Infinite Matroid Theory

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This document, along with exercises and general information on the lecture, can be found at

`http://www.math.uni-hamburg.de/spag/dm/IMT/IMT.html`.

Comments and corrections are welcome. Please email `N.Bowler1729@gmail.com`. So far the authors did not include most of the pictures. Contribution of pictures would be greatly appreciated.

1 Finite Matroids

There are at least two reasons why one would want to generalize the notion of a graph.

(1) Usual benefits of abstraction: Proofs become simpler and more robust.

(2) Duality: Two proofs for the price of one.

The following two theorems of finite graph theory are dual to each other.

> A graph is bipartite iff it has no odd cycles.

> A graph has an eulerian cycle iff all vertices have even degree.

Implementation 1.1. For a finite set E, a set $\mathcal{I}$ is the set of *independent sets* of a matroid on E iff

(I1) $\emptyset \in \mathcal{I}$

(I2) For $I \subseteq J \in \mathcal{I}$ we have $I \in \mathcal{I}$.

(I3) For $I_1, I_2 \in \mathcal{I}$, $x \in I_1 \setminus I_2$ such that $I_2 + x \notin \mathcal{I}$, there is $y \in I_2 \setminus I_1$ with $I_1 - x + y \in \mathcal{I}$.

A set not in $\mathcal{I}$ is *dependent*.

Examples 1.2. (1) Given a family $(\varphi_e : e \in E)$ of vectors in some vector space, let I be independent when $(\varphi_e : e \in I)$ is linearly independent. (*Representable* matroids)

(2) If E is the edge set of a graph, take I to be independent if it is the edge set of some forest in G. (*Graphic* matroids)

(3) For $m \leq n \in \mathbb{N}$, $U_{m,n}$, the *uniform matroid*, has ground set $E = [n]$, and I independent iff $|I| \leq m$.

Exercise 1.3. Which uniform matroids are also graphic?

Definition 1.4. A maximal independent set in a matroid M is called a *base*, the set of bases is denoted $\mathcal{B}(M)$. A minimal dependent set is called a *circuit*, and the set of circuits is denoted $\mathcal{C}(M)$.

Lemma 1.5. Any two bases in a finite matroid are the same size.

Proof. Suppose not for a contradiction, and consider as a counter-example $B_1, B_2 \in \mathcal{B}(M)$ such that $|B_1| > |B_2|$ and $|B_1 \cap B_2|$ maximal. Pick $x \in B_1 \setminus B_2$. Since $B_2 \in \mathcal{B}(M)$ we have $B_2 + x \notin \mathcal{I}$. By (I3) there is $y \in B_2 \setminus B_1$ such that $B_1 - x + y \in \mathcal{I}$. Now let B_3 be a base extending $B_1 - x + y$. We then have $|B_3| \geq |B_1| > |B_2|$ but $|B_3 \cap B_2| \geq |B_1 \cap B_2| + 1$, a contradiction. □

Theorem 1.6. A set $\mathcal{B} \subseteq \mathcal{P}E$ is the set of bases of some finite matroid M iff $\mathcal{B}$ satisfies

(B1) $\mathcal{B}$ is nonempty.

(B2) For $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \setminus B_2$ there has to be $y \in B_2 \setminus B_1$ with $B_1 - x + y \in \mathcal{B}$.

In these circumstances a set is independent in M iff it is a subset of some base in $\mathcal{B}$.

Proof. Suppose $\mathcal{B}$ is the set of bases in some matroid. (B1) is clear since $\emptyset$ extends to some base in the matroid. For (B2) suppose we have $B_1, B_2 \in \mathcal{B}$, $x \in B_1 \setminus B_2$. Since B_2 is a base we know that $B_2 + x$ is not independent in M. By (I3) there is $y \in B_2 \setminus B_1$ such that $B_1 - x + y \in \mathcal{I}(M)$. Let B_3 be a base extending $B_1 - x + y$. By lemma 1.5 it follows that $B_1 - x + y = B_3 \in \mathcal{B}$.

For the converse let $\mathcal{B}$ satisfy (B1) and (B2), and let $\mathcal{I}$ be the set of subsets of elements of $\mathcal{B}$. It is clear that $\mathcal{I}$ satisfies (I1) and (I2). For (I3) let $I_1, I_2 \in \mathcal{I}$, $x \in I_1 \setminus I_2$, such that $I_2 + x \notin \mathcal{I}$. Let $B_1 \supseteq I_1$, $B_2 \supseteq I_2$ be bases extending I_1, I_2 such that $|B_1 \cap B_2|$ is as large as possible. Then $x \in B_1 \setminus B_2$ and $B_2 + x \notin \mathcal{I}$. So there is some $y \in B_2 \setminus B_1$ with $B_1 - x + y \in \mathcal{B}$. If $y \in I_2$ we are done. So suppose $y \notin I_2$ for a contradiction. Using (B2) again we get $z \in B_1 \setminus B_2$ with $B_2 - y + z \in \mathcal{B}$ with $I_2 \subseteq B_2 - y + z$. This contradicts the maximality of $|B_1 \cap B_2|$. □

The *circuit axioms* for a set $\mathcal{C}$ of subsets of a finite set E are

(C1) $\emptyset \notin \mathcal{C}$.

(C2) For $C_1 \subseteq C_2 \in \mathcal{C}$ we must have $C_2 = C_1$.

(C3) For $C_1, C_2 \in \mathcal{C}$, $z \in C_1 \setminus C_2$, $x \in C_1 \cap C_2$ there is $C \in \mathcal{C}$ with $z \in C \subseteq (C_1 \cup C_2) - x$.

Lemma 1.7. If $\mathcal{C}$ satisfies (C3), then is also satisfies

$(C3)_\infty$ Given $z \in C \in \mathcal{C}$ and $X \subseteq C - z$ and for each $x \in X$ a $C_x \in \mathcal{C}$ with $C_x \cap (X + z) = \{x\}$, there is $C' \in \mathcal{C}$ with $z \in C' \subseteq (C \cup \bigcup_{x \in X} C_x) \setminus X$.

Proof. Suppose not and take a counter-examples $(C, z, X, (C_x : x \in X))$ with X of minimal size. Then $X \neq \emptyset$, so pick $x_0 \in X$ and apply (C3) to C, z, x_0 and C_{x_0} to get C' with $z \in C' \subseteq (C \cup C_{x_0}) - x_0$. Let $X' := X \cap C'$. Applying $(C3)_\infty$ to $(C', z, X', (C_x : x \in X'))$ we get the following contradiction

$$\begin{aligned} z \in C'' &\subseteq (C' \cup \bigcup_{x \in X'} C_x) \setminus X' \\ &\subseteq (C \cup \bigcup_{x \in X} C_x) \setminus X. \end{aligned}$$

□

Theorem 1.8. A set $\mathcal{C} \subseteq \mathcal{P}E$ is the set of circuits of a matroid M iff it satisfies the circuit axioms. In such a case a set is independent iff it includes no element of $\mathcal{C}$.

Proof. First suppose $\mathcal{C}$ is the set of circuits of a matroid M. (C1) and (C2) are clear. For (C3) suppose we have $C_1, C_2 \in \mathcal{C}$, $z \in C_1 \setminus C_2$, $x \in C_1 \cap C_2$. Let I be a maximal independent subset of $C_1 \cup C_2 - x - z$. Suppose for a contradiction that $I + z \in \mathcal{I}$. Then by (I3) applied to $I + z, C_1 - z$ and z there is $y \in C_1 - z \setminus (I + z)$ with $I + y = I + z - z + z \in \mathcal{I}$. By maximality of I we have $y = x$ and thus $I + x \in \mathcal{I}$. By (I3) applied to $I + x, C_2 - x$ and x there is $y' \in C_2 - x \setminus (I + x)$ with $I + y' \in \mathcal{I}$. This contradicts the maximality of I. So $I + z \notin \mathcal{I}$ and thus it contains some minimal dependent set C. We have $z \in C \subseteq C_1 \cup C_2 - x$.

Conversely, if $\mathcal{C}$ satisfied the circuit axioms let $\mathcal{I}$ be the set of sets not including an element of $\mathcal{C}$. (I1) and (I2) are clear. For (I3) suppose we have $I_1, I_2 \in \mathcal{I}$, $z \in I_1 \setminus I_2$ with $I_2 + z \notin \mathcal{I}$. Let $C \in \mathcal{C}$ with $z \in C$. Let $X := C \setminus I_1$. If there is some $x \in X$ with $I_1 - z + x \in \mathcal{I}$ we are done. Let us suppose not for a contradiction. Then for each $x \in X$ pick $C_x \in \mathcal{C}$ with $x \in C_x \subseteq I_1 - z + x$. By lemma 1.7 we can apply $(C3)_\infty$ to get a $C' \in \mathcal{C}$ with $z \in C' \subseteq (C \cup \bigcup_{x \in X} C_x) \setminus X \subseteq I_1$, a contradiction. □

Definition 1.9. For $X \subseteq E$ the *restriction* of M to X has as independent sets those independent sets of M which are subsets of X. A *base of X in M* is a base of the restriction of M to X for which we write $M|X := M \backslash (E \setminus X)$. The *rank* of X is the size of bases of X.

Exercise 1.10. If a set $\mathcal{C}$ satisfies (C3) then minimal nonempty subsets of $\mathcal{C}$ give the circuits of a matroid.

Exercise 1.11. A function $r\colon \mathcal{P}E \to \mathbb{Z}_{\geq 0}$ is the *rank function* of a matroid M on E iff:

(R1) $\forall X \subseteq E : r(X) \leq |X|$

(R2) $\forall X \subseteq E \forall x \notin X : r(X) \leq r(X + x) \leq r(X) + 1$

(R3) For $X, Y \subseteq E$ we have $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$.

Definition 1.12. If I is independent but $I + x$ is not, a *fundamental circuit* C_x^I is a circuit with $x \in C_x^I \subseteq I + x$.

Lemma 1.13. In the above context, there is at most one fundamental circuit.

Proof. Suppose for a contradiction that there are two such circuits C and C' with $C \neq C'$, so there is $z \in C \setminus C'$. Since $x \in C \cap C'$ there is by (C3) some $C'' \in \mathcal{C}(M)$ with $z \in C'' \subseteq C \cup C' - x \subseteq I$, a contradiction. □

Theorem and Definition 1.14. If $\mathcal{B}$ is the set of bases of a finite matroid M, then the set $\mathcal{B}^*$ of complements in E of elements of $\mathcal{B}$ is the set of bases of a matroid M^*, the *dual* of M.

Proof. We check the base axioms. (B1) is clear by the fact that $\mathcal{B}$ satisfies (B1). For (B2) suppose we have $B_1, B_2 \in \mathcal{B}^*$ and $x \in B_1 \setminus B_2$. Let $B_1' := E \setminus B_1$ and $B_2' := E \setminus B_2$. So $x \in B_2' \setminus B_1'$. Now $C_x^{B_1'} \not\subseteq B_2'$ since B_2' is independent, so we can choose $y \in C_x^{B_1'} \setminus B_2'$. Then $B' := B_1' + x - y$ is independent by lemma 1.13, so it is a base because it is the same size as B_1'. So letting $B := E \setminus B' = B_1 - x + y$, we have $B \in \mathcal{B}^*$. □

Definition 1.15. The *span* $\mathrm{Sp}_M(X)$ of a set X in a matroid M consists of X together with those $x \in E \setminus X$ such that there is some independent set I with $I \subseteq X$ and $I + x$ dependent. (Equivalently there is a circuit C with $x \in C \subseteq X + x$.) We say a set X is *spanning* iff $\mathrm{Sp}_M(X) = E$.

Examples 1.16. (1) A set of edges in a connected graph is spanning in the associated matroid iff it is connected and it meets every vertex of the graph.

(2) If $(\varphi_e : e \in E)$ is a family of vectors in a vector-space, $X \subseteq E$ is spanning iff $\langle \varphi_e : e \in X \rangle = \langle \varphi_e : e \in E \rangle$.

Lemma 1.17. If $X \subseteq E$ and I is a maximal independent subset of X in M then $\mathrm{Sp}_M(I) = \mathrm{Sp}_M(X)$.

Proof. Evidently $\mathrm{Sp}_M(I) \subseteq \mathrm{Sp}_M(X)$. Suppose for a contradiction that there is $x \in \mathrm{Sp}_M(X) \setminus \mathrm{Sp}_M(I)$. Then there is an independent set $J \subseteq X$ with $J + x$ dependent, but $I + x$ independent. Applying (I3) to $I + x$ and J we get $y \in J \setminus (I + x)$ with $I + y$ independent, contradicting the maximality of I. □

Lemma 1.18. For $A \subseteq B \subseteq E$ we have $\mathrm{Sp}_M(A) \subseteq \mathrm{Sp}_M(B)$, and for $A \subseteq E$, $\mathrm{Sp}_M(A) = \mathrm{Sp}_M(\mathrm{Sp}_M(A))$.

Proof. The first part is clear. For the second part, let I be a maximal independent subset of A. Then I is also a maximal independent subset of $\mathrm{Sp}_M(A) = \mathrm{Sp}_M(I)$, by lemma 1.17. So $\mathrm{Sp}_M(A) = \mathrm{Sp}_M(I) = \mathrm{Sp}_M(\mathrm{Sp}_M(A))$ again by lemma 1.17. □

Lemma 1.19. A set $I \subseteq E$ is independent in M^* iff $E \setminus I$ is spanning in M.

Proof. Equivalently, we just need to show that a set is spanning iff it includes a base. This is immediate from lemma 1.17. □

Note. It is also possible to axiomatise matroids in terms of closure axioms, that is in terms of the endofunctor $\mathrm{Sp}\colon E \to E$.

Definition 1.20. A *cocircuit* of a matroid M is a circuit of M^*.

Examples 1.21. (1) A cocircuit in the matroid associated to a graph is a *bond*, a minimal nonempty cut.

(2) A cocircuit in the matroid associated to a family of vectors $(\varphi_e : e \in E)$ is the complement of a hyperplane in $\langle \varphi_e : e \in E \rangle$.

Lemma 1.22. No circuit C and cocircuit D of the same matroid M can intersect in just one element.

Proof. Suppose for a contradiction that $C \cap D = \{e\}$. Let $I \supseteq C - e$ be a maximal independent subset of $E \setminus D$. Then $\mathrm{Sp}_M(I) \supseteq (E - D) + e$, and $\mathrm{Sp}_M((E-D)+e) = \mathrm{Sp}_M(E \setminus (D-e)) = E$ by lemma 1.19. By 1.18, $\mathrm{Sp}_M(I) = E$, so I is a base of M. Thus $D \subseteq E \setminus I$ is independent in M^*, contradicting that D is a cocircuit. □

Lemma 1.23. For any circuit C of M and any distinct elements e, f of C, there is a cocircuit of M with $C \cap D = \{e, f\}$.

Proof. Let $I \supseteq C - f$ be a maximal independent subset of E. So $f \notin I$ and $C = C_f^I$. Let D be the fundamental circuit of e in $E \setminus I$ with respect to M^*. Then $C \cap D \subseteq \{e, f\}$. By lemma 1.22 it follows that $\{e, f\} \subseteq C \cap D$. □

Definition 1.24. If X is a subset of the set of the ground set E of a matroid M, and $Y = E \setminus X$, then the *restriction* of M to X, denoted $M|_X$ or $M \setminus Y$, is a matroid on the set $E \setminus Y = X$ with independent sets those independent sets of M which are subset of X. The *contraction* of M onto X, denoted $M.X$ or M/Y, is given by $(M^*|_X)^*$.

Examples 1.25. (1) If e is an edge in a graph G, the matroid we get by contracting e in the matroid corresponding to G is the matroid corresponding to G/e.

Lemma 1.26. Let B be a base of a matroid M with ground set $E = P\dot{\cup}Q$. Then $P \cap B$ is a base of $M\backslash Q$ iff $B \cap Q$ is a base of M/P.

Proof. By duality, it suffices to prove that if $B \cap Q$ is a base of M/P then $B \cap P$ is a base of $M\backslash Q$. Suppose for a contradiction that $B \cap Q$ is a base but $B \cap P$ is not. Then $B \cap P$ is independent, so it cannot be maximal in P. There is $x \in P \setminus B$ such that $(P \cap B) + x$ is independent. Let B' be a maximal independent subset of $B + x$ including $(B \cap P) + x$. Then $B + x \subseteq \mathrm{Sp}(B')$, so B' is spanning, so B' is a base, so B' is of the form $B + x - y$ for some $y \in B \cap Q$. Now let $B'' := E \setminus B' \in \mathcal{B}(M^*)$, so $B'' \cap Q \in \mathcal{I}(M^*)$, so $Q \setminus B' \in \mathcal{I}(M^*)$. But $Q \setminus B \nsubseteq Q \setminus B'$, so $Q \setminus B \notin \mathcal{B}(M^*\backslash P)$, so $B \cap Q \notin \mathcal{B}((M^*\backslash P)^*) = \mathcal{B}(M/P)$. $\square$

Lemma 1.27. Let M be a matroid on ground set $E = P\dot{\cup}Q$, B_P a base of $M\backslash Q$ and B_Q a base of M/P. Then $B_P \cup B_Q \in \mathcal{B}(M)$.

Proof. By duality it is enough to prove that $B_P \cup B_Q$ is spanning. But $P \subseteq \mathrm{Sp}_M(B_P)$, so $P \cup B_Q \subseteq \mathrm{Sp}_M(B_P \cup B_Q)$, and $P \cup B_Q$ is spanning, so $B_P \cup B_Q$ is spanning by lemma 1.19. $\square$

Corollary 1.28. Let M be a matroid in $E = P\dot{\cup}Q$. For $B_Q \subseteq Q$, the following are equivalent:

(1) B_Q is a base of M/P

(2) $\exists B_P \in \mathcal{B}(M\backslash Q)$ such that $B_P \cup B_Q \in \mathcal{B}(M)$

(3) $\forall B_P \in \mathcal{B}(M\backslash Q) : B_P \cup B_Q \in \mathcal{B}(M)$

Corollary 1.29. In the above context, $I \subseteq Q$, the following are equivalent:

(1) $I \in \mathcal{I}(M/P)$

(2) $\exists B_P \in \mathcal{B}(M\backslash Q)$ such that $B_P \cup I \in \mathcal{I}(M)$

(3) $\forall B_P \in \mathcal{B}(M\backslash Q) : B_P \cup I \in \mathcal{I}(M)$.

Corollary 1.30. $(M\backslash C_1)\backslash C_2 = M\backslash(C_1 \cup C_2)$, $M/C_1/C_2 = M/(C_1 \cup C_2)$, if C and D are disjoint, then $(M/C)\backslash D = (M\backslash D)/C$.

Proof. The first two statements are clear. For the third, the independent sets on both sides are those I for which there is a base B of $M\backslash C$ with $B \cup I \in \mathcal{I}(M)$. $\square$

Lemma 1.31. Let $C \subseteq E$, M a matroid on E, $C \in \mathcal{C}(M/X)$. Then there is a circuit C' of M with $C \subseteq C' \subseteq C \cup X$.

Proof. Let $p \in C$. Then $C - p \in \mathcal{I}(M/X)$, so by corollary 1.29 there is a base B of X with $B \cup (C - p) \in \mathcal{I}(M)$. Let $C' := C_e^{B\cup(C-p)}$. Then $C' \subseteq C \cup X$. It suffices to show $C \subseteq C'$: for any $f \in C$, $(C - f) \cup B$ is independent in the original matroid by corollary 1.29, so $f \in C'$ and thus $C \subseteq C'$. $\square$

Lemma 1.32. Let M be a matroid on $E = P\dot{\cup}Q\dot{\cup}\{e\}$. Then precisely one of the following happens:

(1) $\exists C \in \mathcal{C}(M)$ with $e \in C \subseteq P + e$

(2) $\exists D \in \mathcal{C}(M^*)$ with $e \in D \subseteq Q + e$

Proof. We can't have both, as if we did we would have $C \cap D = \{e\}$. Let $M' := (M/P)\backslash Q$. The ground set of M' is $\{e\}$. Either $\{e\}$ is a circuit or else it is a circuit in M'. If it is a circuit, then we get the first condition by applying lemma 1.31. If it is a cocircuit, we get the second condition by applying the dual of lemma 1.31. $\square$

Lemma 1.33. Let C be a circuit of M, and $X \subseteq E$. Then $C \setminus X$ is a union of circuits of M/X.

Proof. It is enough to show that for any $e \in C \setminus X$, there is some circuit C' of M/X with $e \in C' \subseteq C \setminus X$. But by lemma 1.32 there is such a C' unless there is a cocircuit D of M/X with $e \in D \subseteq (E \setminus X) \setminus (C \setminus X) + e$. There can't be such a cocircuit, because it would be a cocircuit of M and $C \cap D = \{e\}$. So there is such a C'. $\square$

Definition 1.34. A *scrawl* is a union of circuits in a matroid.

Lemma 1.35. A subset S of E is a scrawl iff it never meets a cocircuit of M in just one place.

Proof. We apply lemma 1.32 as in the proof of lemma 1.33. $\square$

Corollary 1.36. The scrawls of $M/P\backslash Q$ are the sets of the form $S \setminus P$, for S a scrawl of M not meeting Q.

1.1 Matroid Union and Intersection

In this section things **may not** be true in the infinite case.

Definition 1.37. Let B be a base of M and $e \in B$. Then the *fundamental cocircuit* D_e^B of e with respect to B is the fundamental circuit in M^* of e with respect to $E \setminus B$.

Lemma 1.38. Let B be a base of M, $e \notin B$, $f \in B$. Then the following are equivalent. If any of the following condition is true we write eBf.

(1) $B - f + e \in \mathcal{B}(M)$

(2) $f \in C_e^B$

(3) $e \in D_f^B$

Proof. If $B - f + e \in \mathcal{B}(M)$ is independent it cannot include C_e^B, so $f \in C_e^B$. But for $f \in C_e^B$, $B - f + e$ is independent because of the uniqueness of fundamental circuits. The equivalence of the third condition is dual. $\square$

Lemma 1.39. Let B be a base of M, and $(x_i \notin B : i \in [n])$ and $(y_i \in B : i \in [n])$ be such that $x_i B y_i$ for each i but $x_i B y_j \implies i \geq j$. Then

$$B' := B - \{y_i : i \in [n]\} + \{x_i : i \in [n]\}$$

is a base of M.

Proof. By duality, it suffices to show B' is spanning. Let $F := \mathrm{Sp}_M(B')$. We will proof by induction: for each $i \in [n]$, $y_i \in F$. This is true since by the induction hypothesis $C^B_{x_i} - y_i \subseteq F$. So $y_i \in \mathrm{Sp}_M(C^B_{x_i} - y_i) \subseteq \mathrm{Sp}_M(F) = F$. So $B \subseteq F$ and thus $E \subseteq \mathrm{Sp}_M(B) \subseteq \mathrm{Sp}_M(F) = F$. So B' is spanning. □

At some point we will need the following equivalence.

$$\begin{aligned} & r_{M^*}(X) = r_M(E \setminus X) + |X| - r_M(E) \\ \iff & r_M(E \setminus X) + |X| - r_{M^*}(X) = r_M(E) \end{aligned}$$

Definition 1.40. Let M_1 and M_2 be matroids on the same ground set E. Let $B_i \in \mathcal{B}(M_i)$, $i = 1, 2$. An *exchange chain* for B_1 and B_2 is a sequence of $(z_i : k \leq i \leq l)$ such that for all i odd with $k \leq i < l$, $z_i B_1 z_{i+1}$ and for i even with $k \leq i < l$, $z_i B_2 z_{i+1}$.

Lemma 1.41. In the above context, if there is an exchange chain from x to y with $x \in E \setminus (B_1 \cup B_2)$ but $y \in B_1 \cap B_2$. Then there are $B_1' \in \mathcal{B}(M_1), B_2' \in \mathcal{B}(M_2)$ with

$$B_1' \cup B_2' = B_1 \cup B_2 + x$$

Proof. Let $(z_i : k \leq i \leq l)$ be such an exchange chain of minimal length. For each odd i with $k \leq i < l$, we have $z_i B_i z_{i+1}$, but for any other odd j with $k \leq j < l$, we have

$$z_i B_i z_{j+1} \implies i \geq j$$

by minimality of the length. So

$$B_1' := B_1 \cup \{z_i : i \text{ odd}, k \leq i \leq l\} \setminus \{z_{i+1} : i \text{ odd}, k \leq i \leq l\} \in \mathcal{B}(M_1)$$

by lemma 1.39 and similarly

$$B_2' := B_2 \cup \{z_i : i \text{ even}, k \leq i < l\} \setminus \{z_{i+1} : i \text{ even}, k \leq i < l\} \in \mathcal{B}(M_2)$$

and $B_1' \cup B_2' = B_1 \cup B_2 + x$. □

Theorem 1.42. Let M_1 and M_2 be matroids of the same ground set E. Then the size of any largest set of the form $I_1 \cup I_2$ with $I_i \in \mathcal{I}(M_i)$, $i = 1, 2$, is

$$\min_{P \dot\cup Q} (r_{M_1}(P) + r_{P_2}(P) + |Q|)$$

Proof. First let $I_1 \in \mathcal{I}(M_1)$ and $I_2 \in \mathcal{I}(M_2)$, $E = P \dot\cup Q$. Then

$$\begin{aligned} |I_1 \cup I_2| &\leq |(I_1 \cup I_2) \cap P)| + |(I_1 \cup I_2) \cap Q| \\ &\leq |I_1 \cap P| \cup |I_2 \cap P| + |Q| \\ &\leq r_{M_1}(P) + r_{M_2}(P) + |Q|. \end{aligned}$$

It remains to show that if $I_1 \cup I_2$ is largest possible, then there is $E = P \dot\cup Q$ for which the above inequalities are equalities. Without loss of generality I_1 and I_2 are bases of the respective matroids.

We will take Q to be the set of x's for which there is an exchange chain for I_1 and I_2 from x to some $y \in I_1 \cap I_2$ and we take $P := E \setminus Q$. Then

(1) $I_1 \cap I_2 \cap P = \emptyset$ by definition

(2) $I_1 \cap P$ spans P in M_1 since for any $x \in I_1 \setminus P$ we have $C_x^{B_1} \subseteq P$

(3) $I_2 \cap P$ spans P in M_2 similarly

(4) $I_1 \cup I_2 \supseteq Q$ by lemma lemma 1.41

So

$$\begin{aligned} |I_1 \cup I_2| &= |(I_1 \cup I_2) \cap P| + |(I_1 \cup I_2) \cap Q| \\ &= |I_1 \cap P| + |I_2 \cap P| + |Q| \\ &= r_{M_1}(P) + r_{M_2}(P) + |Q|. \end{aligned}$$

□

Theorem and Definition 1.43. Let M_1 and M_2 be matroids on the same ground set E. The *union* $M_1 \wedge M_2$ of M_1 and M_2 is the matroid whose independent sets are those of the form $I_1 \cup I_2$ with $I_i \in \mathcal{I}(M_i)$, $i = 1, 2$.

Proof. Define the function $r \colon \mathcal{P}E \to \mathbb{Z}_{\geq 0}$ by

$$r(X) := \max_{X \supseteq I_i \in \mathcal{I}(M_i)} |I_1 \cup I_2| = \min_{X = P \dot\cup Q} r_{M_1}(P) + r_{M_2}(P) + |Q|$$

by lemma 1.42 applied to $M_1|_X$ and $M_2|_X$.

The function r satisfies (R1) and (R2) by considering the left-hand side and r satisfies (R3) by considering the right-hand side of the above expression. So r is the rank function of some matroid on E. A set I is independent in this matroid iff $r(I) = |I|$, and this is equivalent to $I = I_1 \cup I_2$ for some $I_i \in \mathcal{I}(M_i)$ by the left-hand side. □

Theorem 1.44 (Edmond's intersection theorem)**.** Let M_1 and M_2 be matroids on the same ground set E. Then any largest set I independent in both matroids has size $\min_{E = P \dot\cup Q}(r_{M_1}(P) + r_{M_2}(Q))$.

Proof. Let $I_i \in \mathcal{I}(M_i)$, $i = 1, 2$, as large as possible and with $I_1 \cap I_2 = I$ also as large as possible. Then I_1, I_2 are bases of the respective matroids. But then $I_1 \cup (E \setminus I_2)$ is a base of $M_1 \wedge M_2^*$. So by lemma 1.42

$$\begin{aligned}|I_1 \cap I_2| &= |I_1 \cup (E \setminus I_2)| - (|E| - |I_2|) \\ &= \min_{E = P \dot\cup Q} (r_{M_1}(P) + r_{M_2^*}(P) + |Q| - (|E| - r_{M_2}(E)) \\ &= r_{M_1}(P) + r_{M_2}(Q).\end{aligned}$$

□

Theorem 1.45 (Menger's theorem)**.** Let $A, B \subseteq V(G)$ for some graph G. Then the size of the largest set of vertex-disjoint paths from A to B is the same as the size of the smallest $A - B$-separator.

We will only sketch the proof since we will repeat the proof with more detail for the infinite case.

Proof. Without loss of generality G is connected and so are $G[A]$ and $G[B]$. Let M be the matroid associated to G. Let $M_1 := M/E(G[A])\backslash E(G[B])$ and $M_2 := M\backslash E(G[A])/E(G[B])$. If T is a maximal set independent in both matroids, then

$$|T| = |V(G)| - |A| - |B| + |\{A - B\text{-paths in } T\}|.$$

Now suppose we have a partition $E(G) = P \dot\cup Q$. Let X be the set of vertices joined to A by edges in P. Let $P' := E(G[A])$, $Q' := E(G) \setminus P'$. Then

$$\begin{aligned}\mathrm{rk}_{M_1}(P) + \mathrm{rk}_{M_2}(Q) &\geq \mathrm{rk}_{M_1}(P') + \mathrm{rk}_{M_1}(P \setminus P') + \mathrm{rk}_{M_2}(Q \setminus P') \\ &\geq \mathrm{rk}_{M_1}(P) + \mathrm{rk}_{M_2}(P \setminus P') + \mathrm{rk}_{M_2}(Q \setminus P') \\ &\geq \mathrm{rk}_{M_1}(P') + \mathrm{rk}_{M_2}(Q') \\ &= |X| - |A| + |\{\text{ vertices adjacent to an edge of } Q'\}| - |B| \\ &= |V(G)| + |\{\text{ boundary of } X\}| - |A| - |B|\end{aligned}$$

with the boundary of X being an $A - B$-separator. □

This document has only been proof-read until here. The rest of the document probably contains several errors. You have been warned!

2 Basic examples and axioms

From now on matroids are considered to be infinite. Proofs from section 1 still apply to infinite matroid theory.

(C3)$_\infty$ For all $z \in C \in \mathcal{C}$, $X \subseteq C - z$, $(C_x \in \mathcal{C} : x \in X)$ s.t. $C_x \cap (X + z) = \{z\}$, there exists $C' \in \mathcal{C}$ with $z \in C' \subseteq (C \cup \bigcup_{x \in X} C_x) \setminus X$.

Definition 2.1. A *scrawl-system* on a set is a set $\mathcal{C}$ of subsets of E closed under taking unions and satisfying $(\mathrm{C3})_\infty$.

Lemma and Definition 2.2. If a set $\mathcal{C}$ of subset of E satisfies $(\mathrm{C3})_\infty$, then $\langle\mathcal{C}\rangle$, the closure of $\mathcal{C}$ under taking unions, is a scrawl-system, called the scrawl-system *generated* by $\mathcal{C}$.

Proof. Suppose we have $C, z, X, (C_x : x \in X)$ as in the setup of $(\mathrm{C3})_\infty$. Let $\bar{C} \in \mathcal{C}$ with $z \in \bar{C} \subseteq C$, let $\bar{X} := X \cap \bar{C}$ and for $x \in \bar{X}$ let $\bar{C_x} \in \mathcal{C}$ with $x \in \bar{C_x} \subseteq C_x$. By $(\mathrm{C3})_\infty$ applied to $\mathcal{C}$ there is $C' \in \mathcal{C} \subseteq \langle\mathcal{C}\rangle$ with $z \in C' \subseteq \bar{C} \cup \bigcup_{x\in\bar{X}} \bar{C_x} \setminus \bar{X} \subseteq C \cup \bigcup_{x\in X} C_x \setminus X$. □

Remark 2.3. If $\mathcal{C}$ is a set of finite subset of E satisfying (C3), then is also satisfies $(\mathrm{C3})_\infty$ as in lemma lemma 1.7, so $\langle\mathcal{C}\rangle$ is a scrawl-system. Scrawl-systems occurring in this way are called *finitary.*

Examples 2.4. (1) $U_{n,E}$ is generated by the set of subsets of E of size $n+1$.

(2) If G is a graph, $M_{FC}(G)$ is generated by the edge-sets of the finite cycles in G, and $M_{FB}(G)$ is generated from the finite cuts in G.

Some potential unions for sets $\mathcal{C}$, $\mathcal{D}$ of subsets of E.

(O1) There do not exist $C \in \mathcal{C}$ and $D \in \mathcal{D}$ with $|C \cap D| = 1$.

(O2) For any partition $E = P \dot\cup \{e\} \dot\cup Q$, at least one of the following holds

- $\exists C \in \mathcal{C} : e \in C \subseteq P + e$
- $\exists D \in \mathcal{D} : e \in D \subseteq Q + e$

Definition 2.5. For any set $\mathcal{C}$ of subsets of E, $\mathcal{C}^* := \{D \subseteq E : \forall C \in \mathcal{C} : |C \cap D| \neq 1\}$.

Remark 2.6. $\mathcal{C}$ and $\mathcal{C}^*$ satisfy (O1).

Theorem 2.7. Let $\mathcal{C} \subseteq \mathcal{P}E$. Then the following are equivalent:

(1) $\mathcal{C}$ satisfies $(\mathrm{C3})_\infty$

(2) $\mathcal{C}$ and $\mathcal{C}^*$ satisfy (O2)

(3) $\exists \mathcal{D} \subseteq \mathcal{P}E : \mathcal{C}$ and $\mathcal{D}$ satisfy (O1) and (O2)

Proof. It is clear that (2) $\implies$ (3).

(1) $\implies$ (2) Suppose $\mathcal{C}$ satisfies $(\mathrm{C3})_\infty$, and $E = P \dot\cup \{e\} \dot\cup Q$ such that there is no $C \in \mathcal{C}$ with $e \in C \subseteq P + e$. We must show $\exists D \in \mathcal{C}^*$ with $e \in D \subseteq Q + e$. We set $D := \{f \in Q + e : \nexists C \in \mathcal{C} : f \in C \subseteq P + f\}$. By definition $e \in D \subseteq Q + e$. So we just have to show that $D \in \mathcal{C}^*$. Suppose not for a contradiction. Then there is $C \in \mathcal{C}$ with $C \cap D = \{z\}$, say.

Let $X := (C \setminus P) - z$. Then since $X \subseteq (Q + e) \setminus D$, for each $x \in X$ there is $C_x \in \mathcal{C}$ with $x \in C_x \subseteq P + x$, and $C_x \cap (X + z) = \{x\}$. So we can apply $(\mathrm{C3})_\infty$ to get $C' \in \mathcal{C}$ with $z \in C' \subseteq (C \cup \bigcup_{x\in X} C_x) \setminus X \subseteq P + z$, contradicting $z \in \mathcal{D}$.

(3) $\implies$ (1) Suppose $\mathcal{C}$ and $\mathcal{D}$ satisfy (O1) and (O2). Let $C, z, X, (C_x : x \in X)$ as in the setup of $(C3)_\infty$. Let $P = (C \cup \bigcup_{x \in X} C_x \setminus X) - z$, $e := z$ and $Q := (E \setminus P) - e$. Suppose for a contradiction that there is $D \in \mathcal{D}$ with $z \in D \subseteq Q + z$. Since $\mathcal{C}$ and $\mathcal{D}$ satisfy (O1), there is some $z \neq x \in C \cap D$, so $x \in X$, so $D \cap C_x = \{x\}$, contradicting (O1). So by (O2) there is $C \in \mathcal{C}$ with $z \in C' \subseteq O + z = C \cup \bigcup_{x \in X} C_x \setminus X$.

□

Example 2.8. Let $\mathcal{C}$ be the set of infinite and $\mathcal{D}$ the set of *cofinite*, a set whose complement is finite, subsets of E. Then $\mathcal{C}$ and $\mathcal{D}$ satisfy (O1) and (O2), so $\mathcal{C}$ satisfies $(C3)_\infty$. We will write $U_{\infty,E}$ for $\langle \mathcal{C} \rangle$.

Lemma 2.9 (König)**.** Let T be an infinite tree in which all vertices have finite degree. Then T includes a ray. □

Definition 2.10. Let G be a graph. Then $C_{AC}(G)$ is the set of edges of finite cycles and doublerays in G, and $\mathcal{D}_{AC}(G)$ is the set of cuts in G of which at least one side is rayless, called a *skew cut*.

Remark 2.11. $\mathcal{C}_{AC}(G)$ and $\mathcal{D}_{AC}(G)$ satisfy (O1).

Lemma 2.12. Let H be the *Bean graph* as given by figure 2. Then $\mathcal{C}_{AC}(H)$ does not satisfy $(C3)_\infty$.

Proof. Take C to be the doubleray of G, and X to be the right side of C. For each $x \in X$ take C_x to be the K^3 containing x in H. By applying $(C3)_\infty$ we get a graph which does not contain a cycle or a double ray through z.

□

Remark 2.13. $\mathcal{C}_{AC}(G)$ does not satisfy $(C3)_\infty$ if G includes a subdivision of the Bean graph.

Lemma 2.14. If $\mathcal{C}$ does not include a subdivision of the Bean graph, then $\mathcal{C}_{AC}(G)$ and $\mathcal{D}_{AC}(G)$ satisfy (O2).

Proof. Let $E = P \dot\cup \{e\} \dot\cup Q$. Let $e = vw$ and let K be the subgraph of G with edge set P. If v and w are in the same component of K, there is some path joining v and w, which together with e gives a finite cycle whose edge set C satisfies $e \in C \subseteq P + e$. Otherwise, if the components of K containing v and w each include a ray, we get a double ray whose edge set C satisfies $e \in C \subseteq P + e$. Finally, if one of the components (say the one containing u), includes a ray, let A be the vertex set of the component, and let D be the cut with one side given by A. Then $e \in D \subseteq Q + e$ and it remains to show that $D \in \mathcal{D}_{AC}(G)$. Suppose not for a contradiction. Let R be a ray in $G[A]$, T be a spanning tree of the component of K containing u, and let T' be the subtree of T whose edges lie on paths from u to R through T. T' is infinite and rayless, so by lemma 2.9, it has some vertex w of infinite degree. Now we get a subdivision of H, c.f. the following figure.

$\square$

Definition 2.15. $M_{AC}(G) =< \mathcal{C}_{AC}(G) >$

Theorem 2.16. Let $\mathcal{C}$ and $\mathcal{D}$ be sets of subsets of E. The following are equivalent:

(a) $\mathcal{C}$ is a scrawl system and $\mathcal{D} = \mathcal{C}^*$.

(b) $\mathcal{C}$ and $\mathcal{D}$ are closed under taking unions, and they satisfy (O1) and (O2).

Proof.

(a) $\implies$ (b) $\mathcal{C}$ and $\mathcal{D}$ are closed under unions by definiton and satisfy (O1) by definition and (O2) by theorem 2.7**??**.

(b) $\implies$ (a) $\mathcal{C}$ is a scrawl system by theorem 2.7**??**. $\mathcal{D} \subseteq \mathcal{C}^*$ by (O1). Now let $\mathcal{D} \in \mathcal{C}^*$ and let $e \in \mathcal{D}$ and $\mathcal{P} = E \setminus \mathcal{D}$, $Q := \mathcal{D} - e$. Since $D \in \mathcal{C}^*$, there cannot be $C \in \mathcal{C}^*$ with $e \in C \subseteq P + e$, so by (O2) there is some $D' \in \mathcal{D}$ with $e \in D' \subseteq Q + e = D$. So $\mathcal{D}$ is a union of elements of $\mathcal{D}$, so $D \in \mathcal{D}$.

$\square$

Definition 2.17. If $\mathcal{C}$ is a scrawl system on a set E, then the *dual* scrawl system is $\mathcal{C}^*$. If $X \subseteq E$, $Y \subseteq E \setminus X$, then the *restriction* $\mathcal{C}|_X$ (or $\mathcal{C}\backslash Y$) of $\mathcal{C}$ to X is $\{C \in \mathcal{C} : C \subseteq X\}$. The *contraction* $\mathcal{C}.X$ (or $\mathcal{C}/Y$) of $\mathcal{C}$ into X is $(\mathcal{C}^*|_X)^*$.

Lemma 2.18. $\mathcal{C}.X = \{C \cap X : C \in \mathcal{C}\}$

Proof. $\mathcal{C}^*|_X$ and $\{C \cap X | C \in \mathcal{C}\}$ are both closed under unions and satisfy (O1) and (O2), so by theorem 2.16**??** $\{C \cap X | C \in \mathcal{C}\} = (\mathcal{C}^*|_X)^* = \mathcal{C}.X$. $\square$

Corollary 2.19. $\mathcal{C}/X/Y = \mathcal{C}/(X \cup Y)$, $\mathcal{C}\backslash X\backslash Y = \mathcal{C} \setminus (X \cup Y)$ and if X and Y are disjoint then $\mathcal{C}/X\backslash Y = \mathcal{C}/X\backslash Y$.

Examples 2.20. (1) $U_{n,E}|_X = U_{n,X}$

$$U_{n,E}/Y = \begin{cases} U_{0,E\setminus Y} & \text{if } |Y| > n \\ U_{n-|Y|,E\setminus Y} & \text{if } |Y| \leq n \end{cases}$$

(2) $U_{\infty,E}|_X = U_{\infty,X}$

$$U_{\infty,E}/Y = \begin{cases} U_{0,E\setminus Y} & \text{if } Y \text{ is infinite} \\ U_{\infty,E\setminus Y} & \text{if } Y \text{ is finite} \end{cases}$$

Definition 2.22. Let G be a graph and X a set of G. We say two vertices of G are X-*equivalent* if they are in the same connected component of X. The *contraction* G/X has as vertices equivalence classes of vertices of G under X-equivalence, the edge set $E(G) \setminus X$ and e *joins* the equivalence classes of its G-endpoints.

Lemma 2.23.

$$\begin{aligned}
M_{FC}(G)\backslash X &= M_{FC}(G \setminus X),\\
M_{FC}(G)/X &= M_{FC}(G/X),\\
M_{FB}(G)\backslash X &= M_{FB}(G/X),\\
M_{FB}(G)/X &= M_{FB}(G \setminus X),\\
M_{AC}(G)\backslash X &= M_{AC}(G \setminus X).
\end{aligned}$$

Proof. All of these are clear except the second. For the second, any finite cycle in G/X extends using edges in X to a finite cycle of G. Conversely, let C be a finite cycle in G and let $c \in C \setminus X$. Then C never meets a cut of G/X just once, so let $\mathcal{D}$ be the set of cuts in G/X, we have $C \in \mathcal{D}^*$. But $C_{FC}(G/X)$ and $\emptyset$ satisfy (O1) and (O2), so $\mathcal{D}^* = \langle \mathcal{C}_{FC}(G) \rangle = M_{FC}(G)$. $\square$

Remark 2.24. Any minor of a finitary scrawl system is finitary.

Definition 2.25. If $\mathcal{C}$ is a scrawl-system, then a set I is *$\mathcal{C}$-independent* iff it does not include any nonempty elements of $\mathcal{C}$.

Lemma 2.26. The $\mathcal{C}$-independent sets satisfy (I1), (I2) and (I3).

Proof. (I1) and (I2) are clear. For (I3), let I_1 and I_2 be $\mathcal{C}$-independent and $x \in I_1 \setminus I_2$ with $I_2 + x$ $\mathcal{C}$-dependent. Let $C \in \mathcal{C}$ with $x \in C \subseteq I_2 + x$. There is no $C' \in \mathcal{C}$ with $x \in C' \subseteq I_1$, so we can apply (O2) to get $D \in \mathcal{C}^*$ with $x \in \mathcal{D} \subseteq (E \setminus I_1) + x$. Since $C \cap D \neq \{x\}$, there is $x \neq y \in C \cap D$. We will show $I_1 - x + y$ is $\mathcal{C}$-independent. Suppose not, then there is a $C' \in \mathcal{C}$ with $y \in C' \subseteq I_1 - x + y$. But then $C' \cap D = \{y\}$. $\square$

Lemma 2.27. Let I be a subset of E, $\mathcal{C}$ a scrawl-system on E. Then the following are equivalent:

(1) I is a maximal $\mathcal{C}$-independent set.

(2) $E \setminus I$ is a maximal $\mathcal{C}^*$-independent set.

(3) I is $\mathcal{C}$-independent and $E \setminus I$ is $\mathcal{C}^*$-independent.

Proof.

(1) $\implies$ (3) Let I be a maximal $\mathcal{C}$-independent set. Suppose for a contradiction that there is $\emptyset \neq D \in \mathcal{C}^*$ with $D \in E \setminus I$. Let $e \in D$. Since $I + e$ is $\mathcal{C}$-dependent, there is $C \in \mathcal{C}$ with $e \in C \subseteq I + e$. But then $C \cap D = e$, a contradiction.

(3) $\implies$ (1) Let I be $\mathcal{C}$-independent and $E \setminus I$ be $\mathcal{C}^*$-independent. Suppose for a contradiction that I is not maximal. Then there is $e \in E \setminus I$ with $I + x$ $\mathcal{C}$-independent. So there is $D \in \mathcal{C}^*$with $e \in D \subseteq E \setminus I$.

The equivalence of (1) and (2) is dual to the equivalence of (1) and (3). $\square$

Definition 2.28. In these circumstances I is a *base* of $\mathcal{C}$ and $\mathcal{C}$ is *based*.

Remark 2.29. $U_{\infty,E}$ is not based if E is infinite.

Implementation 2.30. A matroid is a scrawl-system all of whose minors are based.

Remark 2.31. Since $\mathcal{C}$ is based iff $\mathcal{C}^*$ is, the collection of matroids is closed under duality and under taking minors.

Exercise 2.32. Let G be a graph not including a subdivision of the Bean graph $C \in \mathcal{C}_{AC}(G)$, D a cut of G with at least 1 side connected, then $C \cap D$ is finite.

Lemma 2.33. Let G be as in the exercise above, $C \in \mathcal{C}_{AC}(G)$, X a connected rayless subgraph of G. Then the number of pairs (e, x) with $e \in C \setminus E(X)$ and X an endpoint of e in X is finite and even.

Proof. Without loss of generality, all edges of G are in either $E(X)$ or C, and by subdividing edges in $C \setminus E(X)$ we can also assume that no edges in $C \setminus E(X)$ have both endvertices in X. Now let D be the cut induced by $(V(X), V(G) \setminus V(X))$. By Exercise 2.32**??**, D is finite. The pairs in the statement can be identified with elements of D. D has even size because each connected component of $C \cap X$ meets precisely two or none of the edges in D. □

Definition 2.34. Let G be a graph. K a set of vertices of G. Then the algebraic cycles of (G, K) are edge set of:

- finite cycles in G
- finite paths in G with both endpoints in K
- rays in G starting in K
- double rays in G

Lemma 2.35. Let (G, K) be as above, and let $C \subseteq E(G)$ be nonempty and such that any vertex at which C has degree 1 is in K. Then C includes an algebraic cycle of (G, K).

Proof. Suppose not. Then let $xy = e \in C$. Since C includes no finite cycles, x and y are in disjoint components of $G \setminus C$. Both of these components are bases, and one of them, the on containing x say, is rayless and contains no vertex of K. but then all vertices of the component have degree ≥ 2 in C. So we can recursively build a ray from x through this component. □

Lemma 2.36. Let (G, K) be as above, and let $C \subseteq E(G)$ such that the degree in C of any vertex not in K is both finite and even. Then C is the disjoint union of algebraic cycles of (G, K).

Proof. Let $\mathcal{F}$ be the set of sets of disjoint algebraic cycles of (G, K) in C ordered by inclusion. F is nonempty and for any chain in $\mathcal{F}$, the union of that chain is still in $\mathcal{F}$. So by Zorn's Lemma $\mathcal{F}$ has a maximal element F. Since for any algebraic cycle C' of (G, K) and any vertex $v \notin K$, the degree in C' of v is zero or two, $C \setminus \bigcup F$ satisfies the premise of Lemma 2.35**??**. So, if it were nonempty, it would include an algebraic cycle of (G, K), contradicting the maximality of F. So it is empty, so $C = \cup F$, which is a union of disjoint algebraic cycles of (G, K) as required. □

Lemma 2.37. Let G be a graph not including a subdivision of the Bean graph. Let X be a set of edges of G and let K be the set of vertices of G/X that include a ray of X. Then $M_{AC}(G)$ is generated by the algebraic cycles of $(G/X, K)$.

Proof. Any algebraic cycles of (G, K) can be extended to some $C \in \mathcal{C}_{AC}(G)$ using only extra edges from X. Conversely for any $C \in \mathcal{C}_{AC}(G)$, $C \setminus X$ has finite even degree at every vertex of G/X not in K, by Lemma 2.33**??**. So $C \setminus X$ is a disjoint union of algebraic cycles of $(G/X, K)$ by Lemma 2.36**??**. □

Theorem 2.38. Any finitary scrawl-system is a matroid.

Proof. Since the class of finitary scrawl-systems is closed under taking minors, it is enough to prove that any finitary scrawl-system $\langle C \rangle$ is based. Since any union of a chain of $\mathcal{C}$-independent sets is $\mathcal{C}$-independent, by Zorn's Lemma there is a maximal $\mathcal{C}$-independent set. □

Theorem 2.39. Let G be a graph not including a subdivision of the Bean graph. Then $M_{AC}(G)$ is a matroid.

Proof. By Lemma 2.23**??** and 2.37**??** it suffices to show there is a maximal element of the set $\mathcal{I}$ of subsets of $E(G')$ not including an algebraic cycle of (G', K) for any graph G' ad set K of vertices of G'. Let $\mathcal{R}$ be a maximal collection of disjoint rays in $G \setminus K$ which exists by Zorn's Lemma. Let I be the union of the edge-sets of the rays in $\mathcal{R}$. Let $\mathcal{I}'$ be the set of sets in $\mathcal{I}$ including I. Let $(I_l | l \in L)$ be a chain in $\mathcal{I}'$, and suppose for a contradiction that $\cup_{l \in L} I_l \notin \mathcal{I}'$. Let C be an algebraic cycle included in $\cup_{l \in L} I_l$. By maximality of $\mathcal{R}$, C includes a finite set P which is neither the edge set of a finite cycle or the edge-set of a path in G with both endpoints in $K \cup V(E)$. Each edge of P is in some I_l, so since $(I_l : l \in L)$ is a chain and P is finite there is some l with $P \subseteq I_l$. But then I_l includes an algebraic cycle of (G, K). So any such chain has an upper bound, so by Zorn's Lemma $\mathcal{I}'$ has a maximal element, which is also a maximal element of $\mathcal{I}$. □

Lemma 2.40. Let $\mathcal{I}$ be the set of M-independent subsets of E for some matroid M. Then $\mathcal{I}$ satisfies:

(IM) For any $I \subseteq X \subseteq E$ with $I \in \mathcal{I}$, there is a maximal element of $\{J \in \mathcal{I} : I \subseteq J \subseteq X\}$.

Proof. Take a base B of $(M/I)|_X$. We will show that taking $J = I \cup B$ works. Suppose for a contradiction that there is a nonempty scrawl C of M with $C \subseteq J$. $C \notin I$, since $I \in \mathcal{I}$, so $C \setminus I$ is a nonempty scrawl of $(M/I)|_X$ with $C \setminus I \subseteq B$. So $J \in \mathcal{I}$. Suppose for a contradiction that there is $x \in X \setminus J$ with $J + x \in P$. Then $B + x$ is not independent in $(M/I)|_X$, so there is a nonempty scrawl C' of the matroid with $C' \subseteq B + x$. Let C be a scrawl of M with $C' = C \setminus I$. Then $C \subseteq J + x$. So there is no such x, so J is maximal. $\square$

Convention 2.41. For $n \in \mathbb{N}$ we will refer by $1.n_\infty$ to $1.n$ not restricted to finite sets. $1.n_\infty$ is usually proved in the same way as $1.n$ and we will most of the time omit the proof.

Definition 2.42. An *I-matroid* on ground set E is a set $\mathcal{I}$ of subsets of E, satisfying (I1), (I2), (I3) and (IM). Elements of $\mathcal{I}$ are *independent*, other subsets of E are *dependent*, maximal independent sets are *bases* and minimal dependent sets are *circuits*.

Lemma 2.43. Let B_1, B_2 be bases of an I-matroid $\mathcal{I}$ with $B_1 \triangle B_2$ finite. Then $|B_1 \setminus B_2| = |B_2 \setminus B_1|$.

Proof. Suppose not for a contradiction, and let B_1, B_2 be a counter-example with $|B_1 \setminus B_2| > |B_2 \setminus B_1|$ and $|B_2 \setminus B_1|$ as small as possible. Since $|B_1 \setminus B_2| > 0$, there is $x \in B_1 \setminus B_2$. So by (I3) there is $y \in B_2 \setminus B_1$ with $B_1 - x + y \in \mathcal{I}$. Let B_1' be a base including $B_1 - x + y$ which exists by (IM). Then B_1', B_2 is a counter-example with $|B_1' \setminus B_2| > |B_2 \setminus B_1'|$ but $|B_2 \setminus B_1'| < |B_2 \setminus B_1|$, contradicting the minimality of $|B_2 \setminus B_1|$. $\square$

(IM) For any $I \subseteq X \subseteq E$ with $I \in \mathcal{I}$, the set $\{J \in \mathcal{I} : I \subseteq J \subseteq X\}$ has a maximal element.

I-matroid: $\mathcal{I}$ satisfying (I1), (I2), (I3) and (IM).

Theorem. A set $\mathcal{B} \subseteq \mathcal{P}E$ is the set of bases of an I-matroid $\mathcal{I}$ iff it satisfies (B1), (B2) and

(BM) The set of subsets of sets in $\mathcal{B}$ satisfies (IM).

In these circumstances, $\mathcal{I}$ is the set of subsets of elements of $\mathcal{B}$.

Proof. We only prove the 'only if' direction. The 'if' direction is proved as in chapter 1, using lemma 2.43**??** instead of lemma 1.5**??**.

For the 'only if' direction let $\mathcal{B}$ satisfy (B1), (B2) and (BM). Let $\mathcal{I}$ be the set of subsets of elements of $\mathcal{B}$. $\mathcal{I}$ satisfies (IM) by definition, and it clearly satisfies (I1) and (I2). For (I3), let $I_1, I_2 \in \mathcal{I}$ and $x \in I_1$ with $I_2 + x \notin \mathcal{I}$. Chose $B_1 \in \mathcal{B}$ extending I. By (IM) there is a maximal set $J \in \mathcal{I}$ with $I_2 \subseteq J \subseteq B_1 \cup I_2$. Let $B_2 \in \mathcal{B}$ with $J \subseteq B_2$. Notice that $x \in B_1 \setminus B_2$, so by (B2) there is $y \in B_2 \setminus B_1$ with $B_1 - x + y \in \mathcal{B}$. Suppose for a contradiction that $y \notin I_2$. Now apply (B2) to B_2, B_1 and y to get $z \in B_1 \setminus B_2$ with $B_3 := B_2 - y + z \in \mathcal{B}$, so $B_3 \cap (I_2 \cup B_1)$ extends $J + z$, contradicting the maximality of J. So $y \in I_2$ and $I_1 - x + y \subseteq B_1 - x + y$, so $I_1 - x + y \in \mathcal{I}$. $\square$

Lemma 2.44. Any dependent set X of an I-matroid $\mathcal{I}$ includes a circuit.

Proof. Pick a maximal independent set $I \subseteq X$. Since X is dependent, there is $z \in X \setminus I$. Let $C := \{x \in I : I - x + z \in \mathcal{I}\} + z \subseteq X$. Suppose for a contradiction that C is independent. By (IM) we can extend C to a maximal independent subset J of $I + z$. Since $I + z$ is dependent by the maximality of I, there is some $x \in I + z \setminus J$. By the maximality of J, $J + x \notin \mathcal{I}$. So applying (I3) to I, J and x we get $y \in J \setminus I$ with $I - x + y \in \mathcal{I}$. But $J \setminus I = \{z\}$, so $y = z$, so $I - x + z \in \mathcal{I}$. So $x \in C \subseteq J$. This contradicts the fact that $J + x$ is dependent. To prove minimality of C note that every set $C - x$ is independent. If $x = z$, since $C - z \subseteq I$, and if $x \neq z$ since $C - x \subseteq I + z - x \in \mathcal{I}$. So C is minimal. $\square$

Definition 2.45. If $X \subseteq E$, E the ground set of an I-matroid $\mathcal{I}$, the *span* $\mathrm{Sp}_{\mathcal{I}}(X)$ is $X \cup \{x \notin X : \exists I \subseteq X : I \in \mathcal{I} \text{ but } I + x \notin \mathcal{I}\}$.

Lemma. If $X \subseteq E$ and I is a maximal independent subset of X then $\mathrm{Sp}_{\mathcal{I}}(I) = \mathrm{Sp}_{\mathcal{I}}(X)$.

Lemma 2.46. Let $x \notin X$. Then $x \in \mathrm{Sp}_{\mathcal{I}}(X)$ iff there is some circuit C with $x \in C \subseteq X + x$.

Proof. The 'if' direction follows from the fact that $C - x$ is an independent subset of X. For the 'only if' direction, let I be a maximal independent subset of X. Then $x \in \mathrm{Sp}_{\mathcal{I}}(I)$ by lemma 1.16_∞**??**, so $I + x \notin \mathcal{I}$. So by lemma 2.44**??** there is a circuit C with $C \subseteq I + x$. Since $I \in \mathcal{I}$, $C \not\subseteq I$, so $x \in C$. $\square$

Lemma. For $A \subseteq B \subseteq E$, then $\mathrm{Sp}_{\mathcal{I}}(A) \subseteq \mathrm{Sp}_{\mathcal{I}}(B)$, and $\mathrm{Sp}_{\mathcal{I}}(A) = \mathrm{Sp}_{\mathcal{I}}(\mathrm{Sp}_{\mathcal{I}}(A))$.

Theorem. A set $\mathcal{C} \subseteq \mathcal{P}E$ is the set of circuits of an I-matroid $\mathcal{I}$ iff it satisfies (C1), (C2) and $(\mathrm{C3})_\infty$ and

(CM) The set of $\langle \mathcal{C} \rangle$-independent sets satisfies (IM).

In these circumstances, $\mathcal{I}$ is the set of $\langle \mathcal{C} \rangle$-independent sets.

Proof. We may use the proof of theorem 1.18**??** to prove all but the fact that the circuits of any I-matroid satisfy $(\mathrm{C3})_\infty$. For this, let C, z, X and $(C_x : x \in X)$ as in the setup of $(\mathrm{C3})_\infty$. Let $P := (C \cup \bigcup_{x \in X} C_x) \setminus X - z$. For each $x \in X$, we have $C_x - x \subseteq P$, so $x \in \mathrm{Sp}_{\mathcal{I}}(P)$. So $C - z \subseteq \mathrm{Sp}_{\mathcal{I}}(P)$, so $z \in \mathrm{Sp}_{\mathcal{I}}(\mathrm{Sp}_{\mathcal{I}}(P)) = \mathrm{Sp}_{\mathcal{I}}(P)$ by lemma 1.17_∞**??**. By lemma 2.46**??** there is a circuit C' with $z \in C' \subseteq P + z = (C \cup \bigcup_{x \in X} C_x) \setminus X$. $\square$

Remark 2.47. We now get all the results of chapter 1 for I-matroids up to lemma 1.33_∞**??**, except lemma 1.5. In particular, we get a theory of duality for these I-matroids, which coincides with the duality of matroids and the same is true for minors.

Theorem 2.48. A set $\mathcal{I}$ is the set of independent sets of some matroid M iff it is an I-matroid. In these circumstances, M is generated by the circuits of $\mathcal{I}$.

Proof. The 'only if' direction is lemma 2.26**??** and 2.40**??**. For the 'if' direction, let M be the scrawl-system generated by the circuits of $\mathcal{I}$. By theorem 1.18_∞**??**, $\mathcal{I}$ is the set of independent sets of M. Given disjoint subsets P, Q of E, let I be a maximal independent subset of P and let $J \in \mathcal{I}$ with be maximal with $I \subseteq J \subseteq E \setminus Q$. Then by corollary 1.25_∞**??**, J is a base of $M/P\backslash Q$, so M is a matroid. Every $\mathcal{I}$ cocircuit is in $M*$ and so by lemma 1.32_∞**??**, M is generated by the circuits of $\mathcal{I}$. $\square$

Definition 2.49. The sets of independent sets, bases, circuits and scrawls of a matroid M are $\mathcal{I}(M)$, $\mathcal{B}(M)$, $\mathcal{C}(M)$ and $\mathcal{S}(M)$ respectively.

Lemma 2.50 (The Characterisation Lemma)**.** Let M be a matroids on a ground set E, and $\mathcal{C}, \mathcal{D} \subseteq \mathcal{P}E$. Suppose every circuit of M is a union of elements of $\mathcal{C}$, and every cocircuit of M is a union of elements of $\mathcal{D}$ and that $\mathcal{C}$ and $\mathcal{D}$ satisfy (O1). Then $\mathcal{C}(M) \subseteq \mathcal{C} \subseteq \mathcal{S}(M)$, $\mathcal{C}(M^*) \subseteq \mathcal{D} \subseteq \mathcal{S}(M^*)$. In particular, the circuits of M is the set of minimal nonempty elements of $\mathcal{C}$ and $\mathcal{S}(M)$ is the set of unions of elements of $\mathcal{C}$, similarly for the duals.

Proof. By definition, no element of $\mathcal{C}$ ever meets a cocircuit of M just once. So $\mathcal{C} \subseteq (\mathcal{C}(M^*))^* = \mathcal{S}(M)$. Now let $C \in \mathcal{C}(M)$, and suppose $C \notin \mathcal{C}$. Pick $C' \in \mathcal{C}$ with $\emptyset \neq C' \subsetneq C$. Let $e \in C'$. Let $P = C' - e$, $Q = E \setminus C'$. There is no $D \in \mathcal{C}(M^*)$ with $e \in D \subseteq Q + e$ since then $C' \cap D = \{e\}$. So there is a circuit C'' of M with $e \in C'' \subseteq P + e$, by (O2) for $\mathcal{C}(M)$ and $\mathcal{C}(M^*)$. But then $C'' \subseteq C$, a contradiction. So $C \in \mathcal{C}$. That is $\mathcal{C}(M) \subseteq \mathcal{C}$. The last sentence is clear. $\square$

Dual pairs of matroids asociated to a graph:

- finite cycles and bonds
- algebraic cycles and minimal skew cuts
- ?? and finite bonds

Facts about compact topological sapces:

- Any product of compact spaces is compact.
- A subspace of a compact Hausdorff space is compact iff it is closed.
- Any image of a compact space under a continuous map is compact.
- Any continuous injective map from a compact space to a Hausdorff space is a homeomorphism onto its image.

Definition 2.51. Let $(X, \leq)$ be a partially ordered set. A *diagram of topological spaces* on X consists of a topological space T_x for each $x \in X$, and a continuous map $\varphi_{x,y} \colon T_x \to T_y$ whenever $y \leq x$ in X, such that:

(1) $\varphi_{x,x}$ is the identity on T_x

(2) For $x \geq y \geq z$, $\varphi_{x,z} = \varphi_{y,z} \circ \varphi_{x,y}$

A *cone* for such a diagram consists of a topological space T and a family of maps $(f_x\colon T \to T_x : x \in X)$ of continuous maps such that

$$\text{for } x \geq y \text{ in } X \ f_y = \varphi_{x,y} \circ f_x$$

Theorem 2.52. For any such diagram of topological spaces there is a *universal cone* $(T, (\pi_x : x \in X))$, that is, one such that for any other cone $(T', (f_x : x \in X))$ there is a unique continuous map $T' \to gT$ such that $(\forall x \in X) f_x = \pi_x \circ g$.

Proof. Let T be the subspace of $\prod_{x \in X} T_x$ given by $\{k \in \prod_{x \in X T_x} : (\forall x \geq y \text{ in } X) \varphi_{x,y}(k_x) = k_y\}$, with π_x given by the projection map. So for any cone $(T', (f_x : x \in X))$, there is only one function $T' \to gT$ with $f_x = \pi_x \circ g$ for $x \in X$, namely $l \mapsto (f_x(l) : x \in X)$. We just have to check that g is continuous. So take some basic open subset $U = \{k \in T : (\forall i \leq n) k_{x_i} \in U_i\}$, and observe that $g^{-1}(U) = \cap_{i \leq n} f_{x_i}^{-1}(U_i)$, which is open. We call T the *limit* of this diagram. □

Fact 2.53. Any limit of a diagram of compact Hausdorff spaces is compact and Hausdorff.

Now we fix some connected graph G, and let X be the set of all finite subsets of G, partially ordered by inclusion. For each $x \in X$, let $G_x = G/(E \setminus x)$. Each G_x has a corresponding topological space T_x, whose points are either vertices of G_x or the interior points of edges of G_x. These are compact and Hausdorff, and we ahve continuous maps $\varphi_{x,y}$ from T_x to T_y when $y \subseteq x$. Let $|G|$ be the limit of this diagram: $|G|$ is compact and Hausdorff.

Definition 2.54. Let P be an interior point of an edge of G, then $[\![p]\!] \in |G|$ is given by

$$[\![p]\!]_x = \begin{cases} p & e \in x \\ \text{the component of } G \setminus x \text{ containing } e & \text{otherwise} \end{cases}$$

Lemma 2.55. If $k \in |G|$ such that for some $x \in X$ k_x is an interior point p of an edge of G_x, then $k = [\![p]\!]$.

Proof. For any $x' \in X$ with $x \leq x'$ then $\varphi_{x',x}(k_{x'}) = k_x$, so $k_{x'} = p$. So for any $y \in X$, $k_y = \varphi_{x \cup y, y}(k_{x \cup y}) = \varphi_{x \cup y, y}(p) = [\![p]\!]_y$. □

Remark 2.56. If p, q are interior points of edges of G with $[\![p]\!] = [\![q]\!]$, then $p = q$.

Definition 2.57. If v is a vertex of G, $[\![v]\!] \in |G|$ is given by

$$[\![v]\!] = \text{the component of } G - x \text{ containing } v.$$

Lemma 2.58. Let k be a point of $|G|$ and $x \in X$ such that k_x is a finite component of $G - x$. Then either k is of the form $[\![p]\!]$ for p an interior point of an edge of k_x, or of the form $[\![v]\!]$ for v a vertex of k_x.

Proof. Let y be the set of edges of k_x. Let $x' := x \cup y$. If $k_{x'}$ is an interior point of an edge of k_x, we apply lemma 2.55. Otherwise, $k_{x'} = \{v\}$ for some vertex v of k_x. For any $x'' \geq x'$ in X, $\varphi_{x'',x'}(k_{x''}) = \{v\}$, so $k_{x''} = \{v\}$. So for any $y' \in X$, we have $k_{y'} = \varphi_{x'\cup y',y'}(k_{x'\cup y'}) = \varphi_{x'\cup y',y'}(\{v\}) = [\![v]\!]_y$, so $k = [\![v]\!]$. $\square$

Remark 2.59. If v, w are vertices of G, $[\![v]\!] = [\![w]\!]$ iff there is no finite set of edges separating v from w in G.

Example 2.60. Let G be the graph as in figure **??**, called the *ladder*.

For each x, $G - x$ has precisely one infinite component, so there is precisely one point of $|G|$ not of the form $[\![p]\!]$ or $[\![v]\!]$. The space $|G|$ is given by the following figure with its topologicy as a subspace of the plane.

Definition 2.61. A graph G is *finitely separable* iff and two vertices of it can be separated by removing finitely many edges. For any G, the *finitely separable quotient* $\mathrm{F\_sep}(G)$ of G is obtained from G by identifying any pair of vertices that cannot be finitely separated.

Remark 2.62. A set of edges of G is a finite cut of G iff it is a finite cut of $\mathrm{F\_sep}(G)$. We assume from now on that G is finitely separable.

Lemma 2.63. Let G be a graph, A, B sets of vertices of G. Then there are infinitely many vertex-disjoint $A - B$-paths iff A and B cannot be separated by removing finitely many vertices.

Proof. The 'only if' direction is clear. For the 'if' direction, by Zorn's lemma there is a maximal collection of disjoint $A - B$-paths, whose vertices separate A from B, so which must be infinite. $\square$

Lemma 2.64. Let G be a graph, A, B disjoint sets of vertices of G. Then there are infinitely many edge-disjoint $A - B$-paths iff A and B cannot be separated by removing finitely many edges. $\square$

Definition 2.65. Let R be a ray in G. Then $[\![R]\!] \in |G|$ is given by taking $[\![R]\!]_x$ to be the unique component of $G - x$ which contains a *tail*, an infinite component, of R.