Infinite matroid theory exercise sheet 3

1. Deduce from the matroid intersection theorem that for any two matroids M_1 and M_2 on a common ground set E there is a partition of E into E_1 and E_2 and there is a base I_1 of $M_1\restriction_{E_1}$, and a base I_2 of $M_2\restriction_{E_2}$ such that $I_1 \cup I_2$ is independent in both M_1 and M_2.

2. Deduce König's and Rado's theorem from the matroid intersection theorem.

3. (a) Deduce the tree covering theorem for graphs from the matroid union theorem.

   (b) Deduce the tree packing theorem for graphs from the matroid union theorem.

4.* Let M be a matroid with two bases B_1 and B_2. Prove that there is a bijection $\alpha : B_1 \to B_2$ such that $B_1 - x + \alpha(x)$ is a base of M.

5.** Is it true that for any matroid M and any two of its bases B_1 and B_2 there is some bijection $\alpha : B_1 \to B_2$ such that both $B_1 - x + \alpha(x)$ and $B_2 - \alpha(x) + x$ are bases of M?

Reminder: Some theorems from graph theory

Theorem 0.1 (König)**.** *In any finite bipartite graph, there is a matching together with a set of vertices one from each edge of the matching such that every edge is incident with one of these vertices.*

Theorem 0.2 (Packing theorem: Nash-Williams, Tutte)**.** *A finite multigraph has k edge-disjoint spanning trees if and only if every partition P of its vertex set has at least $k(|P|-1)$ crossing edges.*

Here a *crossing edge* is one with endvertices in different partition classes.

Theorem 0.3 (Covering theorem: Nash-Williams)**.** *A finite multigraph $G = (V, E)$ can be partitioned into at most k forests if and only if $e(U) \leq k(|U| - 1)$ for every nonempty set $U \subseteq V$.*

Here $e(U)$ denotes the number of edges with both endvertices in U.

Theorem 0.4 (Rado)**.** *Let G be a bipartite graph with bipartition (A, B) and let M be a matroid with ground set B, and let r_M denote the rank function of M. Then the whole of A can be matched into B such that the set of endvertices of the matching edges in B is M-independent if and only if, for all $K \subseteq A$,*

$$r_M\left(\bigcup_{j\in K} N(j)\right) \geq |K|$$

Here $N(j)$ denotes the set of neighbours of j.

Hints

Concerning question 4:

Use Hall's marriage theorem.