Infinite matroid theory exercise sheet 9

1. Prove Tutte's Linking Theorem.

2.* Let M be a finitary matroid on a ground set E, and let $\overline{P}, \overline{Q}$ be disjoint subsets of E with $\kappa(\overline{P}, \overline{Q}) = \infty$. Show that there is a partition $E \setminus (\overline{P} \cup \overline{Q}) = I \dot\cup J$ with $\kappa_{M/I \setminus J}(\overline{P}, \overline{Q}) = \infty$.

3.** Is the covering conjecture true for arbitrary families of finitary matroids?

4. Let M be a matroid. Let $\mathcal{J} \subseteq \mathcal{I}(M)$ satisfying (I1), (I2) and (I3) such that for every $I \in \mathcal{I}(M)$ there is some $J \in \mathcal{J}$ such that $|I \setminus J|$ is finite. Prove that $\mathcal{J}$ is the set of independent sets of some matroid.

5.** A matroid is *nearly finitary* if its set of independent sets can be obtained as in Exercise 4 from some finitary matroid. A matroid M is *k-nearly finitary* if for every $B \in \mathcal{B}(M)$ and for every $B' \in \mathcal{B}(M^{fin})$ with $B \subseteq B'$ we have that $|B' \setminus B| \leq k$.

   Is it true that every nearly finitary matroid is k-nearly finitary for some k?

Reminder:

Theorem 0.1 (Tutte's Linking Theorem). *Let M be a finite matroid on a ground set E, and let $\overline{P}, \overline{Q}$ be disjoint subsets of E. Then there is a partition $E \setminus (\overline{P} \cup \overline{Q}) = I \dot\cup J$ with*

$$\kappa_{M/I \setminus J}(\overline{P}, \overline{Q}) = \kappa(\overline{P}, \overline{Q}).$$

Hints

For exercise 2: Use the following facts: If $\kappa(\overline{P}, \overline{Q}) = \infty$ then there is a circuit of M meeting both P and Q. The fact that $\kappa(\overline{P}, \overline{Q}) = \infty$ is preserved when finitely many edges are contracted.