Infinite matroid theory mock exam

Answer 3 of the following 5 questions. You have 90 minutes.

1. Let G be a bipartite graph. The transversal system $\mathcal{I}(G)$ of G is the set of subsets of the left bipartition-class of G that can be matched into the right hand side.

   Find a bipartite graph G such that $\mathcal{I}(G)$ is not the set of independent sets of any matroid.

   Show that $\mathcal{I}(G)$ is the set of independent sets of some matroid if all vertices of the left bipartition-class have finite degree.

2. Let V be a vector space over some field k. A finite set $F \subseteq V$ is *affinely dependent* if there are constants $(\lambda_f \in k | f \in F)$ such that $\sum_{f \in F} \lambda_f = 1$, and $\sum_{f \in F} \lambda_f f = 0$.

   Let $E \subseteq V$. Let $\mathcal{C}$ be the set of minimal nonempty affinely dependent subsets of E. Prove that $\mathcal{C}$ is the set of circuits of some matroid with ground set E.

3. State the axiomatisation of infinite matroids in terms of circuits. A *scrawl* of a matroid is a union of circuits. Show directly that a set $\mathcal{S}$ of subsets of E is the set of scrawls of some matroid M on E if and only if it satisfies $(C3)_\infty$ and (CM), and is closed under taking unions.

   Show that in this case the circuits of M are the minimal nonempty elements of $\mathcal{S}$.

4. A subspace of $|G|$ is a *standard subspace* if it is the closure of some set of edges of G. Prove for any finitely separable graph G that any connected standard subspace of $|G|$ is arc-connected.

5. Show that the set $\mathcal{B}$ of bases of any matroid satisfies the following.

   $$\forall B_1, B_2 \in \mathcal{B} : \ \forall x \in B_1 \setminus B_2 \ \exists y \in B_2 \setminus B_1 : \ B_2 + x - y \in \mathcal{B} \qquad (B2^*)$$

   Show that a set $\mathcal{B}$ satisfying $(B1)$, $(B2^*)$ and (BM) need not be the set of bases of a matroid. Show that any such counterexample must be infinite.