt = dw$. We get (at least formally) with $\alpha = 4\pi v|m|$

$$\begin{aligned}
I_-(v,m) &= \int_0^\infty \int_1^\infty e^{-4\pi v|m|\cosh^2 tr} dr/r \sinh^2 t \cosh t dt \\
&= (1/2) \int_0^\infty \int_1^\infty e^{-\alpha(w+1)r} dr/r dw \sqrt{w} \\
&= (1/2) \int_1^\infty e^{-\alpha r} \int_0^\infty e^{-\alpha w r} dw \sqrt{w} dr/r \\
&= (1/2) \int_1^\infty e^{-\alpha r} (\big[e^{-\alpha r w} w^{3/2} 2/3\big]_0^\infty + \int_0^\infty (\alpha r) e^{-\alpha w r} dw w^{3/2} 2/3) dr/r \\
&= (\alpha/3) \int_0^\infty \int_1^\infty e^{-\alpha(w+1)r} dr dw w^{3/2} \\
&= (\alpha/3) \int_0^\infty \big[(-1/(\alpha(w+1)) e^{-\alpha(w+1)r}\big]_1^\infty dw w^{3/2} \\
&= (1/3) \int_0^\infty (w^{3/2}/((w+1)) e^{-\alpha(w+1)} dw
\end{aligned}$$

$$= (1/3)\, e^{-\alpha} \int_0^\infty w^{3/2} (w+1)^{-1} e^{-\alpha w} dw =: (1/3)\, e^{-\alpha} J_-(3/2, \alpha). \tag{12.18.1}$$

12.19. For $m > 0$ we do a similar evaluation starting by (12.17.2). With $\alpha = 4\pi m v$ and the substitution $\sinh^2 t = w, dw = 2\sinh t \cosh t dt$, we get

$$
\begin{aligned}
I_+(v,m) &= \int_0^\infty \int_1^\infty e^{-4\pi v m \sinh^2 t r} dr/r \sinh t \cosh^2 t dt \\
&= (1/2)\int_0^\infty \int_1^\infty e^{-\alpha w r} dr/r dw (w+1)^{1/2} \\
&= (1/2)\int_1^\infty \int_0^\infty e^{-\alpha w r} dw (w+1)^{1/2} dr/r \\
&= (1/2)\int_1^\infty (\left[e^{-\alpha r w}(w+1)^{3/2} 2/3 \right]_0^\infty + \int_0^\infty (\alpha r) e^{-\alpha w r} dw (w+1)^{3/2} 2/3) dr/r \\
&= (1/2)\int_1^\infty (-2/3 + \int_0^\infty (\alpha r) e^{-\alpha w r} dw (w+1)^{3/2} 2/3) dr/r \\
&= (1/3)\int_0^\infty \int_1^\infty \alpha e^{-\alpha w r} dr dw ((w+1)^{3/2} - 1) \\
&= (1/3)(\int_0^\infty e^{-\alpha w} w^{-1}((w+1)^{3/2}-1) dw) =: (1/3) J_+(3/2, \alpha), \qquad (12.19.1)
\end{aligned}
$$

where we used

$$
\int_0^\infty \int_1^\infty \alpha e^{-\alpha w r} dr dw = \int_1^\infty \int_0^\infty \alpha e^{-\alpha w r} dw dr = \int_1^\infty [-(1/r) e^{-\alpha w r}]_0^\infty dr = \int_1^\infty dr/r.
$$

12.20. For $p = 2$ and $m > 0$, with the same substitution as above, we get

$$
\begin{aligned}
I_+(v,m) &= \int_0^\infty \int_1^\infty e^{-4\pi v m \sinh^2 t r} dr/r \sinh t \cosh t dt \\
&= (1/2)\int_0^\infty \int_1^\infty e^{-\alpha w r} dr/r dw \\
&= (1/(2\alpha))\int_1^\infty dr/r^2 \\
&= (1/(2\alpha)) = 1/(8\pi v m). \qquad (12.20.1)
\end{aligned}
$$

And for $m < 0$ again with $\alpha = 4\pi v|m|$

$$
\begin{aligned}
I_-(v,m) &= \int_0^\infty \int_1^\infty e^{-4\pi v|m| \cosh^2 t r} dr/r \sinh t \cosh t dt \\
&= (1/2)\int_0^\infty \int_1^\infty e^{-\alpha(w+1) r} dr/r dw \\
&= (1/2)\int_1^\infty e^{-\alpha r} \int_0^\infty e^{-\alpha w r} dw dr/r \\
&= (1/2\alpha)\int_1^\infty e^{-\alpha r} dr/r^2 \\
&= (1/2)(e^{-\alpha}/\alpha - \int_1^\infty e^{-\alpha r} dr/r). \qquad (12.20.2)
\end{aligned}
$$

12.21. For $p = 1$ and $m > 0$, with the same substitution as above, we get

$$
\begin{aligned}
I_+(v,m) &= \int_0^\infty \int_1^\infty e^{-4\pi vm\sinh^2 tr} dr/r \sinh t dt \\
&= (1/2) \int_0^\infty \int_1^\infty e^{-\alpha wr} dr/r (w+1)^{-1/2} dw \\
&= (1/2) \int_1^\infty (-1 + \int_0^\infty \alpha e^{-\alpha wr} (w+1)^{1/2} dw) dr/r \\
&= (1/2) \int_0^\infty \int_1^\infty \alpha e^{-\alpha wr} ((w+1)^{1/2} - 1) dr dw \\
&= (1/2) \int_0^\infty e^{-\alpha w} ((w+1)^{1/2} - 1) dw/w. \qquad (12.21.1)
\end{aligned}
$$

And for $m < 0$ again with $\alpha = 4\pi v|m|$

$$
\begin{aligned}
I_-(v,m) &= \int_0^\infty \int_1^\infty e^{-4\pi v|m|\cosh^2 tr} dr/r \cosh t dt \\
&= (1/2) \int_0^\infty \int_1^\infty e^{-\alpha(w+1)r} dr/r w^{-1/2} dw \\
&= (1/2) \int_1^\infty e^{-\alpha r} \int_0^\infty e^{-\alpha wr} w^{-1/2} dw dr/r \\
&= (1/(2\sqrt{\alpha}))\Gamma(1/2) \int_1^\infty e^{-\alpha r} r^{-3/2} dr. \qquad (12.21.2)
\end{aligned}
$$

12.22. Summary (The $I-$Integrals). We sum up what we got (with $a = 4\pi mv$) and (2.3.9)

$$
\begin{aligned}
J_+(s,a) &:= \int_0^\infty e^{-aw} ((w+1)^s - 1) dw/w, \\
J_-(s,a) &:= \int_0^\infty e^{-aw} w^s dw/(w+1),
\end{aligned}
$$

where the first line is for positive and the second line for negative m.

$$\begin{aligned}
p=1: \quad & I_+(v,m) = (1/2)\int_0^\infty e^{-aw}((w+1)^{1/2}-1)dw/w = (1/2)J_+(1/2,a)\\
& I_-(v,m) = (1/(4\sqrt{|m|v}))\int_1^\infty e^{-|a|r}r^{-3/2}dr = (1/2)e^{-|a|}J_-(1/2,|a|)\\
p=2: \quad & I_+(v,m) = (1/2a)\\
& I_-(v,m) = (1/(2|a|)(e^{|a|} - \int_1^\infty e^{-|a|}dr/r)\\
p=3: \quad & I_+(v,m) = (1/3)\int_0^\infty e^{-\alpha w}((w+1)^{3/2}-1)dw/w = (1/3)J_+(3/2,a)\\
& I_-(v,m) = (1/3)e^{-|a|}\int_1^\infty e^{-|a|r}r^{3/2}dr/(r+1) = (1/3)e^{-|a|}J_-(3/2,|a|)\\
& \qquad = (1/(4|a|^{3/2})\sqrt{\pi}\int_1^\infty e^{-|a|r}dr/r^{5/2}.
\end{aligned} \tag{12.22.1}$$

The Generalized divisor sum

12.23. In the formulae for the Fourier coefficients of the Eisenstein series in Bruinier-Kühn [BK] and (implicient) in Bruinier-Kuss [BrKu], the notion of a *generalized divisor sum* $\sigma_{\gamma,m}(s)$ appears which albeit rather complicated can be related to our $\tau(D_0,f)$ from (12.8.4. In [BK] (3.28) (for $r=5$ odd) we find the definition (which already came up in (2.5.3))

$$\sigma_{\gamma,m}(s) := \prod_{p|2d_\gamma^2 m\,\det L} \frac{1-\chi_{D_0}(p)p^{(1/2)-s}}{1-p^{1-2s}} L^{(p)}_{\gamma,m}(p^{-(3/2)-s}). \tag{12.23.1}$$

Here one has a lattice $L \simeq \mathbb{Z}^r, r=5$ with quadratic form $q(x)$ and $L'/L \simeq \mathbb{Z}/2\mathbb{Z}$. For $\gamma \in L'/L$, we take over from [BK] (3.17), (3.23), (3.18), (3.14) and (3.20) (having its source in [BrKu] (22))

$$\begin{aligned}
d_\gamma &:= \min\{b\in\mathbb{Z}_{>0}; b\gamma\in L\}\\
2d_\gamma^2 m\,\det L &= D_0 f^2\\
w_p &:= 1+\nu_p(2md_\gamma), \nu_p \text{ the (additive) } p-\text{adic valuation on } \mathbb{Q}\\
N_{\gamma,m}(a) &:= |\{x\in L/aL; q(x-\gamma)+m\equiv 0 \bmod a\}|, \text{ the } \textit{representation number} \bmod a,\\
L^{(p)}_{\gamma,m}(X) &:= N_{\gamma,m}(p^{w_p})X^{w_p} + (1-p^4X)\sum_{\nu=0}^{w_p-1} N_{\gamma,m}(p^\nu)X^\nu \in \mathbb{Z}[X].
\end{aligned} \tag{12.23.2}$$

From here with $X = p^{1-r/2-s}$ we get with the notion from [BrKu]

(12.23.3)

$$L_{\gamma,m}(s,p) := L^{(p)}_{\gamma,m}(X) = N_{\gamma,m}(p^{w_p})p^{w_p(1-r/2-s)} + (1-p^{r/2-s}) \sum_{\nu=0}^{w_p-1} N_{\gamma,m}(p^{\nu})p^{\nu(1-r/2-s)}.$$

For $X = p^{-4}$ and $s = 5/2$ the definition above specializes to

(12.23.4) $$\sigma_{\gamma,m}(5/2) = \prod_{p|D} \frac{1-\chi_{D_0}(p)p^{-2}}{1-p^{-4}} N_{\gamma,m}(p^{w_p})p^{-4w_p}, \quad D = 2d_\gamma^2 m \det L.$$

In [BrKu] Theorem 6 (a partial reformulation of Siegel [S1] Hilfssatz 16), one finds expressions for the representation numbers $N_{\gamma,m}$ namely for $r > 3$, odd, $D = D_0 f^2$ (where $D_0 \in \mathbb{Q}$ and $f \in \mathbb{N}$) such that $(f, \det L) = 1$ and $\nu_\ell(D_0) \in \{0,1\}$ for all primes ℓ with $(\ell, 2\det L) = 1$. Let $\tilde{D}_0 = D_0 d_\gamma^2$ and $\mathcal{D} = 2(-1)^{(r+1)/2}\tilde{D}_0 \det L$. Then, for a prime p not dividing $2\det L$ and $\alpha \in \mathbb{Z}$ with $\alpha > \nu_p(m)$ one has

(12.23.5)

$$p^{\alpha(1-r)}N_{\gamma,m}(p^\alpha) = \frac{1-p^{1-r}}{1-\chi_{\mathcal{D}}(p)p^{(1-r)/2}} \times (\sigma_{2-r}(p^{\nu_p(f)}) - \chi_{\mathcal{D}}(p)p^{(1-r)/2}\sigma_{2-r}(p^{\nu_p(f)-1})).$$

If we specialize to our case $r = 5, \nu_p(f) =: k_p = k$, and $\alpha = w_p > \nu_p(m)$, we get

$$\begin{aligned} N_{\gamma,m}(p^{w_p}) &= \frac{1-p^{-4}}{1-\chi_{\mathcal{D}}(p)p^{-2}} \times (\sigma_{-3}(p^k) - \chi_{\mathcal{D}}(p)p^{-2}\sigma_{-3}(p^{k-1}))p^{4w_p}, \\ &= \frac{1-p^{-4}}{1-\chi_{\mathcal{D}}(p)p^{-2}} \times (1 + p^{-3} + \cdots + p^{-3k} - \chi_{\mathcal{D}}(p)(p^{-2} + \cdots + p^{-3k+1}))p^{4w_p}, \end{aligned}$$

(12.23.6) $$= \frac{1-p^{-4}}{1-\chi_{\mathcal{D}}(p)p^{-2}} \times p^{4w_p-3k}(1 + \sum_{\nu=1}^{k} p^{3\nu}(1-\chi_{\mathcal{D}}(p)p^{-2})).$$

Remark. For $f = \prod_i p_i^{k_i}$, one has

(12.23.7) $$\tau(D_0, f) = \sum_{d|f} \psi(D_0, f/d) = \prod_i (1 + \sum_{d_i=1}^{k_i} p_i^{3d_i}(1-\chi_{D_0}(p_i)p_i^{-2})).$$

Proof. For primes p and q, one has

$$\tau(D_0, p^k) = \psi(D_0, p^k) + \psi(D_0, p^{k-1}) + \cdots + 1 = \sum_{\nu=1}^{k} p^{3\nu}(1-\chi_{D_0}(p)p^{-2}) + 1$$

$$\tau(D_0, pq) = \tau(D_0, p)\tau(D_0, q)$$

Hence, for $f = \prod_i p_i^{k_i}$, $\tau(D_0, f) = \prod_i (\sum_{\nu_i=1}^{k_i} p^{3\nu_i}(1-\chi_{D_0}(p_i)p_i^{-2}) + 1)$. □

For appropriate m, with (12.23.7) we get from (12.23.4)

$$\sigma_{\gamma,m}(5/2) = \prod_{p|D} \frac{1-\chi_{D_0}(p)p^{-2}}{1-p^{-4}} N_{\gamma,m}(p^{w_p})p^{-4w_p}, \tag{12.23.8}$$

$$= \prod_{p|f} p^{-3k_p}(1+\sum_{\nu=1}^{k} p^{3\nu}(1-\chi_{\mathcal{D}}(p)p^{-2})),$$

$$= f^{-3}\tau(D_0, f).$$

In the case $\mathrm{SO}(2,2)$, there is a similar relation with $\psi'(m) := m\prod_{p|m}(1+1/p)$ and the nice formula

$$\sum_{n^2|m} \psi'(m/n^2) = \sigma(m) = \sum_{d|m} d. \tag{12.23.9}$$

13 The Kudla Green function integral for signature (2,2)

13.1. We treat Kudla's approach to calculate the Green function integral for the case $p = 2$, i.e., here $G = \mathrm{SO}(2.2), G/K \simeq \mathbb{D}, L = M_2(\mathbb{Z})$, $X = \mathrm{SO}(L)\backslash\mathbb{D}^+$. As already in previous sections, we take $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in V \simeq \mathbb{R}^4$, $z = (z_1, z_2) \in \mathbb{H}^2$ resp. X, and from (3.9.8)

$$R(M, z) = \frac{|a - bz_2 - cz_1 + dz_1z_2|^2}{2y_1y_2}.$$

For $m \neq 0$, we have

$$I(v, m) = \int_X \Xi(m, v, z)d\mu(z),\ X = (\Gamma\backslash\mathbb{H})^2, \tag{13.1.1}$$

with Kudla's Green function

$$\Xi(m, v, z) = (1/2)\sum_{M\in L_m} \xi(v, z, M),$$

$$\xi(v, z, M) = \beta(2\pi vR(M, z)), \beta_{\sigma+1}(t) = \int_1^\infty e^{-tu}u^{-\sigma-1}du, \beta = \beta_0,$$

$$L_m = \{M \in M_2(\mathbb{Z}); \det(M) = m\},$$

(13.1.2)

$$d\mu(z) = \Omega^2 = \frac{1}{2\pi^2}\frac{dx_1dy_1dx_2dy_2}{(y_1y_2)^2} = 2dv_{HG}\ \Omega = dd^c\log\left(-(y, y)\right)\ \text{([BY] Lemma 3.3)}.$$

One knows

$$L_m^* = \{M \in L_m; M\,\text{primitive}\} = \Gamma\left(\begin{smallmatrix} m & \\ & 1 \end{smallmatrix}\right)(\Gamma_0(|m|)\backslash\Gamma),\ \Gamma = \mathrm{SL}(2,\mathbb{Z}), \tag{13.1.3}$$

and for $m > 0$

$$\psi(m) := m\prod_{p|m}(1+1/p) = [L_m^* : \Gamma] = [\Gamma : \Gamma_0(m)],$$

(13.1.4) $$\sigma(m) := \sum_{d|m} d = \sum_{n^2|m} \psi(m/n^2).$$

$\bar\Gamma = \Gamma\times\Gamma$ acts transitively on L_m^* via $M \mapsto \gamma_1 M^t\gamma_2$ and we have

(13.1.5) $$L_m^* = (\bar\Gamma/\bar\Gamma_{a_m})a_m,\ a_m = {}^t(1,0,0,m)$$

i.e.,

(13.1.6) $$L_m = \sum_{n^2|m} nL_{m/n^2}^* = \sum_{n^2|m} n(\bar\Gamma/\bar\Gamma_{a_{m/n^2}})a_{m/n^2}.$$

Hence, unfolding, we get

(13.1.7) $$I(v,m) = (1/2)\sum_{n^2|m}\int_{\bar\Gamma_{a_{m/n^2}}\backslash\mathbb{H}^2}\beta(2\pi vR(na_{m/n^2,z}))d\mu(z).$$

13.2. To evaluate this, Kudla proposes the procedure already explained in 12.3. For $m > 0$, choose a basis $\mathbf{v}$ for $V(\mathbb{R})$ such that the inner product has the matrix $I_{2,2}$ i.e., $(x,x) = x_1^2 + x_2^2 - x_3^2 - x_4^2$, and $x = 2\alpha v_1$. Hence, one has $q(x) = 2\alpha^2 = m$ and

$$G = \mathrm{SO}(V(\mathbb{R})) \simeq \mathrm{SO}(2,2) = G_x\cdot A\cdot K$$

and in Flensted-Jensen notation ([FlJ] Section 2) $H = G_x \simeq \mathrm{SO}(1,2), L \simeq \mathrm{SO}(2)$ and $M \simeq \{\pm1\}$. Evaluating (12.2.10) as above in the previous section, we get

$$\int_{\Gamma_x\backslash\mathbb{D}^+}\int_1^\infty e^{-2\pi vR(nx,z)r}dr/rd\mu(z),$$

$$= (1/2\pi^2)\mathrm{vol}(\mathrm{SO}(2)/\mathrm{SO}(1))\mathrm{vol}(\Gamma_x\setminus G_x)\mathrm{vol}(K)\int_0^\infty\int_1^\infty e^{-4\pi vm\sinh^2 tr}\sinh t\cosh t dr/rdt,$$

(13.2.1)
$$= (1/\pi)\mathrm{vol}(\Gamma_x\setminus G_x)\mathrm{vol}(K)I_\pm(v,m).$$

13.3. To evaluate $\mathrm{vol}(\Gamma_x\setminus G_x)$, again with $\mathrm{vol}(\Gamma_x\setminus G_x) = \mathrm{vol}(\Gamma_x\setminus\mathbb{D}_{1,2})$, we relate our orthogonal groups to the SL$-$theory. $\bar G = (\mathrm{SL}(2,\mathbb{R}))^2$ acts on $\tilde V = M_2(\mathbb{R})$ via $M \mapsto g\cdot M = g_1M^tg_2 =: M'$. Hence, we have a map

$$\bar G = (\mathrm{SL}(2,\mathbb{R}))^2 \to G = \mathrm{SO}_0(\tilde Q),\ \tilde Q = \begin{pmatrix} & & & 1\\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}$$

(13.3.1) $$g = (g_1,g_2) \mapsto A(g) = \begin{pmatrix}\alpha_1\alpha_2 & \alpha_1\beta_2 & \beta_1\alpha_2 & \beta_1\beta_2\\ \alpha_1\gamma_2 & \alpha_1\delta_2 & \beta_1\gamma_2 & \beta_1\delta_2\\ \gamma_1\alpha_2 & \gamma_1\beta_2 & \delta_1\alpha_2 & \delta_1\beta_2\\ \gamma_1\gamma_2 & \beta_1\delta_2 & \delta_1\gamma_2 & \delta_1\delta_2\end{pmatrix}.$$

As above, we use Siegel's method to determine $\tilde{\Gamma}_{a_m}$ with

$$\tilde{\Gamma}_{a_m} = \Gamma(\tilde{Q}, a_m) = \{\gamma \in \Gamma(\tilde{Q}); \gamma a_m = a_m\}, a_m = {}^t(1,0,0,m)$$

With $S = \tilde{Q}$ and

$$A = (a_m, B) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ m & & & 1 \end{pmatrix}, (a_m, a_m) = 2m$$

one has from (7.1.1) and (7.1.2)

$$\begin{aligned} \tilde{\Gamma}_{a_m} &= \Gamma(S, a_m) = \{W \in \Gamma(K); {}^tWb \equiv b \bmod 2m\} \\ b &= {}^tBSa_m = {}^tB \begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix} \begin{pmatrix} 1 \\ \\ \\ m \end{pmatrix} = \begin{pmatrix} 0 & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} m \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \\ K &= {}^tBSB - b{}^tb/(2m) = \begin{pmatrix} 0 & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 0 & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix} \begin{pmatrix} 0 & & \\ 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} - (1/(2m)) \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & & \\ 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} - (1/(2m)) \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} = - \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & (1/(2m)) \end{pmatrix}. \end{aligned} \tag{13.3.2}$$

Hence, to simplify, for the following, we assume $m > 0$ and look at

$$(u,u) = {}^tuK_mu = (4mu_1u_2 + u_3^2);\ K_m = \begin{pmatrix} 0 & 2m & \\ 2m & 0 & \\ & & 1 \end{pmatrix} \tag{13.3.3}$$

and want to determine $\Gamma(K_m)$. One has the standard way to relate the signature $(2,1)$ groups to $G_1 := \mathrm{SL}(2,\mathbb{R})$: Take

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G_1 \tag{13.3.4}$$

$$M(\mathbf{a}) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, -\det(M) = a^2 + bc = (1/2){}^t\mathbf{a}\tilde{Q}\mathbf{a}, \tilde{Q} = \begin{pmatrix} 2 & & \\ & & 1 \\ & 1 & \end{pmatrix}.$$

One has a map $\rho : G_1 \to \mathrm{SO}(\tilde{Q})$, $g \mapsto \rho(g)$ where

$$gM(\mathbf{a})g^{-1} = M(\rho(g)\mathbf{a}),\ \rho(g) = \begin{pmatrix} \alpha\delta + \beta\gamma & -\alpha\gamma & \beta\delta \\ -2\alpha\beta & \alpha^2 & -\beta^2 \\ 2\gamma\delta & -\gamma^2 & \delta^2 \end{pmatrix}. \tag{13.3.5}$$

We have $4mu_1u_2 + u_3^2 = a^2 + bc$ and put $a = u_3, b = 2u_2, c = 2mu_1$, i.e.,

$$\mathbf{a} = C\mathbf{u},\ C = \begin{pmatrix} & & 1 \\ & 2 & \\ 2m & & \end{pmatrix}, \mathbf{u} = C^{-1}\mathbf{a}, C^{-1} = \begin{pmatrix} & & 1/2m \\ & 1/2 & \\ 1 & & \end{pmatrix} \tag{13.3.6}$$

and get an isomorphism

$$\nu : \mathrm{SO}(\tilde{Q}) \to \mathrm{SO}(K_m),\ A \mapsto C^{-1}AC.$$

Hence we have the surjection $\rho' = \nu \cdot \rho : G_1 \to \mathrm{SO}_0(K_m), g \mapsto C^{-1}\rho(g)C =: A(g)$ with

$$A(g) = \begin{pmatrix} \delta^2 & -\gamma^2/m & \gamma\delta/m \\ -m\beta^2 & \alpha^2 & -\alpha\beta \\ 2m\beta\delta & -2\alpha\gamma & \alpha\delta+\beta\gamma \end{pmatrix} = W = (w_{ij}). \tag{13.3.7}$$

We want to have $w_{ij} \in \mathbb{Z}$ and Siegel's condition ${}^tWb \equiv b \bmod 2m$, i.e.,

$$w_{31} \equiv w_{32} \equiv 0, w_{33} \equiv 1 \bmod 2m.$$

Now, if $g \in \Gamma_0(m)$, one has $\gamma \equiv 0 \bmod m$ and all these conditions are fulfilled. Thus, we get

$$\tilde{\Gamma}_{a_m} \simeq \Gamma_0(m)/\{\pm 1_2\}. \tag{13.3.8}$$

Similarly to (11.8.1), with $\Gamma^m = \nu(\rho(\Gamma_0(m)))$ and $\Gamma^0 = \sigma(\nu(\rho(\Gamma_0(m))))$ we get

$$\begin{array}{ccccccc} \Gamma_0(m) & \longrightarrow & & & \Gamma^m & \longrightarrow & \Gamma^0 \\ \downarrow & & & & \downarrow & & \downarrow \\ \mathrm{SL}_2(\mathbb{R}) & \xrightarrow{\rho_d} & \mathrm{SO}^+(\tilde{Q}) & \xrightarrow{\nu} & \mathrm{SO}^+(K_m) & \xrightarrow{\sigma} & \mathrm{SO}^+(1,2) \\ \downarrow & & & & \downarrow & & \downarrow \\ \mathbb{H} & & \xrightarrow{\Psi} & & \mathbb{D}_m & \longrightarrow & \mathbb{D}_{1,2} \\ \downarrow & & & & \downarrow & & \downarrow \\ \mathcal{F}_1 \simeq \Gamma_0(m)\backslash\mathbb{H} & & \longrightarrow & & \Gamma^m\backslash\mathbb{D}_m & & \Gamma^0\backslash\mathbb{D}_{1,2} \simeq \mathcal{F} \end{array} \tag{13.3.9}$$

For $x + iy \in \mathbb{H}$ and $X = (x_1, x_2) \in \mathbb{D}_{1,2}$, from (11.6.12) we know

$$dv_{Sie} = \frac{dx_1 dx_2}{(1 - x_1^2 - x_2^2)^{(3/2)}}; dv_{Sie} \circ \Psi^* = \frac{dxdy}{y^2} = dv_{\mathbb{H}}. \tag{13.3.10}$$

One has $\mathrm{vol}(\Gamma\backslash\mathbb{H}) = \pi/3$, $\mathrm{vol}(\Gamma_0(m)\backslash\mathbb{H}) = \psi(m)\pi/3$, and

$$\mathrm{vol}(\Gamma^0\backslash\mathbb{D}_{1,2}) = \psi(m)\pi/3. \tag{13.3.11}$$

13.4. We use this for our calculation of the Green function integral 13.1.1).

$$\begin{aligned} I(v,m) &= \int_X \Xi(m,v,z)d\mu(z) = (1/2)\sum_{x\in L_m}\int_{(\Gamma\backslash\mathbb{H})^2}\int_1^\infty e^{-2\pi v R(z,x) r} dr/r d\mu(z) \\ &= (1/2)\sum_{n^2|m}\int_{\bar{\Gamma}_{a_{m/n^2}}\backslash\mathbb{H}^2}\int_1^\infty e^{-2\pi v n^2 R(z, n a_{m/n^2}) r} dr/r d\mu(z). \end{aligned}$$

With (12.20.1) $I_+(v,m) = 1/(8\pi vm)$ and $\sum_{n^2|m}\psi(m/n^2) = \sigma(m)$ we get

(13.4.1) $\quad I(v,m) = (1/2)(1/\pi)\sigma(m)\pi/3\cdot(1/(2\alpha)) = (1/2)(1/(6\alpha)\sigma(m), \alpha = 4\pi vm > 0.$

The same way, using (12.20.2), we get for $m < 0$

(13.4.2) $$I(v,m) = \pi\sigma(m)\pi/3\cdot(1/(2|\alpha|))(e^{-|\alpha|} + |\alpha|\mathrm{Ei}(-|\alpha|)), \alpha = 4\pi vm.$$

Both formulae are up to a factor from different normalization of measures the same as those from Theorem 4.2 in [BeKII].

14 The Green function integral for signature (1,2)

Here, one has a lot of material in the literature. We take out small pieces which may be related to the procedure we followed above, mainly from the report [Ya] by Yang and the detailed paper [KRY] by Kudla, Rapoport and Yang.

14.1. In [KRY], Kudla, Rapoport and Yang treat the case of an indefinite division quaternion algebra B over $\mathbb{Q}$ with maximal order O_B. Here $D(B)$ is the product of all primes p at which B is division. One takes $V = \{x \in B|\,\mathbf{tr}(x) = 0\}$ with quadratic form $Q(x) = -x^2 = N^0(x)$ given by the restriction of the reduced norm, $H = \mathrm{GSpin}(V), \Gamma = O_B^\times$, and $L = V(\mathbb{Q}) \cap O_B, L(m) = \{x \in L|Q(x) = m\}$.
D is the space of oriented negative planes in $V(\mathbb{R})$ and, as usual, $R(x,z) = (\mathrm{pr}_z(x), \mathrm{pr}_z(x) \geqslant 0$ for $x \in V(\mathbb{R}, z \in D$, and $\Xi(m,v) = \sum_{x\in L_m}\xi(v^{1/2}x,z), \xi(x,z) = -\mathrm{Ei}(-2\pi R(x,z))$.
In Section 12, for $m = Q(x) > 0$, (so that Γ_x is finite) and $D(B) > 1$, they calculate

(14.1.1) $$\begin{aligned}\kappa(m,v) :&= (1/4)\int_{\Gamma\backslash D}\sum_{x\in L(m)}\xi(v^{1/2}x,z)d\mu(z), \quad d\mu(z) = 1/(2\pi)dxdy/y^2,\\ &= (1/4)\sum_{x\in L(m),\mathrm{mod}\,\Gamma}\int_{\Gamma_x\backslash D}\xi(v^{1/2}x,z)d\mu(z)\\ &= (1/2)\sum_{x\in L(m),\mathrm{mod}\,\Gamma}|\Gamma_x|^{-1}\int_D\xi(v^{1/2}x,z)d\mu(z)\\ &= \sum_{x\in L(m),\mathrm{mod}\,\Gamma}|\Gamma_x|^{-1}\int_{D^+}\xi(v^{1/2}x,z)d\mu(z)\end{aligned}$$

Since $\xi(gx,gz) = \xi(x,z)$ for $g \in \mathrm{GL}(2,\mathbb{R})$, one may assume that

$$x = m^{1/2}\cdot x_0 = m^{1/2}\cdot\begin{pmatrix} & 1\\ -1 & \end{pmatrix}.$$

Then, writing $z = k_\theta(e^t i) \in \mathbb{H} \simeq D^+$, Kudla, Rapoport and Yang come to $R(x,z) = 2m\sinh^2(t)$. Hence, with

(14.1.2) $$J(t) = \int_0^\infty e^{-tw}((w+1)^{1/2} - 1)dw/w,$$

they get

$$\begin{aligned}
(14.1.3)\quad \kappa(m,v) &= \sum_{x\in L(m),\mathrm{mod}\,\Gamma} |\Gamma_x|^{-1}\int_{D^+} -\mathrm{Ei}(-2\pi R(v^{1/2}x,z))d\mu(z)\\
&= \sum_{x\in L(m),\mathrm{mod}\,\Gamma} |\Gamma_x|^{-1}(1/(2\pi))\int_0^{\pi}\int_0^{\infty} -\mathrm{Ei}(-4\pi mv\sinh^2(t))2\sinh(t)dtd\theta\\
&= \sum_{x\in L(m),\mathrm{mod}\,\Gamma} |\Gamma_x|^{-1}(1/2)\int_0^{\infty}(\int_1^{\infty} e^{-4\pi mv\sinh^2(t)r}dr/r)2\sinh(t)dt\\
&= \sum_{x\in L(m),\mathrm{mod}\,\Gamma} |\Gamma_x|^{-1}(1/2)\int_0^{\infty}(\int_1^{\infty} e^{-4\pi mvwr}dr/r)(w+1)^{-1/2}dw\\
&= \sum_{x\in L(m),\mathrm{mod}\,\Gamma} |\Gamma_x|^{-1}\int_0^{\infty} e^{-4\pi mvw}((w+1)^{1/2}-1)dw/w\\
&= \sum_{x\in L(m),\mathrm{mod}\,\Gamma} |\Gamma_x|^{-1}J(4\pi mv)\\
&= \delta(d,D)H_0(m;D)J(4\pi mv).
\end{aligned}$$

where the last equality uses [KRY] Lemma 9.2 and one has $4m = n^2d, -d$ a fundamental discriminant, and [KRY] (8.20) and (8.19)

$$\delta(d,D) = \prod_{p|D}(1-\chi_d(p)),\quad H_0(m,D) = \frac{h(d)}{w(d)}(\sum_{c|n,(c,D)=1} c\prod_{\ell|c}(1-\chi_d(\ell)\ell^{-1}))$$

where $h(d)$ is the class number and $w(d) = |\mathcal{O}_{k_d}^{\times}|$.

14.2. For our $\mathrm{SO}(1,2)$, if we follow the usual procedure as in (3.10.1). here we have

$$\tilde{V} = \{M = \begin{pmatrix} a & b\\ c & -a\end{pmatrix}; a,b,c\in\mathbb{R}\},$$

$$\tilde{q}(M) = \det M = -a^2 - bc = (1/2)(M,M),$$

$$(14.2.1)\qquad -(M,M') = 2aa' + bc' + cb'$$

and we identify

$$M = \mathbf{a} = {}^t(a,b,c),\ x = {}^t(x_1,x_2,x_3).$$

With $a = x_3, b = x_2 + x_1, c = x_2 - x_1$ one has

$$\tilde{q}(M) = -a^2 - bc = x_1^2 - x_2^2 - x_3^2 = q_0(x).$$

As fixed in (1.12.2) one has a homomorphism

$$G' = \mathrm{SL}(2,\mathbb{R}) \longrightarrow \tilde{G} = \mathrm{O}_0(\tilde{Q}),\ g\longmapsto A(g)$$

where, in particular, for $z = x + iy$, $g_z = \begin{pmatrix} v^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix}$, we get

$$A(g_z) = \begin{pmatrix} 1 & 0 & x/y \\ -2x & y & -x^2/y \\ 0 & 0 & 1/y \end{pmatrix}, \; A(g_z^{-1}) = \begin{pmatrix} 1 & 0 & -x \\ 2x/y & 1/y & -x^2/y \\ 0 & 0 & y \end{pmatrix},$$

As usual, we take $\mathbb{D}$ as space of oriented negative 2-planes $Y \subset V$, and as base point the plane

$$X_i :=< M_1, M_2 >, \; M_1 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, M_2 = \begin{pmatrix} & -1 \\ -1 & \end{pmatrix}.$$

Via g_z this plane is transported to

$$X_z :=< M_1', M_2' >, \; M_1' = (1/y) \begin{pmatrix} y & -2xy \\ & -y \end{pmatrix}, M_2' = (1/y) \begin{pmatrix} -x & x^2 - y^2 \\ -1 & x \end{pmatrix}.$$

i.e., for $Z = \begin{pmatrix} -\bar{z} & \bar{z}^2 \\ -1 & \bar{z} \end{pmatrix}$ one has $X_z =< \text{Re}\, Z, \text{Im}\, Z >$.

As at the beginning, in (3.10.4), we get the kernel of the majorant

$$R(z, M) = R(X_z, M) = (1/2y^2)|2az + b - cz^2|^2.$$

In particular, one has, as to be expected,

$$(M, M)_i = 2a^2 + b^2 + c^2 = 2(x_1^2 + x_2^2 + x_3^2)$$

14.3. For $m > 0$, we can take over the computation from [KRY] reproduced above. We supplement their reasoning. For $x = m^{1/2} \cdot x_0 = m^{1/2} \cdot (\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix})$, the equation

$$R(x, z) = (1/(2y^2))|2az + b - cz^2|^2 = 2m(\sinh \vartheta)^2$$

comes out in the 'hyperbolic coordinates' from (1.13.1) as in (1.13.4) as well as the relation

$$\frac{dx \wedge dy}{y^2} = \sinh \vartheta \, dr \wedge d\vartheta.$$

Hence, one has

$$\begin{aligned} I(v, m) &= \int_X \Xi(m, v, z) d\mu(z), \; X = (\Gamma \backslash \mathbb{H}), d\mu(z) = \frac{dxdy}{2\pi y^2} \\ &= \sum_{x \in L(m), \bmod \Gamma} |\Gamma_x|^{-1} J(4\pi m v) \\ &= H_0(m, 1) J(a), a = 4\pi m v. \end{aligned} \tag{14.3.1}$$

The last equation follows as Lemma 9.2 from [KRY] works also for $D=1$.

14.4. For $m<0$, again we can follow the computation in [KRY]. Given $x \in V, \tilde{q}(x)=m$, by conjugating with a suitable $g \in \mathrm{SL}_2(\mathbb{R})$, one can assume

$$x' := g\cdot x = \sqrt{|m|}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}.$$

Let Γ' be the corresponding conjugate of Γ in $\mathrm{SL}_2(\mathbb{R})$, and remark that $\Gamma'_{x'}$ will generated by $\pm E_2$ and $\begin{pmatrix} \epsilon(x) & \\ & \epsilon(x)^{-1} \end{pmatrix}$ for $\epsilon(x)>1$ the fundamental unit of norm 1 in the order $i_x^{-1}(M_2(\mathbb{Z}))$ in $\mathbb{Q}(\sqrt{|m|})$. For $x=x'$, one has $R(x,z)=(1/(2y^2))|2az+b-cz^2|^2 = 2|m|/y^2|z|^2$. Using polar coordinates $x=r\cos\varphi, y=r\sin\varphi$ we get $R(g\cdot x,z)=\frac{2|m|}{(\sin\varphi)^2}$ and

$$\begin{aligned} I(v,m)_x &= 1/(2\pi)\int_{\Gamma_x\backslash\mathbb{H}} \beta(2\pi v R(g\cdot x,z))d\mu(z), d\mu(z)=\frac{dxdy}{2\pi y^2} \\ &= 1/(2\pi)\int_{\Gamma_x\backslash\mathbb{H}}(\int_1^\infty e^{-4\pi v|m|\frac{1}{(\sin\varphi)^2}w}dw/w)\frac{drd\varphi}{r(\sin\varphi)^2}. \end{aligned} \tag{14.4.1}$$

As in [Fu] p.309, $x'=\begin{pmatrix} \epsilon & \\ & \epsilon^{-1} \end{pmatrix}$ acts on $z\in\mathbb{H}$ as $z\mapsto |m|z$. Hence, a fundamental domain $\mathcal{F}$ of $\Gamma_{x'}\backslash\mathbb{H}$ is the domain bounded by the semi arcs $|z|=1$ and $|z|=\epsilon(x)^2>1$ in $\mathbb{H}$. So, we get

$$\begin{aligned} I(v,m)_x &= 1/(2\pi)\int_{\Gamma_{x'}\backslash\mathbb{H}} \beta(2\pi v R(x',z))d\mu(z) \\ &= (1/\pi)\log|\epsilon(x)|\int_0^\pi(\int_1^\infty e^{-4\pi v|m|\frac{1}{(\sin\varphi)^2}w}dw/w)\frac{d\varphi}{(\sin\varphi)^2}. \end{aligned} \tag{14.4.2}$$

With $t=(\sin\varphi)^{-2}$, one has $(\sin\varphi)^{-2}d\varphi = -(1/2)(t-1)^{-1/2}dt$, and

$$\begin{aligned} I(v,m)_x &= (1/\pi)\log|\epsilon(x)|\int_1^\infty(\int_1^\infty e^{-4\pi|m|vtw}dw/w)\frac{dt}{\sqrt{t-1}} \\ &= (1/\pi)\log|\epsilon(x)|\int_1^\infty e^{-4\pi mvw}(\int_0^\infty e^{-4\pi|m|vw}dw/w)dt/\sqrt{t})dw/w \\ &= (1/\pi)\log|\epsilon(x)|\Gamma(1/2)(4\pi mv)^{-1/2}\int_1^\infty e^{-4\pi|m|vw}dw/w^{3/2} \\ &= (1/(2\pi))\log|\epsilon(x)|(|m|v)^{-1/2}\int_1^\infty e^{-4\pi|m|vw}dw/w^{3/2}, \end{aligned}$$

$$I(v,m) = \sum_{x\in\Gamma\backslash L_m} I(v,m)_x. \tag{14.4.3}$$

As again Lemma 12.3 from [KRY] is valid for $D=1$, namely

$$(\sum_{x\in\Gamma\backslash L_m} 2\delta_x^{-1}\log|\epsilon(x)|) = 4\delta(d;D)H_0(m,D), \tag{14.4.4}$$

one has

$$(14.4.5)\qquad I(v,m)=\sum_{x\in\Gamma\backslash L_m} I(v,m)_x = H_0(m,1)(1/(2\pi\sqrt{|m|v}))\int_1^\infty e^{-4\pi|m|vw}dw/w^{3/2}.$$

14.5. As an alternative, we treat Kudla's approach to calculate the Green function integral using Kudla's formula [Ku1] (3.23) for the case $p=1$. In the case of signature (1,2) the use of Kudla's formula (12.1.6) resp.Flensted-Jensen's formula (12.2.10) depends on the decomposition

$$G=\mathrm{SO}(1,2)=G_x\cdot A\cdot K,\ K=\mathrm{SO}(2)$$

For $x={}^t(0,0,1)$, one has $G_x=\mathrm{SO}(1,1)$

$$(14.5.1)\qquad G_x=\mathrm{SO}(1,1), A=\{a(t)=\begin{pmatrix}\cosh t & & \sinh t\\ & 1 & \\ \sinh t & & \cosh t\end{pmatrix}=\exp(tX_{1,3}), t\in\mathbb{R}\},$$

while, for $x={}^t(1,0,0)$, one has $G_x=\mathrm{SO}(2)$, i.e., a Cartan decomposition $G=KAK$. For $m<0$, we have

$$(14.5.2)\qquad I(v,m)=\int_X \Xi(m,v,z)d\mu(z),\ X=(\Gamma\backslash\mathbb{H}), d\mu(z)=\frac{dxdy}{2\pi y^2}$$

with Green function

$$\Xi(m,v,z)=(1/2)\sum_{M\in L_m}\xi(v,z,m),$$

$$\xi(v,z,m)=\beta(2\pi vR(z,M)), \beta_{\sigma+1}(t)=\int_1^\infty e^{-tu}u^{-\sigma-1}du, \beta=\beta_0,$$

$$R(z,M)=(1/2y^2)|2az+b-cz^2|^2,$$

$$(14.5.3)\qquad L_m=\{M\in M_2(\mathbb{Z});\mathrm{tr}(M)=0,\det(M)=m\}$$

$\Gamma=\Gamma(L)$ acts on L_m via $M\mapsto\gamma M\gamma^{-1}=\gamma\cdot M$ and on has $L_m=\cup_j\Gamma\cdot M_j$
Hence, unfolding, we get

$$(14.5.4)\qquad I(v,m)=(1/2)\sum_j\int_{\Gamma_{M_j}\backslash\mathbb{H}}\beta(2\pi vR(z,M_j))d\mu(z).$$

To evaluate this, Kudla proposes the procedure already explained in 12.3. For $m<0$, as explained above, choose a basis $\mathbf{v}$ for $V(\mathbb{R})$ such that the inner product has the matrix $I_{1,2}$ i.e., $(x,x)=x_1^2-x_2^2-x_3^2$. and $x=\alpha v_3$. Hence, one has $q(x)=-\alpha^2=m$ and $\mathrm{SO}(V(\mathbb{R})\simeq\mathrm{SO}(1,2)=G$ and $G_x\simeq\mathrm{SO}(1,1)$. Similar to (12.4.1) we get

$$R(x,z)=2|m|\cosh^2 t$$

and with (12.2.10) and (12.22.1)

$$\int_{\Gamma_x\backslash\mathbb{D}^+}\int_1^\infty e^{-2\pi vn^2R(x,z)r}dr/rd\mu(z)$$
$$= (1/2\pi)\mathrm{vol}(\Gamma_x\setminus G_x)\int_0^\infty\int_1^\infty e^{-4\pi v|m|\cosh^2 tr}\cosh t dr/rdt$$
$$= (1/2\pi)\mathrm{vol}(\Gamma_x\setminus G_x)I_-(v,m)$$
$$= (1/2\pi)\mathrm{vol}(\Gamma_x\setminus G_x)(1/(4\sqrt{|m|v}))\int_1^\infty e^{-|a|r}dr/r^{3/2}. \tag{14.5.5}$$

14.6. As above, we apply Siegel's method to determine an example of an isotropy group Γ_x. We choose

$$x = M_m = \begin{pmatrix} & m \\ -1 & \end{pmatrix} \equiv (0,m,-1) =: a_m$$

with $\tilde{q}(M_m) = m$ and want to know

$$\Gamma_{M_m} = \Gamma(\tilde{Q},a_m) = \{\gamma\in\Gamma(\tilde{Q});\gamma a_m = a_m\}.$$

With $S = \tilde{Q} = -\begin{pmatrix} 1 & & \\ & & 1/2 \\ & 1/2 & \end{pmatrix}$ and

$$A = (a_m,B) = \begin{pmatrix} & & 1 \\ m & 1 & \\ -1 & & \end{pmatrix}, 2q(a_m) = (a_m,a_m) = 2m$$

one has from (7.1.1) and (7.1.2)

$$\tilde{\Gamma}_{a_m} = \Gamma(S,a_m) = \{W\in\Gamma(K); {}^tWb\equiv b \bmod 2m\}$$
$$b = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix},$$
$$K = {}^tBSB - b{}^tb/m = -\begin{pmatrix} 1/(4m) & \\ & 1 \end{pmatrix}. \tag{14.6.1}$$

Hence, to simplify, for the following, we look at

$$q_m = {}^tuK_mu = (u_1^2+4mu_2^2);\ K_m = \begin{pmatrix} 1 & \\ & 4m \end{pmatrix}. \tag{14.6.2}$$

Here, we find some information in [S6] p.258, namely Siegel treats the example $q(x) = x_2^2 - Sx_1^2$, S not a square. Let t,u be solutions of Pell's equation $t^2-Su^2=1$ with smallest $t+u\sqrt{S}>1$, then, $0\leqslant x_1<\frac{u}{t}x_2$ describes a fundamental domain with respect to the group of unities with determinant 1 and one has

$$\mathrm{vol}(\Gamma\backslash\mathbb{R}^2) = (\sqrt{S}/2)\int_0^{u/t}\frac{dz}{1-Sz^2} = (1/2)\log(t+u\sqrt{S}). \tag{14.6.3}$$

If we put (14.6.3) together with (14.5.5), we get

$$\int_{\Gamma_x\backslash\mathbb{D}^+}\int_1^\infty e^{-2\pi vn^2R(x,z)r}dr/rd\mu(z)$$
$$= (1/2\pi)\mathrm{vol}(\Gamma_x\setminus G_x)\int_0^\infty\int_1^\infty e^{-4\pi v|m|\cosh^2 tr}\cosh t dr/rdt$$
$$= (1/2\pi)\mathrm{vol}(\Gamma_x\setminus G_x)I_-(v,m)$$
$$= (1/2\pi)\mathrm{vol}(\Gamma_x\setminus G_x)(1/(4\sqrt{|m|v}))\int_1^\infty e^{-|a|r}dr/r^{3/2}.$$
$$= (1/2\pi)(1/2)\log(t+u\sqrt{S})(1/(4\sqrt{|m|v}))\int_1^\infty e^{-|a|r}dr/r^{3/2}. \tag{14.6.4}$$

As Pell's equation produces fundamental units ϵ this is up to a factor the formula

$$I(m,v)_x = (1/(2\pi))\log|\epsilon(x)|(|m|v)^{-1/2}\int_1^\infty e^{-4\pi|m|vw}dw/w^{3/2}$$

from (14.4.3).

Remark. If m is a square, say $m=-n^2$, we have $q_m(u)=(u_1+2nu_2)(u_1-2nu_2)$, i.e., one has $G_x\simeq \mathrm{SO}(1,1)$, $\Gamma_x\simeq\{\pm E_2\}$ and $\mathrm{vol}(\Gamma_x\backslash G_x)=\infty$. Hence the Green integral diverges. This is consistent with the observation at the end of Yang's [Ya] (Part (3) of Proposition 3:1).

For $m>0$, as explained above, choose a basis $\mathbf{v}$ for $V(\mathbb{R})$ with $x=\alpha v_1$. Hence, one has $G_x\simeq \mathrm{SO}(2)$. Similar to (12.4.1) we get

$$R(x,z)=2m\sinh^2 t$$

and with (12.2.10) and (12.22.1)

$$\int_{\Gamma_x\backslash\mathbb{D}^+}\int_1^\infty e^{-2\pi vn^2R(x,z)r}dr/rd\mu(z)$$
$$= (1/2\pi)\mathrm{vol}(\Gamma_x\setminus G_x)\int_0^\infty\int_1^\infty e^{-4\pi vm\sinh^2 tr}\sinh t dr/rdt$$
$$= (1/2\pi)\mathrm{vol}(\Gamma_x\setminus G_x)I_+(v,m)$$
$$= (1/2\pi)\mathrm{vol}(\Gamma_x\setminus G_x)(1/2)\int_1^\infty e^{-ar}((r+1)^{1/2}-1)dr/r. \tag{14.6.5}$$

Hence, one has as above in (14.3.1)

$$I(v,m)=\int_X \Xi(m,v,z)d\mu(z),\ X=(\Gamma\backslash\mathbb{H}), d\mu(z)=\frac{dxdy}{2\pi y^2}$$
$$=\sum_{x\in L(m),\mathrm{mod}\,\Gamma}|\Gamma_x|^{-1}J(4\pi mv)$$
$$=H_0(m,1)J(a), a=4\pi mv.$$

14.7. Eisenstein series. It is natural, to ask how much of these results for the Green function integrals lead to the Fourier coefficients of the derivatives of the 3/2-Eisenstein series $\mathcal{E}(\tau, s)$ as for instance Theorem 3.2 in Yang's report [Ya]. In Theorem 3.4, he affirms: The Eisenstein series $\mathcal{E}(\tau, s)$ in Theorem 4.2 has the following Fourier expansion

$$\mathcal{E}(\tau, s) = \sum_{m \equiv 0, -1 \bmod 4} A_m(v, s) q^m$$

where

$$(14.7.1) \quad A_m(v,s) = \Lambda(1/2 - s, \chi_m)(4\pi m v)^{(s-1/2)/2} \Psi_{-3/2}(s, 4\pi m v) \text{ for } m > 0,$$
$$A_m(v,s) = \frac{(s^2 - 1/4)\Lambda(1/2 - s, \chi_m)(4\pi|m|v)^{(s-1/2)/2}\Psi_{3/2}(s, 4\pi|m|v)}{4\sqrt{\pi}e^{4\pi m v}} \text{ for } m < 0,$$
$$A_0(v,s) = -\frac{1}{2\pi}(G(s) + G(-s)),\ G(s) = (4v)^{(1/2-s)/2}(s + 1/2)\Lambda(1 + 2s).$$

Here, $\Psi_n(s, a)$ is as in (2.3.5) and

$$\Lambda(s, \chi_m) = |m|^{3/2}\pi^{-(s+a)/2}\Gamma((s+a)/2)L(s, \chi_m) = \Lambda(1 - s, \chi_m),\ a = (1 + \text{ sign}(m))/2$$

with

$$L(s, \chi_m) = L(s, \chi_d) \prod_{p|n} b_p(n, s),\ b_p(n, s) = \frac{1 - \chi_d(p)X + \chi_d(p)p^k X^{1+2k} - (pX^2)^{1+k}}{1 - pX^2}$$

where $X = p^{-s}$ and $k = \text{ord}_p n$. In Yang's paper, he writes $m = dn^2$ such that $-d$ is the fundamental discriminant of $K_m = \mathbb{Q}(\sqrt{-m})$. Part 2 of his Theorem 3.2 states that

$$(14.7.2) \qquad \mathcal{E}(\tau, 1/2) = E_{\text{Zagier}}(\tau) = -\frac{1}{12} + \frac{1}{8\pi\sqrt{v}} + \sum_{m=1}^{\infty} H_0(m)q^m + \sum_{n>0} 2g(n, v)q^{n^2}$$

where $H_0(m)$ is the Hurwitz class number of binary quadratic forms of discriminant $-m$, and

$$g(n, v) = \frac{1}{16\pi\sqrt{v}} \int_1^{\infty} e^{-4\pi n^2 v r} dr/r^{3/2}.$$

And Part 3 of Yang's Theorem 3.2 states

$$(14.7.3) \qquad \mathcal{E}'(\tau, 1/2) = \sum_m < \hat{\mathcal{Z}}(m, v), \hat{\omega} > q^m.$$

The Fourier coefficients of $\mathcal{E}'(\tau, 1/2)$ are to be found in Kudla-Yang [KY]. There, in

Theorem 6.6, one has (with $-m = dc^2, \mu = 0, 1/2$)

(14.7.4)

$$E'_0(\tau, 1/2, \Phi^{3/2,\mu}) = \delta_{0,\mu}(\frac{1}{2}\log v - \frac{3}{\pi\sqrt{v}}(\frac{1}{2}\log v - \gamma - \log 8\pi + \frac{\zeta'(-1)}{\zeta(-1)}))$$

when $m > 0$ and $m \in -\mu^2 + \mathbb{Z}$

$$E'_m(\tau, 1/2, \Phi^{3/2,\mu}) = -12H(4m)q^m(1 + \frac{\zeta'(-1)}{\zeta(-1)} - \frac{1}{2}\log\pi - \log d$$
$$- \frac{L'(0,\chi_{-m})}{L(0,\chi_{-m})} - \sum_{p|c}(\frac{b'_p(-m,0)}{b'_p(-m,0)} - 2k_p(c)\log p) + \frac{1}{2}J(\frac{1}{2}, 4\pi mv))$$

when $m < 0$ and $m \in -\mu^2 + \mathbb{Z}$, and $-m$ is not a square,

$$E'_m(\tau, 1/2, \Phi^{3/2,\mu}) = -\frac{3}{\pi\sqrt{v}}\frac{H(4m)}{\sqrt{|m|}}\int_1^\infty e^{4\pi mvr}dr/r^{3/2}\, q^m$$

when $m = -c^2 \in -\mu^2 + \mathbb{Z}$,

$$\frac{E'_{-c^2}(\tau, 1/2, \Phi^{3/2,\mu})}{E_{-c^2}(\tau, 1/2, \Phi^{3/2,\mu})} = \frac{1}{2}\log v + 2 - 2\log 2 + \frac{1}{2}\log 2\pi - \frac{1}{2}\gamma + 2\frac{\zeta'(-1)}{\zeta(-1)} - \sum_{p|2c}\frac{b'_p(-m,0)}{b_p(-m,0)} + \frac{\Psi'_{3/2}}{\Psi_{3/2}}.$$

From (15.2.1) and Proposition 6.5 in [KY], we may deduce

$$\mathcal{E}(\tau, s) = \zeta(-1)(E(4\tau, s, \Phi^{3/2,0}) + E(4\tau, s, \Phi^{3/2,1/2}).$$

14.8. Comparison. We relate this to our results for the Green function integrals. From (14.3.1) we have for $a = 4\pi mv, m > 0$

$$I(m; v) = H_0(m,1)J(s) = H_0(m,1)J(1/2, a), \quad J(n,a) = \int_0^\infty e^{ar}\frac{(1+r)^n - 1}{r}dr,$$

and, for $m < 0, m$ not a square, from (14.4.5)

$$I(v,m) = H_0(m,1)\frac{1}{\pi\sqrt{|m|v}}\int_1^\infty e^{|a|r}dr/r^{3/2}.$$

Using Kudla's approach to the Green integral, we got (14.6.4)

$$\int_{\Gamma_x\backslash\mathbb{D}^+}\int_1^\infty e^{-2\pi vn^2 R(x,z)r}dr/r\,d\mu(z)$$
$$= (1/2\pi)\mathrm{vol}(\Gamma_x \setminus G_x)\mathrm{vol}(K)\int_0^\infty\int_1^\infty e^{-4\pi v|m|\cosh^2 tr}\cosh t\,dr/r\,dt$$
$$= (1/2\pi)\mathrm{vol}(\Gamma_x \setminus G_x)\mathrm{vol}(K)I_\pm(v,m)$$
$$= (1/2\pi)\mathrm{vol}(K)(1/2)\log(t + u\sqrt{S})(1/(4\sqrt{|m|v}))\int_1^\infty e^{-|a|r}dr/r^{3/2},$$

and with Lemma 12.3 from [KRY] and the footnote to (8.23) from [KRY] $2H_0(m,1) = H(4m)$

$$I(m,v) = (1/2\pi)\mathrm{vol}(K)H_0(m,1)(1/(4\sqrt{|m|v}))\int_1^\infty e^{-|a|r}dr/r^{3/2}.$$

Hence, up to some still mysterious (or erroneous?) factors, the same principal values for the integrals and the Fourier coefficients.

15 Epilogue

Here, we look back to give an overview over what we have done up to now. In the main body of the paper, we treated the following situation. We are given an $\mathbb{R}-$vector space V of dimension n with a bilinear form $(.,.)$ of signature $(p,2)$ and a lattice L. Expressed in a pedestrian way, Kudla's conjecture relates the Fourier coefficients of the $s-$derivative of the associated Eisenstein series $E(\tau,s)$ at $s=0$ up to a volume factor B to the integral $I(m,v)$ of an appropriate Green function $\Xi(m,v,z)$. As mentioned in the Introduction, there are already a lot ofl papers to make this precise and extend it to more general situations. Here, we assemble to an overview our explicit calculations above for the cases $p=1,2,3$ following an approach proposed by analyzing the formula (3.23) in Kudla's seminal paper [Ku1]:

Kudla's Green function integral formula.

15.1. In the special cases, the following has to be more refined and adapted. Let

(15.1.1)
$$G = \mathrm{SO}(V) \supset \Gamma = \mathrm{SO}(V,\mathbb{Z}),\ \mathbb{D} = G/K, X = \Gamma\backslash\mathbb{D},$$
$$L_m = \{u \in \mathbb{Z}^n; q(u) = m\},$$
$$2R(u,z) = (u,u)_z - (u.u), (u,u)_z \text{ the majorant of the bilinear form } (.,.) \text{ in } z \in \mathbb{D},$$
$$\beta(s) = \int_1^\infty e^{-sr}dr/r,$$
$$\Xi(m,v)(z) = (1/2)\sum_{x\in L_m}\beta(2\pi mR(x,z)), z \in \mathbb{D}, v > 0,$$
$$I(m,v) = \int_X \Xi(m,v)(z)d\mu(z) = (1/2)\int_{\Gamma\backslash\mathbb{D}}\sum_{x\in L_m}\beta(2\pi mR(x,z))d\mu(z).$$

And, as Γ acts on L_m with finitely many orbits, i.e., $L_m = \sum_a(\Gamma/\Gamma_a)a$, and one has the invariance $R(\gamma x,z) = R(x,\gamma^{-1}z)$ one gets by the usual unfolding

(15.1.2)
$$\begin{aligned} I(m,v) &= (1/2)\int_{\Gamma\backslash\mathbb{D}}\sum_{x\in L_m}\beta(2\pi mR(x,z))d\mu(z) \\ &= (1/2)\sum_{x\in\Gamma\backslash L_m} I_x(m,v), \quad I_x(m,v) := \int_{\Gamma_x\backslash\mathbb{D}}\beta(2\pi mR(x,z))d\mu(z). \end{aligned}$$

Now, one has to determine $I_x(m,v)$. One has a 'double set decomposition'

$$G = G_x A_+ K. \tag{15.1.3}$$

For $x \in V$ with $Q(x) = m > 0$, and $z = g_x a_t \cdot z_0$, in our special cases, one has

$$R(x,z) = 2m \sinh^2 t. \tag{15.1.4}$$

Going back to [FlJ] (and [BY]), we interpreted this by the 'Flensted-Jensen formula' (12.2.10) via

$$\begin{aligned}\int_{\Gamma\backslash\mathbb{D}} f(z)\Omega^p &= (p!/(2\pi)^p)\cdot \mathrm{vol}\,(L/M)\int_K\int_{\Gamma_x\backslash G_x}\int_{\mathfrak{b}^+} f(h\exp X'k)\delta(X)dXdhdk,\\ &= (p!/(2\pi)^p)\cdot \mathrm{vol}\,(L/M)\cdot \mathrm{vol}(\Gamma_x\setminus G_x)\cdot \mathrm{vol}(K)\cdot I_+(p,a),\end{aligned} \tag{15.1.5}$$

For x with

$$Q(x) = m < 0,$$

similar we deduce

$$\begin{aligned} I(m,v) &= \int_{\Gamma_a\backslash\mathbb{D}}\int_1^\infty e^{-2\pi v n^2 R(z,a) r} dr/r d\mu(z)\\ &= (p!/(2\pi)^p)\cdot \mathrm{vol}\,(L/M)\cdot \mathrm{vol}(\Gamma_a\setminus G_a)\cdot \mathrm{vol}(K)\cdot I_-(p,a),\\ I_-(p,a) &:= \int_0^\infty\int_1^\infty e^{-4\pi v|m|\cosh^2 tr} dr/r \sinh^{p-1} t\cosh t dt.\end{aligned} \tag{15.1.6}$$

In (12.22.1) we assembled the I-Integrals (with $a = 4\pi m v$):

$$\begin{aligned} p=1:\quad I_+(1,a) &= (1/2)\int_0^\infty e^{-aw}((w+1)^{1/2}-1)dw/w = (1/2)J_+(1/2,a)\\ I_-(1,a) &= (1/2)e^{|a|}\int_1^\infty e^{-|a|r} r^{1/2} dr/(r+1) =: (1/2)e^{|a|}J_-(1/2,|a|)\\ &= (1/(4\sqrt{|m|v}))\int_1^\infty e^{-|a|r} r^{-3/2} dr.\\ p=2:\quad I_+(2,a) &= (1/2a)\\ I_-(2,a) &= (1/(2|a|)(e^{|a|} - |a|\int_1^\infty e^{-|a|} dr/r)\\ p=3:\quad I_+(3,a) &= (1/3)\int_0^\infty e^{-\alpha w}((w+1)^{3/2}-1)dw/w = (1/3)J_+(3/2,a)\\ I_-(3,a) &= (1/3)e^{|a|}\int_1^\infty e^{-|a|r} r^{3/2} dr/(r+1) = (1/3)e^{|a|}J_-(3/2,|a|)\\ &= (1/(4|a|^{3/2})\sqrt{\pi}\int_1^\infty e^{-|a|r} dr/r^{5/2}.\end{aligned}$$

Below, we will discuss the other factors in Kudla's formula but at first we assemble the terms with which we have to compare the Green integrals.

The Eisenstein series.

15.2. For $\mathbf{p = 1}$, in [Ya] Theorem 3.4 the 3/2-series $\mathcal{E}(\tau, s)$ has the following Fourier expansion

$$\mathcal{E}(\tau, s) = \sum_{m \equiv 0, -1 \bmod 4} A_m(v, s) q^m$$

where

(1) $A_m(v, s) = \Lambda(1/2 - s, \chi_m)(4\pi m v)^{(s-1/2)/2} \Psi_{-3/2}(s, 4\pi m v),\ m > 0$

(2) $A_m(v, s) = \dfrac{(s^2 - 1/4)\Lambda(1/2 - s, \chi_m)(4\pi|m|v)^{(s-1/2)/2} \Psi_{3/2}(s, 4\pi|m|v)}{4\sqrt{\pi} e^{4\pi m v}},\ m < 0$

(3) $A_0(v, s) = -\dfrac{1}{2\pi}(G(s) + G(-s)),\ G(s) = (4v)^{(1/2-s)/2}(s + 1/2)\Lambda(1 + 2s).$

For more details see (14.7.1). In Yang's paper, he writes $m = dn^2$ such that $-d$ is the fundamental discriminant of $K_m = \mathbb{Q}(\sqrt{-m})$. Part 2 of his Theorem 3.2 states that

$$\mathcal{E}(\tau, 1/2) = E_{\text{Zagier}}(\tau) = -\frac{1}{12} + \frac{1}{8\pi\sqrt{v}} + \sum_{m=1}^{\infty} H_0(m) q^m + \sum_{n>0} 2g(n, v) q^{n^2} \tag{15.2.1}$$

where $H_0(m)$ is the Hurwitz class number of binary quadratic forms of discriminant $-m$, and

$$g(n, v) = \frac{1}{16\pi\sqrt{v}} \int_1^{\infty} e^{-4\pi n^2 v r} dr / r^{3/2}.$$

And Part 3 of Yang's Theorem 3.2 states

$$\mathcal{E}'(\tau, 1/2) = \sum_m < \hat{\mathcal{Z}}(m, v), \hat{\omega} > q^m. \tag{15.2.2}$$

The Fourier coefficients of $\mathcal{E}'(\tau, 1/2)$ are to be found in Kudla-Yang [KY]. There, in Theorem 6.6, one has (with $-m = dc^2, \mu = 0, 1/2$)

(15.2.3)

$$E_0'(\tau, 1/2, \Phi^{3/2,\mu}) = \delta_{0,\mu}(\frac{1}{2}\log v - \frac{3}{\pi\sqrt{v}}(\frac{1}{2}\log v - \gamma - \log 8\pi + \frac{\zeta'(-1)}{\zeta(-1)}))$$

when $m > 0$ and $m \in -\mu^2 + \mathbb{Z}$

$$E_m'(\tau, 1/2, \Phi^{3/2,\mu}) = -12H(4m)q^m(1 + \frac{\zeta'(-1)}{\zeta(-1)} - \frac{1}{2}\log\pi - \log d$$

$$- \frac{L'(0,\chi_{-m})}{L(0,\chi_{-m})} - \sum_{p|c}(\frac{b_p'(-m,0)}{b_p'(-m,0)} - 2k_p(c)\log p) + \frac{1}{2}J(\frac{1}{2}, 4\pi mv))$$

when $m < 0$ and $m \in -\mu^2 + \mathbb{Z}$, and $-m$ is not a square,

$$E_m'(\tau, 1/2, \Phi^{3/2,\mu}) = -\frac{3}{\pi\sqrt{v}}\frac{H(4m)}{\sqrt{|m|}}\int_1^\infty e^{4\pi mvr}dr/r^{3/2}\, q^m$$

when $m = -c^2 \in -\mu^2 + \mathbb{Z}$,

$$\frac{E_{-c^2}'(\tau, 1/2, \Phi^{3/2,\mu})}{E_{-c^2}(\tau, 1/2, \Phi^{3/2,\mu})} = \frac{1}{2}\log v + 2 - 2\log 2 + \frac{1}{2}\log 2\pi - \frac{1}{2}\gamma + 2\frac{\zeta'(-1)}{\zeta(-1)} - \sum_{p|2c}\frac{b_p'(-m,0)}{b_p(-m,0)} + \frac{\Psi_{3/2}'}{\Psi_{3/2}}.$$

From (15.2.1) and Proposition 6.5 in [KY], we may deduce

$$\mathcal{E}(\tau, s) = \zeta(-1)(E(4\tau, s, \Phi^{3/2,0}) + E(4\tau, s, \Phi^{3/2,1/2}).$$

15.3. For $\mathbf{p = 2}$, in [BeKII] we took over classical material from Zagier's article [Za1] p.32f. One has the analytic Eisenstein series, respective their modifications

$$E(\tau, s) := (1/2)\sum\nolimits_{c,d}'\frac{v^s}{|\, c\tau + d\,|^{2s}}$$

$$E^*(\tau, s) := \pi^{-s}\Gamma(s)E(\tau, s)$$

$$E_2(\tau, s) := (1/(2\pi i))\partial_\tau E^*(\tau, s) = \sum_{m\in\mathbb{Z}} a(v, s, m)q^m$$

$$\mathbb{E}_2(\tau, s) := -12\psi(s)E_2(\tau, s),\ \psi(s) = -1 + 4(\frac{\zeta'(-1)}{\zeta(-1)} + \frac{1}{2})(s-1) + O((s-1)^2)$$

$$= \sum_{m\in\mathbb{Z}} A(v, s, m)q^m$$

Denoting by $E_2'(\tau, s)$ the derivative of $E_2(\tau, s)$ with respect to s, and so from [BeKII], we take over

$$(15.3.1) \qquad a(v,1,m) = \begin{cases} \sigma(m) = \sum_{d|m} d & \text{for } m>0 \\ -1/24 + 1/(8\pi v) & \text{for } m=0 \\ 0 & \text{for } m<0 \end{cases}$$

$$(15.3.2) \qquad a'(v,1,m) = \begin{cases} \sigma(m)(1/(4\pi m v) + \sigma'(m)/\sigma(m)) & \text{for } m>0 \\ -(1/24)(24\zeta'(-1) + \gamma - 1 + \log(4\pi v)) & \\ \qquad -(1/(8\pi v))(-\gamma + \log(4\pi v)) & \text{for } m=0 \\ \sigma(m)(\mathrm{Ei}(-4\pi|m|v) + 1/(4\pi|m|v)e^{-4\pi|m|v}) & \text{for } m<0. \end{cases}$$

$$A'(v,1,m) = -12 \begin{cases} \sigma(m)(4(\zeta'(-1)/\zeta(-1) + 1/2) + 1/(4\pi m v) & \\ \qquad +{\sigma^*}'_{1/2}(m)/\sigma^*_{1/2}(m))) & \text{for } m>0 \\ 3\zeta'(-1) - (1/8) + (\gamma/24) + (1/24)\log(4\pi v) & \\ \quad +(1/8\pi v)(-48\zeta'(-1) - \gamma + 2 + \log(4\pi v)). & \text{for } m=0 \\ \sigma(|m|)(\mathrm{Ei}(-4\pi|m|v) + 1/(4\pi|m|v)e^{-4\pi|m|v}) & \text{for } m<0 \end{cases}$$

The sigmas in these terms are those from the paper by Zagier

$$\sigma^*_s(n) := |n|^s \sum_{d|n, d>0} d^{-2s} = \sigma^*_{-s}(n).$$

Hence one has $\sqrt{m}\,\sigma^*_{1/2}(m) = \sum_{d|m} d = \sigma(m)$ and $\sigma'(m)/\sigma(m) := {\sigma^*}'(m)_{1/2}/\sigma^*_{1/2}(m) = (\sigma(m)\log m - 2\sum_{d|m} d\log d)/\sigma(m)$.

15.4. For $\mathbf{p=3}$, one can find similar formulae for weight 5/2-series determined by the lattice L and the associated Weil representation in [Ku1] and [BK]. As we did from the beginning, we follow [BK] and from (4.52) have

$$(15.4.1) \qquad \begin{aligned} E_0(\tau,0) &= \sum_{\gamma\in L'/L} \sum_{m\in\mathbb{Z}-q(\gamma),\, m>0} c_0(\gamma,m,0,v)e(mu)e_\gamma \\ &= \sum_{\gamma\in L'/L} \sum_{m\in\mathbb{Z}-q(\gamma), m>0} C_0(\gamma,m,0)e_\gamma(m\tau) \\ &= 2e_0 - \frac{2}{B} \sum_{\gamma\in L'/L} \sum_{m\in\mathbb{Z}-q(\gamma), m>0} \deg(\mathcal{H}(\gamma,-m))e_\gamma(m\tau). \end{aligned}$$

From (2.9.1), (2.11.1) and (2.11.2), we have

$$\begin{aligned} c_0(\gamma,m,0,v) &= C(\gamma,m,0)e^{-a/2}\,\text{for}\,m>0,\\ &=0, \quad \text{for}\,m<0,\\ c_0'(\gamma,m,0,v) &= \frac{\partial}{\partial s}c_0(\gamma,m.s.v)|_{s=0}\\ &= C(\gamma,m,0)e^{-a/2}(J_+(3/2,a)+\frac{C'(\gamma,m,0)}{C(\gamma,m,0)}),\,\text{for}\,m>0,\\ &= C(\gamma,m,0)e^{-|a|/2}\cdot J_-(3/2,a),\,\text{for}\,m<0,\end{aligned}$$

$$C(\gamma,m,0) := -2^6\cdot 3\cdot 5\cdot \pi^{-2}|m|^{3/2}L(\chi_{d_F},2)\sigma_{\gamma,m}(5/2). \tag{15.4.2}$$

Comparisons.

We relate the results for Fourier coefficients to our results for the Green function integrals.

15.5. For $\mathbf{p}=\mathbf{1}$ and $a=4\pi mv, m>0$, from (14.3.1) by the direct calculation of the Green integrals in [KRY] we have

$$I(m;v)=H_0(m,1)J(a)=H_0(m,1)J_+(1/2,a), \quad J_+(n,a)=\int_0^\infty e^{ar}\frac{(1+r)^n-1}{r}dr,$$

and, for $m<0, m$ not a square, from (14.4.5) as well

$$I(v,m)=H_0(m,1)\frac{1}{2\pi\sqrt{|m|v}}\int_1^\infty e^{|a|r}dr/r^{3/2}=e^{-|a|}J_-(1/2,|a|). \tag{15.5.1}$$

For x with $q(x)=m<0$, one may also try to use Kudla's approach to the Green integral. We got (14.6.4)

$$\begin{aligned} &\int_{\Gamma_x\backslash\mathbb{D}^+}\int_1^\infty e^{-2\pi vn^2R(x,z)r}dr/rd\mu(z)\\ &\qquad = (1/2\pi)\mathrm{vol}(\Gamma_x\setminus G_x)(1/(4\sqrt{|m|v}))\int_1^\infty e^{-|a|r}dr/r^{3/2},\\ &\qquad = (1/2\pi)(1/2)\log(t+u\sqrt{S})(1/(4\sqrt{|m|v}))\int_1^\infty e^{-|a|r}dr/r^{3/2}.\end{aligned}$$

Here, for the fundamental domain volume term, we used a result from Siegel [S6]: For $q(x)=x_2^2-Sx_1^2, S$ no square, he comes to the volume of a fundamental domain of the full group of units as $v(S)=\frac{\sqrt{S}}{2}\int_0^{u/t}\frac{dz}{1-z^2}=\frac{1}{2}\log\left(t+u\sqrt{S}\right)$, where t,u the solutions from $t^2-u2S=1$ with smallest $t+u\sqrt{S}>1$. And with Lemma 12.3 from[KRY] and (8.22) from [KRY] $2H_0(m,1)=H(4m)$

$$I(m,v)=(1/2\pi)H_0(m,1)(1/(4\sqrt{|m|v}))\int_1^\infty e^{-|a|r}dr/r^{3/2}.$$

Hence, essentially up to a factor 12 (from somewhere different normalizations?) we get the same principal values for the integrals and the Fourier coefficients of the Eisenstein series. Moreover, we see that, as for $m = -d^2$ one has a finite unit group, the Green function integral diverges and has to be modified as proposed in Yang's paper.

15.6. For $\mathbf{p} = \mathbf{2}$ and $m \neq 0$, as in [BeKI] (2.0.2), we have

$$I(v,m) = \int_X \Xi(m,v,z)d\mu(z),\ X = (\Gamma\backslash\mathbb{H})^2, d\mu(z) = \frac{dx_1dy_1dx_2dy_2}{(y_1y_2)^2} \tag{15.6.1}$$

with $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ and Green function

$$\Xi(m,v,z) = (1/2)\sum_{M\in L_m}\xi(v,z,m),$$

$$\xi(v,z,m) = \beta(2\pi vR(z,M)),\ R(z,M) = \frac{|a-bz_2-cz_1+dz_1z_2|^2}{2y_1y_2},$$

$$L_m = \{M \in M_2(\mathbb{Z}); \det(M) = m\} \tag{15.6.2}$$

In [BeKII] Theorem 4.2, by a direct evaluation of the Green function integral with $d\mu(z) = dv_{\mathbb{H}^2}$, for $m > 0$ resp.$m < 0$, we got

$$\begin{aligned} I(v,m) &= \sigma(m)(2\pi^2/(3a)), \quad a = 4\pi mv, \\ &= \sigma(|m|)(2\pi^2/(3|a|))(e^{-|a|} - |a|\int_1^\infty e^{-|a|r}dr/r). \end{aligned} \tag{15.6.3}$$

Remark. If we relate this to the Fourier coefficients of the Eisenstein series (15.3.1), we get

$$\frac{18}{\pi^2}\cdot I(m,v) = A'(m,1,v) + \dots \tag{15.6.4}$$

where the factor in [BeKII] is identified via the Chern class

$$\hat{c}_1(\bar{\mathcal{L}})^2 = \frac{72}{4\pi^2}\frac{dx_1\wedge dy_1\wedge dx_2\wedge dy_2}{y_1^2y_2^2}$$

i.e., a factor $B' = 36$ if the Green function integral is done with the measure

$$\Omega^2_{\mathbb{H}^2} = \frac{1}{8\pi^2}\frac{dx_1\wedge dy_1\wedge dx_2\wedge dy_2}{y_1^2y_2^2} \tag{15.6.5}$$

from [BK] (5.7) (while in [vdG] p.59 $\omega_{\mathbb{H}^2} = \frac{1}{4\pi^2}\frac{dx_1\wedge dy_1\wedge dx_2\wedge dy_2}{y_1^2y_2^2}$).

Now, as an alternative, above we evaluated the Green function integral using Kudla's formula following the procedure already explained in 12.3.

With $d\mu = \Omega^2 = (1/4\pi^2)dv_{\mathbb{H}^2}$ in (13.4.1) and (13.4.2), we got

$$\begin{aligned} I(v,m) &= \sigma(m)/(6\alpha), \alpha = 4\pi vm > 0. \\ &= \sigma(m)/(6|\alpha|)(e^{-|\alpha|} + |\alpha|\mathrm{Ei}(-|\alpha|)), \alpha = 4\pi vm < 0. \end{aligned}$$

Both formulae are up to a factor 2^2 (from somewhere different normalizations?) the same as those from Theorem 4.2 in [BeKII].

15.7. For $\mathbf{p = 3}$ one has a two-component Green function $\Xi(\gamma, m, v)$ with $\gamma = 0$ for $m \in \mathbb{Z}$ and $\gamma = 1$ for $m \in \mathbb{Z} + 1/4$, and as in (4.6.1) the Green function integrals

$$\begin{aligned} I(0,m,v) &:= \int_X \Xi(0,m,v)(z)d\mu_z = \sum_{x\in L_{0,m}} \int_{\Gamma\backslash\mathbb{H}_2} \beta(2\pi vR(x,z))d\mu_z, \\ I(1,m,v) &:= \int_X \Xi(1,m,v)(z)d\mu_z = \sum_{x\in L_{1,m}} \int_{\Gamma\backslash\mathbb{H}_2} \beta(2\pi vR(x,z))d\mu_z. \end{aligned} \tag{15.7.1}$$

with $L_{\gamma,m} = \{u \in \mathbb{Z}^5; \hat{q}(u) = u_3^2 - 4u_2u_4 - 4u_1u_5 = 4m\}$. As usual, with $a = a_{\gamma,m} = {}^t(1,0,0,0,-m)$ or $= {}^t(0,1,1,-(4m-1)/4,0)$ for $\gamma = 0$ or $= 1$, we infer

(15.7.2)

$$I(0,v,m) = \sum_{n^2|\delta_\gamma m} I(0,v,m,n),\ I(0,v,m,n) = \int_{\Gamma_{a_{0,m/n^2}}\backslash\mathbb{H}_2} \int_1^\infty e^{-2\pi vR(na_{0,m/n^2},z)r}dr/rd\mu(z),$$

and similarly, for $I(1,v,m)$. Finally, we got in (12.16.1) that the integrals over Kudla's Green function and the Green function from Bruinier-Kühn [BK] Definition 4.1 and Theorem 4.10 essentially add up to the coefficient of the derivative of the Eisenstein series.

15.8. Corollary. For $m \neq 0$, and $B = \int_X \Omega^3 = 2^{-5}3^{-2}5^{-1}$, we have

$$c_0'(\gamma,m,0,v) = e^{-a/2}((4/B)\cdot(I(\gamma,m,v) - I^{BK}(\gamma,-m,v)) + *\, c_0(\gamma,m,0,v)). \tag{15.8.1}$$

This is in line with the result from Ehlen-Sankaran [ES] **Theorem 3.6:** For each $z \in \mathbb{D}^0(V)$, in the $q-$series

$$-\log v\varphi_0^\vee + \sum_m (Gr_0^K(m,v) - Gr_0^B(m))q^m \tag{15.8.2}$$

is the $q-$expansion of a modular form in $A^!_\kappa(\rho_L^\vee)$ of weight $\kappa = p/2 + 1$.

16 Appendix: Gauge forms, invariant differentials and measures.

As the evaluation of Kudla's for this text central integration formula depends on the normalization of measures, we assemble some material related to this topic which, though here finally superfluous may also be interesting in some other context.

Measures from Bruinier-Yang and Lie Algebras

16.1. In his famous article [Ta], Tamagawa on his way to translate Siegel's mass formula from [S6] into the adelic language, restates the following Siegel material. Take $S \in \mathrm{Sym}_n(\mathbb{Z})$. Let G be the algebraic group of real $n \times n$ matrices $X = (x_{ij})$ with ${}^tXSX = S$ and $\det X = 1$. For $X \in \mathrm{M}_n(\mathbb{R})$, let ${}^tXSX = T = (t_{ij})$ and let $t_{ij}, 1 \leqslant i \leqslant j \leqslant m$ be the coordinate functions of the $n(n+1)/2-$dimensional affine space of all $n \times n$ symmetric matrices. Then the t_{ij} are polynomials of x_{ij}, so one has a $n(n-1)/2-$form $\bar{\omega}$ such that

$$\wedge_{i,j=1}^n dx_{ij} = \wedge_{i \leqslant j} dt_{ij} \wedge \bar{\omega}. \tag{16.1.1}$$

One has the injection map $\iota : G \to \mathrm{M}_n(\mathbb{R})$, so that $\iota^*(\bar{\omega}) = \omega$ is a $n(n-1)/2-$form on G.

16.2. As allready used above in (12.6.2), in [BY], Bruinier and Yang extending part of Section 5.3 from [KRY] propose a kind of refinement which comes down to the following. Let F be a (local) field and $V = F^n$ with a quadratic form $q(x) = (1/2)^txSx$ of signature (p,q). Let α be a gauge form for V^{n-1}, i.e., a highest order exterior differential form

$$\alpha = \wedge_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n-1} dx_{ij} \in (\wedge^{n(n-1)} V^{n-1})^* \tag{16.2.1}$$

and β a gauge form for $\mathrm{Sym}_{n-1}(F)$, i.e.,

$$\beta = \wedge_{1 \leqslant i < j \leqslant n-1} dt_{ij} \in (\wedge^{n(n-1)/2} \mathrm{Sym}_{n-1}(\mathbb{R}))^*. \tag{16.2.2}$$

Take

$$Q : V^{n-1} \to \mathrm{Sym}_{n-1}(F), \quad x \mapsto (1/2)((x_i, x_j)) \tag{16.2.3}$$

and let V_{reg}^{n-1} be the subset of $x \in V^{n-1}$ with $\det Q(x) \neq 0$, and $\mathrm{Sym}_{n-1}^{\mathrm{reg}}(F)$ be the subset of $T \in \mathrm{Sym}_{n-1}(F)$ with $\det T \neq 0$. Then Q reduces to a regular map from V_{reg}^{n-1} to $\mathrm{Sym}_{n-1}^{\mathrm{reg}}(F)$.

Fix an $x = (x_1, \dots, x_{n-1}) \in V_{\mathrm{reg}}^{n-1}$ with $Q(x) = T$ and identify the tangent space $T_x(V_{\mathrm{reg}}^{n-1})$ with V^{n-1}. Then the differential dQ_x is given by

$$dQ_x(v) = (1/2)((x, v) + (v, x)) \in \mathrm{Sym}_{n-1}(F), \quad v \in V^{n-1}.$$

Let

$$j_x : \mathrm{Sym}_{n-1}(F) \to V^{n-1}, \quad u \mapsto j_x(u) = (1/2)xQ(x)^{-1}u. \tag{16.2.4}$$

Then one has $dQ_x \circ j_x(u) = u$ and the decomposition

$$T_x(V^{n-1}_{\mathrm{reg}}) = \mathrm{Im}(j_x) \oplus \ker(dQ_x).$$

Choose $u = (u_1, \ldots, u_{n(n-1)/2}) \in \mathrm{Sym}^{\mathrm{reg}}_{n-1}(F)$ with $\beta(u) \neq 0$ and define a $n(n-1)/2-$form $\nu \in ((\wedge^{n(n-1)/2} V^{n-1})^*$ as follows: for $t = (t_1, \ldots, t_{n(n-1)/2}) \in V^{n(n-1)/2}$, put

$$\nu(t) := \alpha(j_x(u), t)\beta(u)^{-1}. \tag{16.2.5}$$

One can verify that this ν is independent of u and (see [KRY] Lemma 5.3.1) that one has

$$\alpha = Q^*(\beta) \wedge \nu \quad \text{and} \quad \nu = (h, g)^*\nu \tag{16.2.6}$$

for $h \in \mathrm{SO}(V)$ and $g \in \mathrm{GL}_{n-1}$, where $\mathrm{SO}(V) \times \mathrm{GL}_{n-1}$ acts on V^{n-1} via $(h, g)x = hxg^{-1}$. One may identity $\ker dQ_x$ with the tangent space $T_x(Q^{-1}(T))$ of $Q^{-1}(T)$, and, hence, ν defines a gauge form on $Q^{-1}(T)$. Finally, using the isomorphism

$$i_x : \mathrm{SO}(V) \to Q^{-1}(T), \quad h \mapsto i_x(h) = hx \tag{16.2.7}$$

one obtains a gauge form $i_x^*(\nu)$ on $\mathrm{SO}(V)$ which is again denoted by ν. This form does not depend on x or T and gives a Haar measure $dh = d_\nu h$ on $H = \mathrm{SO}(V)$. In [BY] Section 2.3, this is made more explicit: Let $\mathbf{e} = (e_1, \ldots, e_n)$ be an ordered basis of V and $J := Q(\mathbf{e}) = (1/2)((e_i, e_j)) \in \mathrm{Sym}_n(F)$. Let E_{ij} denote a matrix whose $(ij)-$entry is one and all other entries are zero. Then one has $V^{n-1} = \langle E_{ij}, 1 \leqslant i \leqslant n; 1 \leqslant j \leqslant n-1\rangle$. Let de_{ij} be the dual basis and $\alpha = \wedge_{ij} de_{ij}$ with

$$\alpha((E_{ij})) = \alpha(E_{11}, E_{12}, \ldots, E_{n,n-1}) = 1.$$

Hence, $Y_{ij} = E_{ij} + E_{ji}; 1 \leqslant i \leqslant j \leqslant n-1$ is a basis of $\mathrm{Sym}_{n-1}(F)$. Let dy_{ij} be the dual and $\beta = \wedge_{ij} dy_{ij}$ with

$$\beta(Y_{11}, Y_{12}, \ldots, Y_{n-1,n-1}) = 1.$$

Now, assume $J = \mathrm{diag}(a_1, \ldots, a_n)$ and let be $X_{ij} := a_j E_{ij} - a_i E_{ji}$. Hence, one has $\mathfrak{h} = \mathfrak{so}(V) = \langle X_{ij}; 1 \leqslant i < j \leqslant n\rangle$ and ([BY] Prop. 2.5)

$$\nu(X_{12}, X_{13}, \ldots, X_{n-1,n}) = \pm 1. \tag{16.2.8}$$

In section 3, Bruinier and Yang proceed as follows. Let V have signature $(m, 2)$ with Witt decomposition $V = V_0 + e\mathbb{R} + f\mathbb{R}$, e, f isotropic with $(e, f) = 1$. Let $\mathbb{D}$ be the Grassmannian of oriented negative 2-planes in V and

$$\mathcal{H} = \{z = x + iy \in V_{0,\mathbb{C}};\ (y, y) < 0\}$$

the associated tube domain where $G = \mathrm{SO}(V)$ acts as

$$gw(z) = j(w, z) \cdot w(gz), g \in G, w(z) = z + e - q(z)f \in V_{\mathbb{C}}. \tag{16.2.9}$$

Hence, the first Chern form of the dual of the tautological bundle over $\mathbb{D}$ is the (1,1)-form on $\mathbb{D} \simeq \mathcal{H}$

$$\Omega = dd^c \log(-(y,y)) \tag{16.2.10}$$

as in [Ku1] Prop.4.11, given by

$$\Omega = -\frac{1}{2\pi i}\Big(-\frac{(y,dz)\wedge(y,d\bar{z})}{(y,y)^2} + \frac{(dz,d\bar{z})}{2(y,y)}\Big). \tag{16.2.11}$$

As in [Ku1] (5.8) or [BK] (4.50), the measure $d\mu(z)$ on $\mathbb{D} \simeq \mathcal{H}$ is given by Ω^3 with

$$\Omega^3 = -\frac{3}{16\pi^3}\det(y)^{-3}\Big(\frac{i}{2}\Big)^3 dz_1\wedge d\bar{z}_1\wedge dz_2\wedge d\bar{z}_2\wedge dz_3\wedge d\bar{z}_3. \tag{16.2.12}$$

In the paper [BY] by Bruinier and Yang, there is another formula relating differential forms and measures. From Proposition 3.4 in [BY], we have

$$(d\ell_x)^*(-\Omega)^m = \pm\frac{m!}{(2\pi)^m}\nu_{\mathfrak{p}} \tag{16.2.13}$$

Here, fixing a base point $z \in \mathcal{H}$, in [BY] (3.10) and (3.11) one has explicit isomorphisms $\ell_z : G(\mathbb{R})/K_\infty \simeq \mathcal{H}$ and $d\ell_z : \mathfrak{p} \simeq V_{0,\mathbb{C}}$ inducing

$$(d\ell_x)^*(dx_2\wedge dy_2\wedge\cdots\wedge dx_{n-1}\wedge dy_{n-1}) = \pm\nu_{\mathfrak{p}}. \tag{16.2.14}$$

$\nu_{\mathfrak{p}}$ is part of the decomposition $\nu = \nu_+\wedge\nu_-\wedge\nu_{\mathfrak{p}}$ corresponding to the decomposition

$$\mathfrak{g} = \mathfrak{k}_+ \oplus \mathfrak{k}_- \oplus \mathfrak{p}. \tag{16.2.15}$$

To get back to our case, we look at:

16.3. The Lie algebra $\mathfrak{g} = \mathfrak{so}(3,2)$. We have

$$\begin{aligned}\mathfrak{g} &= \mathfrak{k} + \mathfrak{p}\\ &= \{\begin{pmatrix} A & \\ & B\end{pmatrix}, A \in M_3(\mathbb{R})\,\text{skew}, B \in M_2(\mathbb{R})\,\text{skew}\} + \{\begin{pmatrix} & C\\ {}^tC & \end{pmatrix}, C \in M_{3,2}(\mathbb{R})\}\\ &= \langle X_{1,2}, X_{1,3}, X_{2,3}, X_{4,5}\rangle + \langle X_{1,4}, X_{1,5}, X_{2,4}, X_{2,5}, X_{3,4}, X_{3,5}\rangle\end{aligned} \tag{16.3.1}$$

Here we put $X_{12} = E_{12} - E_{21}$ etc. and $X_{14} = E_{14} + E_{41}$ etc,, i.e., the negative of the X_{ij} from [BY] above.

There are the Lie relations

$$
\begin{aligned}
&[X_{1,2}, X_{1,3}] = -X_{2,3}, \quad [X_{1,2}, X_{2,3}] = X_{1,3},\\
&[X_{1,2}, X_{1,4}] = -X_{2,4}, \quad [X_{1,2}, X_{2,4}] = X_{1,4},\\
&[X_{1,2}, X_{1,5}] = -X_{2,5}, \quad [X_{1,2}, X_{2,5}] = X_{1,5},\\
&[X_{1,2}, X_{3,4}] = 0, \quad [X_{1,2}, X_{3,5}] = 0, \quad [X_{1,2}, X_{4,5}] = 0,\\
&[X_{1,3}, X_{1,4}] = -X_{3,4}, \quad [X_{1,3}, X_{2,4}] = 0,\\
&[X_{1,3}, X_{1,5}] = -X_{3,5}, \quad [X_{1,3}, X_{2,5}] = 0,\\
&[X_{1,3}, X_{3,4}] = X_{1,4}, \quad [X_{1,3}, X_{3,5}] = X_{1,5}, \quad [X_{1,3}, X_{4,5}] = 0,\\
&[X_{1,4}, X_{1,5}] = X_{4,5}, \quad [X_{1,4}, X_{2,3}] = 0,\\
&[X_{1,4}, X_{2,4}] = X_{1,2}, \quad [X_{1,4}, X_{2,5}] = 0,\\
&[X_{1,4}, X_{3,4}] = X_{1,3}, \quad [X_{1,4}, X_{3,5}] = 0, \quad [X_{1,4}, X_{4,5}] = X_{1,5},\\
&[X_{1,5}, X_{2,3}] = 0, \quad [X_{1,5}, X_{3,4}] = 0,\\
&[X_{1,5}, X_{2,4}] = 0, \quad [X_{1,5}, X_{3,5}] = X_{1,3},\\
&[X_{1,5}, X_{2,5}] = X_{1,2}, \quad [X_{1,5}, X_{4,5}] = -X_{1,4},\\
&[X_{2,3}, X_{2,4}] = -X_{3,4}, \quad [X_{2,3}, X_{2,5}] = -X_{3,5},\\
&[X_{2,3}, X_{3,4}] = X_{2,4}, \quad [X_{2,3}, X_{3,5}] = X_{2,5}, \quad [X_{2,3}, X_{4,5}] = 0,\\
&[X_{2,4}, X_{3,4}] = X_{2,3}, \quad [X_{2,4}, X_{3,5}] = 0,\\
&[X_{2,4}, X_{2,5}] = X_{4,5}, \quad [X_{2,4}, X_{4,5}] = X_{2,5},\\
&[X_{2,5}, X_{3,4}] = 0, \quad [X_{2,5}, X_{3,5}] = X_{2,3}, \quad [X_{2,5}, X_{4,5}] = -X_{2,4}\\
&[X_{3,4}, X_{3,5}] = X_{4,5}, \quad [X_{3,4}, X_{4,5}] = X_{3,5}, \quad [X_{3,5}, X_{4,5}] = -X_{3,4}.
\end{aligned}
\tag{16.3.2}
$$

Here $\mathfrak{k}, \mathfrak{p}$ are the ± 1 eigenspaces of the Cartan involution τ with $\tau X = -{}^t X$, Moreover, we look at three involutions σ commuting with τ,

$$
\begin{aligned}
\sigma_1 X &= \begin{pmatrix} 1 & & & & 0 \\ & -1 & & & \\ & & -1 & & \\ & & & -1 & \\ 0 & & & & -1 \end{pmatrix} X \begin{pmatrix} 1 & & & & 0 \\ & -1 & & & \\ & & -1 & & \\ & & & -1 & \\ 0 & & & & -1 \end{pmatrix}, \ (\textbf{case I})\\
\sigma_3 X &= \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & -1 & & \\ & & & 1 & \\ 0 & & & & 1 \end{pmatrix} X \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & -1 & & \\ & & & 1 & \\ 0 & & & & 1 \end{pmatrix}, \ (\textbf{case I'})\\
\sigma_5 X &= \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ 0 & & & & -1 \end{pmatrix} X \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ 0 & & & & -1 \end{pmatrix}, \ (\textbf{case II})
\end{aligned}
\tag{16.3.3}
$$

$\mathfrak{h}, \mathfrak{q}$ denote (respectively) the ± 1 eigenspaces of the involution σ, $\mathfrak{b}$ a maximal abelian subalgebra of $\mathfrak{p} \cap \mathfrak{q}$, and $\mathfrak{m}$ the centralizer of $\mathfrak{b}$ in $\mathfrak{l} = \mathfrak{k} \cap \mathfrak{h}$ and $\mathfrak{l}'$ the orthogonal complement of $\mathfrak{m}$ in $\mathfrak{l}$.

Hence, we have $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$

$$\mathfrak{k} = \langle X_{1,2}, X_{1,3}, X_{2,3}, X_{4,5}\rangle \; \mathfrak{p} = \langle X_{1,4}, X_{1,5}, X_{2,4}, X_{2,5}, X_{3,4}, X_{3,5}\rangle \tag{16.3.4}$$

and $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ for the respective cases I,I' and II

$$\begin{array}{rclcrcl} \mathfrak{h} & = & \langle X_{2,3}, X_{4,5}, X_{2,4}, X_{2,5}, X_{3,4}, X_{3,5}\rangle & \mathfrak{q} & = & \langle X_{1,2}, X_{1,3}, X_{1,4}, X_{1,5}\rangle, \\ & = & \langle X_{1,2}, X_{4,5}, X_{1,4}, X_{1,5}, X_{2,4}, X_{2,5}\rangle & & = & \langle X_{1,3}, X_{2,3}, X_{3,4}, X_{3,5}\rangle, \\ & = & \langle X_{1,2}, X_{1,3}, X_{2,3}, X_{1,4}, X_{2,4}, X_{3,4}\rangle & & = & \langle X_{1,5}, X_{2,5}, X_{3,5}, X_{4,5}\rangle, \end{array} \tag{16.3.5}$$

with

$$\begin{array}{rclrclrclrcl} \mathfrak{l} = \mathfrak{k}\cap\mathfrak{h} & = & \langle X_{2,3}, X_{4,5}\rangle & \mathfrak{p}\cap\mathfrak{q} & = & \langle X_{1,4}, X_{1,5}\rangle, & \mathfrak{b} & = & \langle X_{1,5}\rangle, & \mathfrak{m} & = & \langle X_{2,3}\rangle, \\ & = & \langle X_{1,2}, X_{4,5}\rangle, & & = & \langle X_{3,4}, X_{3,5}\rangle, & & = & \langle X_{3,5}\rangle & & = & \langle X_{1,2}\rangle, \\ & = & \langle X_{1,2}, X_{1,3}, X_{2,3}\rangle, & & = & \langle X_{1,5}, X_{2,5}, X_{3,5}\rangle, & & = & \langle X_{1,5}\rangle, & & = & \langle X_{2,3}\rangle \end{array}$$

and

$$\begin{array}{rclrcl} \mathfrak{h}\cap\mathfrak{p} & = & \langle X_{2,4}, X_{2,5}, X_{3,4}, X_{3,5}\rangle, & \mathfrak{l}' & = & \langle X_{4,5}\rangle, \\ & = & \langle X_{1,4}, X_{1,5}, X_{2,4}, X_{2,5}\rangle, & & = & \langle X_{4,5}\rangle, \\ & = & \langle X_{1,4}, X_{2,4}, X_{3,4}\rangle, & & = & \langle X_{1,2}, X_{1,3}\rangle. \end{array}$$

Corresponding groups are $G = \mathrm{SO}(3,2), K = \mathrm{SO}(3)\times\mathrm{SO}(2)$ and

$$\begin{array}{rclrclrcl} H & = & \mathrm{SO}(2,2), & L & = & \mathrm{SO}(2)\times\mathrm{SO}(2), & L/M & \simeq & \mathrm{SO}(2) \\ & = & \mathrm{SO}(2,2), & & = & \mathrm{SO}(2)\times\mathrm{SO}(2), & & \simeq & \mathrm{SO}(2) \\ & = & \mathrm{SO}(3,1), & & = & \mathrm{SO}(3), & & \simeq & \mathrm{SO}(3)/\mathrm{SO}(2). \end{array}$$

16.4. Guided by Heckman-Schlichtkrull [HS] p.109f, we also look at the following (here we restrict to the cases I and II). Take

$$\begin{array}{rclcl} \mathfrak{g}_+ & = & \mathfrak{k}\cap\mathfrak{h} + \mathfrak{p}\cap\mathfrak{q} & = & \langle X_{2,3}, X_{4,5}\rangle + \langle X_{1,4}, X_{1,5}\rangle, \\ & & & = & \langle X_{1,2}, X_{1,3}, X_{2,3}\rangle + \langle X_{1,5}, X_{2,5}, X_{3,5}\rangle, \\ \mathfrak{g}_- & = & \mathfrak{k}\cap\mathfrak{q} + \mathfrak{p}\cap\mathfrak{h} & = & \langle X_{1,2}, X_{1,3}\rangle + \langle X_{2,4}, X_{2,5}, X_{3,4}, X_{3,5}\rangle, \\ & & & = & \langle X_{4,5}\rangle + \langle X_{1,4}, X_{2,4}, X_{3,4}\rangle, \end{array}$$

and, for $\alpha \in \mathfrak{b}^*$, with $\mathfrak{g}_\alpha = \{Y \in \mathfrak{g}; [B,Y] = \alpha(B)Y \text{ for all } B \in \mathfrak{b}\}$ one has in case I

$$\begin{aligned} \mathfrak{g}_0 &= \langle X_{2,3}, X_{1,5}, X_{2,4}, X_{3,4}\rangle \\ \mathfrak{g}_1 &= \langle X_{1,2}+X_{2,5}, X_{1,3}+X_{3,5}, X_{4,5}-X_{1,4}\rangle \\ \mathfrak{g}_{-1} &= \langle X_{1,2}-X_{2,5}, X_{1,3}-X_{3,5}, X_{4,5}+X_{1,4}\rangle. \end{aligned} \tag{16.4.1}$$

We decompose $\mathfrak{g}_\alpha = \mathfrak{g}_\alpha^+ \oplus \mathfrak{g}_\alpha^-$ where $\mathfrak{g}_\alpha^\pm = \mathfrak{g}_\alpha \cap \mathfrak{g}_\pm$, put $m_\alpha^\pm = \dim \mathfrak{g}_\alpha^\pm$ and

$$J(Y) = \prod_{\alpha\in\Sigma^+(\mathfrak{b},\mathfrak{g})} \sinh^{m_\alpha^+}\alpha(Y)\cosh^{m_\alpha^-}\alpha(Y) \tag{16.4.2}$$

Hence, in our situation we get

$$\begin{aligned} J(B) &= \sinh t \cosh^2 t \quad \text{in Case I,} \\ &= \sinh^2 t \cosh t \quad \text{in Case II.} \end{aligned} \tag{16.4.3}$$

16.5. For $\mathfrak{a} := \langle X_{1,5}, X_{24} \rangle$ and $\Psi := \Delta(\mathfrak{a}, \mathfrak{g})$ one has the root spaces $\mathfrak{g}_{\alpha_j} = G_j \mathbb{R}$

$$\begin{aligned} G_{1,5} &= X_{1,3} \pm X_{3,5}, \quad G_{3,7} = X_{2,3} \pm X_{3,4}, \\ G_{2,4} &= X_{1,2} \pm X_{2,5} \pm X_{4,5} - X_{1,4}, \\ G_{3,8} &= X_{1,2} \pm X_{2,5} \mp X_{4,5} + X_{1,4}, \end{aligned} \tag{16.5.1}$$

i.e., one has the roots $\Psi = \{(0, \pm 1), (\pm 1, 0), (1, \pm 1), (-1, \pm 1)\}$.

16.6. For a moment, we stay with [HS]. As a refinement of the Cartan decomposition $G = K \exp \mathfrak{p}$ from Heckman-Schlichtkrull [HS] Theorem 2.4, we have the decomposition

$$G = KA^+H, \quad A = \exp \mathfrak{b}^+ \tag{16.6.1}$$

which also appears in Kudla [Ku1] p.318. As already remarked above, this formula looks as if the right hand side doesn't cover the elements of the left hand side, i.e., in our Case I, $\{\exp tX_{1,4};\ t \in \mathbb{R}\}$ does not appear in one of the three factors on the right hand side but one easily verifies $\mathrm{Ad}(\ell)\exp X_{1,5} = \exp X_{1,4}$, for $\ell = \{\begin{pmatrix} 1 &&&& \\ & 1 &&& \\ && 1 && \\ &&& 0 & -1 \\ &&& 1 & 0 \end{pmatrix} \in K$. Similarly in Case II, on the Lie algebra level, one has on the right hand side

$$\langle X_{12}, X_{13}, X_{23}, X_{45} \rangle,\ \langle X_{15} \rangle,\ \langle X_{12}, X_{13}, X_{23}, X_{14}, X_{24}, X_{34} \rangle$$

but one has $[X_{1,2}, X_{1,5}] = -X_{2,5}$ and $[X_{1,3}, X_{1,5}] = -X_{3,5}$.

Oda-Tsuzuki's integral formula

16.7. In [OT] p.49, we find the relation

$$\mathfrak{g} = (\mathrm{Ad}\, a_t)\mathfrak{h} + \mathbb{R}X_{15} + \mathfrak{k}, \quad a_t = \exp tX_{15}. \tag{16.7.1}$$

There is still more background. Oda and Tsuzuki [OT] have the fundamental integration formula (1.3.2)

$$\begin{aligned} \int_G \varphi(g)dg &= \int_H d\mu_H(h) \int_K dk \int_0^\infty \varphi(h \exp(tY_0)k)\gamma_{H\backslash G}(t)dt, \\ \gamma_{H\backslash G}(t) &= (\sinh(t))^{m_\lambda^+}(\cosh(t))^{m_\lambda^-}(2^{-1}\sinh(2t))^{m_{2\lambda}^+}(\cosh(2t))^{m_{2\lambda}^-} \end{aligned} \tag{16.7.2}$$

for which they refer to Heckmann-Schlichtkrull [HS] p.110,Th.2.5. This theorem says that an invariant measure dx on $X = G/H$ is given by

$$\int_X f(x)dx = \int_K \int_{\mathfrak{a}_q^+} f(k \exp(Y) \cdot o) J(Y) dY dk \tag{16.7.3}$$

where $J(Y)$ is given by (16.4.2) is the γ in the formula above. There is no proof of the general result in [HS] but the discussion of the example $G = \mathrm{SO}_e(p,q), H = \mathrm{SO}_e(p,q-1)$, i.e.,

$$X = G/H = \{x \in \mathbb{R}^{p+q} : \, x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 = 1\}.$$

The Lebesgue measure $dx = dx_1 \ldots dx_{p+q}$ on $\mathbb{R}^{p+q}$ is $G-$invariant. One takes polar coordinates $(v,r) \in \mathbb{S}^{p-1} \times \mathbb{R}^+$ and $(w,s) \in \mathbb{S}^{q-1} \times \mathbb{R}^+$ on the first p and last q entries, respectively, to get

$$dx = dvdwr^{p-1}drs^{q-1}ds,$$

where dv and dw are rotation invariant measures on the two spheres. Restricting to the open set where $r > s$ one can write the pair (r,s) as $(\xi\cosh t, \xi\sinh t)$, and one has $drds = \xi d\xi dt$. Hence,

$$dx = dvdw\xi^{p+q-1}d\xi \cosh^{p-1}t \sinh^{q-1}tdt,$$

and as X is given by $\xi = 1$, we get that the measure

$$dvdw \cosh^{p-1}t \sinh^{q-1}tdt$$

is invariant on X and this is said to be in accordance with the theorem above.

16.8. Another way to the proof of Theorem 2.5 in [HS] is indicated by their Example 2.2. For

$$\begin{aligned} X &= \mathrm{SO}_e(p,q)/\mathrm{SO}_e(p-1,q) \\ &= \{x \in \mathbb{R}^{p+q}; x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 = 1\}, x_1 > 1 \text{ if } p = 1, \end{aligned} \tag{16.8.1}$$

$\mathfrak{b} = \mathbb{R}X_{1,p+q}$ is maximal abelian in $\mathfrak{p} \cap \mathfrak{q}$ and the centralizer $C_{K\cap H}(\mathfrak{b})$ of $X_{1,p+q}$ in $K \cap H$ consists of elements of the form

$$\begin{pmatrix} 1 & & & \\ & V & & \\ & & W & \\ & & & 1 \end{pmatrix}, V \in \mathrm{SO}(p-1), W \in \mathrm{SO}(q-1).$$

Hence $K/C_{K\cap H}(\mathfrak{b})$ can be identified with $\mathbf{S}^{p-1} \times \mathbf{S}^{q-1}$ and one has a polar coordinate map Φ given by

$$\begin{aligned} \mathbf{S}^{p-1} \times \mathbf{S}^{q-1} \times \mathbb{R} = K/C_{K\cap H}(\mathfrak{b}) \times \mathbb{R} \to X \\ (v,w,t) \mapsto (v_1 \mathrm{ch}\, t, \ldots, v_p \mathrm{ch}\, t, w_1 \mathrm{ch}\, t, \ldots, w_q \mathrm{ch}\, t) \end{aligned} \tag{16.8.2}$$

This leads to a transformation formula

$$\int_X f(x)dx = \int_{K/C_{K\cap H}(\mathfrak{b})} \int_{\mathfrak{b}} f(\dot{k}\exp B \cdot o)J(v,w,t)dtd\dot{k}, \tag{16.8.3}$$

where $J(v,w,t)$ is the functional determinant.

16.9. Remark. This is still not the formula appearing in [HS] Theorem 2.5 where the integral on the right is done over K. (I would like to know how this is done.)

Tsuzuki's Formula.

16.10. Finally, in a paper by Tsuzuki [Ts], we found a refinement of the fundamental integration formula from [OT] (1.3.2) (here cited by (16.7.2)) resp. Kudla's Green-integral formula (12.1.6). We take over from [Ts] Section 4. Let Γ be a discrete subgroup of $G = \mathrm{U}(n,1)$ and $H \subset G, K_H = H \cap K$, such that H/K_H is of codimension r in G/K, and $\Gamma_H = \Gamma \cap H$. Moreover, let dk and dk_0 be the Haar measures of the compact groups K and K_H with total volume 1. There is a unique Haar measure dg on G such that the quotient measure dg/dk corresponds to the measure on the symmetric space G/K determined by the invariant volume form vol. Define dh on H analogously: dh/dk_0 corresponds to the measure on H/K_H determined by vol_H. Then Tsuzuki has his
Lemma 4.1. For any measurable function f on G we have

$$\int_G f(g)dg = \int_H dh \int_K dk \int_0^\infty f(ha_tk)\rho(t)dt \tag{16.10.1}$$

with dt the usual Lebesgue measure on $\mathbb{R}$ and

$$\rho(t) = 2c_r(\mathrm{sinh}t)^{2r-1}(\mathrm{cosh}t)^{2n-2r+1}, \; c_r = \pi^r/(r-1)!.$$

Proof: For closed subgroups $Q_1 \subset Q_2$ of G with Lie algebras $\mathfrak{q}_i, i = 1,2$ regard $(\mathfrak{q}_2/\mathfrak{q}_1)^* \subset \mathfrak{q}^+$ by the dual map of the orthogonal projection $\mathfrak{q} \to \mathfrak{q}_2$ and the canonical surjection $\mathfrak{q}_2 \to \mathfrak{q}_2/\mathfrak{q}_1$. Let $\mathrm{vol}_{\mathfrak{q}_2/\mathfrak{q}_1}$ be the element $\xi_1 \wedge \cdots \wedge \xi_s \in \wedge(\mathfrak{q}_2/\mathfrak{q}_1)^*$ with (ξ_i) any orthogonal basis of $(\mathfrak{q}_2/\mathfrak{q}_1)^*$. Assume Q_1 is compact, then there exists a unique left Q_2-invariant $s-$form Z_{Q_2/Q_1} on Q_2/Q_1 whose value at $o = eQ_2$ is $\mathrm{vol}_{\mathfrak{q}_2/\mathfrak{q}_1}$. Let dZ_{Q_2/Q_1} be the Q_2-invariant measure on Q_2/Q_1 corresponding to Z_{Q_2/Q_1}. For example $\mathrm{vol}_{\mathfrak{g}/\mathfrak{k}} = \mathrm{vol}$ and $\mathrm{vol}_{\mathfrak{h}/\mathfrak{m}} = \mathrm{vol}_H \wedge \mathrm{vol}_{\mathfrak{h}\cap\mathfrak{k}/\mathfrak{m}}$.

From [HS] Theorem 2.4 p.108 resp. [FlJ] p.262, one takes that the decomposition $G = HAK$ jields a diffeomorphism

$$j : H/M \times (0,\infty) \to (G - HK)/K, \quad (\dot{h}, t) \mapsto ha_tK \tag{16.10.2}$$

Here, for $G = U(n,1)$, from [Ts] p. 314, one has

$$\begin{aligned} A &= \{a_t = \exp(tY_0) = \mathrm{diag}(I_{n-1}, \left(\begin{smallmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{smallmatrix}\right)), t \in \mathbb{R}\} \\ M &= \{\mathrm{diag}(u_1, u_2, u_0, u_0); u_1 \in \mathrm{U}(n-r), u_2 \in \mathrm{U}(r-1), u_0 \in \mathrm{U}(1)\} \end{aligned} \tag{16.10.3}$$

Analyzing the differential forms resp. the Lie algebra relations, one can prove

$$j^* Z_{G/K} = 2(\sinh t)^{2r-1}(\cosh t)^{2n-2r+1} Z_{H/M} \wedge dt. \tag{16.10.4}$$

Using j as an identification, we then have for the corresponding measures

$$dg/dk = dZ_{G/K}(\dot{g}) = 2(\sinh t)^{2r-1}(\cosh t)^{2n-2r+1} dZ_{H/M}(\dot{h})dt. \tag{16.10.5}$$

Let dm be the Haar measure of M with total volume one. The resulting quotient measure dh/dm is proportional to $dZ_{H/M}(\dot{h})$, i.e.,

$$dh/dm = C_0 dZ_{H/M}(\dot{h}) \tag{16.10.6}$$

Since $dZ_{H/H\cap K}(\dot{h}) = dh/dk_0$, one has

$$dZ_{H/M}(\dot{h}) = dh/dk_0 \cdot dZ_{H\cap K/M}(\dot{k_0})$$

and, using $dh/dm = (dh/dk_0)(dk_0/dm)$, one comes to

$$dk_0/dm = C_0 dZ_{H\cap K/M}(\dot{k_0}). \tag{16.10.7}$$

From $1 = \int_{K\cap H} dk_0 = \int_{K\cap H/M} dk_0/dm \int_M dm$, one has $\int_{K\cap H/M} dk_0/dm = 1$ and obtains

$$C_0^{-1} = \int_{K\cap H/M} dZ_{H\cap K/M}(\dot{k_0}). \tag{16.10.8}$$

To compute this integral, use the diffeomorphism $K\cap H/M \mapsto \mathbb{S}^{2r-1}$ to get $C_0^{-1} = \pi^r/\Gamma(r)$. Putting all this skillfully together should lead to the above proposed formula:

(16.10.9)

$$\begin{aligned}
\int_G f(g)dg &= \int_{G/K} dg/dk \int_K dk f(\dot{g}k) \\
&= \int_{G/K} dZ_{G/K}(\dot{g}) \int_K dk f(\dot{g}k) \\
&= \int_{H/M} dZ_{H/M}(\dot{h}) \int_K dk f(ha_t k) \int_0^\infty 2(\sinh t)^{2r-1}(\cosh t)^{2n-2r+1} dt \\
&= \int_{H/M} dh/dm(\dot{h})(1/C_0) \int_K dk \int_0^\infty f(ha_t k) 2(\sinh t)^{2r-1}(\cosh t)^{2n-2r+1} dt.
\end{aligned}$$

16.11. Remark. For $G = \mathrm{SO}(3,2)$ and $H = \mathrm{SO}(2,2)$ (Case I) resp. $= \mathrm{SO}(3,1)$ (Case II) the same procedure leads to

$$\int_G f(g)dg = \int_H dh \int_K dk \int_0^\infty f(ha_t k)\rho(t)dt \tag{16.11.1}$$

with

$$\rho(t) = 2(\sinh t)(\cosh t)^2 2\pi, \text{Case I}, \qquad (16.11.2)$$
$$= 2(\sinh t)^2(\cosh t)4\pi, \text{Case II}.$$

Namely, here one has $K \cap H/M = (\mathrm{SO}(2)\times\mathrm{SO}(2))/\mathrm{SO}(2)$ resp. $= \mathrm{SO}(3)/\mathrm{SO}(2)$, i.e.,$c = 2\pi$, resp. $= 4\pi$.

From p.313 in [Ts], in our case we have $dg/dk = \mathrm{vol} = (1/3!)\omega^3$ (and $dh/dk_0 = \mathrm{vol}_H = (1/2!)\omega_H$. for Case II). If we apply the fomula (16.11.2) with these normalizations to Kudla's formula for the Green integral, we get the formulae used above in the main text.

Flensted-Jensen's integration formula

16.12. The decomposition formula (16.6.1) is also the background for Flensted-Jensen's important integration formula. To be careful, we reproduce still more from Section 2 of [FlJ]. There the Killing form defines Riemannian (i.e., Euclidean) structures on $\mathfrak{p} \cap \mathfrak{h}, \mathfrak{b}^+$, and L/M, and one lets the measure on L/M be $\mathrm{vol}(L/M)^{-1}$ times the volume element. Via Killing form, one has Riemannian structures on G/K and H/L, and by their volume elements also measures.
Remark. $\mathfrak{g} = \mathfrak{so}(p, q)$ has the Killing form

$$B(X; Y) = (p + q - 2)\,\mathrm{tr}\,(XY). \qquad (16.12.1)$$

Hence, for $(p, q) = (3, 2)$), and the X_{ij} from above, we have $B(X_{ij}, X_{ij}) = 6$.
Moreover, take measures on G and H such that

$$\int_G f(x)dx = \int_{G/K}\int_K f(xk)dkdxK, \quad \int_K dk = 1, \quad \text{for } f \in C_c(G)$$
$$\int_H f(x)dx = \int_{H/L}\int_L f(xk)dkdxL, \quad \int_L dk = 1, \quad \text{for } f \in C_c(H). \qquad (16.12.2)$$

One has the standard diffeomorphism ([FlJ] (2.1))

$$\Phi_0 : \mathfrak{p} \cap \mathfrak{h} \times \mathfrak{p} \cap \mathfrak{q} \times K \to G, \quad (X, Y, k) \mapsto \exp X \cdot \exp Y \cdot k. \qquad (16.12.3)$$

In [FlJ] p.261, from Helgason ([He] X Lemma 1.16) one has that the map

$$\Psi : L/M \times \mathfrak{b} \to \mathfrak{p} \cap \mathfrak{q}, \quad (lM, B) \mapsto \mathrm{Ad}(l)B \qquad (16.12.4)$$

is a diffeomorphism onto an open dense set. Therefore, the maps

$$\Phi_1 : \mathfrak{p} \cap \mathfrak{h} \times L/M \times \mathfrak{b} \times K \to G,\ (X, lM, B) \mapsto (\exp X\, l \exp B), \qquad (16.12.5)$$

and

$$\Phi : \mathfrak{p} \cap \mathfrak{h} \times L/M \times \mathfrak{b} \to G/K \tag{16.12.6}$$

given by

$$\Phi(X, lM, B) = \pi(\exp X \, l \exp B),$$

where $\pi : G \to G/K$ is the canonical map, are diffeomorphisms unto open dense sets.

Taking the Jacobians $J(X, lM, B) = |\det d\Phi_{(X,lM,B)}|$ and $J_1(X) = |\det d\Phi^1_{(X)}|$ with reference to the respective Riemannian structures, one has for $f \in C_c(G)$ and $f_1 \in C_c(H)$

$$\int_{H/M} f_1(x)dx = \int_{\mathfrak{p}\cap\mathfrak{h}} \int_{L/M} f_1(\exp x \, lM) J_1(X) dlM dX$$

$$\int_{G/K} f(x)dx = \mathrm{vol}\,(L/M) \int_{\mathfrak{p}\cap\mathfrak{h}} \int_{L/M} \int_{\mathfrak{b}^+} f(\Phi(X, lM, B)) J(X, lM, B) dB dlM dX$$

$$= \mathrm{vol}\,(L/M) \int_{H/M} \int_{\mathfrak{b}^+} f(h \exp B)\delta_1(B) dB dh \tag{16.12.7}$$

where $\delta_1(B) = |\det d\Phi_{(0,eM,B)}|$, $B \in \mathfrak{b}^+$. From here (his formula (2.9)), Flensted-Jensen comes to the formula (2.14) in his Theorem 2.6

$$\int_G f(x)dx = \mathrm{vol}\,(L/M) \int_K \int_H \int_{\mathfrak{b}^+} f(k \exp Bh)\delta(B) dB dh dk \quad \text{for } f \in C_c(G)$$

where δ given by Flensted-Jensen's formula (2.12) comes from the δ_1.
We observe that in this formula the number of variables to be integrated may be different on both sides. For our case A of SO(3, 2) we have 10 on the left and 11 on the right. By private mail of Flensted-Jensen this is explained as follows. As M is compact, in (16.12.7) one can write H instead of H/M in his formula (2.9) and use the rest of his proof.

16.13. The following here helpful observation to reduce the number of integrations is owed to Jens Funke: If M is compact, there is an invariant measure dx and one has

$$\int_M \int_M f(x_1 \cdot x_2) dx_1 dx_2 = \int_M (\int_M f(x)dx) dx_2$$

$$= \mathrm{vol}(M) \int_M f(x)dx. \tag{16.13.1}$$

In formula (16.15.12) M is the centralizator of A in $H \cap K$ and one has members of M in H and K which can be brought together by commuting with $\exp B \in A$. Apparently, Flensted-Jensen had this in mind.

16.14. In addition, we reproduce two integration formulas from Knapp's book [Kn]:
Theorem 8.32. Let G be a Lie group, let S and T be closed subgroups such that $S \cap T$ is compact, multiplication $S \times T \to G$ is an open map, and the set of products ST exhausts G except possibly for a set of Haar measure 0. Let Δ_T and Δ_G denote the modular functions of T and G. Then the left Haar measures on G, S, and T can be normalized so that

$$\int_G f(x) d_l x = \int_{S \times T} f(st) \frac{\Delta_T(t)}{\Delta_G(t)} d_l s d_l t \tag{16.14.1}$$

for all Borel functions $f \geqslant 0$ on G.
Theorem 8.36. Let G be a Lie group, let H be a closed subgroup, and let Δ_G and Δ_H be the respective modular functions. Then the necessary and sufficient condition for G/H to have a nonzero G invariant Borel measure is that the restriction to H of Δ_G is equal to Δ_H. In this case such a measure $d\mu(gH)$ is unique up to a scalar, and it can be normalized so that

$$\int_G f(g) d_l g = \int_{G/H} [\int_H f(gh) d_l h] d\mu(gH) \quad \text{for all } f \in C_c(G). \tag{16.14.2}$$

16.15. Now, we follow Flensted-Jensen for $G = \mathrm{SO}(p,q)$: There, the Killing form defines Riemannian (i.e., Euclidean) structures on $\mathfrak{p} \cap \mathfrak{h}, \mathfrak{b}^+$, and L/M, and Flensted-Jensen lets the measure on L/M be $\mathrm{vol}(L/M)^{-1}$ times the volume element. Via Killing form, one has Riemannian structures on G/K and H/L, and by their volume elements also measures.
Remark. $\mathfrak{g} = \mathfrak{so}(p,q)$ has the Killing form

$$B(X;Y) = (p+q-2)\,\mathrm{tr}\,(XY). \tag{16.15.1}$$

Hence, for $(p,q) = (3,2))$, and the X_{ij} from above, we have $B(X_{ij}, X_{ij}) = 6$.
Moreover, take measures on G and H such that

$$\begin{aligned} \int_G f(x) dx &= \int_{G/K} \int_K f(xk) dk dxK, \quad \int_K dk = 1, \quad \text{for } f \in C_c(G) \\ \int_H f(x) dx &= \int_{H/L} \int_L f(xk) dk dxL, \quad \int_L dk = 1, \quad \text{for } f \in C_c(H). \end{aligned} \tag{16.15.2}$$

One has the standard diffeomorphism ([FlJ] (2.1))

$$\Phi_0 : \mathfrak{p} \cap \mathfrak{h} \times \mathfrak{p} \cap \mathfrak{q} \times K \to G, \quad (X,Y,k) \mapsto \exp X \cdot \exp Y \cdot k. \tag{16.15.3}$$

In [FlJ] p.261, from Helgason ([He] X Lemma 1.16) one has that the map

$$\Psi : L/M \times \mathfrak{b} \to \mathfrak{p} \cap \mathfrak{q}, \quad (lM, B) \mapsto \mathrm{Ad}(l) B \tag{16.15.4}$$

is a diffeomorphism onto an open dense set. Therefore, the maps

$$\Phi_1 : \mathfrak{p} \cap \mathfrak{h} \times L/M \times \mathfrak{b} \times K \to G, \; (X, lM, B) \mapsto (\exp X \, l \exp B), \tag{16.15.5}$$

and

$$\Phi : \mathfrak{p} \cap \mathfrak{h} \times L/M \times \mathfrak{b} \to G/K \tag{16.15.6}$$

given by

$$\Phi(X, lM, B) = \pi(\exp X \, l \exp B),$$

where $\pi : G \to G/K$ is the canonical map, are diffeomorphisms unto open dense sets.

Taking the usual transformation formula, Flensted-Jensen gets

$$\int_{G/K} f(x)dx = \mathrm{vol}\,(L/M) \int_{\mathfrak{p}\cap\mathfrak{h}} \int_{L/M} \int_{\mathfrak{b}^+} f(\Phi(X, lM, B))J(X, lM, B)dBdlMdX \tag{16.15.7}$$

with the Jacobian $J(X, lM, B) = |\det d\Phi_{(X,lM,B)}|$. As above, one has the diffeomorphism

$$\Phi' : \mathfrak{p} \cap \mathfrak{h} \to H/L, \quad X \mapsto \exp XL \tag{16.15.8}$$

and the appropriate transformation formula

$$\int_{H/L} f(x)dx = \int_{\mathfrak{p}\cap\mathfrak{h}} f(\Phi'(X))J_1(X)dX \; J_1(X) = |\det d\Phi'_{(X)}|. \tag{16.15.9}$$

Via $\int_{H/M} = \int_{H/L} \int_{L/M}$ this extends to Flensted-Jensens formula

$$\int_{H/M} f_1(x)dx = \int_{\mathfrak{p}\cap\mathfrak{h}} \int_{L/M} f_1(\exp XlM)J_1(X)dlMdX, \; J_1(X) = |\det d\Phi'_{(X)}|. \tag{16.15.10}$$

Since the measure on G/K is $H-$invariant, he has $J(X, lM, B) = J_1(X)\delta_1(B)$ and joins (16.15.7) and (16.15.10) to his equation (2.9)

$$\int_{G/K} f(x)dx = \mathrm{vol}\,(L/M) \int_{L/M} \int_{\mathfrak{b}^+} f(h \exp B)\delta_1(B)dBdh. \tag{16.15.11}$$

From here (his formula (2.9)), Flensted-Jensen comes to the formula (2.14) in his Theorem 2.6

$$\int_G f(x)dx = \mathrm{vol}\,(L/M) \int_K \int_H \int_{\mathfrak{b}^+} f(k \cdot \exp Bh)\delta(B)dBdhdk \quad \text{for } f \in C_c(G) \tag{16.15.12}$$

where δ given by Flensted-Jensen's formula (2.12) comes from the δ_1.
In his text he provides the calculation of $d\Phi_{(0,eM,B)}$. We will reproduce part of this below.

Our application revisited

16.16. We can use this for our interpretation of Kudla's formula for our Green integral. Here, our integrand is a function which from the left is $H-$invariant and from the right $K-$invariant, hence essentially depends on the group A. Hence, we can apply (16.12.7) to write

$$\int_{G/K} f(x)dx = \mathrm{vol}\,(L/M) \int_H \int_{\mathfrak{b}^+} f(h \exp B)\delta_1(B) dB dh \tag{16.16.1}$$

with

$$\begin{aligned} dx &= dx_{14}dx_{15}dx_{24}dx_{25}dx_{34}dx_{35} \\ dBdh &= dx_{15}dx_{23}dx_{45}dx_{24}dx_{25}dx_{34}dx_{35} \\ &= dx_{15}dx_{12}dx_{13}dx_{23}dx_{14}dx_{24}dx_{34} \end{aligned} \tag{16.16.2}$$

in case I resp II. And one has to determine $\delta_1(B)$, i.e., the derivative of $\Phi_{(0,eM,B)}$.

In [FlJ] this is done as follows. We have by [FlJ] (2.8)

$$\phi : \mathfrak{p} \cap \mathfrak{h} \times L/M \times \mathfrak{b} \to G/K,\ \phi(X, lM, B) = \pi(\exp X\, l \exp B).$$

Fix an $B \in \mathfrak{b}^+$, put $a = \exp B$, $s = (0, eM, B)$ and, for the canonical map $\pi : G \to G/K$, one has the differential at e $d\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{k} \simeq \mathfrak{p}$, i.e., for $X \in \mathfrak{g}$ we have $d\pi(X) = (1/2)(X - \tau X) = (1/2)(X + {}^tX)$. Then $\Phi(s) = \pi(a)$ and the tangent spaces are given by

$$T_s = \mathfrak{p} \cap \mathfrak{h} \times \mathfrak{l}' \times \mathfrak{b}^+, \quad T_{\pi(a)} \simeq \mathfrak{p}.$$

Flensted-Jensen now chooses orthonormal bases in T_s and $T_{\pi(a)}$. In our cases, for T_s, this comes to the following.
- For

$$\begin{aligned} \mathfrak{p} \cap \mathfrak{h} &= \oplus_{\beta \in \Sigma^+ \cup \{0\}} V_\beta \\ V_\beta &= \{X \in \mathfrak{p} \cap \mathfrak{h}; (\mathrm{ad}B)^2 X = \langle \beta, X \rangle^2 X \text{ for all } B \in \mathfrak{b}\} \end{aligned}$$

using from above 16.3, we have in Case I $\mathfrak{p} \cap \mathfrak{h} = \langle X_{24}, X_{25} X_{34}, X_{35} \rangle$

$$V_\beta = V_1 = \langle X_{25}, X_{35} \rangle,\ V_0 = \langle X_{24}, X_{34} \rangle \tag{16.16.3}$$

and in Case II $\mathfrak{p} \cap \mathfrak{h} = \langle X_{14}, X_{24} X_{34} \rangle$

$$V_\beta = V_1 = \langle X_{14} \rangle,\ V_0 = \langle X_{24}, X_{34} \rangle \tag{16.16.4}$$

- Similarly, for

$$\begin{aligned} \mathfrak{l}' &= \oplus_{\alpha \in \Delta^+ \cup \{0\}} \mathfrak{l}'_\alpha \\ \mathfrak{l}'_\alpha &= \{T \in \mathfrak{l}'; (\mathrm{ad}B)^2 T = \langle \alpha, T \rangle^2 X \text{ for all } B \in \mathfrak{b}\}, \end{aligned}$$

in Case A we have $\mathfrak{l}'_\alpha = \mathfrak{l}'_1 = \langle T := X_{45}\rangle$, and $\mathfrak{l}'_\alpha = \mathfrak{l}'_1 = \langle T_1 := X_{12}, T_2 := X_{13}\rangle$ in Case B.
- For $\mathfrak{b}$, in both cases, we have $B = X_{15}$ Hence, taking together the three types of elements X_{ij}, we have an orthogonal base for T_s where the Killing form (16.12.1) says, we should apply a factor 1/6 to get normalization.
Concerning $T_{\pi(a)} \simeq \mathfrak{p}\cap\mathfrak{h} + \mathfrak{p}\cap\mathfrak{q}$, one has the following general remark. Since B is in $\mathfrak{b}$ and $\mathfrak{b}$ is maximal abelian in $\mathfrak{p}\cap\mathfrak{q}$, it follows that $\mathrm{ad}B(\mathfrak{l}') + \mathfrak{b} = \mathfrak{p}\cap\mathfrak{q}$ and that

$$Y^i_\alpha = -\langle\alpha, B\rangle^{-1}[B, T^i_\alpha],\ \alpha\in\Delta^+_0$$

and $i = 1,\dots,m_\alpha$ is a basis for $\mathrm{ad}\,B(\mathfrak{l}')$. In particular, in Case A, we only have $Y = -\langle\alpha, B\rangle^{-1}X_{14}$, and in Case B we come to $Y_1 = -\langle\alpha, B\rangle^{-1}X_{25}$ and $Y_= - \langle\alpha, B\rangle^{-1}X_{35}$. We get a basis of $T_{\pi(a)}$ if we take $B = X_{15}$, these Y and the X from (16.16.3) resp. (16.16.4).

Again, in the general case, Flensted-Jensen has his result for the factor δ in the integral, if he verifies

$$\begin{aligned} d\phi_s(B) &= B \quad \text{for } B\in\mathfrak{b}\\ d\phi_s(T^i_\alpha) &= \sinh\langle\alpha, B\rangle Y^i_\alpha \quad \text{for}\,\alpha\in\Delta^+_0, i = 1,\dots,m_\alpha\\ d\phi_s(X^j_\beta) &= \cosh\langle\beta, B\rangle X^j_\beta \quad \text{for}\,\beta\in\Delta^+_b, i = 1,\dots,q_\beta. \end{aligned} \tag{16.16.5}$$

If $(X, T, B')\in T_s$ then $d\phi_s(X, T, B')$ is the tangent vector $\gamma^\cdot(0)$ to the curve through $\pi(a), a = \exp B$,

$$\begin{aligned} \gamma(t) &= \pi(\exp tX\exp tT\exp(B + tB'))\\ &= a\pi(\exp t\mathrm{Ad}(a^{-1})X\exp t\mathrm{Ad}(a^{-1})T\exp(tB')) \end{aligned} \tag{16.16.6}$$

Then, he gets
!.) If $X = T = 0$ then $\gamma^\cdot(0) = B'$,
2.) If $X = B' = 0$ and $T = T^i_\alpha$ then

$$\begin{aligned} \gamma^\cdot(0) &= d\pi(\mathrm{Ad}(a^{-1})T^i_\alpha) = (1/2)(e^{-\mathrm{ad}\,B}T^i_\alpha - \tau(e^{-\mathrm{ad}\,B}T^i_\alpha))\\ &= (1/2)(e^{-\mathrm{ad}\,B}T^i_\alpha - e^{\mathrm{ad}\,B}T^i_\alpha) = -\frac{\sinh\langle\alpha, B\rangle}{\langle\alpha, B\rangle}[B, T^i_\alpha] = \sinh\langle\alpha, B\rangle Y^i_\alpha, \end{aligned} \tag{16.16.7}$$

3.) If $T = B' = 0$ and $X = X^j_\beta$ then

$$\begin{aligned} \gamma^\cdot(0) &= d\pi(\mathrm{Ad}(a^{-1})X^j_\beta) = (1/2)(e^{-\mathrm{ad}\,B}X^j_\beta - \tau(e^{-\mathrm{ad}\,B}X^j_\beta))\\ &= (1/2)(e^{-\mathrm{ad}\,B}X^j_\beta + e^{\mathrm{ad}\,B}X^j_\beta) = \cosh\langle\beta, B\rangle X^j_\beta. \end{aligned} \tag{16.16.8}$$

Here, apparently, one uses $\mathrm{Ad}(\exp X) = e^{\mathrm{ad}X}$ and the fact that the $\mathrm{ad}\,B\,X$ in the formulae above are skew while the $\mathrm{ad}\,B\,T$ are symmetric.
In our situation this restricts to the following. In Case A we have

$$\begin{aligned} d\phi_s(B) &= B,\\ d\phi_s(T) &= \sinh(t)Y,\\ d\phi_s(X_{25}) &= \cosh(t)X_{25} \text{ and } \quad d\phi_s(X_{35}) = \cosh(t)X_{35}, \end{aligned} \tag{16.16.9}$$

i.e., for the missing factor in (16.16.1) we have

$$\delta_1(B) = |\sinh(t)|\cosh^2(t) \tag{16.16.10}$$

and analogously in Case B

$$\delta_1(B) = \sinh^2(t)\cosh(t). \tag{16.16.11}$$

As they should, these formulae are consistent with Kudla's formulae (3.23) in [Ku1]. Hence, in our situation, we have from (16.16.1), for a discrete $\Gamma_x \subset H$

$$\begin{aligned}\int_{\Gamma_x\backslash G/K} f(x)dx &= \mathrm{vol}\,(L/M)\int_{\Gamma_x\backslash H}\int_{\mathfrak{b}^+} f(h\exp B)\delta_1(B)dBdh \\ \delta_1(B) &= |\sinh(t)|\cosh^2(t) \quad \text{in Case A} \\ &= \sinh^2(t)\cosh(t) \quad \text{in Case B}\end{aligned} \tag{16.16.12}$$

16.17. Here, to show some more background, we reproduce some standard material from Helgason's book on differential geometry [He]. Given a manifold M with affine connection, on p.32/3 he denotes by $\mathrm{Exp} = \mathrm{Exp}_p$ the exponential mapping from an open neighborhood of 0 ot the tangent space T_pM to an open neighborhood of p in M given by $X \mapsto \gamma_X(1)$ as described in Theorem 6.1. Theorem 6.5 (with a long proof) gives a formula for the differential of the map Exp. In Chapter II Theorem 1.7, this is used for the derivative of the map

$$\exp : \mathfrak{g} = \mathrm{Lie}\,G \to G, \quad X \mapsto \exp X$$

with $L_h g = hg$ to get

$$d\exp_X = d(L_{\exp_X})_e \circ \frac{1 - e^{-\mathrm{ad}\,X}}{\mathrm{ad}\,X}. \tag{16.17.1}$$

On p.179 in Chapter IV, this is applied to take $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ with $sX = -X$ and for $X \in \mathfrak{p}$, let

$$T_X := (\mathrm{ad}\,X)^2|_{\mathfrak{p}},$$

and $\pi : G \to G/K$ the natural mapping, $o = \pi\, e$ and $\tau(g)$ the mapping $xK \to gxK$ of G/K onto itself. $d\pi$ identifies $\mathfrak{p}$ with the tangent space $T_o(G/K)$. Hence, Theorem 4.1 says that the Exponential mapping of $\mathfrak{p}$ to G/K is independent of the choice of the Riemannian structure and its differential is given by

$$d\,\mathrm{Exp}_X = d\pi(\exp_X)_o \circ \sum_{n=0}^{\infty} \frac{(T_X)^n}{(2n+1)!}, \quad X \in \mathfrak{p}. \tag{16.17.2}$$

Injections and measures

16.18. In the application of Kudla's central formula, we have the problem of the relation between the measures for the different volume integrals, namely for the signature (3,2) and

the signature (2,2) and (3,1). We have already discussed Siegel's approach to parametrize his representation spaces. It is tempting to use this here:
We have

(16.18.1)

$$G_0 = \mathrm{SO}(2,1) \to G_- = \mathrm{SO}(3,1) \to G = \mathrm{SO}(3,2) \leftarrow G_+ = \mathrm{SO}(2,2) \leftarrow G_0 = \mathrm{SO}(2,1)$$

and use the embeddings ι given by

$$G_0 \ni g_0 = \begin{pmatrix} A & b \\ {}^tc & d \end{pmatrix} \mapsto \begin{pmatrix} A & b\, 0_2 \\ {}^tc & d\, 0 \\ {}^t0_2 & 0\, 1 \end{pmatrix} \in G_+ \quad A \in M_{22}, b, c \in M_{21}, d \in M_{11}$$

$$G_0 \ni g_0 = \begin{pmatrix} A & b \\ {}^tc & d \end{pmatrix} \mapsto \begin{pmatrix} 1\, {}^t0_2 & 0 \\ 0_2\, A & b \\ 0\, {}^tc & d \end{pmatrix} \in G_- \quad A \in M_{22}, b, c \in M_{21}, d \in M_{11}$$

$$G_- \ni g_- = \begin{pmatrix} A & b \\ {}^tc & d \end{pmatrix} \mapsto \begin{pmatrix} A & b\, 0_3 \\ {}^tc & d\, 0 \\ {}^t0_3 & 1 \end{pmatrix} \in G \quad A \in M_{33}, b, c \in M_{31}, d \in M_{11}$$

(16.18.2) $$G_+ \ni g_+ = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} 1\, {}^t0_2 & {}^t0_2 \\ 0_2\, A & B \\ 0_2\, C & D \end{pmatrix} \in G \quad A, B, C, D \in M_{22}.$$

As one has the maps to the homogeneous spaces, Siegel's representation spaces $\mathbb{D}$

$$G \ni g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto BD^{-1} = Z \in \mathbb{D},$$

we get injections

(16.18.3) $$\mathbb{D}_{2,1} \to \mathbb{D}_{3,1} \to \mathbb{D}_{3,2} \leftarrow \mathbb{D}_{2,2} \leftarrow \mathbb{D}_{2,1}$$

and one could hope that this helps in the understanding how to normalize in the factor $\mathrm{vol}(\Gamma_x \backslash G_x), G_x = G_-$ or $= G_+$ in Kudla's formula.
We already discussed the (3,1)-case and for the parametrizing by the space $\mathbb{H}^+$ got in (11.4.18)

$$dv_{\mathrm{Sie}} = (\det(E - X^tX)^{-m/2} \prod_{k=1}^{n} \prod_{l=1}^{m-n} dx_{kl} = \frac{dx \wedge dy \wedge dr}{r^3} = dv_{\mathbb{H}^+}$$

and in the (2,2)-case for $\mathbb{H}^2$ we got in (11.3.8)

$$dv_{\mathrm{Sie}} = (1/(4y_1^2 y_2^2))\, dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 = (1/4) dv_{\mathbb{H}^2}.$$

Unfortunately, the (3,2)-case is a bit more tiresome to compute.

For the parametrization Ψ of the representation space $\mathbb{D}_{32}$ by the elements $z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}$ of Siegel's half space $\mathbb{H}_2$, and the map $\tilde{A} : \mathrm{Sp}(2, \mathbb{R}) \to \tilde{G} = \mathrm{SO}(\tilde{Q})$, we have from (1.8.16)

$$\tilde{A}(g_z) = \begin{pmatrix} \eta & \eta x_3/y_3 & 2(x_3y_2 - x_2y_3)/y_3 & (x_3y_2^2 - 2x_2y_2y_3 + x_1y_3^2)/(\eta y_3) & -\zeta^2/\eta \\ & \eta/y_3 & 2y_2/y_3 & y_2^2/(\eta y_3) & -x_1/\eta \\ & & 1 & y_2/\eta & -x_2/\eta \\ & & & y_3/\eta & -x_3/\eta \\ & & & & 1/\eta \end{pmatrix}$$

$$(16.18.4) \qquad =: \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ & b_2 & b_3 & b_4 & b_5 \\ & & 1 & c_4 & c_5 \\ & & & d_4 & d_5 \\ & & & & e_5 \end{pmatrix}$$

One has

$$G = \mathrm{SO}(3, 2) = C^{-1}\tilde{G}C$$

with (3.8.1)

$$C = (1/\sqrt{2}) \begin{pmatrix} 1 & & & & 1 \\ & 1 & & 1 & \\ & & 1 & & \\ & -1 & & 1 & \\ -1 & & & & 1 \end{pmatrix}, \quad C^{-1} = (1/\sqrt{2}) \begin{pmatrix} 1 & & & & -1 \\ & 1 & & -1 & \\ & & 2 & & \\ & 1 & & 1 & \\ 1 & & & & 1 \end{pmatrix},$$

Hence, with

$$\eta^2 = y_1y_3 - y_2^2, \zeta^2 = x_1x_3 - x_2^2, I = x_3y_1 - 2x_2y_2 + x_1y_3, II = \eta^2 - \zeta^2,$$

$$
\begin{aligned}
A(g_z) &= C^{-1}\tilde{A}(g_z)C =: \begin{pmatrix} A & B \\ C & D \end{pmatrix} \\
(16.18.5) \qquad &= (1/2)\begin{pmatrix} 1 &&&& -1 \\ & 1 && -1 & \\ && 2 && \\ & 1 && 1 & \\ 1 &&&& 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ & b_2 & b_3 & b_4 & b_5 \\ && 1 & c_4 & c_5 \\ &&& d_4 & d_5 \\ &&&& e_5 \end{pmatrix} \begin{pmatrix} 1 &&&& 1 \\ & 1 && 1 & \\ && 1 && \\ & -1 && 1 & \\ -1 &&&& 1 \end{pmatrix} \\
&= (1/2)\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 - e_5 \\ & b_2 & b_3 & b_4 - d_4 & b_5 - d_5 \\ && 2 & 2c_4 & 2c_5 \\ & b_2 & b_3 & b_4 + d_4 & b_5 + d_5 \\ a_1 & a_2 & a_3 & a_4 & a_5 + e_5 \end{pmatrix} \begin{pmatrix} 1 &&&& 1 \\ & 1 && 1 & \\ && 1 && \\ & -1 && 1 & \\ -1 &&&& 1 \end{pmatrix} \\
&= (1/2)\begin{pmatrix} a_1 - a_5 + e_5 & a_2 - a_4 & a_3 & a_2 + a_4 & a_1 + a_5 - e_5 \\ -b_5 + d_5 & b_2 - b_4 + d_4 & b_3 & b_2 + b_4 - d_4 & b_5 - d_5 \\ && 2 & 2c_4 & 2c_5 \\ -b_5 - d_5 & b_2 - b_4 - d_4 & b_3 & b_2 + b_4 + d_4 & b_5 + d_5 \\ a_1 - a_5 - e_5 & a_2 - a_4 & a_3 & a_2 + a_4 & a_1 + a_5 + e_5 \end{pmatrix} \\
B &= (1/\eta)\begin{pmatrix} I & II - 1 \\ y_1 - y_3 & x_3 - x_1 \\ 2y_2 & -2x_2 \end{pmatrix} \\
D &= (1/\eta)\begin{pmatrix} y_1 + y_3 & -x_1 - x_3 \\ I & II + 1 \end{pmatrix} \\
Z &= BD^{-1} \\
&= (1/\hat{D})\begin{pmatrix} 2I & I(x_1 + x_3) + (II - 1)(y_1 + y_3) \\ (y_1 - y_3)(II + 1) + (x_1 - x_3)I & 2(y_1 x_3 - x_1 y_3) \\ 2y_2(II + 1) + 2x_2 I & 2y_2(x_1 + x_3) - 2x_2(y_1 + y_3) \end{pmatrix}, \\
&= (1/\hat{D})\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \\ z_{31} & z_{32} \end{pmatrix}, \\
\hat{D} &= \eta^2 \det D = (y_1 + y_3)(1 + \eta^2 - \zeta^2) + (x_3 y_1 - 2x_2 y_2 + x_1 y_3)(x_1 + x_3). \\
z_{11} &= 2(x_3 y_1 - 2x_2 y_2 + x_1 y_3) \\
z_{12} &= y_1(|z_3|^2 - 1 + x_2^2 - y_2^2) + y_3(|z_1|^2 - 1 + x_2^2 - y_2^2) - 2x_2 y_2(x_1 + x_3) \\
z_{21} &= -y_1(|z_3|^2 - 1 + x_2^2 - y_2^2) + y_3(|z_1|^2 - 1 + x_2^2 - y_2^2) - 2x_2 y_2(x_3 - x_1) \\
z_{22} &= 2(y_1 x_3 - x_1 y_3) \\
z_{31} &= 2(y_1 y_2 y_3 - y_2^3 - x_1 x_3 y_2 + y_2 + x_1 x_3 y_1 + x_1 x_3 y_3 - x_2^2 y_2) \\
z_{32} &= 2(x_1 y_2 - x_2 y_1 - x_2 y_3 + x_3 y_2).
\end{aligned}
$$

Though perhaps this won't help here much, but for the sake of some completeness, we add the corresponding results for the other cases.

i) For $A:\mathrm{SL}_2(\mathbb{R})\to\mathrm{SO}(1,2)$ and $\Psi:\mathbb{H}\to\mathbb{D}_{12}$ we have from (11.6.7)

$$A(g_z)=(1/2y)\cdot\begin{pmatrix}x^2+y^2+1 & -x^2+y^2-1 & -2xy\\ x^2+y^2-1 & -x^2+y^2+1 & -2xy\\ -2x & +2x & 2y\end{pmatrix}=\begin{pmatrix}a & b\\ {}^tc & D\end{pmatrix}$$

and from (11.6.9)

$$\begin{aligned}\Psi(z)=Z_z&=bD^{-1}\\ &=(-x^2+y^2-1,-2xy)\begin{pmatrix}2y & 2xy\\ -2x & y^2-x^2+1\end{pmatrix}(1/\xi),\quad \xi=x^2+y^2+1\\ &=(1/(|z|^2+1))(|z|^2-1,-2x)\end{aligned}$$

and $\Psi^*dv_{\mathrm{Sie}}=dv_{\mathbb{H}}$.

Similarly, one has for $A:\mathrm{SL}_2(\mathbb{R})\to\mathrm{SO}(2,1)$ and $\Psi:\mathbb{H}\to\mathbb{D}_{21}$

$$A(g_z)=(1/2y)\cdot\begin{pmatrix}2y & -2xy & -2xy\\ 2x & -x^2+y^2+1 & -x^2+y^2-1\\ -2x & x^2+y^2-1 & x^2+y^2+1\end{pmatrix}=\begin{pmatrix}A & b\\ {}^tc & d\end{pmatrix}$$

and this time

$$\Psi(z)=Z_z=b\cdot d^{-1}=(1/(x^2+y^2+1))\begin{pmatrix}-2xy\\ -x^2+y^2-1\end{pmatrix}:=\begin{pmatrix}x_1\\ x_2\end{pmatrix}. \tag{16.18.6}$$

Here, with $\xi=x^2+y^2+1$, we get

$$dx_1\wedge dx_2=-8(y^2/\xi^3)dx\wedge dy,\quad \text{and}\quad \det(E_2-Z_z{}^tZ_z)=4y^2/\xi^2, \tag{16.18.7}$$

hence, for $dv_{\mathrm{Sie}}=\det(E_2-Z_z{}^tZ_z)^{-3/2}dx_1\wedge dx_2$ (strangely enough?)

$$\Psi^*dv_{\mathrm{Sie}}=\frac{dx\wedge dy}{y}=y\cdot dv_{\mathbb{H}} \tag{16.18.8}$$

ii) For $A:\mathrm{SL}_2(\mathbb{C})\to\mathbb{SO}(1,3)$ and the map $\Psi:\mathrm{H}^+\to\mathbb{D}_{13}$, we have from (11.4.12)

$$A_P:=A(g_P)=\begin{pmatrix}(|z|^2+1+r^2)/(2r) & x & y & (r^2-|z|^2-1)/(2r)\\ x/r & 1 & & -x/r\\ y/r & & 1 & -y/r\\ (|z|^2-1+r^2)/(2r) & x & y & (r^2-|z|^2+1)/(2r)\end{pmatrix}$$

If here we take (as in (11.4.16))

$${}^tb=(x,y,(r^2-|z|^2-1)/(2r)),\ D=\begin{pmatrix}1 & & -x/r\\ & 1 & -y/r\\ x & y & (r^2-|z|^2+1)/(2r)\end{pmatrix}.$$

and put $\Xi := \det(D) = (1 + r^2 + x^2 + y^2)/(2r)$ we get

$$X_P = {}^t b \cdot D^{-1} = (x, y, r - \Xi) \cdot \begin{pmatrix} \Xi - x^2/r & -xy/r & x/r \\ -xy/r & \Xi - y^2/r & y/r \\ -x & -y & 1 \end{pmatrix} (1/\Xi)$$
$$= (1/(r\Xi))(x, y, r\Xi - 1)$$

One has $1 - X_P {}^t X_P = \Xi^{-2}$ and with $r\Xi = (1/2)(x^2 + y^2 + r^2)$

$$d(x/(r\Xi)) \wedge d(y/(r\Xi)) \wedge d(1 - 1/(r\Xi)) = (r\Xi)^{-4} r dx \wedge dy \wedge dr.$$

Hence, from (11.4.18) for $n = 1, m = 4$ Siegel's volume element (11.2.7) comes out as

$$dv_{\mathrm{Sie}} = (\det(E - X^t X)^{-m/2} \prod_{k=1}^{n} \prod_{l=1}^{m-n} dx_{kl},$$
$$= \Xi^4 \cdot (r\Xi)^{-4} r dx \wedge dy \wedge dr,$$
$$= \frac{dx \wedge dy \wedge dr}{r^3}, \tag{16.18.9}$$

i.e., exactly the standard volume element $dv_{\mathbb{H}^+}$ for the hyperbolic three-space.

Similarly, for $A : \mathrm{SL}_2(\mathbb{C}) \to \mathbb{SO}(3, 1)$ and the map $\Psi : \mathrm{H}^+ \to \mathbb{D}_{31}$, we have

$$A_P := A(g_P) = (1/2r) \begin{pmatrix} -|z|^2 + 1 + r^2 & 2xr & 2yr & r^2 + |z|^2 - 1 \\ -2y & 2r & & 2y \\ -2x & & 2r & 2x \\ -|z|^2 - 1 + r^2)/(2r) & 2yr & 2xr & r^2 + |z|^2 + 1 \end{pmatrix}$$

and

$$X_P = {}^t b \cdot d^{-1} = \begin{pmatrix} r^2 + |z|^2 - 1 \\ 2y \\ 2x \end{pmatrix} (1/\Xi) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \Xi = r^2 + x^2 + y^2 + 1$$

We get

$$dx_1 \wedge dx_2 \wedge dx_3 = (4r/\Xi^4) dx \wedge dy \wedge dr,$$
$$\det(E_3 - X_P {}^t X_P) = 4r^2 \Xi^{-2},$$

hence, for $dv_{\mathrm{Sie}} = (\det(E_3 - X_P {}^t X_P))^{-2} dx_1 \wedge dx_2 \wedge dx_3$ again

$$\Psi^* dv_{\mathrm{Sie}} = \frac{dx \wedge dy \wedge dr}{r^3} = dv_{\mathbb{H}^+} \tag{16.18.10}$$

iii) For $A : \mathrm{SL}_2(\mathbb{R})^2 \to \mathbb{SO}(2,2) = G$ and the map $\Psi : \mathrm{H}^2 \to \mathbb{D}_{22}$, with $z = (z_1, z_2) \in \mathbb{H}^2, q_1 = \sqrt{y_1 y_2}, q_2 = \sqrt{y_1/y_2}$ we have from (11.3.3)

(16.18.11)

$$A(z) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$$
$$= (1/2) \begin{pmatrix} q_1 + 1/q_1 + x_1x_2/q_1 & -x_2q_2 + x_1/q_2 & -x_2q_2 - x_1/q_2 & q_1 - 1/q_1 - x_1x_2/q_1 \\ (-x_1 + x_2)/q_1 & q_2 + 1/q_2 & -q_2 + 1/q_2 & (x_1 - x_2)/q_1 \\ (x_1 + x_2)/q_1 & -q_2 + 1/q_2 & q_2 + 1/q_2 & -(x_1 + x_2)/q_1 \\ q_1 - 1/q_1 + x_1x_2/q_1 & -x_2q_2 + x_1/q_2 & x_2q_2 + x_1/q_2 & q_1 + 1/q_1 - x_1x_2/q_1 \end{pmatrix}, .$$

We have $\zeta = \det D = (1/(4y_1y_2))\xi$, $\xi := ((|z_1|^2 + 1)y_2 + (|z_2|^2 + 1)y_1)$, and, hence, from (11.3.5) we get the map $\Psi : \mathbb{H}^2 \to \mathbb{D}_{2,2}$

$$\mathbb{H}^2 \ni z = (z_1, z_2) \mapsto Z = BD^{-1}$$
$$= (1/\xi) \begin{pmatrix} 2(x_1y_2 + x_2y_1) & (|z_1|^2 - 1)y_2 + (|z_2|^2 - 1)y_1 \\ (|z_1|^2 - 1)y_2 - (|z_2|^2 - 1)y_1 & 2(x_1y_2 - x_2y_1) \end{pmatrix}$$
$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{D}_{2,2}.$$

With $\delta = ad - bc = \frac{-(|z_1|^2+1)y_2+(|z_2|^2+1)y_1}{(|z_1|^2+1)y_2+(|z_2|^2+1)y_1}$, we have

$$\Delta := \det(E - Z^tZ) = 1 - a^2 - b^2 - c^2 - d^2 + \delta^2$$
$$= 2^4 y_1^2 y_2^2/(((|z_1|^2 + 1)y_2 + (|z_2|^2 + 1)y_1)^2 = \zeta^{-2}$$

and

$$da \wedge db \wedge dc \wedge dd = 2^6 y_1^2 y_2^2/(|z_1|^2 + 1)y_2 + (|z_2|^2 + 1)y_1)^4 \, dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2.$$

Hence, for our case, Siegel's formula (11.2.7) shows up to the factor 1/4 the usual volume element for $\mathbb{H}^2$

(16.18.12) $$\Psi^* dv_{\mathrm{Sie}} = (1/(4y_1^2 y_2^2)) \, dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 = (1/4) dv_{\mathbb{H}^2}.$$

Similary to (16.18.11), we have

(16.18.13)

$$A(z)^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$$
$$= (1/(2q_1)) \begin{pmatrix} 1 + x_1x_2 + y_1y_2 & -x_1 + x_2 & -x_1 - x_2 & 1 - x_1x_2 - y_1y_2 \\ (x_1 - x_2)y_2 & y_1 + y_2 & y_1 - y_2 & (x_1 - x_2)y_2 \\ -(x_1 + x_2)y_2 & y_1 - y_2 & y_1 + y_2 & (x_1 + x_2)y_2 \\ 1 + x_1x_2 - y_1y_2 & -x_1 + x_2 & -x_1 - x_2 & 1 - x_1x_2 + y_1y_2 \end{pmatrix}, .$$

16.19. Summary. using the embeddings (16.18.2) we realize the injections from (16.18.3)

(16.19.1)

$$\mathbb{D}_{12} \ni \Psi_{12}(z) = \frac{1}{|z|^2+1}(|z|^2-1, 2x) \mapsto \frac{1}{|z|^2+1}\begin{pmatrix} 0 & 0 \\ |z|^2-1 & 2x) \end{pmatrix} \in \mathbb{D}_{22},$$

$$\mathbb{D}_{21} \ni \Psi_{21}(z) = \frac{1}{|z|^2+1}\begin{pmatrix} -2xy \\ -x^2+y^2-1 \end{pmatrix} \mapsto \frac{1}{|z|^2+1}\begin{pmatrix} -2xy & 0 \\ -x^2+y^2-1 & 0 \end{pmatrix} \in \mathbb{D}_{22},$$

$$\mathbb{D}_{31} \ni \Psi_{31}(P) = \frac{1}{r^2+|z|^2+1}\begin{pmatrix} r^2+|z|^2-1 \\ 2y \\ 2x \end{pmatrix} \mapsto \frac{1}{r^2+|z|^2+1}\begin{pmatrix} r^2+|z|^2-1 & 0 \\ 2y & 0 \\ 2x & 0 \end{pmatrix} \in \mathbb{D}_{32},$$

$$\mathbb{D}_{22} \ni \Psi_{22}(z_1, z_2) = \frac{1}{\Xi}\begin{pmatrix} 2(x_1y_2+x_2y_1) & (|z_1|^2-1)y_2+(|z_2|^2-1)y_1 \\ -(|z_1|^2-1)y_2+(|z_2|^2-1)y_1 & 2(x_1y_2-x_2y_1) \end{pmatrix}$$

$$\mapsto \frac{1}{\Xi}\begin{pmatrix} 0 & 0 \\ 2(x_1y_2+x_2y_1) & (|z_1|^2-1)y_2+(|z_2|^2-1)y_1 \\ -(|z_1|^2-1)y_2+(|z_2|^2-1)y_1 & 2(x_1y_2-x_2y_1) \end{pmatrix} \in \mathbb{D}_{32},$$

$$\mathbb{D}_{32} \ni \Psi_{32}(z_1, z_2, z_3) = \frac{1}{\hat{D}}\begin{pmatrix} 2I & I(x_1+x_3)+(II-1)(y_1+y_3) \\ (y_1-y_3)(II+1)+(x_1-x_3)I & 2(y_1x_3-x_1y_3) \\ 2y_2(II+1)+2x_2I & 2y_2(x_1+x_3)-2x_2(y_1+y_3) \end{pmatrix}$$

with $\Xi = (|z_1|^2+1)y_2 + (|z_2|^2+1)y_1$, $\hat{D} = (y_1+y_3)(II+1)+(x_1+x_3)I$, and

$$\eta^2 = y_1y_3 - y_2^2, \zeta^2 = x_1x_3 - x_2^2, I = x_3y_1 - 2x_2y_2 + x_1y_3, II = \eta^2 - \zeta^2.$$

In particular, we get

(16.19.2)

$$\mathbb{D}_{32} \ni \Psi_{32}(z_1, 0, z_3) = \frac{1}{\hat{D}}\begin{pmatrix} 2(x_1y_3+x_3y_1) & y_1(|z_3|^2-1)+y_3(|z_1|^2-1) \\ -y_1(|z_3|^2-1)+y_3(|z_1|^2-1) & 2(y_1x_3-x_1y_3) \\ 0 & 0 \end{pmatrix}$$

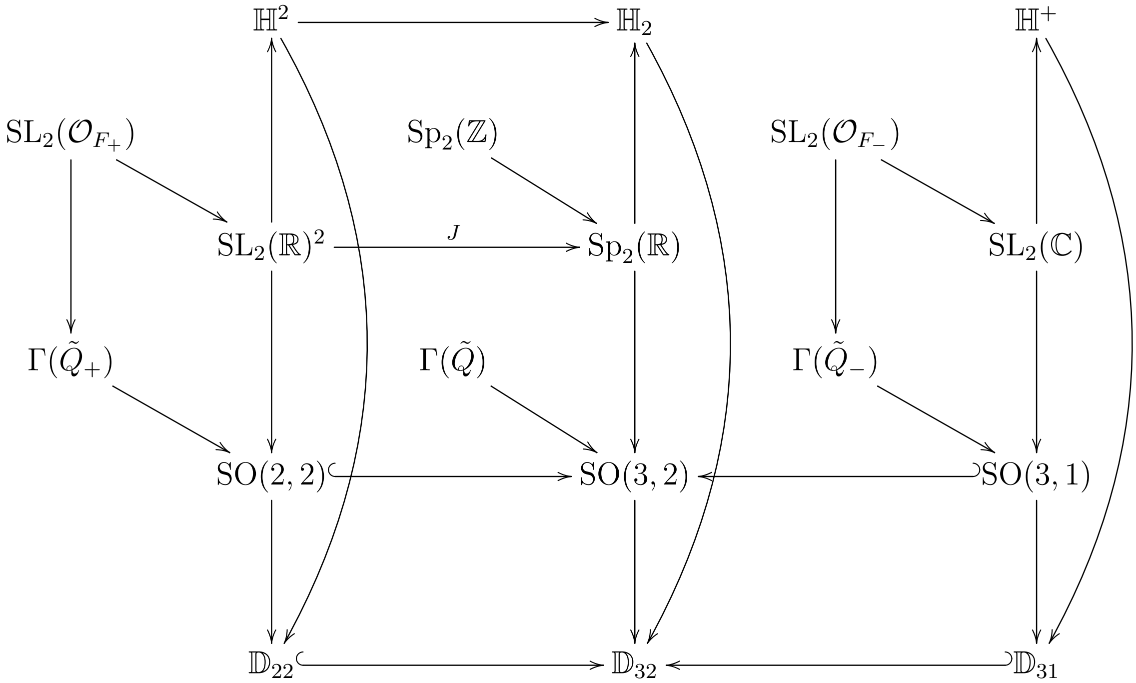
with $\hat{D} = y_1(|z_3|^2+1) + y_3(|z_1|^2+1)$.

16.20. Remark. This is very near to what one would expect from the embedding of $\mathbb{D}_{22}$ in $\mathbb{D}_{32}$ given above (the zero-line has the wrong place). This can be cured by changing the embedding above of SO(2, 2) into SO(3, 2) to

(16.20.1)
$$\mathrm{SO}(2,2) \ni \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A & & B \\ & 1 & \\ C & & D \end{pmatrix} \in \mathrm{SO}(2,2)$$

Done this, one has the following picture of our orthogonal world

(16.20.2)



Hence, with

$$J : \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \to \mathrm{Sp}(2,\mathbb{R}),\ (g_1, g_2) \mapsto \begin{pmatrix} a_1 & & b_1 & \\ & a_2 & & b_2 \\ c_1 & & d_1 & \\ & c_2 & & d_2 \end{pmatrix},$$

and $Z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}$, from (16.19.1) one has maps

(16.20.3)

$$\begin{array}{ccccc} \mathbb{H}^2 \ni (z_3, z_1) & \longmapsto & Z \in \mathbb{H}_2 & & P = (x,y,r) \in \mathbb{H}^+ . \\ \downarrow & & \downarrow & & \downarrow \\ \Psi_{22}(z_3, z_1) & \longrightarrow & \Psi_{32}(Z) & \longleftarrow & \Psi_{31}(P) \end{array}$$

The square on the left hand side is commutative, while we have not found a nice formula for a map from $\mathbb{H}^+$ to $\mathbb{H}_2$ resp. from $\mathrm{SL}_2(\mathbb{C})$ to $\mathrm{Sp}_2(\mathbb{R})$.

16.21. . Trying to find some, we attempted the following: We have from (11.4.12)

$$
\begin{aligned}
A(g_P) &= \begin{pmatrix} (|z|^2+1+r^2)/(2r) & x & y & (r^2-|z|^2-1)/(2r) \\ x/r & 1 & & -x/r \\ y/r & & 1 & -y/r \\ (|z|^2-1+r^2)/(2r) & x & y & (r^2-|z|^2+1)/(2r) \end{pmatrix} \in \mathrm{SO}(3,1) \\
&=: \begin{pmatrix} \alpha_1 & x & y & \alpha_4 \\ x/r & 1 & & -x/r \\ y/r & & 1 & -y/r \\ \delta_1 & x & y & \delta_4 \end{pmatrix}
\end{aligned}
$$

and this nor as

$$
\begin{pmatrix} \alpha_1 & x & y & \alpha_4 & \\ x/r & 1 & & -x/r & \\ y/r & & 1 & -y/r & \\ \delta_1 & x & y & \delta_4 & \\ & & & & 1 \end{pmatrix} \text{ neither as } \begin{pmatrix} \alpha_1 & x & y & & \alpha_4 \\ x/r & 1 & & & -x/r \\ y/r & & 1 & & -y/r \\ & & & 1 & \\ \delta_1 & x & y & & \delta_4 \end{pmatrix}
$$

fits into an element (16.18.5) from SO(3, 2)

$$
A(g_z) = (1/2) \begin{pmatrix} a_1-a_5+e_5 & a_2-a_4 & a_3 & a_2+a_4 & a_1+a_5-e_5 \\ -b_5+d_5 & b_2-b_4+d_4 & b_3 & b_2+b_4-d_4 & b_5-d_5 \\ & & 2 & 2c_4 & 2c_5 \\ -b_5-d_5 & b_2-b_4-d_4 & b_3 & b_2+b_4+d_4 & b_5+d_5 \\ a_1-a_5-e_5 & a_2-a_4 & a_3 & a_2+a_4 & a_1+a_5+e_5 \end{pmatrix}.
$$

16.22. Above in (16.3.1) we treated the Lie algebra of SO(3,2). It is isomorphic to the Lie algebra of $G = \mathrm{Sp}(2,\mathbb{R})$ and we discuss this here:
One has

$$
\begin{aligned}
G &= \{g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right); {}^tAD - {}^tCB = E, {}^tCA = {}^tAC, {}^tDB = {}^tBD\}, \\
K &= \{g = \left(\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right); (A+iB) \in U(2)\}, \\
\mathrm{Lie}\, G = \mathfrak{g} &= \mathfrak{sp}(2,\mathbb{R}) = \{X = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right), A = -{}^tD, B = {}^tB, C = {}^tC \in M(2,\mathbb{R})\} \\
&= \mathfrak{k} + \mathfrak{p}, \\
\mathfrak{k} &= \{X = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right), A = D = -{}^tD, B = {}^tB = -C = -{}^tC \in M(2,\mathbb{R})\}, \\
\mathfrak{p} &= \{X = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right), A = {}^tA = -{}^tD = -D, B = {}^tB = C = {}^tC \in M(2,\mathbb{R}), \}. \qquad (16.22.1)
\end{aligned}
$$

We take as basis for $\mathfrak{k}$

$$
\begin{aligned}
K_1 &= \begin{pmatrix} & 1 & & \\ -1 & & & \\ & & & 1 \\ & & -1 & \end{pmatrix}, K_2 = \begin{pmatrix} 0 & & 0 & 1 \\ 0 & & & \\ -1 & & & \\ 0 & & & \end{pmatrix}, \\
K_3 &= \begin{pmatrix} & & & 0 \\ & & & 1 \\ & & & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, K_4 = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}, \qquad (16.22.2)
\end{aligned}
$$

and for $\mathfrak{p}$

$$A_1 = \begin{pmatrix} 1 &&& \\ & 0 && \\ && -1 & \\ &&& 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 &&& \\ & 1 && \\ && 0 & \\ &&& -1 \end{pmatrix},$$

$$G_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 &&& \\ 1 &&& \\ 0 &&& \end{pmatrix}, G_2 = \begin{pmatrix} &&& 0 \\ &&& 1 \\ &&& 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$G_3 = \begin{pmatrix} &&& 1 \\ && 1 & \\ & 1 && \\ 1 &&& \end{pmatrix}, G_4 = \begin{pmatrix} & 1 && 0 \\ 1 &&& \\ &&& -1 \\ 0 && -1 & 0 \end{pmatrix}. \tag{16.22.3}$$

16.23. A way to realize the isomorphism between $\mathfrak{sp}(2,\mathbb{R})$ and $\mathfrak{so}(3,2)$ is hidden in the problems in [Kn] p.208. We propose to proceed as follows.
Here (clearly going back to Siegel [S3, S4] and as in [GN]), we realize $V = \mathbb{R}^5$ as the space $\mathcal{V}$ of skew-symmetric matrices

$$M = M(u) = \begin{pmatrix} u_1 J & XJ \\ JX & -u_5 J \end{pmatrix} \in \mathrm{M}_4(\mathbb{R}) \tag{16.23.1}$$

with

$$X = \begin{pmatrix} u_2 & u_3 \\ u_3 & u_4 \end{pmatrix}, J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{M}_2(\mathbb{R}). \tag{16.23.2}$$

The symplectic group $\check{G} = \mathrm{Sp}(2,\mathbb{R})$ acts (transitively) on $\mathcal{V}$ via

$$(g, M(u)) \longmapsto gM(u)^t g =: M(A(g)u) =: M(u') \tag{16.23.3}$$

preserving the quadratic form $\tilde{q} = (1/2)^t u \tilde{Q} u = u_3^2 - u_2 u_4 - u_1 u_5$. As usual, this leads to a homomorphism $\mathrm{Sp}(2,\mathbb{R}) \longrightarrow \tilde{G}$ where $g \in \check{G}$ is mapped to the matrix $\tilde{A}(g)$ with $u' = \tilde{A}(g)u$. With $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, this leads to

$$\begin{aligned} gM(u)^t g = & \\ = & \begin{pmatrix} u_1 AJ^tA + BJX^tA + AXJ^tB - u_5 BJ^tB & u_1 AJ^tC + BJX^tC + AXJ^tD - u_5 BJ^tD \\ u_1 CJ^tA + DJX^tA + CXJ^tB - u_5 DJ^tB & u_1 CJ^tC + DJX^tC + CXJ^tD - u_5 DJ^tD \end{pmatrix} \\ = & \begin{pmatrix} u'_1 J & X'J \\ JX' & -u'_5 J \end{pmatrix}. \end{aligned} \tag{16.23.4}$$

If we apply this to the element $g = \exp tX$ from the 1-parameter subgroup of $\mathrm{Sp}(2,\mathbb{R})$ generated by an $X \in \mathfrak{sp}(2,\mathbb{R})$, we come to an element $\tilde{A}(g) \in \mathrm{SO}(\tilde{Q})$ and via conjugation with C we get $C^{-1}\tilde{A}(g)C = A(g) \in \mathrm{SO}(3,2)$, leading to an element $\phi(X) \in \mathfrak{so}(3,2)$.
As an example, we take $X = G_4 = \begin{pmatrix} & 1 && 0 \\ 1 &&& \\ &&& -1 \\ 0 && -1 & 0 \end{pmatrix}$, get

$$g = \exp tG_4 = \begin{pmatrix} C & S && 0 \\ S & C && \\ && C & -S \\ 0 && -S & C \end{pmatrix}, C := \mathrm{ch}\, t, S := \mathrm{sh}\, t$$

and, using $C^2 - S^2 = 1$,

$$\tilde{A}(g) = \begin{pmatrix} 1 & & & & 0 \\ & C^2 & 2CS & S^2 & \\ & CS & C^2+S^2 & CS & \\ & S^2 & 2CS & C^2 & \\ 0 & & & & 1 \end{pmatrix}, \text{ and } A(g) = C^{-1}\tilde{A}(g)C = \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & C^2+S^2 & 2CS & \\ & & 2CS & C^2+S^2 & \\ 0 & & & & 1 \end{pmatrix}.$$

Remembering $C := \operatorname{ch} t, S := \operatorname{sh} t$, and deriving, with $\frac{d}{dt}(\operatorname{sh} t)|_{t=0} = 0$, we get

$$\begin{pmatrix} 0 & & & & 0 \\ & 0 & & & \\ & & 0 & 2 & \\ & & 2 & 0 & \\ 0 & & & & 0 \end{pmatrix} = 2X_{34} \in \mathfrak{so}(3,2).$$

In the same way, we get the map $\phi : \mathfrak{sp}(2) \to \mathfrak{so}(3,2)$ given by

(16.23.5)
$$\begin{aligned} &K_1 \mapsto X_{23}, && A_1 \mapsto X_{15} + X_{24}, \quad G_3 \mapsto -2X_{35}, \\ &K_2 \mapsto -X_{12} - X_{45}, && A_2 \mapsto X_{15} - X_{24}, \quad G_4 \mapsto 2X_{34}. \\ &K_3 \mapsto X_{12} - X_{45}, && G_1 \mapsto X_{14} - X_{25}, \\ &K_4 \mapsto -X_{13}, && G_2 \mapsto X_{14} + X_{25}. \end{aligned}$$

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