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SOME DIFFERENTIAL OPERATORS IN THE THEORY OF JACOBI-FORMS

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This is the report of someone who, coming from arithmetic and function theory, got lost in representation theory while trying to consider the joint roof for elliptic modular forms, Maaß wave-forms, elliptic functions, Jacobi-forms and theta-functions. Thus, the following is not yet a summary of results but rather of some observations and questions to the experts, hoping they find the function-theoretic background interesting enough to look again at a - from a general point of view - very simple example, and take these notes as a base for further discussions.

All those functions mentioned above are automorphic forms or functions living on

$$X = \mathbb{C} \times \mathbb{H} \qquad (\mathbb{H} = \{\omega \in \mathbb{C},\ \mathrm{Im}\,\omega > 0\})$$

This space is a homogeneous space for the semi-direct product

$$\mathbb{R}^2 \rtimes SL_2(\mathbb{R})$$

But as becomes clear for instance in Igusa [I] or Eichler-Zagier [E+Z] the right group to look at is the semidirect product of a three-dimensional Heisenberg group $(H\mathbb{R})$ with $SL_2(\mathbb{R})$. This group has been introduced by several authors, but there still seems to be no generally accepted name. In Igusa's notation it is denoted by $\mathbb{B}(\mathbb{R})$. Kirillov([Ki]p.287) uses $St(1,\mathbb{R})$, and this notation will be followed here.

The general method of representation theory, so very successful for the study of automorphic forms for semisimple or reductive groups, tells to look for the irreducible unitary representations of the group and try to find appropriate models living on the group or the space X. And these models can be characterized as eigenfunctions of differential operators (the Laplace Operator in simple cases). Now the problem here is: While the types of irreducible unitary representations of $St(1,\mathbb{R})$ are known by Mackey's method (and will be reviewed here in a first part), the space X is not symmetric, not even weakly symmetric but only a generalized symmetric space in the sense of Kowalski [Ko]. Consequently the invariant differential operators on X do not form a commutative ring. But by Helgason [He_1],[He_2] they may be characteri ed from within the Lie algebra $St(1,\mathbb{R})$ of the group $St(1,\mathbb{R})$ and several features from the theory of $SL(2,\mathbb{R})$ extend to this case: There is a natural Laplacian, which breaks up into two nice (noncommuting) invariant operators. One can try to use these operators to classify the functions mentioned at the beginning. This will be done in the second part of these notes. One further observation to be made here still mystifies the author but must have a very simple explanation: there are joint eigenfunctions for those two differential operators of second order (in their action on a certain bundle), which are just the exponentials of the potential of the Kähler-metric generating the Laplacian.

As an application of the formalism developed here, there will be given an explanation for the fact that Jacobi's theta function satisfies the heat equation.

I owe much to the inspiring atmosphere of the IHES and to hints by so many people at the IHES, in Paris and in Hamburg, and it is better that I don't mention any names here and instead give my thanks globally.

1. Notations

The following groups will be used:

i) The semi-direct product

$$G' := \mathbb{R}^2 \rtimes SL(2,\mathbb{R})$$

with the multiplication law(for $\lambda,\mu,\lambda',\mu' \in \mathbb{R}; M,M' \in SL(2,\mathbb{R})$)

$$(\lambda,\mu;M)(\lambda',\mu';M')=((\lambda',\mu')+(\lambda,\mu)M';MM'),$$

which can be understood by the identification

$$G' \longleftrightarrow GL(3,\mathbb{R})$$

$$(\lambda,\mu,M) \longmapsto \begin{pmatrix} 1 & \lambda & \mu \\ 0 & a & b \\ 0 & c & d \end{pmatrix}, \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

ii) The Heisenberg Group $H(\mathbb{R})$ with the multiplication law (for $\lambda,\mu,\varkappa,\lambda',\mu',\varkappa' \in \mathbb{R}$)

$$(\lambda,\mu;\varkappa)(\lambda'\mu';\varkappa')=(\lambda+\lambda',\mu+\mu';\varkappa+\varkappa' + \det\begin{pmatrix} \lambda & \mu \\ \lambda' & \mu' \end{pmatrix})$$

(Here and in the following the notation of Eichler and Zagier([E+Z]) is adopted, other authors use a factor one half in the third term of the multiplication law).

iii) The semi-direct product

$$St(1,\mathbb{R}) = H(\mathbb{R}) \rtimes SL(2,\mathbb{R})$$

with the multiplication law

$$(*)[\lambda,\mu,\varkappa;M][\lambda'\mu',\varkappa';M']=[(\lambda',\mu')+(\lambda,\mu)M,\varkappa+\varkappa'+\det\begin{pmatrix}(\lambda,\mu)M' \\ (\lambda'\mu')\end{pmatrix}; MM']$$

which also can be understood as a multiplication law of a subgroup of $GL(4,\mathbb{R})$ (e.g.[E+Z] p.71).

iv) The Jacobi group

$$G^J := St'(1,\mathbb{R}) := (H(\mathbb{R})/\mathbb{Z}) \rtimes SL(2,\mathbb{R})$$
$$= \mathbb{R}^2 . S^1 \rtimes SL(2,\mathbb{R})$$

whose multiplication law can be obtained easily from (*) by the change of variable

$$\zeta = e^{2\pi i\kappa}$$

for the 1-sphere S^1.

As usual the following groups will be looked at as subgroups of the above mentioned groups (when possible):

$$N = \{n(x)=\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix};\ x \in \mathbb{R}\},\ T = \{t(a)=\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}, a \in \mathbb{R}, a > 0\}$$

$$B = \{b(x,y):=\begin{pmatrix}y^{\frac{1}{2}} & xy^{-\frac{1}{2}}\\ 0 & y^{-\frac{1}{2}}\end{pmatrix}, x \in \mathbb{R}; y \in \mathbb{R}, y > 0\} = N\,T$$

$$K = SO(2) = \{r(\vartheta) = \begin{pmatrix}\cos\vartheta & \sin\vartheta\\ -\sin\vartheta & \cos\vartheta\end{pmatrix}\}$$

$$K^J = S^1 \times SO(2) = \{(\zeta, r(\vartheta))\}$$

K and K^J are maximal compact subgroups of G' resp. G^J with

$$G^J/K^J = G'/K = X .$$

In the three cases i, iii and iv the groups are semi-direct products of a well known normal subgroup $\mathfrak{n}$ with $SL(2,\mathbb{R}) =: G_0$. This allows a complete classifacation of their equivalence classes of irreducible unitary representations by "Mackey's method"[Ma_1, Ma_2], which has been applied especially by Duflo [Du] to get a parametrization of the unitary dual $\hat{G}$ for an arbitrary algebraic group G (see too the thesis of Sliman [SL]). Though in principle well known, this parametrization will be developed in the next paragraph.

Representation means here a continuous representation in some Hilbertspace .

2. Mackey's method for $\mathbb{R}^2 \rtimes SL(2,\mathbb{R})$, $St(1,\mathbb{R})$ and G^J.

The case of $G' = \mathbb{R}^2 \rtimes SL(2,\mathbb{R})$ is the simplest, because here the normal subgroup $\mathfrak{n} = \mathbb{R}^2$ is abelian.
The unitary dual $\hat{\mathfrak{n}}$ of $\mathbb{R}^2$ is again $\mathbb{R}^2$ and $G_0 = SL(2,\mathbb{R})$ operates by

$$\hat{\mathfrak{n}} \ni \chi = \chi^{\,{}^t(P,Q)} \longmapsto \chi M = \chi^{M^{-1}\cdot\,{}^t(P,Q)} \quad (P,Q \in \mathbb{R}, M \in G_0) .$$

$SL(2,\mathbb{R})$ operates transitively on $\mathbb{R}^2 \setminus (0,0)$, so there are two G_o-orbits in $\hat{\mathfrak{N}}$, represented by $\chi_o = \chi^{\pm(0,0)}$ and say by $\chi_1 = \chi^{\pm(1,0)}$. The isotropy subgroups in G_o of these characters are

$$G_{o\chi_o} = SL(2,\mathbb{R}) \quad \text{resp.} \quad G_{o\chi_1} = N$$

Therefore, by [Ma_1]p.77 there exist exactly two kinds of irreducible unitary representations of G', namely

I, the usual irreducible unitary representations of $SL(2,\mathbb{R})$ extended trivially to G', that is to say the principal series, the supplementary series and the discrete series (see for instance Lang [La_1]p.17)

II the induced representations $U^{\chi_1 L_S}$ of G' induced for each $S \in \mathbb{R}$ by the representation

$$\chi_1 L_S : (\lambda,\mu;1)(0;n(x)) \longmapsto e^{i(\lambda+xS)}$$

of the subgroup

$$\mathbb{R}^2 \rtimes N =: N^J$$

of G'.

In the case of $G = St(1,\mathbb{R})$ or $G^J = St'(1,\mathbb{R})$ the normal subgroup $\mathfrak{N}$ is no longer abelian, so by [Ma_2] projective representations of G_o enter:

The unitary dual of the Heisenberg-group $\mathfrak{N} = H(\mathbb{R})$ may be parametrized by $\mathbb{R}^2 \cup \mathbb{R}^*$ where the elements (P,Q) of $\mathbb{R}^2$ belong to the representations $\pi = \chi(P,Q)$ trivial on the center $C(\mathbb{R}) = \{(0,0,\varkappa);\ \varkappa \in \mathbb{R}\}$ of $H(\mathbb{R})$ and the elements m of $\mathbb{R}^*$ belong to representations π_m of $\mathfrak{N}$, which restricted to $C(\mathbb{R})$ give the character

$$\pi_m(0,0;\varkappa) = e^{2\pi i m\varkappa}.$$

This and three models of this representation (the Schrödinger-representation, the lattice-representation and the Fock-representation) are discussed in Cartier [Ca].

G operates on $\hat{\mathfrak{N}}$ by

$$\pi \to \pi^g \text{ with } (\pi^g)(n) = \pi((\lambda_o,\mu_o)M^{-1}, \varkappa_o + 2\det\begin{pmatrix}\lambda\mu\\ \lambda_o\mu_o\end{pmatrix})$$

$$\text{for } g = (\lambda,\mu,\varkappa;M) \text{ and } n = (\lambda_o,\mu_o,\varkappa_o).$$

For the representations π of the type $\chi(P,Q)$ there are the two G_o-orbits, which give the two types of representations of $St(1,\mathbb{R})$ being simply the trivial extensions of the representations I and II of G' discussed above.

For each representation π of the type $\pi_m (m \neq 0)$ there is a type III of representations of $St(1,\mathbb{R})$: The orbit $\pi_m{}^G$ consists only of π_m itself, because $\pi_m{}^g$ and π_m giving the same character on $C(\mathbb{R})$

$$\pi_m{}^g(0,0;\varkappa_o) = \pi_m(0,0;\varkappa_o) = e^{2\pi i m \varkappa_o}$$

are equivalent by the theorem of Stone - von Neumann.
So there exists for each $g \in G$ a unitary operator M_g with

$$\pi^g(n) = M_g \pi(n) M_g^{-1} \quad \text{for all } n \in \mathfrak{N}.$$

There is $M_g = \pi(g)$ for $g \in \mathfrak{N}$ and M_g is uniquely determined up to a multiplicative constant of modulus one. Thus M gives a projective representation of G which composed with the canonical projection $G \to G/\mathfrak{N} = G_o$ gives for all m (see [Ki] p.288/9) the Segal-Shale-Weil representation $\mathbb{R}$ of G_o for an - up to a trivial multiplicator system - well-defined cocycle ω of order two determined for instance in Lion-Vergne [L + V] p.56). All unitary irreducible representations of G can be obtained by tensoring M with the $1/\omega$-representations of $G_o = SL(2,\mathbb{R})$. These are for instance listed in the discussion of the metaplectic group $Mp(1,\mathbb{R})$ in Gelbart [Ge_1] p.77.

For the unitary dual of the group G^J there are to be retained from the representations just discussed exactly those which are trivial on $C(\mathbb{Z})$. Thus in the discussion above nothing is changed in type I and II, and for type III only $m \in \mathbb{Z} \setminus 0$ is to be taken.

The aim of the following paragraphs is to identify models of these representations with spaces of functions characterized as eigenfunctions of differential operators on the group G^J or on the space $X = G^J/K^J$. The base for this discussion will be the study of the Liealgebra $st(1,\mathbb{R})$ of $St(1,\mathbb{R})$ and some of its representations in different spaces.

3. Coordinization, a G^J-invariant metric and Laplacian on X

It seems to be convenient to use several coordinizations:
The space

$$X = \mathbb{C} \times \mathbb{H}$$

will be described as a four-dimensional $\mathbb{R}$-space either by the complex coordinates

$$v = \alpha+i\beta \in \mathbb{C},\ \omega = x+iy \in \mathbb{H} \quad \text{and their conjugates } \bar{v},\bar{\omega}$$

or the real coordinates

$$\alpha,\ \beta,\ x,\ y\ ,$$

or the real coordinates

$$p,\ q,\ x,\ y \quad \text{with} \quad v = p\omega+ q\ .$$

G^J operates on X by

$$(v,\omega) \mapsto \Big(\frac{v+\lambda\omega+\mu}{c\omega+d},\ \underbrace{\frac{a\omega+b}{c\omega+d}}_{=:M(\omega)}\Big) = g(v,\omega)$$

for

$$g = [(\lambda,\mu),\zeta,M] \in G^J\ ,\ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$$

This gives

$$(p,q,\omega) \longmapsto (((p,q)+(\lambda,\mu))M^{-1}\ ,\ M(\omega))$$

G^J/K^J can be identified naturally with X via

$$gK^J = [(\lambda,\mu),\zeta,M]K^J \longmapsto \Big(\frac{\lambda i+\mu}{ci+d}\ ,\ \frac{ai+b}{ci+d}\Big)$$

in the v,ω coordinates. In the (p,q,ω)-coordinates this means

$$gK^J \longmapsto ((\lambda,\mu)M^{-1},\ M(i))$$

and suggests the following coordinization of G^J

$$G^J \ni g = [(p,q),\zeta;1]\,[0,0,0;n(x)t(y^{1/2})r(\vartheta)]\ .$$

Here the notations of the first paragraph are used and give

$$gK^J \longmapsto (p,q;\omega = x+iy)\ .$$

As communicated to the author by Kähler and already mentioned in [Be_1] there is a family of G^J-invariant Kähler metrics given by the closed exterior two form

$$\Omega = d\bar{d}F \quad \text{with} \quad F = c_1 \log(\omega-\bar{\omega}) - ic_2 \frac{(v-\bar{v})^2}{\omega-\bar{\omega}},\ c_1,c_2 > 0$$

namely

$$ds^2 = -c_1 \frac{d\omega_1 d\bar{\omega}}{(\omega-\bar{\omega})^2} + \frac{2ic_2}{\omega-\bar{\omega}}\left(dv - \frac{v-\bar{v}}{\omega-\bar{\omega}}\,d\omega\right)\wedge\left(d\bar{v} - \frac{v-\bar{v}}{\omega-\bar{\omega}}\,d\bar{\omega}\right)$$

or

$$ds^2 = c_1 \frac{dx^2 - dy^2}{4y^2} + c_2 \frac{1}{y}\left((x^2+y^2)dp^2 + 2y\,dp\,dq + dq^2\right).$$

To this metric belongs the Laplace-Operator

$$\Delta = \frac{1}{c_1} 4y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{1}{c_2}\,\frac{1}{y}\left(\frac{\partial^2}{\partial p^2} - 2x\frac{\partial^2}{\partial p\,\partial q} + (x^2+y^2)\frac{\partial^2}{\partial q^2}\right)\ .$$

By the way, X is a "generalized symmetric space" in the sense of Kowalski [Ko] : for each point $(v,\omega) \in X$ there exists an isometry of order 4 fixing precisely the point (v,ω). For $(0,i)$ this automorphism is given by $g = (0;(-{}^{0\,1}_{1\,0}))$.

4. The Liealgebra $\mathfrak{g} = st(1,\mathbb{R})$ of $St(1,\mathbb{R})$ and G^J and a realization by differential operators

The Liealgebra $\mathfrak{g}$ of $St(1,\mathbb{R})$ may be generated by elements T_i $(i=0,\dots,5)$ with the multiplication table

	T_o	T_1	T_2	T_3	T_4	T_5
T_o	0	0	0	0	0	0
T_1	0	0	$2T_o$	T_1	T_2	T_2
T_2	0	$-T_o 2$	0	$-T_2$	T_1	$-T_1$
T_3	0	$-T_1$	T_2	0	$2T_5$	$2T_4$
T_4	0	$-T_2$	$-T_1$	$-2T_5$	0	$-2T_3$
T_5	0	$-T_2$	T_1	$-2T_4$	$2T_3$	0

Here the T_i are chosen to give the one parameter subgroups

$$\exp tT_o = [0,0,t;1],\ \exp tT_1 = [t,0,0;1], \exp tT_2 = [0,t,0;1]$$

$$\exp tT_3 = [0,0,0;\begin{pmatrix} e^t & 0 \\ 0 & e^{-t}\end{pmatrix}],\ \exp tT_4 = [0,0,0;\begin{pmatrix}\cosh t & \sinh t \\ \sinh t & \cosh t\end{pmatrix}],$$

$$\exp tT_5 = [0,0,0;\begin{pmatrix}\cos t & \sin t \\ -\sin t & \cos t\end{pmatrix}] .$$

By the realization of $\mathfrak{g}$ as the subalgebra of polynomials of degree ≤ 2 in the Weylalgebra of non-commutative polynomials in p, q with the commutation law $pq-qp=1$ (see [Ki]p.28) these elements T_i may be identified in the following way

$$T_o = 1/2, \qquad T_1 = p, \qquad T_2 = q,$$

$$T_3 = \tfrac{1}{2}(pq+qp),\ T_4 = \tfrac{1}{2}(q^2-p^2),\ T_5 = \tfrac{1}{2}(p^2+q^2)$$

The T_i for $i=1,\dots,5$ go by the canonical map in the Liealgebra $\mathfrak{g}'$ of $\mathbb{R}^2 \rtimes SL(2,\mathbb{R})$ to elements T_i' given by the matrices

$$T_1' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad T_2' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$T_3' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad T_4' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad T_5' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} .$$

For

$$\mathfrak{k} = \langle T_o, T_5 \rangle \quad \text{resp.} \quad \mathfrak{k}' = \langle T_5' \rangle$$

the Liealgebra of K^J resp. $K = SO(2)$ and

$$\mathfrak{m} = \langle T_1, T_2, T_3, T_4 \rangle \quad \text{resp.} \quad \mathfrak{m}' = \langle T_1', T_2', T_3', T_4' \rangle$$

there are vector-space direct sum decompositions

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m} \quad \text{resp.} \quad \mathfrak{g}' = \mathfrak{k}' + \mathfrak{m}'$$

with

$$[k, \mathfrak{m}] \subset \mathfrak{m} \quad \text{resp.} \quad [k', \mathfrak{m}'] \subset \mathfrak{m}' \; .$$

In consequence $X = G^J/K^J$ is a reductive space, and Helgason's theory of invariant differential operators can be applied and this will be done in the next paragraph.

For several purposes it is convenient to look, too, at the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$ and introduce the operators

$$Z_o = i\, T_o \qquad\qquad Z = i\, T_5$$

$$X_\pm = \frac{1}{\sqrt{2}} (T_3 \mp i\, T_4) \; , \quad Y_\pm = \frac{1}{\sqrt{2}} (T_1 \mp i\, T_2)$$

Then these elements are eigenvectors for ad k and the multiplication table becomes

	Z_o	Y_+	Y_-	X_+	X_-	Z
Z_o	0		————			0
Y_+		0	$2Z_o$	0	$\sqrt{2}Y_-$	$-Y_+$
Y_-		$-2Z_o$	0	$\sqrt{2}Y_+$	0	Y_-
X_+		0	$-\sqrt{2}Y_+$	0	$2Z$	$-2X_+$
X_-		$-\sqrt{2}Y_-$	0	$-2Z$	0	$2X_-$
Z	0	Y_+	$-Y_-$	$2X_+$	$-2X_-$	0

The T_i define left-invariant vector-fields on St(1,n) (and G^J) given by the following formula (where the coordinization $g = (p,q,\kappa;1)(0,M)$ has to be kept in mind): For a function $\phi = \phi(p,q,\kappa,x,y,\vartheta)$ we have

$$\mathcal{L}_{T_i}\phi = \frac{d}{dt}\phi(g\ \exp tT_i)\Big|_{t=0}$$

and this gives after some calculations (for the last three operators see for instance [La]p.114/5)

$$\mathcal{L}_{T_0} = \frac{\partial}{\partial\kappa}$$

$$\mathcal{L}_{T_1} = y^{-1/2}\Big[\cos\vartheta\frac{\partial}{\partial p} - (y\sin\vartheta + x\cos\vartheta)\frac{\partial}{\partial q} - (p(y\sin\vartheta + x\cos\vartheta) + q\cos\vartheta)\frac{\partial}{\partial\kappa}\Big]$$

$$\mathcal{L}_{T_2} = y^{-1/2}\Big[\sin\vartheta\frac{\partial}{\partial p} + (y\cos\vartheta - x\sin\vartheta)\frac{\partial}{\partial q} - (p(y\cos\vartheta - x\sin\vartheta) - q\sin\vartheta)\frac{\partial}{\partial\kappa}\Big]$$

$$\mathcal{L}_{T_3} = -2y\sin 2\vartheta\frac{\partial}{\partial x} + 2y\cos 2\vartheta\frac{\partial}{\partial y} + \sin 2\vartheta\frac{\partial}{\partial\vartheta}$$

$$\mathcal{L}_{T_4} = 2y\cos 2\vartheta\frac{\partial}{\partial x} + 2y\sin 2\vartheta\frac{\partial}{\partial y} - \cos 2\vartheta\frac{\partial}{\partial\vartheta}$$

$$\mathcal{L}_{T_5} = \frac{\partial}{\partial\vartheta}$$

and

$$\sqrt{2}\cdot\mathcal{L}_{Y_+} = y^{-1/2}\Big[e^{-i\vartheta}\Big(\frac{\partial}{\partial p} - (x+iy)\frac{\partial}{\partial q}\Big) - e^{-i\vartheta}(p(x+iy)+q)\frac{\partial}{\partial\kappa}\Big]$$

$$\sqrt{2}\cdot\mathcal{L}_{Y_-} = y^{-1/2}\Big[e^{i\vartheta}\Big(\frac{\partial}{\partial p} - (x-iy)\frac{\partial}{\partial q}\Big) - e^{i\vartheta}(p(x-iy)+q)\frac{\partial}{\partial\kappa}\Big]$$

$$\sqrt{2}\cdot\mathcal{L}_{X_+} = -ie^{-2i\vartheta}\Big(2y\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big) - \frac{\partial}{\partial\vartheta}\Big)$$

$$\sqrt{2}\cdot\mathcal{L}_{X_-} = ie^{2i\vartheta}\Big(2y\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big) - \frac{\partial}{\partial\vartheta}\Big).$$

5. $U(\mathfrak{m})^{\mathfrak{k}}$ and G^J-invariant differential operators on $\mathcal{X}$

In the usual theory (see for instance [Bo] or [Ge_2]) automorphic forms for a discrete subgroup Γ of a semisimple group G with maximal compact subgroup K are characterized as left Γ-invariant, right K-covariant functions on G with a certain growth condition which are eigenfunctions of the differential operators on G corresponding to the center of the enveloping algebra $U(\mathfrak{g})$ of the Lie algebra of G. The center of $U(\mathfrak{g})$ in the case of the group G^J resp. $St(1,\mathbb{R})$ is generated by T_0 and not rich enough to give anything useful here.*) But there is the possibility of looking at the G^J-invariant differential operators $\mathbb{D}(G^J/K^J)$ on $\mathcal{X} = G^J/K^J$, which may be thought of as left-G-invariant and right-K-invariant vector fields on G. More precisely, in the case of a reductive homogeneous space there is by Helgason ([He_1] or [He_2] Th.2.8) a linear bijection of $\mathbb{D}(G^J/K^J)$ with $I(\mathfrak{m})$, the set of $Ad_{G^J}(K^J)$-invariants in the symmetric algebra $S(\mathfrak{m})$ of the complement $\mathfrak{m} = \langle T_1,\dots,T_4\rangle$ of $\mathfrak{k} = \langle T_0,T_5\rangle$ in $\mathfrak{g}$. Explicitly, if $Q \in I(\mathfrak{m})$, $f \in C^\infty(X)$ and Φ is the canonical lifting of f to G then

$$(D_{\lambda(Q)}f)(gK^J) = Q\left(\frac{\partial}{\partial t_1},\dots,\frac{\partial}{\partial t_4}\right)\Phi(g\exp(t_1T_1+\dots+t_4T_4))\Big|_{t=0}.$$

Here λ stands for the symmetrizing operator: $I(\mathfrak{m})$ can be described as the $\mathrm{ad}\,\mathfrak{k}$-invariants of $U(\mathfrak{m})$ and there is the following

Lemma: $U(\mathfrak{m})^{\mathfrak{k}}$ may be generated by

---

*) Borho tells me to be careful here: after a suitable localization the center is generated by two elements.

$$Q_1 = T_1^2 + T_2^2 \quad , \quad Q_2 = T_3^2 + T_4^2$$
$$Q_3 = \lambda((T_1^2 - T_2^2)T_3 + (T_1T_2 + T_2T_1)T_4)$$
$$Q_4 = \lambda((T_1^2 - T_2^2)T_4 + (T_1T_2 + T_2T_1)T_3) .$$

Proof: Elements $P \in U(\mathfrak{g})$ will be thought of as symmetrized polynomials in $T_0,\ldots,T_5$ and similarly $Q \in U(\mathfrak{m})$ as polynomials in $T_1,\ldots,T_4$. $Q \in U(\mathfrak{m})^{\mathfrak{k}}$ is equivalent to $\mathrm{ad}Z_0Q = \mathrm{ad}ZQ = 0$. Z_0 being in the center of $\mathfrak{g}$, the first equality gives no condition. To evaluate the second equality, we can deduce from the multiplication table that

$$(*) \qquad \mathrm{ad}Z(Y_+^jY_-^kX_+^\ell X_-^m) = (j-k+2\ell-2m)(Y_+^jY_-^kX_+^\ell X_-^m)$$

holds. This gives zero for the four "basic" combinations

$$j=k=1 \ ; \quad \ell=m=1 \ ; \quad j=2,\ m=1 \ ; \quad k=2,\ \ell=1 \ ,$$

i.e. the following elements of $U(\mathfrak{m}^{\mathbb{C}})$ are $\mathfrak{k}$-invariant

$$P_1 = Y_+Y_- + Y_-Y_+$$
$$P_2 = X_+X_- + X_+X_-$$
$$P_3 = Y_+^2X_- + Y_+X_-Y_+ + X_-Y_+^2$$
$$P_4 = Y_-^2X_+ + Y_-X_+Y_- + X_+Y_-^2 .$$

And it follows from (*) that all $\mathfrak{k}$-invariant elements of $U(\mathfrak{m}^{\mathbb{C}})$ may be algebraically combined from $P_1,\ldots,P_4$. In view of the equations $\overline{Y}_\pm = Y_\mp$ and $\overline{X}_\pm = X_\mp$, the operators P_1 and P_2 are already real (and equal to Q_1 resp. Q_2). Thus together with

$P_3+P_4 = 2\mathrm{Re}P_3$ and $-i(P_3-P_4) = 2\mathrm{Jm}P_3$ (equal up to constant factor to Q_3 resp. Q_4) they generate the ring of $\mathfrak{k}$-invariants in $U(\mathfrak{m})$.

Using the expressions at the end of paragraph 4, this translates into the differential operators on G^J:

$$\Delta_1 = \mathcal{L}_{P_1} = \frac{1}{y}\Big[\frac{\partial^2}{\partial p^2} - (\omega+\bar\omega)\frac{\partial^2}{\partial p\partial q} + |\omega|^2\frac{\partial^2}{\partial q^2} - (v+\bar v)\frac{\partial^2}{\partial p\partial\kappa} + (\omega\bar v+\bar\omega v)\frac{\partial^2}{\partial q\partial\kappa} + |v|^2\frac{\partial^2}{\partial\kappa^2}\Big]$$

$$\Delta_2 = \mathcal{L}_{P_2} = 4y^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) - 4y\frac{\partial^2}{\partial x\partial\vartheta} + \frac{\partial^2}{\partial\vartheta^2}$$

$$\Delta_3 = \mathcal{L}_{\mathrm{Re}P_3} = \frac{3}{\sqrt{2}}\Big[\frac{\partial^2}{\partial p^2} - 2\mathrm{Re}\,\omega\frac{\partial^2}{\partial p\partial q} + \mathrm{Re}(\omega^2)\frac{\partial^2}{\partial q^2} - 2\mathrm{Re}\,v\frac{\partial^2}{\partial p\partial\kappa} + 2\mathrm{Re}(v\omega)\frac{\partial^2}{\partial q\partial\kappa} + \mathrm{Re}(v^2)\frac{\partial^2}{\partial\kappa^2}\Big]\frac{\partial}{\partial y} - \frac{3}{\sqrt{2}}\Big[-2\mathrm{Jm}\,\omega\frac{\partial^2}{\partial p\partial q} + \mathrm{Jm}(\omega^2)\frac{\partial^2}{\partial q^2} - 2\mathrm{Jm}\,v\frac{\partial^2}{\partial p\partial\kappa} + 2\mathrm{Jm}(v\omega)\frac{\partial^2}{\partial q\partial\kappa} + \mathrm{Jm}(v^2)\frac{\partial^2}{\partial\kappa^2}\Big]\Big[\frac{\partial}{\partial x} - \frac{1}{2y}\frac{\partial}{\partial\vartheta}\Big] - \frac{3}{2}\Delta_1$$

$$\Delta_4 = \mathcal{L}_{\mathrm{Jm}P_3} = \frac{3}{\sqrt{2}}\Big[\frac{\partial^2}{\partial p^2} - 2\mathrm{Re}\,\omega\frac{\partial^2}{\partial p\partial q} + \ldots\Big]\Big[\frac{\partial}{\partial x} - \frac{1}{2y}\frac{\partial}{\partial\vartheta}\Big] + \frac{3}{\sqrt{2}}\Big[-2\mathrm{Jm}\,\omega\frac{\partial^2}{\partial p\partial q} + \ldots\Big]\frac{\partial}{\partial y} - \frac{\partial}{\partial\kappa}\,.$$

Applied to functions Φ on G^J, which are constant on the classes gK^J, Δ_1 and Δ_2 give the operators

$$\Delta_1^\circ = \frac{1}{y}\Big[\frac{\partial^2}{\partial p^2} - 2x\frac{\partial^2}{\partial p\partial q} + (x^2+y^2)\frac{\partial^2}{\partial q^2}\Big]$$

$$\Delta_2^\circ = 4y^2\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right],$$

which combine to give the Laplace operator given in paragraph 3. In the same way the operators Δ_3 and Δ_4 go down to G^J-invariant operators Δ_3° and Δ_4° on $\mathcal{X}$, which generate the ring $\mathbb{D}(G^J/K^J)$ of G^J-invariant operators on $\mathcal{X}$.

In the spirit of Selberg ([Se])and Helgason ([He_1],[He_2]) it is now natural to define for $\lambda = (\lambda_1,\dots,\lambda_4) \in \mathbb{C}^4$

$$\mathcal{E}_\lambda = \{f \in \mathcal{C}^\infty(\mathcal{X});\ \Delta_i^\circ f = \lambda_i f \text{ for } i=1,\dots,4\}$$

and the eigenspace representation T_λ of G^J given by

$$T_\lambda(g)f(z) = f(g^{-1}(z)) \text{ for all } f \in \mathcal{E}_\lambda \text{ and } z \in \mathcal{X}.$$

Then there is of course Helgason's problem ([He_1] p. 241): determine the joint eigenspaces $\mathcal{E}_\lambda$ and identify the representation T_λ; in particular, for what λ is T_λ irreducible? Lacking an answer here, which goes beyond the usual SL_2-theory, the problem will be broadened by replacing the functions on $\mathcal{X}$ by sections of certain vector bundles over $\mathcal{X}$ (i.e. certain automorphic forms) and replacing the Δ_i° by differential operators on the space of sections invariant under the action of G^J.

6. Jacobi Forms

Jacobi-forms recently have been thoroughly studied by Eichler and Zagier [E+Z]. Other references in this context are (see the introduction of [E+Z]) Pyatetskii-Shapiro [Pya], Shimura [Shi_1], [Shi_2], Berndt [Be_1]-[Be_3] and Frenkel [Fe]. Eichler and Zagier introduce for complex functions h living on $\mathcal{X} = \mathbb{C}\times\mathbb{H}$ and elements $g = [(\lambda,\mu),\zeta;M] \in G^J$ the operator

$$h \longrightarrow h|_{k,m}[g]$$

defined by *)

$$(h|_{k,m}[(\lambda,\mu),\zeta,M])(v,\omega)$$

$$= \underbrace{\zeta^m(c\omega+d)^{-k}e^m\left(-\frac{c(v+\lambda\omega+\mu)^2}{c\omega+d}+\lambda^2\omega+2\lambda v+\lambda\mu\right)}_{=\, j_{k,m}(g;v,\omega)}h\left(\frac{v+\lambda\omega+\mu}{c\omega+d},\frac{a\omega+b}{c\omega+d}\right). \qquad (e^m(z):=e^{2\pi imz})$$

For a subgroup $\Gamma\subset SL(2,\mathbb{R})$ of finite index they put

$$\Gamma^J = \mathbb{Z}^2\rtimes\Gamma$$

and give the

<u>Definition</u>: <u>A Jacobi form of weight</u> k <u>and index</u> m ($k,\ m \in \mathbb{N}$) on Γ^J is a complex function h on $\mathcal{X} = \mathbb{C}\times\mathbb{H}$ with the properties

i) h is holomorphic on $\mathcal{X}$

ii) $h|_{k,m}[\gamma] = h$ for all $\gamma \in \Gamma^J$

iii) for each $M \in SL(2,\mathbb{Z})$ $h|_{k,m}[0,M]$ has a Fourier development of the form

$$\Sigma\ c(n,r)q^n\varepsilon^r \text{ with } c(n,r) = 0 \text{ unless } n\geq r^2/4m$$

$(q = e(\omega),\ \varepsilon = e(v))$.

If h satisfies a stronger condition iii') with $c(n,r) = 0$ unless

---

*) The order of the variables in $\mathbb{C}$ and $\mathbb{H}$ has been reversed here with respect to [E+Z], because for the author the prototype of functions with this transformation property is the Weierstrass - $\wp$ function, which still usually is written as $\wp = \wp(v,\omega)$.

$n>r^2/4m$, it is called a <u>cuspform</u>.

To get something non-trivial for $m = 0$ (and even $k = 0$) the condition i) has to be weakened to the condition

i') h is meromorphic.

This gives the notion of meromorphic Jacobi forms and Jacobi functions (see [Be_2]).

The first main result of Eichler and Zagier (th.1.1) is the finite-dimensionality of the space $J_{k,m}(\Gamma)$ of Jacobi forms of weight k and index m on Γ^J. Among many other things they also look at the usual lifting of functions h on $\mathfrak{X}$ to functions Φ on G^J given by

$$h \xrightarrow[\varphi_{km}]{} \Phi_h \text{ with } \Phi_h(g) = (h|_{km}[g])(0,i),$$

and thus establish an identification (th.1.4) between the space

$\mathcal{X}_{k,m}(\Gamma^J)$ of complex functions h on $\mathfrak{X}$ satisfying the condition ii) in the definitions of the Jacobi forms

and the space

$\mathcal{X}_{k,m}(\Gamma^J)$ of complex functions Φ on G^J left-invariant under Γ^J and transforming on the right by the representation

$$(*) \qquad \Phi(g[(0,0),\zeta;r(\vartheta)]) = \zeta^m e^{ik\vartheta}\Phi(g)$$

of the maximal compact subgroup $K^J = S^1 \times SO(2)$ of G^J.

A further step in the characterization of Jacobi forms as functions on G^J is given by the following observation:
Applied to functions Φ_h on G^J coming from the lifting φ_{km}, that is functions of the type

$$\Phi_h(g) = h((g(0,i))j_{km}(g;0,i) = \zeta^m e^{ik\vartheta} y^{k/2} e^{2\pi impv} f(p,q;x,y),$$

the operators from paragraph 5 give (remembering $\frac{\partial}{\partial k} = 2\pi i \zeta \frac{\partial}{\partial \zeta}$)

$$\mathcal{L}_{Y_+}\Phi = Ce^{-i\vartheta}(2y)^{-\frac{1}{2}}(f_p - \omega f q) \qquad (C = \zeta^m e^{ik} y^{k/2} e^{2\pi impv})$$

$$\mathcal{L}_{Y_-}\Phi = Ce^{i\vartheta}(2y)^{-\frac{1}{2}}(f_p - \overline{\omega} f g + 2p(\omega - \overline{\omega})(2\pi im)f)$$

$$\mathcal{L}_{X_+}\Phi = \frac{i}{\sqrt{2}}Ce^{-2i\vartheta}(-2y(f_x + if_y))$$

$$\mathcal{L}_{X_-}\Phi = \frac{i}{\sqrt{2}}Ce^{2i\vartheta}(2y(f_x - if_y) + (4yp^2(2\pi im) - 2ik)f).$$

and consequently

$$\Delta_i \Phi = C\Delta_i^{(k,m)} f$$

with

$$\Delta_1^{(k,m)} = y^{-1}\left(\frac{\partial^2}{\partial p^2} - 2x\frac{\partial^2}{\partial p \partial q} + (x^2+y^2)\frac{\partial^2}{\partial q^2}\right) + 4pi(2\pi im)\left(\frac{\partial}{\partial p} - \omega\frac{\partial}{\partial q}\right) + 2i(2\pi im)$$

and

$$\Delta_2^{(k,m)} = 4y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - 2y(2ik - 4yp^2(2\pi im))\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) - 2k.$$

This shows

Remark: The lifting φ_{km} of a holomorphic Jacobi form h goes to an eigenfunction $\Phi_h = \Phi_{km}(h)$ of the operators $\Delta_i^{(k,m)}$ (i=1,2).

Here the following formalism is used: For functions

$$h(v,\bar{v};\omega,\bar{\omega}) = h(p\omega+q,p\bar{\omega}+q;\omega,\bar{\omega}) = f(p,q;\omega,\bar{\omega})$$

we have

$$\frac{\partial f}{\partial p} - \omega\frac{\partial f}{\partial q} = \frac{\partial h}{\partial \bar{v}}(\bar{\omega}-\omega) \;;\quad \frac{\partial f}{\partial p} - \bar{\omega}\frac{\partial f}{\partial q} = \frac{\partial h}{\partial v}(\omega-\bar{\omega})$$

$$\frac{\partial f}{\partial \omega} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = \frac{\partial h}{\partial \omega} + \frac{v-\bar{v}}{\omega-\bar{\omega}}\frac{\partial h}{\partial v}$$

$$\frac{\partial f}{\partial \bar{\omega}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{\partial h}{\partial \bar{\omega}} + \frac{v-\bar{v}}{\omega-\bar{\omega}}\frac{\partial h}{\partial \bar{v}}$$

(To simplify matters, there is no distinction made between functions of $(\omega,\bar{\omega})$ and (x,y).).

It is tempting to try to characterize the lifted Jacobi forms completely as automorphic forms on the group G^J. To do this the condition iii) concerning the Fourier development has to be understood as a boundedness condition for the function Φ_h. This is still an open problem to me. The answer should be connected with the representation-theoretic motivation for the automorphy-factor j_{km} (see Satake [S]). Anyway, the space $J_{k,m}(\Gamma^J)$ with the scalar product ([E+Z]p.25)

$$(h,h') = \int_{\Gamma^J\backslash\mathcal{X}} e^{-4\pi m\beta^2/y} h(v,\omega)\overline{h'(v,\omega)}y^{k-3}\,d\alpha\, d\beta\, dx\, dy$$

is a first candidate for a ~~(finite dimensional)~~ discrete series representation T_{km} of G^J given by

$$T_{km}(g)h = h|_{km}[g^{-1}] \quad \text{for } h \in J_{k,m}(\Gamma^J).$$

Starting from here something non-discrete is easily gotten by the procedure in the next paragraph.

7. Maass-Jacobi forms

In the classical SL_2-theory not only automorphic forms for the automorphy factor

$$j_k(M;\omega) = (c\omega+d)^{-k}$$

are studied; but also forms for the factor

$$\frac{j_k}{|j_k|}(M;\omega) = e^{ik\vartheta}$$

(See particularly: Petersson [Pe], Maass [Ma], Selberg [Se], Roelcke [$Ro_{1,2}$] and Elstrodt [El]; care has to be taken with the index k whose significance is sometimes different with different authors). These forms give for $k \in \mathbb{R}$ in general projective representations of $SL(2,\mathbb{R})$ resp. ordinary representations of the universal covering group $\widetilde{SL}(2,\mathbb{R})$ of $SL(2,\mathbb{R})$ (Sally [Sa]).

For the Jacobi group it is now natural to take as a factor

$$\mu_{km}(g;p,q,x,y,) = \frac{j_{km}}{|j_{km}|}(g;v,\omega) = \zeta^m\left(\frac{c\omega+d}{c\bar{\omega}+d}\right)^{k/2} e^{2\pi i m(\dots)}$$

and to define for complex functions f on $\mathcal{X}$ with $z = (p,q,x,y) \in \mathcal{X}$

$$f|[g;k,m](z) = f(g(z))\mu_{k,m}(g,z) \text{ for } k \in \mathbb{R} \text{ and } m \in \mathbb{Z}.$$

If $\vartheta = \arg z$ is to be chosen from the interval $(-\pi,\pi]$ there is a factor system (see Roelcke [Ro$_1$]p.295)

$$\sigma_k(M,N) = e^{2\pi i k w(M,N)},$$

w a cocycle taking values in $\{-1,0,1\}$ such that

$$f|[g_1g_2;k,m] = \sigma_k(M_1,M_2)(f|[g_1:k,m])|[g_2;k,m]$$

holds. Let further ν be a unitary multiplicator system for a discrete group $\Gamma^J = \mathbb{Z}^2 \rtimes \Gamma$ in G^J (with $-1 \in \Gamma$) such that

$$\nu(g_1g_2) = \sigma_k(M_1,M_2)\nu(g_1)\nu(g_2) \text{ and } \nu((0,-1)) = e^{-\pi i k},$$

then $\mathcal{J}_{k,m}(\Gamma)$ may be defined as the space of complex valued functions f on $\mathcal{X}$ with

i) f is real-analytic in all four variables p, q, x, y $\in X$

ii) $f|[\gamma;k,m] = \nu(\gamma)f$ for all $\gamma \in \Gamma$

iii) f takes a certain growth condition for the cusps, which will have to be specified elsewhere.

The automorphy factor μ_{km} gives in the usual way a <u>lifting map</u> $\Psi_{k,m}$ from functions f on $\mathcal{X}$ to functions Φ_f on G^J with

$$(*) \quad \begin{aligned}\Phi_f(g) &= f|[g;k,m](0,0;0,1)\\ &= \zeta^m e^{ik\vartheta} e^{2\pi i m(p^2x+pq)} f(p,q;x,y) .\end{aligned}$$

Here again the coordinization of G^J

$$g = [p,q;\zeta;1][0,0,0;n(x)t(y^{1/2})r(\vartheta)]$$

is used. With

$$D = \zeta^m e^{ik} e^{2\pi i m(p^2 x+pq)}$$

the operators from paragraph 5 give here applied to functions of the type (*)

$$\mathcal{L}_{Y_+}\Phi = De^{-i\vartheta}\tilde{\Lambda}_m f,\quad \tilde{\Lambda}_m = (2y)^{-1/2}\Big[\frac{\partial}{\partial p} - \omega\frac{\partial}{\partial q} - 2pyi(2\pi i m)\Big]$$

$$\mathcal{L}_{Y_-}\Phi = De^{i\vartheta}\tilde{K}_m f,\quad \tilde{K}_m = (2y)^{-1/2}\Big[\frac{\partial}{\partial p} - \overline{\omega}\frac{\partial}{\partial q} + 2pyi(2\pi i m)\Big]$$

$$\mathcal{L}_{X_+}\Phi = \frac{-i}{\sqrt{2}}De^{-2i\vartheta}\Lambda_{k,m} f,\quad \Lambda_{k,m} = 2y\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big) + 2yp^2(2\pi i m) - ik$$

$$\mathcal{L}_{Y_-}\Phi = \frac{i}{\sqrt{2}}De^{2i\vartheta}K_{k,m} f,\quad K_{k,m} = 2y\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big) + 2yp^2(2\pi i m) - ik ,$$

and consequently

$$\Delta_i\Phi = D\Delta_i^{[k,m]} f$$

with

$$\Delta_1^{[k,m]} = y^{-1}\Big[\frac{\partial^2}{\partial p^2} - 2x\frac{\partial^2}{\partial p\,\partial q} + (x^2+y^2)\frac{\partial^2}{\partial q^2} + 4y^2 p(2\pi i m)\frac{\partial}{\partial q} + (2py)^2(2\pi i m)^2\Big]$$

$$= \tilde{\Lambda}_m\tilde{K}_m + \tilde{K}_m\tilde{\Lambda}_m = 2\tilde{K}_m\tilde{\Lambda}_m + 2i(2\pi i m) = 2\tilde{\Lambda}_m\tilde{K}_m - 2i(2\pi i m)$$

and

$$\Delta_2^{[k,m]} = (2y)^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) + 4y(2yp^2(2\pi i m) - ik)\frac{\partial}{\partial x} + (2yp^2(2\pi i m) - ik)^2$$

$$= \frac{1}{2}(\Lambda_{k+2,m}K_{k,m} + K_{k-2,m}\Lambda_m) = \Lambda_{k+2,m}K_{k,m} + 2k$$

$$= K_{k-2,m}\Lambda_m - 2k .$$

As is to be expected, the operators $\tilde{\Lambda}_m$, $\tilde{K}_m$, $\Lambda_{k,m}$, $K_{k,m}$ have each a simple commutation property with respect to the operators $|[g;k,m]$. By straightforward computations it can be shown

<u>Lemma</u>: For complex functions f on $\mathcal{X}$ and all $g \in G^J$, $k,m \in \mathbb{R}$ the relations

$$\tilde{K}_m(f|[g;k,m\ ) = (\tilde{K}_m f)|[g;k+1,m]$$

$$\tilde{\Lambda}_m(f|[g;k,m]) = (\tilde{\Lambda}_m f)|[g;k-1;m]$$

$$K_{k,m}(f|[g;k,m]) = (K_{k,m}f)|[g;k+2,m]$$

$$\Lambda_{k,m}(f|[g;k,m]) = (K_{k,m}f)|[g;k-2,m]$$

hold.

As a corollary the $\Delta_i^{[k,m]}$ commute with the operators $|[g,k,m]$. So here again the question of eigenspace representations arises. This time there will be gotten only σ_k-representations of G^J, but there are quite nice joint eigenfunctions for the $\Delta_i^{[k,m]}$ (i = 1,2):

<u>Remark</u>: For

$$f = y^{k/2} e^{-2\pi m p^2 y}$$

there is

$$\Delta_1^{[k,m]} f = -4\pi m f \text{ and } \Delta_2^{[k,m]} f = -2kf\ .$$

It is further to be remarked that these eigenfunctions are of the form

$$f = e^F,$$

where F appeared in paragraph 3 as the potential of the G^J-invariant Kähler-metric, which is fundamental for this theory. This fact should leave a simple explanation, which should be got by further looking into Selberg [Se].

Starting from this f, one can in the usual way (see [E+Z]p.16ff) construct Eigenstein series which give automorphic forms on $\mathcal{X}$ and, being non-holomorphic eigenfunctions of $\Delta_i^{[k,m]}$ $(i = 1,2)$, will be called here <u>Maass-Jacobi forms</u>:

Formally it may be put for each $k, m \in \mathbb{R}$

$$E(p,q,x,y;k,m) = \sum_{\gamma \in \Gamma_\infty^J \setminus \Gamma_1^J} y^{k/2} e^{-2\pi m\, p^2} |[\gamma;k,m]$$

with $\Gamma_1^J = \mathbb{Z}^2 \rtimes SL(2, \quad) \supset \Gamma_\infty^J = \{[(0,\mu), \pm\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}], \mu_1 n \in \quad\}$.

The automorphy factor $\mu_{k,m}$ being of modulus one, the series converges for positive m at least as well as the series

$$\sum_{(c,d)=1} |c\omega + d|^{-k},$$

i.e. at least for $k \geqslant 4$, Here, too, further research has to be done.

8. Concluding remarks

The author is aware that the material presented here is rather preliminary and fragmentary. Real work should start now putting some of the fragments together, studying models of the representations given in the second paragraph by using the differential operators derived later on, and answer the questions arising. The author hopes though to have shown that something may be done this way for the Jacobi group, and he would be very glad if someone got interested enough to join forces with him in further studies.

At last, it is perhaps worthwhile to also state the following two observations:

More nonholomorphic eigenfunctions

Weierstrass $\wp$-function and their derivatives are annihilated by operators Δ_1° and Δ_2°. Another family of eigenfunctions for Δ_1 and Δ_2 (with eigenvalues $(0,-2k)$) is given by

$$\Phi = v^\ell y^{k/2} e^{ik\vartheta} \qquad (\ell \in \mathbb{Z}) .$$

This too may be used to get automorphic forms by the Eigenstein series procedure.

Jacobi's theta function and the heat equation

Jacobi's theta function

$$\vartheta(v,\omega) = \sum_{n \in \mathbb{Z}} e^{2\pi i(n^2\omega + 2nv)}$$

satisfies the heat equation

$$(8\pi i\frac{\partial}{\partial\omega} - \frac{\partial^2}{\partial v^2})\vartheta = 0.$$

ϑ being a Jacobi form of type (k,m) = (1/2,1) (see, for instance, [Mu]) and being in a space realizing the Segal-Shale-Weil representation of G^J (see [S]), this fact may be understood as a consequence of the formalism developed in paragraph 6:

Let h be a holomorphic Jacobi form on $\mathcal{X}$ of type (k,m) and Φ = Ch its lifting to a function on G^J. With the operators from paragraph 6, then there holds

$$\mathcal{L}_{X_+}\Phi = \mathcal{L}_{Y_+}\Phi = 0, \quad \mathcal{L}_{Z_0}\Phi = -2\pi m\Phi, \quad \mathcal{L}_{Z}\Phi = -k\Phi$$

and

$$\mathcal{L}_{X_+X_-}\Phi = -2k\Phi, \quad \mathcal{L}_{Y_+Y_-}\Phi = -4\pi m\Phi.$$

Using the formula given in 6 to change from the variables (p,q,x,y) to the variables (v,ω), it may be calculated

$$\mathcal{L}_{X_-}\Phi = \frac{i}{\sqrt{2}}Ce^{2i\vartheta}(2y(f_x-if_y)+(2\pi im)4p^2f-2ikf)$$

$$= \sqrt{2}Ce^{2i\vartheta}((\omega-\bar{\omega})h_\omega+(v-\bar{v})h_v+(2\pi im)\frac{(v-\bar{v})^2}{\omega-\bar{\omega}}h+kh)$$

and

$$\mathcal{L}_{Y_-}\Phi = Ce^{i\vartheta}(2y)^{-1/2}(f_p-\bar{\omega}f_q+(2\pi im)2p(\omega-\bar{\omega})f)$$

$$\mathcal{L}_{Y_-^2}\Phi = Ce^{2i\vartheta}i((\omega-\bar{\omega})h_{vv}+8\pi im(v-\bar{v})h_v+(2\pi im)^2 4\frac{(v-\bar{v})^2}{\omega-\bar{\omega}}h$$

$$+2(2\pi im)h)$$

Comparization gives for $k = \frac{1}{2}$ just

$$(-\sqrt{2}4\pi m \mathcal{L}_{X_-} - \mathcal{L}_{Y_-^2})\Phi = (\omega-\bar{\omega})C(-8\pi m h_{\omega} - i h_{vv}).$$

And as W. Borho kindly pointed out to the author, this expression is seen to be zero simply by looking at the Weil representation of $\mathfrak{g}$ realized as a subalgebra of the Weyl algebra $A_1(\mathbb{R})$ given in paragraph 4. There we have (see, for instance, [Ki]p.288) for a space $L^2(\mathbb{R})$ with coordinate u the representation

$$p \mapsto \frac{d}{du}, \qquad q \mapsto i\lambda u, \qquad 1 \mapsto i\lambda$$

$$p^2 \mapsto \frac{1}{i\lambda}\frac{d^2}{du^2}, \qquad qp \mapsto u\frac{d}{du}, \qquad q^2 \mapsto i\lambda u^2.$$

Introducing the operators defined in 4. gives, after a small calculation with $\lambda = 4\pi m$, the desired result

$$-\sqrt{2}\lambda X_- - Y_-^2 \mapsto 0.$$

This calculation may even be avoided by looking at [B] Lemma 3.4 (which is worthwhile anyway in this context), where the structure of $U(\mathfrak{g}^{\mathbb{C}})$ is described.

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