Cones over pseudo-Riemannian manifolds and their holonomy

D. V. Alekseevsky*, V. Cortés, A. S. Galaev, and T. Leistner†

Abstract

By a classical theorem of Gallot (1979), a Riemannian cone over a complete Riemannian manifold is either flat or has irreducible holonomy. We consider metric cones with reducible holonomy over pseudo-Riemannian manifolds. First we describe the local structure of the base of the cone when the holonomy of the cone is decomposable. For instance, we find that the holonomy algebra of the base is always the full pseudo-orthogonal Lie algebra. One of the global results is that a cone over a compact and complete pseudo-Riemannian manifold is either flat or has indecomposable holonomy. Then we analyse the case when the cone has indecomposable but reducible holonomy, which means that it admits a parallel isotropic distribution. This analysis is carried out, first in the case where the cone admits two complementary distributions and, second for Lorentzian cones. We show that the first case occurs precisely when the local geometry of the base manifold is para-Sasakian and that of the cone is para-Kählerian. For Lorentzian cones we get a complete description of the possible (local) holonomy algebras in terms of the metric of the base manifold.

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1 Introduction

Let (M, g) be a (connected) pseudo-Riemannian manifold of signature

$$(p, q) = (-, \cdots, -, +, \cdots, +).$$

We denote by H the holonomy group of (M, g) in a point $p \in M$ and by $\mathfrak{h} \subset \mathfrak{so}(V)$, $V = T_pM$, the corresponding holonomy algebra. We say that $\mathfrak{h}$ is *decomposable* if V contains a proper non-degenerate $\mathfrak{h}$-invariant subspace. By Wu's theorem [20] this means that M is locally decomposed as a product of two pseudo-Riemannian manifolds. In the opposite case $\mathfrak{h}$ is called *indecomposable*. We say that $\mathfrak{h}$ is *reducible* if it preserves a (possibly degenerate) proper subspace of V. Then $\mathfrak{h}$ is of exactly one of the following types:

(i) decomposable,

(ii) reducible indecomposable, or

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(iii) irreducible.

Let $(\widehat{M} = \mathbb{R}^+ \times M, \widehat{g} = dr^2 + r^2 g)$ be the (space-like) metric cone over (M, g). We denote by $\widehat{H}$ the holonomy group of $(\widehat{M}, \widehat{g})$ and by $\widehat{\mathfrak{h}} \subset \mathfrak{so}(\widehat{V})$ $(\widehat{V} = T_p\widehat{M},\ p \in \widehat{M})$ the corresponding holonomy algebra. In the present article we shall describe the geometry of the base (M, g) for each of the three possibilities (i-iii) for the holonomy algebra of the cone $\widehat{M}$. Our first result describes the holonomy algebra and local structure of a manifold (M, g) with decomposable holonomy $\widehat{\mathfrak{h}}$ of the cone.

Theorem 4.1. *Let (M, g) be a pseudo-Riemannian manifold with decomposable holonomy algebra $\widehat{\mathfrak{h}}$ of the cone $\widehat{M}$. Then the manifold (M, g) has full holonomy algebra $\mathfrak{so}(p, q)$, where (p, q) is the signature of the metric g. Furthermore, there exists an open dense submanifold $M' \subset M$ such that any point $p \in M'$ has a neighborhood isometric to a pseudo-Riemannian manifold of the form $(a, b) \times N_1 \times N_2$ with the metric given either by*

$$g = ds^2 + \cos^2(s) g_1 + \sin^2(s) g_2 \quad \textit{or} \quad g = -ds^2 + \cosh^2(s) g_1 + \sinh^2(s) g_2,$$

where g_1 and g_2 are metrics on N_1 and N_2 respectively.

Let us recall the following fundamental theorem of Gallot which settles the problem for Riemannian cones over complete Riemannian manifolds.

Theorem 1 (S. Gallot, [14]). *Let (M, g) be a complete Riemannian manifold of dimension ≥ 2 with decomposable holonomy algebra $\widehat{\mathfrak{h}}$ of the cone $\widehat{M}$. Then (M, g) has constant curvature 1 and the cone is flat. If, in addition, (M, g) is simply connected, then it is equal to the standard sphere.*

For pseudo-Riemannian manifolds (M, g) the completeness assumption yields only the following generalisation of Gallot's result:

Theorem 7.1. *Let (M, g) be a complete pseudo-Riemannian manifold of dimension ≥ 2 with decomposable holonomy $\widehat{\mathfrak{h}}$ of the cone $\widehat{M}$. Then there exists an open dense submanifold $M' \subset M$ such that each connected component of M' is isometric to a pseudo-Riemannian manifold of the form*

(1) a pseudo-Riemannian manifold M_1 of constant sectional curvature 1 or

(2) a pseudo-Riemannian manifold $M_2 = \mathbb{R}^+ \times N_1 \times N_2$ with the metric

$$-ds^2 + \cosh^2(s) g_1 + \sinh^2(s) g_2,$$

where (N_1, g_1) and (N_2, g_2) are pseudo-Riemannian manifolds and (N_2, g_2) has constant sectional curvature -1 or $\dim N_2 \leq 1$.

Moreover, the cone $\widehat{M_2}$ is isometric to the open subset $\{r_1 > r_2\}$ in the product of the space-like cone $(\mathbb{R}^+ \times N_1, dr^2 + r^2 g_1)$ over (N_1, g_1) and the time-like cone $(\mathbb{R}^+ \times N_2, -dr^2 + r^2 g_2)$ over (N_2, g_2).

For compact and complete pseudo-Riemannian manifolds (M, g) we are able to establish the same conclusion as in Theorem 1:

Theorem 6.1. *Let (M, g) be a compact and complete pseudo-Riemannian manifold of dimension ≥ 2 with decomposable holonomy group $\widehat{H}$ of the cone $\widehat{M}$. Then (M, g) has constant curvature 1 and the cone is flat.*

We remark that for indefinite pseudo-Riemannian manifolds compactness does not imply completeness, see for example [19], p. 193, for a geodesically incomplete Lorentz metric on the 2-torus (the so-called Clifton-Pohl torus).

Since there is no simply connected compact indefinite pseudo-Riemannian manifold of constant curvature 1, we obtain the following corollary.

Corollary 1. *If (M, g) is a simply connected compact and complete indefinite pseudo-Riemannian manifold, then the holonomy algebra of the cone $(\widehat{M}, \widehat{g})$ is indecomposable.*

Now we consider the case (ii) when the holonomy algebra $\widehat{\mathfrak{h}}$ of the cone $\widehat{M}$ is indecomposable but reducible. We completely analyse the situation in the following two cases:

(ii.a) $\widehat{\mathfrak{h}}$ preserves a decomposition $T_p\widehat{M} = V \oplus W$ $(p \in \widehat{M})$ into two complementary subspaces V and W.

(ii.b) $\widehat{M}$ is Lorentzian.

In the case (ii.a) one can show that $\widehat{M}$ admits (locally) a para-Kähler structure, which means that the holonomy algebra $\widehat{\mathfrak{h}}$ preserves two complementary *isotropic* subspaces. The following theorem characterises para-Kählerian cones as cones over para-Sasakian manifolds.

Theorem 8.1. *Let (M, g) be a pseudo-Riemannian manifold. There is a one-to-one correspondence between para-Sasakian structures (M, g, T) on (M, g) and para-Kähler structures $(\widehat{M}, \widehat{g}, \widehat{J})$ on the cone $(\widehat{M}, \widehat{g})$. The correspondence is given by $T \mapsto \widehat{J} := \widehat{\nabla} T$.*

Similarly, we have the following characterisation of the case when the cone $\widehat{M}$ admits (locally) a para-hyper-Kähler structure, which means that the holonomy algebra $\widehat{\mathfrak{h}}$ preserves two complementary isotropic subspaces $T^{\pm}$ and a skew-symmetric complex structure J such that $JT^+ = T^-$. In particular, it preserves the para-hyper-complex structure $(\widehat{J}_1, \widehat{J}_2, \widehat{J}_3 = \widehat{J}_1\widehat{J}_2)$, where $\widehat{J}_1|_{T^{\pm}} = \pm Id$ and $\widehat{J}_3 = J$.

Theorem 8.2. *Let (M, g) be a pseudo-Riemannian manifold. There is a one-to-one correspondence between para-3-Sasakian structures (M, g, T_1, T_2, T_3) on (M, g) and para-hyper-Kähler structures $(\widehat{M}, \widehat{g}, \widehat{J}_1, \widehat{J}_2, \widehat{J}_3 = \widehat{J}_1\widehat{J}_2)$ on the cone $(\widehat{M}, \widehat{g})$. The correspondence is given by $T_\alpha \mapsto \widehat{J}_\alpha := \widehat{\nabla} T_\alpha$.*

Finally, we consider the case (ii.b) when the cone is Lorentzian with indecomposable reducible holonomy algebra. Lorentzian holonomy algebras are classified, [17] and [13], and the following theorems describe which of these are holonomies of cones. In addition, the local geometry is described.

Theorem 9.1. *Let (M, g) be a Lorentzian manifold of signature $(1, n-1)$ or a negative definite Riemannian manifold and $(\widehat{M} = \mathbb{R}^+ \times M, \widehat{g})$ the cone over M with Lorentzian signature $(1, n)$ or $(n, 1)$ respectively. If the holonomy algebra $\widehat{\mathfrak{h}}$ of $\widehat{M}$ is indecomposable reducible (i.e. preserves an isotropic line) then it annihilates a non-zero isotropic vector.*

The next theorem treats the case of a Lorentzian cone $\widehat{M}$ over a negative definite Riemannian manifold M.

Theorem 9.1a. *Let (M, g) be a negative definite Riemannian manifold and $(\widehat{M}, \widehat{g})$ the cone over M equipped with the Lorentzian metric of signature $(+, -, \cdots, -)$. If $\widehat{M}$ admits a non-zero parallel isotropic vector field then M is locally isometric to a manifold of the form*

$$(M_0 = (a, b) \times N, g = -ds^2 + e^{-2s} g_N), \tag{1.1}$$

where $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R} \cup \{+\infty\}$, $a < b$ and (N, g_N) is a negative definite Riemannian manifold. Furthermore, if $\mathfrak{hol}(\widehat{M_0})$ is indecomposable then

$$\mathfrak{hol}(\widehat{M_0}, \widehat{g}) \cong \mathfrak{hol}(N, g_N) \ltimes \mathbb{R}^{\dim N}.$$

If the manifold (M, g) is complete then the isometry is global, (N, g_N) is complete and $(a, b) = \mathbb{R}$.

Finally, we deal with a Lorentzian cone $\widehat{M}$ over Lorentzian manifold M.

Theorem 9.1b. *Let (M, g) be a Lorentzian manifold and $(\widehat{M}, \widehat{g})$ the cone over M equipped with the Lorentzian metric. If $\widehat{M}$ admits a non-zero parallel isotropic vector field then there exists an open dense submanifold $M' \subset M$ such that any point of M' has a neighborhood isometric to a manifold of the form (1.1), where (N, g_N) is a positive definite Riemannian manifold. Furthermore, if $\mathfrak{hol}(\widehat{M_0})$ is indecomposable then*

$$\mathfrak{hol}(\widehat{M_0}, \widehat{g}) \cong \mathfrak{hol}(N, g_N) \ltimes \mathbb{R}^{\dim N}.$$

If the manifold (M, g) is complete then each connected component of M' is isometric to a manifold of the form (1.1), where (N, g_N) is a positive definite Riemannian manifold and $(a, b) = \mathbb{R}$.

Let us conclude this introduction with some brief remarks about applications of these. The theorem of Gallot was used by C. Bär in the classification of Riemannian manifolds admitting a real Killing spinor [5]. In general, a pseudo-Riemannian manifold admits a real/imaginary Killing spinors if and only if its space-like/time-like cone admits a parallel spinor (details in [9]). Hence, a strategy for studying manifolds with Killing spinors is to study their cones with parallel spinors. Now, in order to classify manifolds with parallel spinors the knowledge of their holonomy group is essential. In the Riemannian situation Gallot's result reduces the problem to irreducible holonomy groups of cones. With our results this strategy becomes applicable to arbitrary signature.

Another applications in the same spirit — solutions to an overdetermined system of PDE's correspond to parallel sections for a certain connection — comes from conformal geometry. Here, to a conformal class of a metric one can assign the so-called Tractor bundle with Tractor connection. Parallel sections for this connection correspond to metrics in the conformal class which are Einstein. For conformal classes which contain an Einstein metric the holonomy of the conformal Tractor connection reduces to the Levi-Civita holonomy of the Fefferman-Graham ambient metric [12]. For conformal classes containing proper Einstein metrics with positive/negative Einstein constant the ambient metric reduces to the space-like/time-like cone over a metric in the conformal class [18, 3, 4]. Again, our results enable us to describe the holonomy of the conformal Tractor connection by the holonomy of the cone.

To carry out the details of both applications lies beyond the scope of this paper and will be subject to future research.

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2 Doubly warped products

Let (N_1, g_1) and (N_2, g_2) be two pseudo-Riemannian manifolds, $\varepsilon = \pm 1$ and f_1, f_2 two nowhere vanishing smooth functions on an open interval $I = (a, b) \subset \mathbb{R}$. The manifold $M = I \times N_1 \times N_2$ with the metric

$$g = \varepsilon ds^2 + f_1(s)^2 g_1 + f_2(s)^2 g_2 \tag{2.1}$$

is called a *doubly warped product*. We will consider the coordinate vector field $\partial_0 = \partial_s$ on the interval I and vector fields $X, X', \ldots$ on N_1 and $Y, Y', \ldots$ on N_2 as vector fields on M.

Proposition 2.1. *(i) The Levi-Civita connection ∇ of the pseudo-Riemannian manifold (M, g) is given by:*

$$\begin{array}{rcl}
\nabla_{\partial_0}\partial_0 &=& 0 \\
\nabla_{\partial_0}X &=& \nabla_X\partial_0 = \dfrac{f_1'}{f_1}X \\
\nabla_{\partial_0}Y &=& \nabla_Y\partial_0 = \dfrac{f_2'}{f_2}Y \\
\nabla_X X' &=& -\varepsilon\dfrac{f_1'}{f_1}g(X,X')\partial_0 + \nabla^1_X X' \\
\nabla_Y Y' &=& -\varepsilon\dfrac{f_2'}{f_2}g(Y,Y')\partial_0 + \nabla^2_Y Y' \\
\nabla_X Y &=& \nabla_Y X = 0,
\end{array}$$

where ∇^1 and ∇^2 are the Levi-Civita connections of g_1 and g_2.

(ii) A curve $I \ni t \mapsto (s(t), \gamma_1(t), \gamma_2(t)) \in I \times N_1 \times N_2 = M$ is a geodesic if and only if

it satisfies the equations:

$$\begin{aligned}
\ddot{s} &= \varepsilon(f_1'(s)f_1(s)g_1(\dot{\gamma_1},\dot{\gamma_1}) + f_2'(s)f_2(s)g_2(\dot{\gamma_2},\dot{\gamma_2})) \\
\nabla^i_{\dot{\gamma_i}}\dot{\gamma_i} &= -2\frac{f_i'}{f_i}\dot{s}\dot{\gamma_i},\ i=1,2.
\end{aligned}$$

(iii) In terms of the arclength parameters u_1, u_2 of the curves γ_1, γ_2 in N_1, N_2 the equations (ii) take the form:

$$\ddot{s} = \varepsilon\varepsilon_1 f_1'(s)f_1(s)\dot{u_1}^2 + \varepsilon\varepsilon_2 f_2'(s)f_2(s)\dot{u_2}^2 \tag{2.2}$$

$$\ddot{u_i} = -2\frac{f_i'}{f_i}\dot{s}\dot{u_i},\ i=1,2. \tag{2.3}$$

where $\varepsilon_i = g_i(\frac{d\gamma_i}{du},\frac{d\gamma_i}{du}) \in \{\pm1,0\}$.

Corollary 2.1. *(i) The submanifolds $I\times N_i \subset M$, $i=1,2$, are totally geodesic, as well as their intersection I.*

(ii) Any geodesic $\Gamma\subset M$ is contained in a totally geodesic submanifold $I\times\Gamma_1\times\Gamma_2\subset M$, where $\Gamma_i = pr_{N_i}(\Gamma)\subset N_i$ are geodesics in N_i, $i=1,2$.

Proposition 2.2. *The curvature tensor of the doubly warped product (2.1) is given by:*

$$\begin{aligned}
R(\partial_0,X) &= -\varepsilon\frac{f_1''}{f_1}\partial_0\wedge X \\
R(\partial_0,Y) &= -\varepsilon\frac{f_2''}{f_2}\partial_0\wedge Y \\
R(X,X') &= -\varepsilon\left(\frac{f_1'}{f_1}\right)^2 X\wedge X' + R^1(X,X') \\
R(Y,Y') &= -\varepsilon\left(\frac{f_2'}{f_2}\right)^2 Y\wedge Y' + R^2(Y,Y') \\
R(X,Y) &= -\varepsilon\frac{f_1'f_2'}{f_1f_2}X\wedge Y
\end{aligned}$$

Recall that a pseudo-Riemannian metric g has constant curvature κ if and only if its curvature tensor has the form

$$R(X,Y) = \kappa X\wedge Y = \kappa(X\otimes g(Y,\cdot) - Y\otimes g(X,\cdot)).$$

Corollary 2.2. *A doubly warped product (2.1) has constant curvature $\kappa = -\varepsilon c$ if and only if the metrics g_1 and g_2 have constant curvature κ_1, κ_2 and the warping functions satisfy the following system of equations:*

$$\frac{f_1''}{f_1} = c \quad \text{if} \quad \dim N_1 > 0,$$

$$\frac{f_2''}{f_2} = c \quad \text{if} \quad \dim N_2 > 0,$$

$$\frac{f_1' f_2'}{f_1 f_2} = c, \quad \textit{if} \quad \dim N_1 > 0 \quad \textit{and} \quad \dim N_2 > 0,$$
$$\left(\frac{f_1'}{f_1}\right)^2 - \varepsilon \frac{\kappa_1}{f_1^2} = c \quad \textit{if} \quad \dim N_1 > 1,$$
$$\left(\frac{f_2'}{f_2}\right)^2 - \varepsilon \frac{\kappa_2}{f_2^2} = c \quad \textit{if} \quad \dim N_2 > 1.$$

Solving these equations we get the following corollary. We will denote by g_k, g_k', g_k'' pseudo-Riemannian metrics of constant curvature $k \in \{\pm 1, 0\}$.

Corollary 2.3. *Then the following doubly warped product metrics g_k have constant curvature k:*

$$\begin{aligned}
g_{-\varepsilon} &= \varepsilon ds^2 + \cosh^2(s) g'_{-\varepsilon} + \sinh^2(s) g''_{\varepsilon} \\
g_{\varepsilon} &= \varepsilon ds^2 + \cos^2(s) g'_{\varepsilon} + \sin^2(s) g''_{\varepsilon} \\
g_{-\varepsilon} &= \varepsilon ds^2 + e^{2s} g'_0 \\
g_0 &= \varepsilon ds^2 + s^2 g'_{\varepsilon} + g''_0 \\
g_{-\varepsilon} &= \varepsilon ds^2 + \cosh^2(s) dt^2 + \sinh^2(s) g''_{\varepsilon} \\
g_{-\varepsilon} &= \varepsilon ds^2 + \cosh^2(s) g'_{-\varepsilon} + \sinh^2(s) du^2 \\
g_{\varepsilon} &= \varepsilon ds^2 + \cos^2(s) dt^2 + \sin^2(s) g''_{\varepsilon} \\
g_{\varepsilon} &= \varepsilon ds^2 + \cos^2(s) g'_{\varepsilon} + \sin^2(s) du^2 \\
g_0 &= \varepsilon ds^2 + s^2 dt^2 + g''_0 \\
g_{-\varepsilon} &= \varepsilon ds^2 + \sinh^2(s) g'_{\varepsilon} \\
g_{-\varepsilon} &= \varepsilon ds^2 + \cosh^2(s) g'_{-\varepsilon} \\
g_{\varepsilon} &= \varepsilon ds^2 + \sin^2(s) g'_{\varepsilon} \\
g_0 &= \varepsilon ds^2 + g'_0 \\
g_{-\varepsilon} &= \varepsilon ds^2 \pm \cosh^2(s) dt^2 \pm \sinh^2(s) du^2 \\
g_{\varepsilon} &= \varepsilon ds^2 \pm \cos^2(s) dt^2 \pm \sin^2(s) du^2 \\
g_{-\varepsilon} &= \varepsilon ds^2 \pm \sinh^2(s) dt^2 \\
g_{-\varepsilon} &= \varepsilon ds^2 \pm \cosh^2(s) dt^2 \\
g_{\varepsilon} &= \varepsilon ds^2 \pm \sin^2(s) dt^2
\end{aligned}$$

Any doubly warped product of pseudo-Riemannian manifolds (N_1, g_1), (N_2, g_2) which has constant curvature ± 1 or 0 belongs to the above list up to a shift $s \mapsto s + s_0$ and rescaling

$$(f_i^2, g_i) \mapsto (\lambda_i^2 f_i^2, \frac{1}{\lambda_i^2} g_i).$$

Geometric realisation of doubly warped products of constant curvature

Now we give a realisation of the above doubly warped products in terms the pseudo-sphere models of the spaces of constant curvature.

The standard pseudo-spheres as models of spaces of curvature ± 1

Let $\mathbb{R}^{t,s} = (\mathbb{R}^{t+s}, \langle \cdot, \cdot \rangle - \sum_{i=1}^{t} dx_i^2 + \sum_{i=t+1}^{t+s} dx_i^2)$ be the standard pseudo-Euclidian vector space of signature (t, s). We denote by

$$\begin{aligned} S_+^{t,s} &:= \{x \in \mathbb{R}^{t,s+1} | \langle x, x \rangle = +1\} \\ S_-^{t,s} &:= \{x \in \mathbb{R}^{t+1,s} | \langle x, x \rangle = -1\} \end{aligned}$$

the two unit pseudo-spheres. The induced metric $g_\pm = g_{S_\pm^{t,s}}$ of $S_\pm^{t,s}$ has signature (t, s) and constant curvature ± 1. More precisely the curvature tensor is given by

$$R(X, Y)Z = \pm(\langle Y, Z \rangle X - \langle X, Z \rangle Y).$$

Notice that $S_+^{0,n} = S^n$ is the standard unit n-sphere, $S_-^{0,n} = H^n$ is hyperbolic n-space, $S_+^{1,n-1} = dS^n$ is de Sitter n-space and $S^{1,n-1} = AdS^n$ is anti de Sitter n-space.

Flat space as cone over the pseudo-spheres

The domains

$$\mathbb{R}_\pm^{t,s} := \{x \in \mathbb{R}^{t,s} | \pm \langle x, x \rangle > 0\} \subset \mathbb{R}^{t,s}$$

are isometrically identified via the map $(r, x) \mapsto rx$ with the space-like or time-like cone over $S_+^{t,s-1}$ or $S_-^{t-1,s}$ endowed with the metric $\pm dr^2 + r^2 g_\pm$, respectively. In particular, the space-like cone over a space of constant curvature 1 and the time-like cone over a space of constant curvature -1 are flat.

Realisation of doubly warped products by double polar coordinates

Now we show that any splitting of a pseudo-Euclidian vector space as an orthogonal sum of two pseudo-Euclidian subspaces induces local parametrisations of the pseudo-spheres. Using these 'double polar' parametrisations (more precisely, polar equidistant parametrisations [1]) we will show that the spaces of constant curvature can be locally presented as doubly warped products with trigonometric or hyperbolic warping functions over spaces of appropriate constant curvature.

We consider the pseudo-spheres $S_+(V) = S_+^{t,s} \subset V = \mathbb{R}^{t,s+1}$ and $S_-(V) = S_-^{t,s} \subset V = \mathbb{R}^{t+1,s}$. Any orthogonal decomposition $V = V_1 \oplus V_2 = \{v = x + y | x \in V_1, y \in V_2\}$, $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_1 + \langle \cdot, \cdot \rangle_2$ defines a diffeomorphism

$$(s, \bar{x}, \bar{y}) \mapsto x + y, \quad x = \cos(s)\bar{x}, \quad y = \sin(s)\bar{y}, \tag{2.4}$$

of $(0, \frac{\pi}{2}) \times S_\varepsilon(V_1) \times S_\varepsilon(V_2)$ onto the (not necessarily connected) domain

$$D = \{v = x + y \in S_\varepsilon(V) | 0 < \varepsilon \langle x, x \rangle_1 < 1\}.$$

Similarly the map

$$(s, \bar{x}, \bar{y}) \mapsto x + y, \quad x = \cosh(s)\bar{x}, \quad y \sinh(s)\bar{y}, \tag{2.5}$$

is a diffeomorphism of $\mathbb{R}^+ \times S_\varepsilon(V_1) \times S_{-\varepsilon}(V_2)$ onto the domain

$$D' = \{v = x + y \in S_\varepsilon(V) | \varepsilon \langle x, x \rangle_1 > 1\}.$$

Proposition 2.3. *With respect to the diffeomorphisms (2.4) and (2.5) the metric g_ε of $S_\varepsilon(V)$ is given by*

$$\begin{aligned} g_\varepsilon|_D &= \varepsilon ds^2 + \cos^2(s) g_{S_\varepsilon(V_1)} + \sin^2(s) g_{S_\varepsilon(V_2)} \\ g_\varepsilon|_{D'} &= -\varepsilon ds^2 + \cosh^2(s) g_{S_\varepsilon(V_1)} + \sinh^2(s) g_{S_{-\varepsilon}(V_2)}. \end{aligned}$$

Horospherical coordinates and corresponding warped products

Let $(V, \langle\cdot,\cdot\rangle)$ be an indefinite pseudo-Euclidian vector space, $p, q \in V$ two isotropic vectors such that $\langle p, q\rangle = 1$ and $W = \mathrm{span}\{p,q\}^\perp$. Then

$$\mathbb{R}^+ \times W \ni (s,\xi) \mapsto y = up + vq + x \in S_\varepsilon(V), \quad u = \pm\frac{1}{2}e^{-s}(\varepsilon - e^{2s}\langle\xi,\xi\rangle), \quad v = \pm 2e^s, \quad x = e^s\xi$$

is a diffeomorphism onto the domain $S_\varepsilon(V) \cap \{y \in V | \pm v > 0\}$. In the coordinates (s,ξ) the hypersurfaces $s = const$ correspond to the hyperplane sections (horospheres) $\{y \in S_\varepsilon(V) | \langle y, p\rangle = \pm e^s\}$ and the curves $\xi = \xi_0 = const$ are geodesics perpendicular to the horospheres. A direct calculation shows that:

Proposition 2.4. *The induced metric of the pseudo-sphere $S_\varepsilon(V)$ in horospherical coordinates (s,ξ) is given by:*

$$g_\varepsilon = \varepsilon ds^2 + e^{2s} g_0,$$

where $g_0 = d\xi^2$ is the induced pseudo-Euclidian metric on W.

Completeness of some doubly warped products

Proposition 2.5. *Let (N_1, g_1), (N_2, g_2) be pseudo-Riemannian manifolds and $(M = I \times N_1 \times N_2, g = \varepsilon ds^2 + f_1(s)^2 g_1 + f_2(s)^2 g_2)$ a doubly warped product with non-constant warping functions as in Corollary 2.3. Then (M, g) is complete only in the following cases:*

(i)
$$g = \varepsilon ds^2 + \cosh^2(s) g_1,$$
where $I = \mathbb{R}$ and g_1 is complete, and

(ii)
$$g = \varepsilon ds^2 + e^{2s} g_1$$
where $I = \mathbb{R}$ and εg_1 is complete and positive definite.

Proof. The system (2.2-2.3) has solutions $u_i = u_i^0 = const$, $s = at + b$, which are complete if and only if $I = \mathbb{R}$. This excludes all the warping functions which have a zero. It remains to check that the metric (i) is complete for any complete metric g_1 and that (ii) is complete only if εg_1 is complete and positive definite. In fact, in both cases the squared velocity $l = g(\dot\gamma, \dot\gamma)$ is constant. In the second case, for instance, it is given by $l = \varepsilon \dot s^2 + \varepsilon_1 e^{2s} \dot u^2$, $u := u_1$, which yields $\ddot y = \varepsilon l y$ after the substitution $y = e^s$. The differential equation $\ddot y = \varepsilon l y$ admits solutions which are positive on the real line if and only if $\varepsilon l > 0$. The positivity is necessary since $y = e^s$. This shows that g is positive or negative definite, i.e. εg_1 is positive definite. The other case is similar, see [9], where the case of Lorentzian signature is considered. □

3 Examples of cones with reducible holonomy

Let $\widehat{g} = cdr^2 + r^2 g$ be the cone metric on $\widehat{M} := \mathbb{R}^+ \times M$, where (M, g) is a pseudo-Riemannian manifold. Depending on the sign of the constant c the cone is called space-like ($c > 0$) or time-like ($c < 0$). Later on we will assume, without restriction of generality, that $c = 1$. In fact, as we allow g to be of any signature we can rescale $\widehat{g}$ by $\frac{1}{c} \in \mathbb{R}^*$.

We denote by ∂_r the radial vector field. The Levi-Civita connection of the cone $(\widehat{M}, \widehat{g})$ is given by

$$\left.\begin{array}{rcl} \widehat{\nabla}_{\partial_r}\partial_r &=& 0, \\ \widehat{\nabla}_X \partial_r &=& \frac{1}{r}X, \\ \widehat{\nabla}_X Y &=& \nabla_X Y - \frac{r}{c} g(X,Y)\partial_r, \end{array}\right\} \tag{3.1}$$

for all vector fields $X, Y \in \Gamma(T\widehat{M})$ orthogonal to ∂_r. The curvature $\widehat{R}$ of the cone is given by the following formulas including the curvature R of the base metric g:

$$\left.\begin{array}{rcl} \partial_r \lrcorner \widehat{R} &=& 0, \\ \widehat{R}(X,Y)Z &=& R(X,Y)Z - \frac{1}{c}\left(g(Y,Z)X - g(X,Z)Y\right), \text{ or} \\ \widehat{R}(X,Y,Z,U) &=& r^2\left(R(X,Y,Z,U) - \frac{1}{c}\left(g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\right)\right), \end{array}\right\} \tag{3.2}$$

for $X, Y, Z,\ U \in TM$. This implies that if (M, g) is a space of constant curvature κ, i.e.

$$R(X,Y,Z,U) \;=\; \kappa\left(g(X,U)g(Y,Z) - g(X,Z)g(Y,U)\right),$$

the cone has the curvature $r^2\left(\kappa - \frac{1}{c}\right)\left(g(X,U)g(Y,Z) - g(X,Z)g(Y,U)\right)$. In particular, if $\kappa = \frac{1}{c}$, then the cone is flat, as it is the case for the $c = 1$ cone over the standard sphere of radius 1 or the $c = -1$ cone over the hyperbolic space.

From now on we assume $c = \pm 1$. We denote by $(\widehat{M} = \mathbb{R}^+ \times M, \widehat{g} = dr^2 + r^2 g)$, the space-like cone over (M, g) and by $(\widehat{M}^- = \mathbb{R}^+ \times M, \widehat{g}^- = -dr^2 + r^2 g)$ the time-like cone. Notice that the metric $\widehat{g}^-$ of the time-like cone $\widehat{M}^-$ over (M, g) is obtained by multiplying the metric $dr^2 - r^2 g$ of the space-like cone over $(M, -g)$ by -1. Thus it is sufficient to consider only space-like cones.

We will now present some examples which illustrate that Gallot's statement is not true in arbitrary signature, and that the assumption of completeness is essential even in the Riemannian situation.

Example 3.1. Let (F, g_F) be a complete pseudo-Riemannian manifold of dimension at least 2 and which is not of constant curvature 1. Then the pseudo-Riemannian manifold

$$(M = \mathbb{R} \times F, g = -ds^2 + \cosh^2(s) g_F)$$

is complete, the restricted holonomy group of the cone over (M, g) is non-trivial and admits a non-degenerate invariant proper subspace.

Proof. The manifold (M, g) is complete by Proposition 2.5. The non-vanishing terms of the Levi-Civita connection ∇ of (M, g) are given by

$$\left.\begin{array}{rcl} \nabla_X \partial_s &=& \tanh(s) X, \\ \nabla_{\partial_s} X &=& \partial_s X + \tanh(s) X, \\ \nabla_X Y &=& \nabla^F_X Y + \cosh(s)\sinh(s) g_F(X,Y)\partial_s, \end{array}\right\} \tag{3.3}$$

where $X, Y \in TF$ are vector fields depending on the parameter s and ∇^F is the Levi-Civita connection of the manifold (F, g_F). Consider on $\widehat{M}$ the vector field $X_1 = \cosh^2(s)\partial_r - \frac{1}{r}\sinh(s)\cosh(s)\partial_s$. We have $\widehat{g}(X_1, X_1) = \cosh^2(s) > 0$. It is easy to check that the distribution generated by the vector field X_1 and by the distribution $TF \subset T\widehat{M}$ is parallel.

For the curvature tensor R of (M, g) we have

$$R(X,Y)Z \quad = \quad R_F(X,Y)Z + \tanh^2(s)\left(g_F(Y,Z)X - g_F(X,Z)Y\right), \tag{3.4}$$

where $X, Y, Z,\ U \in TF$ and R_F is the curvature tensor of (F, g_F). This shows that (M, g) cannot have constant sectional curvature, unless F has constant curvature 1 (see Corollary 2.3). Thus the cone $(\widehat{M}, \widehat{g})$ is not flat. □

Example 3.2. Let M be a manifold of the form $\mathbb{R}\times N$ with the metric $g = -(dt^2 + e^{-2t}g_N)$, where (N, g_N) is a pseudo-Riemannian manifold. Then

1. The light-like vector field $e^{-t}(\partial_r + \frac{1}{r}\partial_t)$ on the space-like cone $\widehat{M}$ is parallel.
2. If $(N = N_1 \times N_2, g_N = g_1 + g_2)$ is a product of a flat pseudo-Riemannian manifold (N_1, g_1) and of a non-flat pseudo-Riemannian manifold (N_2, g_2), then $\widehat{M}$ is locally decomposable and not flat[1]. In fact, there is a parallel non-degenerate *flat* distribution of dimension $\dim N_1$ on $\widehat{M}$.

The manifold (M, g) in Example 3.2 is complete if and only if g_N is complete and positive definite, see Proposition 2.5. Notice that g is the hyperbolic metric in horospherical coordinates if (N, g_N) is Euclidian space.

Example 3.3. Let (M_1, g_1) and (M_2, g_2) be two pseudo-Riemannian manifolds. Then the product of the cones

$$(\widehat{M_1} \times \widehat{M_2} = (\mathbb{R}^+ \times M_1) \times (\mathbb{R}^+ \times M_2), \widehat{g} = (dr_1^2 + r_1^2 g_1) + (dr_2^2 + r_2^2 g_2))$$

is the space-like cone over the manifold

$$(M = \left(0, \frac{\pi}{2}\right) \times M_1 \times M_2, g = ds^2 + \cos^2(s)g_1 + \sin^2(s)g_2).$$

Proof. Consider the functions

$$r = \sqrt{r_1^2 + r_2^2} \in \mathbb{R}^+, \quad s = \operatorname{arctg}\left(\frac{r_2}{r_1}\right) \in \left(0, \frac{\pi}{2}\right).$$

Since $r_1, r_2 > 0$, the functions r and s give a diffeomorphism $\mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \times \left(0, \frac{\pi}{2}\right)$. For $\widehat{M_1} \times \widehat{M_2}$ we get

$$\widehat{M_1} \times \widehat{M_2} \cong \mathbb{R}^+ \times \left(0, \frac{\pi}{2}\right) \times M_1 \times M_2$$

and

$$\widehat{g_1} + \widehat{g_2} = dr^2 + r^2(ds^2 + \cos^2(s)g_1 + \sin^2(s)g_2).$$

□

[1] We learned this from Helga Baum.

Suppose that the manifolds (M_1, g_1) and (M_2, g_2) are Riemannian. Then the manifold (M, g) is Riemannian and incomplete. The cone over M is decomposable. Moreover, it is not flat, unless the manifolds (M_1, g_1), (M_2, g_2) are of dimension less than 2 or of constant curvature 1, see Corollary 2.3. Example 3.3 shows that the completeness assumption in Theorems 1 and 6.1 is necessary.

Example 3.4. Let (M_1, g_1) and (M_2, g_2) be two pseudo-Riemannian manifolds. Then the space-like cone over the manifold

$$(M = \mathbb{R}^+ \times M_1 \times M_2, g = -ds^2 + \cosh^2(s) g_1 + \sinh^2(s) g_2)$$

is isometric to the open subset $\Omega = \{r_1 > r_2\}$ in the product of the cones

$$(\widehat{M_1} \times \widehat{M_2} = (\mathbb{R}^+ \times M_1) \times (\mathbb{R}^+ \times M_2), \widehat{g_1} + \widehat{g_2}^- = (dr_1^2 + r_1^2 g_1) + (-dr_2^2 + r_2^2 g_2)).$$

Proof. Consider the functions

$$r = \sqrt{r_1^2 - r_2^2} \in \mathbb{R}^+, \quad s = \operatorname{artanh}\left(\frac{r_2}{r_1}\right) \in \mathbb{R}^+.$$

The functions r and s give a diffeomorphism $\{(r_1, r_2) \in \mathbb{R}^2 | 0 < r_2 < r_1\} \to \mathbb{R}^+ \times \mathbb{R}^+$. For $\widehat{M_1} \times \widehat{M_2}$ we get

$$\Omega \cong \mathbb{R}^+ \times \mathbb{R}^+ \times M_1 \times M_2$$

and

$$\widehat{g_1} + \widehat{g_2}^- = dr^2 + r^2(-ds^2 + \cosh^2(s) g_1 + \sinh^2(s) g_2).$$

□

Example 3.5. Let $(t, x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n})$ be coordinates on $\mathbb{R}^{2n+1}$. Consider the metric g given by

$$g = \left(\begin{array}{c|c|c} -1 & 0 & u^t \\ \hline 0 & 0 & H^t \\ \hline u & H & G \end{array}\right)$$

where

- $u = (u_1, \ldots, u_n)$ is a diffeomorphism of $\mathbb{R}^n$, depending on $x_1, \ldots, x_n$,
- $H = \frac{1}{2}\left(\frac{\partial}{\partial x_j}(u_i)\right)_{i,j=1}^n$ its non-degenerate Jacobian, and
- G the symmetric matrix given by $G_{ij} = -u_i u_j$.

Then the space-like cone over $(\mathbb{R}^{2n+1}, g)$ is not flat but its holonomy representation decomposes into two totally isotropic invariant subspaces. For the proof of this see Proposition 8.3 in Section 8.

4 Local structure of decomposable cones

In this section we assume that the holonomy group of the cone $(\widehat{M}, \widehat{g})$ is decomposable and we give a local description of the manifold (M, g), independently of completeness.

Suppose that the holonomy group Hol_x of $(\widehat{M}, \widehat{g})$ at a point $x \in \widehat{M}$ is decomposable, that is $T_x\widehat{M}$ is a sum $T_x\widehat{M} = (V_1)_x \oplus (V_2)_x$ of two non-degenerate Hol_x-invariant orthogonal subspaces. They define two parallel non-degenerate distributions V_1 and V_2. Denote by X_1 and X_2 the projections of the vector field ∂_r to the distributions V_1 and V_2 respectively. We have

$$\partial_r = X_1 + X_2. \tag{4.1}$$

We decompose the vectors X_1 and X_2 with respect to the decomposition $T\widehat{M} = T\mathbb{R}^+ \oplus TM$,

$$X_1 = \alpha\partial_r + X, \quad X_2 = (1-\alpha)\partial_r - X, \tag{4.2}$$

where α is a function on $\widehat{M}$ and X is a vector field on $\widehat{M}$ tangent to M. We have

$$\widehat{g}(X, X) = \alpha - \alpha^2, \quad \widehat{g}(X_1, X_1) = \alpha, \quad \widehat{g}(X_2, X_2) = 1 - \alpha. \tag{4.3}$$

Lemma 4.1. *The open subset $U = \{x|\alpha(x) \neq 0, 1\} \subset \widehat{M}$ is dense.*

Proof. Suppose that $\alpha = 1$ on an open subspace $V \subset \widehat{M}$. We claim that $\partial_r \in V_1$ on V. Indeed, on V we have

$$X_1 = \partial_r + X, \;\; X_2 = -X \quad \text{and} \quad \widehat{g}(X, X) = 0.$$

We show that $X = 0$. Let $Y_2 \in V_2$. We have the decomposition $Y_2 = \lambda\partial_r + Y$, where λ is a function on $\widehat{M}$ and $Y \in TM$. It is

$$\widehat{\nabla}_{Y_2} X_1 = \widehat{\nabla}_{\lambda\partial_r + Y}(\partial_r + X) = \frac{1}{r}Y + \widehat{\nabla}_{Y_2} X.$$

Note that $Y = Y_2 - \lambda X_1 + \lambda X$. Hence,

$$\widehat{\nabla}_{Y_2} X_1 = \frac{1}{r}(Y_2 - \lambda X_1 + \lambda X) + \widehat{\nabla}_{Y_2} X.$$

Since $X, \widehat{\nabla}_{Y_2} X, Y_2 \in V_2$ and $\widehat{\nabla}_{Y_2} X_1 \in V_1$, we see that

$$\frac{1}{r}(Y_2 + \lambda X) + \widehat{\nabla}_{Y_2} X = 0.$$

From $\widehat{g}(X, X) = 0$ it follows that $\widehat{g}(\widehat{\nabla}_{Y_2} X, X) = 0$. Thus we get $\widehat{g}(Y_2, X) = 0$ for all $Y_2 \in V_2$. Since V_2 is non-degenerate, we conclude that $X = 0$. Thus $\partial_r \in V_1$.

Let $Y_2 \in V_2$, then $\widehat{\nabla}_{Y_2}\partial_r = \frac{1}{r}Y_2$. Since the distribution V_1 is parallel and $\partial_r \in V_1$, we see that $Y_2 = 0$ and $V_2 = 0$. Contradiction. $\square$

We now consider the dense open submanifold $U \subset \widehat{M}$. The vector fields X_1, X_2 and X are nowhere isotropic on U. For $i = 1, 2$ let $E_i \subset V_i$ be the subdistribution of V_i orthogonal to X_i. Denote by L the distribution of lines on U generated by the vector field X. We get on U the orthogonal decomposition

$$T\widehat{M} = T\mathbb{R} \oplus L \oplus E_1 \oplus E_2.$$

Lemma 4.2. *Let $Y_1 \in E_1$ and $Y_2 \in E_2$, then on U we have*

1. $Y_1\alpha = Y_2\alpha = \partial_r\alpha = 0.$
2. $\widehat{\nabla}_{Y_1}X = \frac{1-\alpha}{r}Y_1$, $\widehat{\nabla}_{Y_2}X = -\frac{\alpha}{r}Y_2.$
3. $\widehat{\nabla}_{\partial r}X = \partial_r X + \frac{1}{r}X = 0.$
4. $\widehat{\nabla}_X X = \left(\frac{(1-\alpha)^2}{r} - X\alpha\right)X_1 + \left(\frac{\alpha^2}{r} - X\alpha\right)X_2.$

Proof. Using (4.2), we have

$$\begin{aligned}\widehat{\nabla}_{Y_1}X_1 &= (Y_1\alpha)\partial_r + \tfrac{\alpha}{r}Y_1 + \nabla_{Y_1}X,\\ \widehat{\nabla}_{Y_1}X_2 &= -(Y_1\alpha)\partial_r + \tfrac{1-\alpha}{r}Y_1 - \nabla_{Y_1}X.\end{aligned}$$

Since $Y_1 \in E_1 \subset V_1$ and the distributions V_1, V_2 are parallel, projecting these equations onto V_2 and adding them yields $\widehat{\nabla}_{Y_1}X_2 = 0$. Then, from the second equation, we see that $Y_1\alpha = 0$ and $\widehat{\nabla}_{Y_1}X = \nabla_{Y_1}X = \frac{1-\alpha}{r}Y_1$. The other claims can be proved similarly. □

Since $\partial_r\alpha = 0$, the function α is a function on M. Note that $U = \mathbb{R}^+ \times U_1$, where

$$U_1 = \{x \in M|\alpha(x) \neq 0, 1\} \subset M.$$

Claim 3 of Lemma 4.2 shows that $X = \frac{1}{r}\tilde{X}$, where $\tilde{X}$ is a vector field on the manifold M. Hence the distributions L and $E = E_1 \oplus E_2$ do not depend on r and can be considered as distributions on M. Claim 2 of Lemma 4.2 shows that the distributions E_1 and E_2 also do not depend on r. We get on U_1 the orthogonal decompositions

$$TM = L \oplus E, \quad E = E_1 \oplus E_2.$$

Lemma 4.3. *The function α satisfies on U_1 the following differential equation*

$$\tilde{X}\alpha = 2(\alpha - \alpha^2).$$

Proof. From

$$r^2\widehat{\nabla}_X X = \widehat{\nabla}_{\tilde{X}}\tilde{X} = \nabla_{\tilde{X}}\tilde{X} - rg(\tilde{X},\tilde{X})\partial_r = \nabla_{\tilde{X}}\tilde{X} - r(\alpha - \alpha^2)\partial_r$$

and Claim 4 of Lemma 4.2 we conclude that $\nabla_{\tilde{X}}\tilde{X} \in TM$ is a linear combination of X_1 and X_2 and hence proportional to $X = (1-\alpha)X_1 - \alpha X_2$. This implies

$$(2\alpha - 1)\left(X\alpha - \frac{2}{r}(\alpha - \alpha^2)\right) = 0.$$

If $\alpha = \frac{1}{2}$, then $\widehat{\nabla}_X X = \frac{1}{4r}\partial_r$ and $\nabla_{\tilde{X}}\tilde{X} = \frac{r}{2}\partial_r$. The last equality is impossible. □

Corollary 4.1. *On M we have*

$$\tilde{X} = \frac{1}{2}grad(\alpha).$$

Lemma 4.3 implies that if t is a coordinate on M corresponding to the vector field $\tilde{X}$, then

$$\alpha(t) = \frac{e^{2t}}{e^{2t}+c},$$

where c is a constant.
From Lemmas 4.2 and 4.3 it follows that

$$\widehat{\nabla}_X X = -\frac{\alpha-\alpha^2}{r}\partial_r + \frac{1-2\alpha}{r}X.$$

On the subset $U_1 \subset M$ we get the following

$$\nabla_{\tilde{X}}\tilde{X} = (1-2\alpha)\tilde{X}, \quad \nabla_{Y_1}\tilde{X} = (1-\alpha)Y_1, \quad \nabla_{Y_2}\tilde{X} = -\alpha Y_2, \tag{4.4}$$

for any $Y_1 \in \Gamma(E_1)$ and $Y_2 \in \Gamma(E_2)$.

Theorem 4.1. *Let (M,g) be a pseudo-Riemannian manifold. If the holonomy group of the metric cone over (M,g) admits a non-degenerate invariant subspace, then $\mathfrak{hol}(M,g) = \mathfrak{so}(p,q)$, where (p,q) is the signature of the metric g.*

Proof. Since the distributions V_1 and V_2 are parallel, for any $Y_1 \in V_1$ and $Y_2 \in V_2$ we have $\widehat{R}(Y_1,Y_2) = 0$. From (3.2) it follows that $\widehat{R}(X,Y_2) = \widehat{R}(X,Y_1) = 0$. Hence, $R(\tilde{X},Y) = \tilde{X}\wedge Y$ for all vector fields Y on M. By Lemma 4.1, there exists $y \in M$ such that $g_y(\tilde{X},\tilde{X}) \neq 0$. The holonomy algebra of the manifold M at the point y contains the subspace $\tilde{X}_y \wedge T_yM$. Since $g_y(\tilde{X},\tilde{X}) \neq 0$, this vector subspace generates the whole Lie algebra $\mathfrak{so}(T_yM, g_y)$. □

Theorem 4.2. *Let (M,g) be a pseudo-Riemannian manifold and $(\widehat{M} = \mathbb{R}^+ \times M, \widehat{g} = dr^2 + r^2 g)$ the cone over M. Suppose that the holonomy group of $(\widehat{M},\widehat{g})$ admits a non-degenerate proper invariant subspace. Then there exists a dense open submanifold $U_1 \subset M$ such that each point $x \in U_1$ has an open neighborhood $W \subset U_1$ that satisfies one of the following conditions*

(1.) For W we have a decomposition

$$W = (a,b)\times N_1 \times N_2, \quad (a,b) \subset \left(0, \frac{\pi}{2}\right)$$

and for the metric $g|_W$ we have

$$g|_W = ds^2 + \cos^2(s)g_1 + \sin^2(s)g_2,$$

where (N_1,g_1) and (N_2,g_2) are pseudo-Riemannian manifolds;
Moreover, any point $(r,x) \in \mathbb{R}^+ \times W \subset \widehat{M}$ has a neighborhood of the form

$$((a_1,b_1)\times N_1)\times((a_2,b_2)\times N_2), \quad (a_1,b_1),(a_2,b_2) \subset \mathbb{R}^+$$

with the metric

$$(dt_1^2 + t_1^2 g_1) + (dt_2^2 + t_2^2 g_2).$$

(2.) For W we have a decomposition

$$W = (a,b) \times N_1 \times N_2, \quad (a,b) \subset \mathbb{R}^+$$

and for the metric $g|_W$ we have

$$g|_W = -ds^2 + \cosh^2(s) g_1 + \sinh^2(s) g_2,$$

where (N_1, g_1) and (N_2, g_2) are pseudo-Riemannian manifolds.
Moreover, any point $(r,x) \in \mathbb{R}^+ \times W \subset \widehat{M}$ has a neighborhood of the form

$$((a_1,b_1) \times N_1) \times ((a_2,b_2) \times N_2), \quad (a_1,b_1),(a_2,b_2) \subset \mathbb{R}^+$$

with the metric

$$(dt_1^2 + t_1^2 g_1) + (-dt_2^2 + t_2^2 g_2).$$

Proof. We need the following

Lemma 4.4. *(i) The distributions $E_1, E_2, E = E_1 \oplus E_2 \subset TM$ defined on $U_1 \subset M$ are involutive and the distributions $E_1 \oplus L, E_2 \oplus L \subset TM$ are parallel on U_1.*

(ii) Let $x \in U_1$ and $M_x \subset U_1$ the maximal connected integral submanifold of the distribution E. Then the distributions $E_1|_{M_x}, E_2|_{M_x} \subset TM_x = E|_{M_x}$ are parallel.

Proof. (i) On $U \subset \widehat{M}$, the distribution $E_i = V_i \cap TM$ is the intersection of two involutive distributions and hence involutive, for $i = 1,2$. The corresponding distributions E_1, E_2 of $U_1 \subset M$ are therefore involutive. The involutivity of E follows from Corollary 4.1. Next we prove that $E_i \oplus L$ is involutive. The formulas (4.4) show that $\nabla_{E_i} \tilde{X} = E_i$. Now we check that $\nabla_{Y_1} Y_1' \in \Gamma(E_1 \oplus L)$ for all $Y_1, Y_1' \in \Gamma(E_1)$. Calculating the scalar product with $Y_2 \in \Gamma(E_2)$ we get

$$g(\nabla_{Y_1} Y_1', Y_2) = -g(Y_1', \nabla_{Y_1} Y_2) = -g(Y_1', \widehat{\nabla}_{Y_1} Y_2) = 0,$$

since the distribution V_2 is parallel.
(ii) The fact that $E_i \oplus L \subset TM$ is parallel implies that $E_i \subset TM_x$ is parallel. □

Now we return to the Examples 3.3 and 3.4.
In Example 3.3 we have

$$V_1 = T\widehat{M_1}, \quad V_2 = T\widehat{M_2}, \quad E_1 = TM_1, \quad E_2 = TM_2,$$

$$X_1 = \cos(s)\partial_{r_1}, \quad X_2 = \sin(s)\partial_{r_2}, \quad \alpha = \cos^2(s), \quad X = -\frac{1}{r}\sin(s)\cos(s)\partial_s.$$

Note that $0 < \alpha < 1$.
In Example 3.4 we have

$$V_1 = T\widehat{M_1}, \quad V_2 = T\widehat{M_2}^{-}, \quad E_1 = TM_1, \quad E_2 = TM_2,$$

$$X_1 = \cosh(s)\partial_{r_1}, \quad X_2 = \sinh(s)\partial_{r_2}, \quad \alpha = \cosh^2(s), \quad X = -\frac{1}{r}\sinh(s)\cosh(s)\partial_s.$$

Note that $\alpha > 1$.

Let $x \in U_1$. we have two cases: (1.) $0 < \alpha(x) < 1$; (2.) $\alpha(x) < 0$ or $\alpha(x) > 1$.

Case (1.) Suppose that $0 < \alpha(x) < 1$. Then $0 < \alpha < 1$ on some open subset $W \subset U_1$ containing the point x. Thus $g(\tilde{X}, \tilde{X}) = \hat{g}(X, X) = \alpha - \alpha^2 > 0$ on W. Recall that $\tilde{X}$ is a gradient vector field, see Corollary 4.1. Hence we can assume that W has the form $(a, b) \times N$, where $(a, b) \subset \mathbb{R}$ and N is the level set of the function α. Note also that the level sets of the function α are integral submanifolds of the involutive distribution E. Since $\tilde{X}$ is orthogonal to E and $Z(g(\tilde{X}, \tilde{X})) = 0$ for all $Z \in E$, the metric $g|_W$ can be written as

$$g|_W = ds^2 + g_N,$$

where g_N is a family of pseudo-Riemannian metrics on N depending on the parameter s. We can assume that $\partial_s = -\frac{\tilde{X}}{\sqrt{g(\tilde{X},\tilde{X})}}$.

By Lemma 4.4 and the Wu theorem, the manifold W is locally a product of two pseudo-Riemannian manifolds. For $Y_1, Z_1 \in E_1$ and $Y_2, Z_2 \in E_2$ in virtue of Lemma 4.2 we have

$$(L_{\tilde{X}}g)(Y_1, Z_1) = 2(1-\alpha)g(Y_1, Z_1), \quad (L_{\tilde{X}}g)(Y_1, Y_2) = 0, \quad (L_{\tilde{X}}g)(Y_2, Z_2) = -2\alpha g(Y_2, Z_2).$$

This means that the one-parameter group of local diffeomorphisms of W generated by the vector field $\tilde{X}$ preserves the Wu decomposition of the manifolds W. Hence the manifold $(N, g|_N)$ can be locally decomposed into a direct product of two manifolds N_1 and N_2 which are integral manifolds of the distributions E_1 and E_2 such that

$$g_N = h_1 + h_2,$$

where h_i, $i = 1, 2$ is a metric on N_i which depends on s.

From Lemmas 4.2,4.3 it follows that the function α depends only on s and satisfies the following differential equation

$$\partial_s \alpha = -2\sqrt{\alpha - \alpha^2}.$$

Hence,

$$\alpha = \cos^2(s + c_1),$$

where c_1 is a constant. We can assume that $c_1 = 0$. Since on W we have $0 < \alpha < 1$ and $\partial_s \alpha < 0$, we see that $(a, b) \subset \left(0, \frac{\pi}{2}\right)$.

Let $Y_1, Z_1 \in E_1$ be vector fields on W such that $[Y_1, \partial_s] = [Z_1, \partial_s] = 0$. From (4.4) it follows that $\nabla_{Y_1}\partial_s = -\frac{\sqrt{\alpha-\alpha^2}}{\alpha}Y_1$. The Koszul formula implies that $2g(\nabla_{Y_1}\partial_s, Z_1) = \partial_s g(Y_1, Z_1)$. Thus we have

$$-2\tan(s)g(Y_1, Z_1) = \partial_s g(Y_1, Z_1).$$

This means that

$$h_1 = \cos^2(s)g_1,$$

where g_1 does not depend on s. Similarly,

$$h_2 = \sin^2(s)g_2,$$

where g_2 does not depend on s.

For the cone over W we get

$$\mathbb{R}^+ \times W = \mathbb{R}^+ \times (a,b) \times N_1 \times N_2$$

and

$$\widehat{g}|_{\mathbb{R}^+\times W} = dr^2 + r^2(ds^2 + \cos^2(s)g_1 + \sin^2(s)g_2).$$

Consider the functions $t_1 = r\cos(s)$, $t_2 = r\sin(s)$. They define a diffeomorphism from $\mathbb{R}^+ \times (a,b)$ onto a subset $V \subset \mathbb{R}^+ \times \mathbb{R}^+$.

Let $(r,y) \in \mathbb{R}^+ \times W \subset \widehat{M}$, then there exist a subset $(a_1,b_1)\times(a_2,b_2) \subset V$, where $(a_1,b_1),(a_2,b_2) \subset \mathbb{R}^+$ and $r \in (a_1,b_1)$.

On the subset

$$((a_1,b_1)\times N_1)\times((a_2,b_2)\times N_2) \subset \mathbb{R}^+ \times W$$

the metric $\widehat{g}$ has the form

$$(dt_1^2 + t_1^2 g_1) + (dt_2^2 + t_2^2 g_2).$$

Case (2.) Suppose that $\alpha(x) > 1$. Now $\partial_s = -\frac{\tilde{X}}{\sqrt{\alpha^2-\alpha}}$ and the function α satisfies

$$\partial_s\alpha = 2\sqrt{\alpha^2-\alpha}.$$

Hence,

$$\alpha = \cosh^2(s + c_1).$$

Again we can assume $c_1 = 0$ and from $\partial_s\alpha > 0$ we get $(a,b) \subset \mathbb{R}^+$. For the metric $g|_W$ on $W = (a,b)\times N_1 \times N_2$ we have

$$g|_W = -ds^2 + g_N = -ds^2 + \cosh^2(s)g_1 + \sinh^2(s)g_2.$$

The case $\alpha(x) < 0$ is equivalent to the case $\alpha(x) > 1$ by interchanging the roles of V_1 and V_2, which interchanges α with $1-\alpha$ and X with $-X$. Theorem 4.2 is proved. □

5 Geodesics of cones

Using Proposition 2.1, we now calculate the geodesics on the cone. Suppose $\Gamma(t) = (r(t),\gamma(t))$ is a geodesic on the cone $(\widehat{M},\widehat{g})$, where $\gamma(t)$ is a curve on (M,g). Suppose we have the initial conditions

$$\Gamma(0) = (r,x) \text{ and } \dot{\Gamma}(0) = (\rho, v),$$

for some $x \in M$, $v \in T_xM$.

Then $r(t)$ and $\gamma(t)$ satisfy

$$\begin{aligned} 0 &= \ddot{r}(t) - r(t)g(\dot{\gamma}(t),\dot{\gamma}(t)), &(5.1)\\ 0 &= 2\,\dot{r}(t)\dot{\gamma}(t) + r(t)\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t). &(5.2)\end{aligned}$$

Now one makes the following ansatz. Suppose that γ is given as a reparametrisation of a geodesic $\beta : \mathbb{R} \to M$ of g:

$$\gamma(t) \quad = \quad \beta(f(t))$$

where β is a geodesic of g with initial condition

$$\beta(0) = x \text{ and } \dot\beta(0) = v \neq 0,$$

implying the initial conditions for f:

$$f(0) = 0 \text{ and } \dot f(0) = 1.$$

As $\dot\gamma(t) = \dot f(t)\cdot\dot\beta(f(t))$, $g\left(\dot\gamma(t), \dot\gamma(t)\right) = \dot f(t)^2 g\left(\dot\beta(f(t)), \dot\beta(f(t))\right)$ and $\nabla_{\dot\gamma(t)}\dot\gamma(t) = \ddot f(t)\dot\beta(f(t))$, we get from (5.1) and (5.2)

$$\begin{aligned} 0 &= \ddot r(t) - r(t)\dot f(t)^2 g\left(v, v\right), && (5.3)\\ 0 &= 2\,\dot r(t)\dot f(t) + r(t)\ddot f(t) && (5.4) \end{aligned}$$

with initial conditions

$$\begin{aligned} r(0) &= r\ , & f(0) &= 0\ ,\\ \dot r(0) &= \rho\ , & \dot f(0) &= 1. \end{aligned}$$

The solution to these equations is straightforward by distinguishing several cases.

From now on we assume that $\rho \neq 0$ and $v \neq 0$ and consider the remaining cases for v being light-like, space-like, or time-like.

1.) v is light-like, $g(v, v) = 0$, i.e. β is a light-like geodesic. Then the equations become

$$\begin{aligned} 0 &= \ddot r(t),\\ 0 &= 2\,\rho \dot f(t) + (\rho t + r)\ddot f(t), \end{aligned}$$

i.e. $r(t) = \rho t + r$ on the one hand, and $f(t) = \frac{rt}{\rho t + r}$ on the other. This implies that f and thus Γ is defined for $t \in [0, -\frac{r}{\rho})$ if $\rho < 0$, and for $t \geq 0$ otherwise.

2.) v is not light-like, $g(v, v) \neq 0$, i.e. β is a space-like or time-like geodesic. Then we set $g(\dot\beta(t), \dot\beta(t)) = g(v, v) =: \pm L^2$ with $L > 0$. The equations (5.3) and (5.4) become

$$\begin{aligned} 0 &= \ddot r(t) \mp r(t)\dot f(t)^2 L^2,\\ 0 &= 2\,\dot r(t)\dot f(t) + r(t)\ddot f(t). \end{aligned}$$

The solutions of these equations are the following

$$\begin{aligned} r_\pm(t) &= \sqrt{(\rho t + r)^2 \pm L^2 r^2 t^2},\\ f_\pm(t) &= \frac{1}{L}\arctan^\pm\left(\frac{Lrt}{\rho t + r}\right), \end{aligned}$$

in which we have introduced the notation $\arctan^+ := \arctan$ and $\arctan^- := \operatorname{artanh}$. Obviously r_+ is defined for all $t \in \mathbb{R}$ whereas f_+ is defined for $t \in [0, -\frac{r}{\rho})$ if $\rho < 0$, and for $t \geq 0$ otherwise. The functions r_- and f_- are defined on an interval $[0, T)$, where T is the first positive zero of the polynomial $((Lr - \rho)t - r)((L + r\rho)t + r)$ or $T = \infty$ if the polynomial has no positive zero. More explicitly, $T = \frac{r}{Lr - \rho}$ if $\rho < Lr$ and $T = \infty$ if $Lr \leq \rho$.

6 Cones over compact complete manifolds

Here we generalise the proof of Gallot [14] for metric cones over compact and geodesically complete pseudo-Riemannian manifolds. We obtain the following result.

Theorem 6.1. *Let (M, g) be a compact and complete pseudo-Riemannian manifold of dimension ≥ 2 with decomposable holonomy group $\widehat{H}$ of the cone $\widehat{M}$. Then (M, g) has constant curvature 1 and the cone is flat.*

Proof. In the Riemannian case, the values of the function α defined in 4.2 are trivially restricted to the interval $[0, 1]$, since $\alpha = \hat{g}(X_1, X_1) \geq 0$ and $1 - \alpha = \hat{g}(X_2, X_2) \geq 0$. We shall now establish the same result for *compact* complete pseudo-Riemannian manifolds (M, g). Example 3.1 shows that completeness does not suffice.

Lemma 6.1. *Under the assumptions of Theorem 6.1, the function α on M satisfies $0 \leq \alpha \leq 1$.*

Proof. On the open dense subset $U_1 \subset M$ we define the vector field

$$\hat{X} = \frac{\tilde{X}}{\sqrt{|g(\tilde{X}, \tilde{X})|}} = \frac{\tilde{X}}{\sqrt{|\alpha^2 - \alpha|}}.$$

From (4.4) and Lemma 4.3 it follows that $\nabla_{\hat{X}} \hat{X} = 0$, i.e. $\hat{X}$ is a geodesic vector field.

Let $x \in M$ and suppose that $\alpha(x) < 0$. Denote by $U_x \subset U_1$ the connected component of the set U_1 containing the point x. Since M is complete, we have a geodesic $\gamma(s)$ such that $\gamma(0) = x$ and $\dot{\gamma}(s) = \hat{X}(\gamma(s))$ if $\gamma(s) \in U_x$. From Lemma 4.3 it follows that

$$\dot{\gamma}(s)\alpha = \hat{X}(\gamma(s))\alpha = -2\sqrt{\alpha^2 - \alpha}$$

for all s such that $\gamma(s) \in U_x$. Hence along the curve $\{\gamma(s) | \gamma(s) \in \bar{U}_x\}$ we have

$$\alpha(s) = \frac{(e^{-2s} + c_1)^2}{4e^{-2s} c_1},$$

where c_1 is a constant. Since $\alpha(\gamma(0)) < 0$, we see that $c_1 < 0$. If $c_1 \leq -1$, then for all $s > 0$ we have $\gamma(s) \in U_x$ and $\alpha(\gamma(s))$ tends to $-\infty$ as s tends to $+\infty$. If $c_1 > -1$, then for all $s < 0$ we have $\gamma(s) \in U_x$ and $\alpha(\gamma(s))$ tends to $-\infty$ as s tends to $-\infty$. Since M is compact, we get a contradiction. The case $\alpha(x) > 1$ is similar. □

Now we can prove the theorem completely analogously to Gallot by verifying the same lemmas as in his proof.

Lemma 6.2. *Let $\Gamma(t) = (r(t), \gamma(t))$ be a geodesic in $(\widehat{M}, \widehat{g})$. Then the vector field along Γ defined by*

$$H(t) \quad := \quad r(\Gamma(t))\partial_r(\Gamma(t)) - t\dot{\Gamma}(t)$$

is parallel along $\Gamma(t)$.

Proof. The lemma follows directly from (3.1):

$$\begin{aligned}\widehat{\nabla}_{\dot\Gamma(t)} H(t) &= \widehat{\nabla}_{\dot\Gamma(t)}\left(r(t)\partial_r - t\dot\Gamma(t)\right)\\ &= \dot r(t)\partial_r + r(t)\widehat{\nabla}_{\dot\Gamma(t)}\partial_r - \dot\Gamma(t) - t\,\underbrace{\widehat{\nabla}_{\dot\Gamma(t)}\dot\Gamma(t)}_{=0}\\ &= \dot r(t)\partial_r + r(t)\underbrace{\widehat{\nabla}_{\dot\gamma(t)}\partial_r}_{=\frac{1}{r(t)}\dot\gamma(t)} - \underbrace{\dot\Gamma(t)}_{=\dot r(t)\partial_r+\dot\gamma(t)} = 0,\end{aligned}$$

where $r(t) := r(\Gamma(t))$. □

For each point $q \in \widehat{M}$ we denote by $\widehat{M}^1_q$ and $\widehat{M}^2_q$ the integral manifolds of the distributions V_1 and V_2 passing through the point q. For $i = 1, 2$ we define the following subsets of $\widehat{M}$:

$$C_i \;:=\; \left\{p \in \widehat{M} \mid \partial_r(p) \in V_i\right\}.$$

Then we can prove the following lemma.

Lemma 6.3. *Let $p_1 \in C_1$ (respectively, $p_2 \in C_2$). Then $\widehat{M}^2_{p_1}$ (respectively, $\widehat{M}^1_{p_2}$) is totally geodesic and flat.*

Proof. The leaves of the foliations induced by V_1 and V_2 are totally geodesic, since both distributions are parallel. It suffices to show that $\widehat{M}^1_{p_2}$ is flat.

Consider a geodesic Γ of $\widehat{M}^1_{p_2}$ starting at $p_2 = (r, x)$. Then the vector field along this geodesic $H(t)$ defined as in Lemma 6.2 is parallel. We have $H(0) = r\partial_r \in V_2$ and $\dot\Gamma(0) \in V_1$ which implies

$$\begin{aligned} H(t) \;=\; r(t)\partial_r - t\,\dot\Gamma(t) &\;\in\; (V_2)_{\Gamma(t)} \text{ and}\\ \dot\Gamma(t) &\;\in\; (V_1)_{\Gamma(t)},\end{aligned}$$

as the distributions V_i are invariant under parallel transport. Since $\partial_r \lrcorner \widehat{R} = 0$, we have

$$\widehat{R}(.,.)H(t) \;=\; -t\,\widehat{R}(.,.)\dot\Gamma(t).$$

Since $\widehat{R}(.,.)$ are elements of the holonomy algebra leaving V_1 and V_2 invariant this implies that

$$\widehat{R}(.,.)\dot\Gamma(t) \;=\; 0.$$

From this we see that the Jacobi fields along Γ are those of a flat manifold, which implies that $\widehat{M}^1_{p_2}$ is flat. □

Recall that we have a dense open subset $U = \{x \in \widehat{M} | \alpha(x) \neq 0, 1\} \subset \widehat{M}$.

Lemma 6.4. *Any point $p \in U$ has a flat neighbourhood.*

Proof. Fix a point $p \in U$. Note that for $i = 1, 2$ we have $C_i \cap U = \emptyset$.

Consider the geodesic $\Gamma(t)$ starting at p and satisfying the initial condition $\dot{\Gamma}(0) = -r(p)X_1(p)$. Let $H(t)$ be the vector field along Γ as in Lemma 6.2. We claim that if the geodesic $\Gamma(t)$ exists for $t = 1$, then $\Gamma(1) \in C_2$. Indeed, suppose that $\Gamma(t)$ exists for $t = 1$. Denote by $\tau : T_p\widehat{M} \to T_{\Gamma(1)}\widehat{M}$ the parallel displacement along $\Gamma(t)$. Since $H(1) = r(\Gamma(1))\partial_r(\Gamma(1)) - \dot{\Gamma}(1)$, we have $r(\Gamma(1))\partial_r(\Gamma(1)) = H(1) + \dot{\Gamma}(1)$. From Lemma 6.2 and the fact that $\Gamma(t)$ is a geodesic it follows that

$$r(\Gamma(1))\partial_r(\Gamma(1)) = \tau(H(0)) + \tau(\dot{\Gamma}(0)) = \tau(r(p)\partial_r(p) - r(p)X_1(p)) = r(p)\tau(X_2(p)) \in V_2(\Gamma(1)).$$

This shows that $\Gamma(1) \in C_2$.

Now we prove that the geodesic $\Gamma(t)$ exists for $t = 1$. We can apply the results of the previous section. In the notations of the previous section we have $v = -r(p)X(p)$ and $\rho = -r(p)\alpha(p)$. Since $0 < \alpha(p) < 1$, we have $0 < L^2 = g(v, v) = \alpha(p) - \alpha^2(p)$ and $r - |\rho| > 0$. Then the function $r(t)$ defining the geodesic $\Gamma(t)$ is defined on $\mathbb{R}$. The other defining function $f(t)$ is given by

$$f(t) = \frac{1}{L} \arctan \frac{Lr(p)t}{\rho t + r(p)} = \frac{1}{L} \arctan \frac{Lt}{1 - \alpha(p)t}.$$

We see that f is defined for $t \in [0, 1]$ as $\alpha(p) < 1$. Thus the geodesic $\Gamma(t)$ is defined for $t \in [0, 1]$.

Since the integral manifolds of the distribution V_1 are totally geodesic and $\dot{\Gamma}(0) \in V_1(p)$, we have $\widehat{M}^1_p = \widehat{M}^1_{\Gamma(1)}$. From Lemma 6.3 it follows that $\widehat{M}^1_p$ is flat. Similarly we show that $\widehat{M}^2_p$ is also flat. Hence, any point p has a flat neighbourhood. □

From Lemma 6.4 it follows that the dense subset $U \subset \widehat{M}$ is flat. Thus $\widehat{M}$ is flat and (M, g) has constant sectional curvature 1. This finishes the proof of Theorem 6.1. □

Since simply connected indefinite pseudo-Riemannian manifolds of constant curvature are never compact, we get the statement of Corollary 1 as a consequence of this theorem.

7 Cones over complete manifolds

Theorem 7.1. *Let (M, g) be a complete pseudo-Riemannian manifold of dimension ≥ 2 with decomposable holonomy $\widehat{\mathfrak{h}}$ of the cone $\widehat{M}$. Then there exists an open dense submanifold $M' \subset M$ such that each connected component of M' is isometric to a pseudo-Riemannian manifold of the form*

(1) a pseudo-Riemannian manifold M_1 of constant sectional curvature 1 or

(2) a pseudo-Riemannian manifold $M_2 = \mathbb{R}^+ \times N_1 \times N_2$ with the metric

$$-ds^2 + \cosh^2(s)g_1 + \sinh^2(s)g_2,$$

where (N_1, g_1) and (N_2, g_2) are pseudo-Riemannian manifolds and (N_2, g_2) has constant sectional curvature -1 or $\dim N_2 \leq 1$.

Moreover, the cone $\widehat{M}_2$ *is isometric to the open subset* $\{r_1 > r_2\}$ *in the product of the space-like cone* $(\mathbb{R}^+ \times N_1, dr^2 + r^2 g_1)$ *over* (N_1, g_1) *and the time-like cone* $(\mathbb{R}^+ \times N_2, -dr^2 + r^2 g_2)$ *over* (N_2, g_2)*.*

Proof. By going over to the universal covering, if necessary, we can assume that (M, g) is simply connected. Then $\widehat{M}$ is simply connected and decomposable. Let $\cup_{i\in I} W_i = U_1$ be the representation of the open subset $U_1 \subset M$ as the union of disjoint connected open subsets. For each W_i we have two possibilities: (1.) $0 < \alpha < 1$ on W_i; (2.) $\alpha < 0$ or $\alpha > 1$ on W_i. Consider these two cases.

(1.) Suppose that $0 < \alpha < 1$ on W_i. Similarly to the proof of Theorem 6.1 we can show that the cone over W_i is flat.

(2.) Suppose that $\alpha > 1$ on W_i. As in the proof of Lemma 6.1 we can show that $\alpha(W_i) = (1, +\infty)$. To proceed we need the following statement which is a generalisation of an argument used in the proof of Theorem 27 in [9].

Proposition 7.1. *Let* (M, g) *be a connected pseudo-Riemannian manifold and* $\alpha \in C^\infty(M)$ *with gradient* Z *such that* $g(Z, Z) \neq 0$ *and* $Z = (f \circ \alpha) \cdot X$ *for a vector field* X *such that* $g(X, X) = h \circ \alpha$*, where* f *and* h *are smooth functions on the open interval* $\mathrm{Im}\ \alpha$*. If the flow* ϕ *of* X *satisifies the following condition,*

(∗) *For all* $c \in \mathrm{Im}\ \alpha$ *exists an open interval* I_c *such that for all* $p \in F_c := \alpha^{-1}(c)$ *the interval* I_c *is the maximal intervall on which the flow* $t \mapsto \phi_t(p)$ *is defined,*

then M *is diffeomorphic to the product of the image of the function* α *and a level set of* α*. In particular, if the manifold* (M, g) *is geodesically complete and the vector field* X *is a geodesic vector field, then* M *is diffeomorphic to* $\mathrm{Im}(\alpha) \times$ level set*.*

Proof. First we notice that $0 \neq g(Z, Z) = (f^2 \cdot h) \circ \alpha$. As M is connected, we may assume that $g(Z, Z) > 0$ and thus $h > 0$. As the sign of f plays no role in what follows we also assume that $f > 0$. Furthermore, α satisfies the following differential equation on M,

$$X(\alpha) = \frac{1}{f \circ \alpha} Z(\alpha) = (f \circ \alpha) g(X, X) = (f \cdot h) \circ \alpha. \tag{7.1}$$

Let $\phi : F_c \times I_c \to M$, $(p, t) \mapsto \phi_t(p)$ be the flow of the vector field X. The proof is now based on the observation that if p and q are in the same level set of α, then

$$\alpha(\phi_t(p)) = \alpha(\phi_s(q)) \iff t = s \tag{7.2}$$

for all $t, s \in \mathbb{R}$. To verify this, for each $p \in F_c$ we consider the real function

$$\varphi_c : I_c \ni t \quad \mapsto \quad \alpha(\phi_t(p)) \in \mathrm{Im}\ \alpha$$

which satisfies the ordinary differential equation

$$\varphi_c'(t) = d\alpha_{\phi_t(p)}(X(\phi_t(p)) = f(\varphi_c(t)) \cdot h(\varphi_c(t)) > 0. \tag{7.3}$$

Hence, for each $p \in F_c$ the function $\varphi_c(t) = \alpha(\phi_t(p))$ is subject to the ordinary differential equation (7.3) with the same initial condition $\varphi_c(0) = \alpha\left(\phi_0(p)\right) = \alpha\left(\phi_0(q)\right) = c$. Uniqueness of the solution implies that $\alpha\left(\phi_t(p)\right) = \alpha\left((\phi_t(q)\right)$ for all t and all $q \in F_c$. This proves

($\Longleftarrow$) of (7.2), and shows that φ_c does not depend on the starting point $p \in F_c$. Having this, (7.3) also shows that φ_c is strictly monotone, and thus injective which gives ($\Longrightarrow$) of (7.2).

(7.2) shows that the flow ϕ of X sends one level set F_c of α to another one F_d, i.e. $\alpha(\phi_t(p)) = \alpha(\phi_t(q))$ for all $t \in I_c$ and p, q in the same level set F_c.

Next, we show that two level sets that are joint by an integral curve of X are diffeomorphic. In fact, if $p \in F_c$ and $q = \phi_t(p)$, ϕ_t is a local diffeomorphism between F_c and F_d. ($\Longrightarrow$) of (7.2) implies that $\phi_t|_{f_c}$ is injective. To verify that it is surjective we notice that $\phi_{-t}|_{F_d}$ is also an injective local diffeomorphism. Hence, $\phi_{-t} \circ \phi_t = id_{F_c}$, which implies that $\phi_t : F_c \to F_d$ is a global diffeomorphism.

Finally, we show that for two level sets there is at least one flow line connecting them. To this end, we set $\phi(F_c) := \{\phi_t(p) \mid p \in F_c,\ t \in I_c\}$ and write

$$M = \bigcup_{c \in \mathrm{Im}\ \alpha} \phi(F_c).$$

We have seen that, if F_c and F_d are connected by an integral curve, then they are diffeomorphic under ϕ_t. But the maximality of I_c and I_d implies that $\phi(F_c) = \phi(F_d)$. If, on the other hand, F_c and F_d are not joined by an integral curve then, by maximality of I_c and I_d, a common point of $\phi(F_c)$ and $\phi(F_d)$ would lie on an integral curve joining F_c and F_d, i.e. $\phi(F_c) \cap \phi(F_d) = \emptyset$. In the latter case M can be written as disjoint union of open sets $\phi(F_c)$ which is not possible as M was supposed to be connected.

Hence, each integral curve meets each level set once, they are all diffeomorphic, i.e. M is diffeomorphic to $I_c \times F_c$. But this implies that M is diffeomorphic to $(\mathrm{Im}\ \alpha) \times F$ where F is a level set of α. $\square$

Resuming the proof of the theorem we notice that the vector field $\widehat{X} = \frac{\tilde{X}}{\sqrt{\alpha^2 - \alpha}}$ is geodesic and proportional to the gradient of α. Since (M, g) is complete and $\widehat{X}$ is geodesic its integral curves are defined for all t. As in the proof of Proposition 7.1 one shows that level sets are mapped onto level sets under the flow of $\widehat{X}$. This shows that $(*)$ in Proposition 7.1 is satisfied for the vector field $\widehat{X}|_{W_i} \in \Gamma(TW_i)$ on the manifold W_i: for F_c the interval I_c is limited by the real number a for which $\phi_a(F_c) \subset F_1$. Hence, we can apply Proposition 7.1 to the manifolds W_i and the vector field $\widehat{X}|_{W_i} \in \Gamma(TW_i)$. Combining the result with the proof of Case (2.) from Theorem 4.2 yields a decomposition

$$W_i = \mathbb{R}^+ \times N_1 \times N_2.$$

For the metric $g|_{W_i}$ we obtain that

$$g|_{W_i} = -ds^2 + \cosh^2(s) g_1 + \sinh^2(s) g_2,$$

where (N_1, g_1) and (N_2, g_2) are pseudo-Riemannian manifolds. The fact that the cone $(\widehat{W_i}, \widehat{g|_{W_i}})$ is isometric to an open subset of the product of a space-like cone over (N_1, g_1) and of a time-like cone over (N_2, g_2) is shown by Example 3.4.

By a variation of the proof of Theorem 6.1 we will show now that the time-like cone over the manifold (N_2, g_2) is flat and we will explain why it is not the case for the manifold (N_1, g_1).

Fix a point $p \in W_i$. Consider the geodesics $\Gamma_1(t)$ and $\Gamma_2(t)$ starting at p and satisfying the initial conditions $\dot{\Gamma}_1(0) = -r(p)X_1(p)$ and $\dot{\Gamma}_2(0) = -r(p)X_2(p)$.

Now we prove that the geodesic $\Gamma_2(t)$ exists for $t = 1$ and the geodesic $\Gamma_1(t)$ does not exist for $t = 1$. We can apply the results of section 5. For Γ_1 we have $v_1 = -r(p)X(p)$ and $\rho_1 = -r(p)\alpha(p)$; for Γ_2 we have $v_2 = r(p)X(p)$ and $\rho_2 = r(p)(\alpha(p) - 1)$.

From Section 5 it follows that the functions $r_1(t)$ and $f_1(t)$ defining the geodesic $\Gamma_1(t)$ are defined on the interval $\left[0, \frac{1}{\sqrt{\alpha^2(p)-\alpha(p)}+\alpha(p)}\right) \subset [0, 1)$. The functions $r_2(t)$ and $f_2(t)$ defining the geodesic $\Gamma_2(t)$ are defined on the interval $\left[0, \frac{1}{\sqrt{\alpha^2(p)-\alpha(p)}-\alpha(p)+1}\right) \supset [0, 1]$.

Thus the geodesic $\Gamma_2(t)$ is defined for $t \in [0, 1]$ and the geodesic $\Gamma_1(t)$ is not defined for all $t \in [0, 1]$.

As in the proof of Theorem 6.1 we get that the manifold $\widehat{M}_p^2$ is flat. This means that the induced connection on the distribution $V_2|_{W_i}$ is flat and the *time-like* cone over the manifold (N_2, g_2) is flat, i.e. (N_2, g_2) has constant sectional curvature -1 or $\dim N_2 \leq 1$.

Note that as in Example 3.1 it can be $\alpha > 1$ on M, then $C_2 = \emptyset$ and the induced connection on V_1 need not be flat.

The case $\alpha|_{W_i} < 0$ is similar, with the roles of V_1 and V_2 interchanged. $\square$

8 Para-Kähler cones

Para-Kähler cones and para-Sasakian manifolds

Now we consider the case where the holonomy algebra $\widehat{\mathfrak{h}}$ of the space-like cone $\widehat{M}$ over (M, g) is indecomposable and preserves a decomposition $T_p\widehat{M} = V \oplus W$ $(p \in \widehat{M})$ into two complementary (necessarily degenerate) subspaces V and W. The next lemma[2] reduces the problem to the case $V = V^\perp$, $W = W^\perp$.

Lemma 1 (cf. Thm. 14.4 [15]). *Let E be a pseudo-Euclidian vector space and $\mathfrak{h} \subset \mathfrak{so}(E)$ an indecomposable Lie subalgebra. If E admits a non-trivial $\mathfrak{h}$-invariant decomposition $E = V \oplus W$ then it admits an $\mathfrak{h}$-invariant decomposition $E = V' \oplus W'$ into a sum of totally isotropic subspaces.*

By the lemma, we can assume that V, W are totally isotropic of the same dimension, which implies that the metric has neutral signature. In this section we use a similar approach as in the previous sections but with different structures coming up. These structures are related to a *para-complex structure*, and to a *para-Sasakian structure*. We recall the basic definitions given in [11] and [10].

Definition 8.1. 1. Let V be a real finite dimensional vector space. A *para-complex structure* on V is an endomorphism $J \in End(V)$, such that $J^2 = \mathrm{Id}$ and the two eigenspaces $V^\pm := \ker(\mathrm{Id} \mp J)$ of J have the same dimension. The pair (V, J) is called a *para-complex vector space*.

[2] communicated to us by Lionel Bérard Bergery

2. Let $\mathcal{V}$ be a distribution on a manifold M. An *almost para-complex structure* on $\mathcal{V}$ is a field $J \in \Gamma(End\,\mathcal{V})$ of paracomplex structures in $\mathcal{V}$. It is called *integrable* or *paracomplex structure on* $\mathcal{V}$ if the eigen-distributions $\mathcal{V}^{\pm} := \ker(\mathrm{Id} \mp J)$ are involutive.

3. A manifold M endowed with a para-complex structure on TM is called a *para-complex manifold*.

Similar to the complex case, the integrability of J is equivalent to the vanishing of the *Nijenhuis tensor* N_J defined by

$$N_J(X,Y) \;:=\; J\left([JX,Y]+[X,JY]\right)-[X,Y]-[JX,JY], \quad X,Y\in\Gamma\mathcal{V}. \tag{8.1}$$

Definition 8.2. 1. Let (V,J) be a para-complex vector space equipped with a scalar product g. (V,J,g) is called *para-hermitian vector space* if J is an anti-isometry for g, i.e.

$$J^*g \;:=\; g(J.,J.) \;=\; -g. \tag{8.2}$$

2. A *(almost) para-hermitian manifold* (M,J,g) is an (almost) para-complex manifold (M,J) endowed with a pseudo-Riemannian metric g such that $J^*g=-g$. The two-form $\omega := g(J,) = -g(,J)$ is called the para-Kähler form of (M,J,g).

3. A *para-Kähler manifold* (M,J,g) is a para-hermitian manifold (M,J,g) such that J is parallel with respect to the Levi-Civita-connection ∇ of g.

As in the complex case, the condition ∇J is equivalent to $N_J=0$ and $d\omega=0$. In contrary to the complex case, a $2n$–dimensional para-hermitian manifold has to be of neutral signature (n,n). Note that eigen-distributions $\mathcal{V}^{\pm}$ of J are totally isotropic and auto-orthogonal, i.e. $(\mathcal{V}^{\pm})^{\perp}=\mathcal{V}^{\pm}$. For a para-Kähler manifold the condition $\nabla J=0$ means that the ±1-eigen-distributions $\mathcal{V}^{\pm}$ are parallel. We get

Proposition 8.1. *A pseudo-Riemannian manifold (M,g) is a para-Kähler manifold if and only if the holonomy group preserves a decomposition of the tangent space into a direct sum of two totally isotropic subspaces.*

In the following we will show that metric cones with para-Kähler structure are precisely cones over para-Sasakian manifolds.

Definition 8.3. A *para-Sasakian manifold* is a pseudo-Riemannian manifold (M,g) of signature $(n+1,n)$, where $n+1$ is the number of time-like dimensions, endowed with a time-like geodesic unit Killing vector field T such that ∇T defines an integrable para-complex structure $J=\nabla T|_E : E\to E$ on $E=T^{\perp}$. The pair (g,T) is called a *para-Sasakian structure.*

Note that the eigen-distributions $E^{\pm}$ of $J=\nabla T|_E$ are totally isotropic and J is an anti-isometry of $g|_E$. Indeed, using the condition that T is a Killing field for $X_{\pm}$ and $Y_{\pm}$ in $\Gamma(E^{\pm})$, we get

$$0 \;=\; (\mathcal{L}_T g)(X_{\pm},Y_{\pm}) \;=\; g(\nabla_{X_{\pm}}T,Y_{\pm})+g(\nabla_{Y_{\pm}}T,X_{\pm}) \;=\; 2g(X_{\pm},Y_{\pm}). \tag{8.3}$$

A para-Sasakian manifold carries several other structures. First of all it has *contact structure* given by the contact form $\theta := g(T, .)$. Indeed, for $d\theta$ we get that

$$d\theta(X_+, X_-) \;=\; -g(T, [X_+, X_-]) \;=\; 2g(X_+, X_-), \tag{8.4}$$

with $X_\pm \in \Gamma(E^\pm)$. Since $E^\pm$ are dual to each other, this implies that $\theta \wedge d\theta^n \neq 0$, hence θ is a contact form. The Reeb vector field of this contact structure is T, because

$$d\theta(T, X) \;=\; -g(T, [T, X]) \;=\; -g(T, \nabla_T X - \nabla_X T) \;=\; 0. \tag{8.5}$$

It also admits a *para-CR structure* (see for example [2]), which is defined on a $(2n+1)$-dimensional manifold M as an n-dimensional subbundle E of TM together with a para-complex structure J on E. For a para-Sasakian manifold this para-CR structure is given by the one-form θ. From (8.4) and from the assumption that $E^\pm$ are involutive we see that the Levi-form $L_\theta \in \Gamma(S^2E)$ of this para-CR structure, defined by $L_\theta(X, Y) := d\theta|_E(X, JY)$, is given by the metric,

$$L_\theta(X, Y) \;=\; d\theta(X, JY) \;=\; -2g(X, Y) \tag{8.6}$$

and is thus non-degenerate. Hence for a para-Sasakian manifold, the metric g can be expressed in terms of the contact form θ and its Levi form:

$$g \;=\; -\theta^2 - \frac{1}{2} L_\theta. \tag{8.7}$$

This is in analogy to *strictly pseudo-convex pseudo-Hermitian structures* (see for example [7] and [6]). Although the definition of a para-Sasakian structure seems rather weak, it entails the following properties.

Lemma 8.1. *Let (M, g, T) be a para-Sasakian manifold with $E = T^\perp$ and eigen-distributions $E^\pm$. Then:*

1. *$E^\pm$ are auto-parallel and $N_J|_{E^\pm} = \nabla J|_{E^\pm} = 0$.*
2. *For $X_\pm \in \Gamma(E^\pm)$ it holds that $\nabla_{X_-} X_+ = -g(X_+, X_-)T \mod E^+$ and $\nabla_{X_+} X_- = g(X_+, X_-)T \mod E^-$.*
3. *For $X_\pm \in \Gamma(E^\pm)$ it holds $[T, X_\pm] \subset \Gamma(E^\pm)$.*

Proof. 1. Let $X_\pm$ and $Y_\pm$ be in $E^\pm$. (8.3) implies that

$$g(\nabla_{X_\pm} Y_\pm, T) \;=\; -g(X_\pm, Y_\pm) \;=\; 0,$$

which ensures that $\nabla_{X_\pm} Y_\pm \in E$. Now, $E^\pm$ are integrable, which implies on the one hand the relation for N_J, and gives on the other hand, using the Koszul formula, that $g(\nabla_{X_\pm} Y_\pm, Z_\pm) = 0$ for all $Z_\pm \in E^\pm$. Hence, $E^\pm$ are auto-parallel, which yields the relation for ∇J.

2. First of all we have that

$$g(\nabla_{X_-} X_+, T) \;=\; -g(X_+, \nabla_{X_-} T) \;=\; g(X_+, X_-).$$

Next we show that $\nabla_{X_-}X_+$ is orthogonal to E^+. In the following equations $g(Y_i^-,Y_j^+) = \delta_{ij}$ and the lower indices $+,-,0$ denote the corresponding component in $E^\pm$ and $\mathbb{R}T$:

$$
\begin{aligned}
2g(\nabla_{Y_i^-}Y_j^+,Y_k^+) &\overset{\text{Koszul}}{=} g\big([Y_i^-,Y_j^+],Y_k^+\big) + g\big([Y_k^+,Y_j^+],Y_i^-\big) + g\big([Y_k^+,Y_i^-],Y_j^+\big)\\
&= g\big([Y_i^-,Y_j^+]_-,Y_k^+\big) + g\big([Y_k^+,Y_j^+],Y_i^-\big) + g\big([Y_k^+,Y_i^-]_-,Y_j^+\big)\\
&\overset{(8.4)}{=} -\frac{1}{2}g\Big(T,\big[Y_k^+,[Y_i^-,Y_j^+]_-\big] + \big[[Y_k^+,Y_j^+],Y_i^-\big] + \big[Y_j^+,[Y_k^+,Y_i^-]_-\big]\Big)\\
&= -\frac{1}{2}g\Big(T,\big[Y_k^+,[Y_i^-,Y_j^+]_-\big] + \big[Y_i^-,[Y_j^+,Y_k^+],\big] + \big[Y_j^+,[Y_k^+,Y_i^-]_-\big]\Big)\\
&= -\frac{1}{2}g\Big(T,\underbrace{\big[Y_k^+,[Y_i^-,Y_j^+]\big] + \big[Y_i^-,[Y_j^+,Y_k^+],\big] + \big[Y_j^+,[Y_k^+,Y_i^-]\big]}_{=0 \text{ Jacobi identity}}\Big)\\
&\quad \underbrace{+\frac{1}{2}\,g\Big(T,\underbrace{\big[Y_k^+,[Y_i^-,Y_j^+]_+\big]}_{\in E^+} + \underbrace{\big[Y_j^+,[Y_k^+,Y_i^-]_+\big]}_{\in E^+}\Big)}_{=0}\\
&\quad +\frac{1}{2}g\Big(T,\big[Y_k^+,[Y_i^-,Y_j^+]_0\big] + \big[Y_j^+,[Y_k^+,Y_i^-]_0\big]\Big)\\
&= -\frac{1}{2}g\Big(T,\big[Y_k^+,\underbrace{g(T,[Y_i^-,Y_j^+])}_{=-2\delta_{ij}}T\big] + \big[Y_j^+,\underbrace{g(T,[Y_k^+,Y_i^-])}_{=2\delta_{ik}}T\big]\Big)\\
&= g\Big(T,\delta_{ij}\big[Y_k^+,T\big] - \delta_{ik}\big[Y_j^+,T\big]\Big)\\
&= 0
\end{aligned}
$$

This implies that $\nabla_{X_-}X_+ \in \mathbb{R}T \oplus E^+$, which proves the second statement.

The last point follows from the general fact:

$$\text{If } T \text{ is a Killing vector field, and } \theta = g(T,.), \text{ then } \mathcal{L}_T\nabla\theta = 0. \tag{8.8}$$

Indeed, the Killing equation for T is equivalent to $\nabla\theta = \frac{1}{2}d\theta \in \Omega^2 M$. This implies for arbitrary tangent vectors X and Y using the skew symmetry of $\nabla\theta$ that

$$
\begin{aligned}
0 &= \frac{1}{2}dd\theta(T,X,Y)\\
&= T\left(\nabla\theta(X,Y)\right) - X\left(\nabla\theta(T,Y)\right) + Y\left(\nabla\theta(T,X)\right)\\
&\quad -\nabla\theta([T,X],Y) + \nabla\theta([T,Y],X) - \nabla\theta([X,Y],T)\\
&= \left(\mathcal{L}_T\nabla\theta\right)(X,Y) - X\left(\nabla\theta(T,Y)\right) + Y\left(\nabla\theta(T,X)\right) - \nabla\theta([X,Y],T)\\
&= \left(\mathcal{L}_T\nabla\theta\right)(X,Y) - X\left(\theta(\nabla_Y T)\right) + Y\left(\theta(\nabla_X T)\right) + \theta(\nabla_{[X,Y]}T)\\
&= \left(\mathcal{L}_T\nabla\theta\right)(X,Y) - \theta(R(X,Y)T)\\
&= \left(\mathcal{L}_T\nabla\theta\right)(X,Y).
\end{aligned}
$$

This can easily be applied to our situation, where we have that

$$\nabla\theta \;=\; g(J.,.).$$

For $X_\pm \in E^\pm$ and $Y \in TM$, (8.8) implies that

$$\begin{aligned} 0 &= (L_T\nabla\theta)(X_\pm, Y) \\ &= T(g(JX_\pm, Y)) - g(J([T, X_\pm]), Y) - g(JX_\pm, [T, Y]) \\ &= \pm\underbrace{(\mathcal{L}_T g)(X_\pm, Y)}_{=0} \pm g([T, X_\pm], Y) - g(J([T, X_\pm]), Y), \end{aligned}$$

which gives $[T, X_\pm] \in E^\pm$. □

Using these properties we obtain a description of para-Sasakian manifolds which might look more familiar.

Proposition 8.2. (M, g, T) *is a para-Sasakian manifold if and only if* (M, g) *is a pseudo-Riemannian manifold of signature* $(n+1, n)$ *and* T *a time-like geodesic unit Killing vector field, such that the endomorphism* $\phi := \nabla T \in \Gamma(End(TM))$ *satisfies:*

$$\phi^2 = id + g(., T)T \tag{8.9}$$

$$(\nabla_U\phi)(V) = -g(U, V)T + g(V, T)U, \ \ \forall U, V \in TM \tag{8.10}$$

Proof. First, let (M, g) be a pseudo-Riemannian manifold of signature $(n+1, n)$ with a time-like geodesic unit Killing vector field T satisfying (8.9) and (8.10). The fact that T is geodesic means that $\phi T = 0$ and implies that ϕ preserves $E := T^\perp$. Putting $J := \phi|_E$, the equation (8.9) shows that $J^2 = id_E$, i.e. ϕ is a skew-symmetric involution and therefore a para-complex structure. Finally (8.10) ensures that $J = \phi|_E$ is integrable because $\nabla J|_{E^\pm} = 0$.

For the converse statement we assume that (M, g, T) is a para-Sasakian manifold. Setting $\phi := \nabla T$ we get $\phi^2|_E = J^2 = id$ and $\phi^2(T) = 0$ which gives (8.9). We have to check (8.10): For $U = V = T$ both sides of (8.10) are zero. For $U = X \in T^\perp$ and $V = T$ the right hand side is given by $g(T, T)X = -X$, but also the left hand side which is $(\nabla_X\phi)(T) = -\phi(\nabla_X T) = -\phi^2(X) = -X$. For $U = T$ and $V = X_\pm \in E^\pm$ the right hand side vanishes, and the left hand side as well because of $[T, E^\pm] \subset E^\pm$:

$$\begin{aligned} (\nabla_T\phi)(X_\pm) &= \nabla_T JX_\pm - J(\nabla_T X_\pm) \\ &= \pm[T, X_\pm] + X_\pm - J([T, X_\pm]) - J^2(X_\pm) \\ &= \pm[T, X_\pm] - J([T, X_\pm]) = 0. \end{aligned}$$

For U and V both in $E^\pm$ both sides vanish because of the integrability of the para-complex structure. For $U = X_+ \in E^+$ and $V = X_- \in E^-$ the right hand side of (8.10) is equal to $-g(X_+, X_-)T$ and the left hand side is given by

$$(\nabla_{X_+}\phi)X_- = -\nabla_{X_+}X_- - \phi(\nabla_{X_+}X_-) = -g(X_+, X_-)T$$

because of the second point of the lemma. □

Now we can formulate the main theorem of this section.

Theorem 8.1. *Let (M, g) be a pseudo-Riemannian manifold. There is a one-to-one correspondence between para-Sasakian structures (M, g, T) on (M, g) and para-Kähler structures $(\widehat{M}, \widehat{g}, \widehat{J})$ on the cone $(\widehat{M}, \widehat{g})$. The correspondence is given by $T \mapsto \widehat{J} := \widehat{\nabla}T$.*

Proof. First assume that (M, g, T) is a para-Sasakian manifold with para-complex structure $J = \nabla T$ on $E := T^\perp$, which splits into eigen-distributions $E^\pm$. The para-complex structure on the metric cone $(\widehat{M}, \widehat{g})$ is defined by

$$\widehat{J} \;:=\; \widehat{\nabla}T.$$

Because of the formula for the covariant derivative of the cone, $\widehat{J}$ is given by

$$\begin{array}{rcccccl} \widehat{J}(\partial_r) & = & \widehat{\nabla}_{\partial_r}T & = & \frac{1}{r}T & & \\ \widehat{J}(T) & = & \widehat{\nabla}_T T & = & \nabla_T T - rg(T,T)\partial_r & = & r\partial_r \\ \widehat{J}(X) & = & \widehat{\nabla}_X T & = & \nabla_X T - rg(X,T)\partial_r & = & J(X), \quad X \in E, \end{array}$$

which implies that $\widehat{J}$ is an almost para-complex structure, and also an almost para-hermitian structure with respect to the cone metric $\widehat{g}$. The eigen-distributions of $\widehat{J}$ are given by

$$V^\pm \;=\; \mathbb{R}(r\partial_r \pm T) \oplus E^\pm.$$

They are involutive because the distributions $E^\pm$ are involutive and

$$[r\partial_r \pm T, X_\pm] \;=\; \pm [T, X_\pm] \in E^\pm$$

for $X_\pm \in \Gamma(E^\pm)$. Hence, $\widehat{\nabla}T$ defines a para-Kähler structure on the cone.

Now assume that the cone $(\widehat{M}, \widehat{g})$ over (M, g) is a para-Kähler manifold with para-complex structure $\widehat{J}$. We consider the decomposition $TM = V^+ \oplus V^-$ into the totally isotropic eigen-distributions of $\widehat{J}$. Then the radial vector field decomposes as follows,

$$\partial_r \;=\; \underbrace{\rho\partial_r + X}_{:=X_+ \in V^+} + \underbrace{(1-\rho)\partial_r - X}_{:=X_- \in V^-}, \tag{8.11}$$

where $X \in \Gamma(\widehat{M})$ is a global vector field tangent to M. This vector field defines a para-Sasakian structure. First of all, we prove

Lemma 8.2. *The vector field $2rX = \widehat{J}(r\partial_r)$ on $\widehat{M}$ is tangent to M and r-independent. Its restriction to the submanifold $M \cong \{1\} \times M \subset \mathbb{R}^+ \times M = \widehat{M}$ defines is a time-like geodesic unit vector field T on (M, g).*

Proof. As V^+ and V^- are totally isotropic, we get for X defined in (8.11)

$$0 \;=\; \rho^2 + r^2 g(X,X) \;=\; (1-\rho)^2 + r^2 g(X,X).$$

This implies $\rho = \frac{1}{2}$ and $g(X,X) = -\frac{1}{4r^2}$. By the holonomy invariance of the distributions $V^\pm$, stated in Proposition 8.1, we get

$$V^+ \;\ni\; \widehat{\nabla}_{\partial_r}\left(\frac{1}{2}\partial_r + X\right) \;=\; \widehat{\nabla}_{\partial_r}X$$

and similar

$$V^- \ \ni \ \widehat{\nabla}_{\partial_r}\left(\frac{1}{2}\partial_r - X\right) \ = \ -\widehat{\nabla}_{\partial_r} X,$$

which implies $\widehat{\nabla}_{\partial_r} X = 0$. Hence, $[\partial_r, X] = -\frac{1}{r}X$, and thus $X = \frac{1}{2r}T$ where T is a vector field on M with $g(T,T) = -1$. It follows that T is a geodesic vector field because:

$$V^\pm \ \ni \ \widehat{\nabla}_T\left(r\partial_r \pm T\right) \ = \ T \pm \left(\nabla_T T + r\partial_r\right) \ = \ \pm \underbrace{(r\partial_r \pm T)}_{=2rX_\pm \in V^\pm} \pm \nabla_T T,$$

i.e. $\nabla_T T \in V + \cap V m = \{0\}$. □

Hence, the vector fields $X_\pm$, defined in (8.11) are given by

$$X_\pm \ = \ \frac{1}{2}\left(\partial_r \pm \frac{1}{r}T\right)$$

for T a time-like geodesic unit vector field on M. We consider now the orthogonal complement of $X_\mp$ in $V^\pm$.

Lemma 8.3. *Let $E^\pm := \{Y \in V^\pm \mid \widehat{g}(Y, X_\mp) = 0\} \subset V^\pm$ be the orthogonal complement of $X_\mp$ in $V^\pm$. Then $E^\pm$ are tangential to M, orthogonal to T, totally isotropic, and $E := E^+ \oplus E^-$ is the orthogonal complement of T in TM.*

Proof. As V^+ is totally isotropic any $U = a\partial_r + Y \in E^+$ ($Y \in TM$) is orthogonal to X_+ and X_-, which is equivalent to $0 = a \pm rg(Y,T)$. Hence, $a = g(Y,T) = 0$. The same holds for $U \in E^-$. Both are totally isotropic with respect to g as $V^\pm$ are totally isotropic with respect to $\widehat{g}$. □

This gives the following decomposition of the tangent bundle into three non-degenerate distributions

$$T\widehat{M} \ = \ \mathbb{R}\cdot\partial_r \oplus^\perp \mathbb{R}\cdot T \oplus^\perp \left(E^+ \oplus E^-\right),$$

where E^+ and E^- are totally isotropic.

Lemma 8.4. *The vector field T satisfies $\widehat{\nabla} T|_{E^\pm} = \pm\mathrm{Id}$.*

Proof. The holonomy invariance of $V^\pm$ implies that

$$\widehat{\nabla} X_\pm \ : \ T\widehat{M} \to V_\pm.$$

But the formulae for $\widehat{\nabla}$ imply that

$$\widehat{\nabla} X_\pm|_{TM} \ = \ \frac{1}{2r}\left(\mathrm{Id}_{TM} \pm \widehat{\nabla} T\right).$$

Applying this to $E^\pm$ gives that $\widehat{\nabla} T$ leaves E^+ and E^- invariant. Hence, $\widehat{\nabla} X_\pm$ is zero on $E^\mp$, and thus $\widehat{\nabla} T|_{E^\mp} = \mp\mathrm{Id}$. □

As T is a vector field on M, its orthogonal complement E does not depend on the radial coordinate r and defines a distributions on M. The same holds for $E^{\pm}$ because $\widehat{\nabla}T$ is an endomorphism on E which does not depend on r and is given as $\pm\mathrm{Id}$ on $E^{\pm}$, which are also denoted by E and $E^{\pm}$. Thus, $\nabla T|_{E^{\pm}} = \widehat{\nabla}T|_{E^{\pm}} = \pm\mathrm{Id}$ defines an almost para-complex structure J on $E = T^{\perp} \subset TM$. As its eigen-spaces $E^{\pm}$ are totally isotropic, J is an anti-isometry, $g(J.,J.) = -g$. This implies that T is a Killing vector field:

Lemma 8.5. *T is a Killing vector field on M.*

Proof. $(\mathcal{L}_T g)(U,V) = g(JU,V) + g(JV,U) = 0$, because J is an anti-isometry. □

Lemma 8.6. *E^+ and E^- are involutive.*

Proof. For Y_+ and Z_+ in E^+ by the holonomy invariance of V^+, it is $[Y_+,Z_+] \in V^+$. Hence, it suffices to show that $[Y_+,Z_+] \perp X_-$. But this is true because $\widehat{\nabla}X_-|_{E^+} = 0$ (see the proof of Lemma 8.4):

$$\widehat{g}([Y_+,Z_+],X_-) = -\widehat{g}(Z_+,\widehat{\nabla}_{Y_+}X_-) + \widehat{g}(Y_+,\widehat{\nabla}_{Z_+}X_-) = 0.$$

We get the same for E^-. □

Summarising we get that T is a geodesic, time-like unit Killing vector field, and ∇T is an integrable para-complex structure on $T^{\perp}$. Hence, (M,g,T) is a para-Sasakian manifold. □

Examples of para-Sasakian manifolds

Now we construct a family of para-Sasakian manifolds (M,g,T) of positive non constant curvature which implies that the associated cone $\hat{M}$ is not flat. We will describe (g,T) in terms of coordinates.

Let (M,g,T) be a para-Sasaki manifold. Consider a filtration of TM by integrable distributions $E^+ \subset \mathbb{R}\cdot T \oplus E \subset TM$. The Frobenius Theorem implies existence of local coordinates on M adapted to this filtration and a hypersurface which contains the leaves of E^+ and is transversal to T. We choose local coordinates on this hypersurface adapted to E^+. Since T is a Killing vector field, its flow can be used to extend these coordinates to coordinates $(t,x_1,\ldots,x_n,x_{n+1},\ldots,x_{2n})$ on some open subset $U \subset M$ such that

$$\frac{\partial}{\partial t} = T|_U \text{ and } \frac{\partial}{\partial x_i} \in \Gamma(E^+|_U).$$

Obviously, ith respect to these coordinates the metric g is given by the matrix of the form

$$\left(\begin{array}{c|c|c} -1 & 0 & u^t \\ \hline 0 & 0 & H^t \\ \hline u & H & G \end{array}\right).$$

Here $u = (u_1, \dots, u_n) \in C^\infty(U, \mathbb{R}^n)$, H a non-degenerate matrix of real functions on U and G a symmetric matrix of real functions on U. We choose a basis Y_i^- of of vector fields on E^- which such that

$$\begin{aligned} 0 &= g(T, Y_i^-), \\ \delta_{ij} &= g(\frac{\partial}{\partial x_i}, Y_j^-), \text{ and} \\ 0 &= g(Y_i^-, Y_j^-), \end{aligned}$$

First of all, these orthogonality relations imply that

$$Y_i^- := \left(H^{ij} u_j\right) \cdot T + b_{ij} \frac{\partial}{\partial x_j} + H^{ij} \frac{\partial}{\partial x_{j+n}}$$

where H^{ij} is the inverse matrix to H_{ij} and

$$b_{ij} + b_{ji} = -H^{ik} \left(u_k u_l + G_{kl}\right) H^{jl}. \tag{8.12}$$

As T is a Killing vector field and $Y_i^- \in \Gamma(E^-)$, we get that $[T, Y_i^-] = 0$ which implies that H, u, and b do not depend on t. Now we consider the condition (8.6) which can be written as $g|_E = -\frac{1}{2} L_\theta$ or

$$\begin{aligned} -2\delta_{ij} &= g([\frac{\partial}{\partial x_i}, Y_j^-], T) \\ &= -\frac{\partial}{\partial x_i}(H^{jk} u_k) + \frac{\partial}{\partial x_i}(H^{jk}) u_k \\ &= -H^{jk} \frac{\partial}{\partial x_i}(u_k). \end{aligned}$$

It implies

$$H_{ij} = \frac{1}{2} \frac{\partial}{\partial x_j}(u_i). \tag{8.13}$$

Then we evaluate the condition that ∇T acts as $-id$ on E^-. Note that the inverse matrix of the metric is given by

$$\left(\begin{array}{c|c|c} -1 & v^t & 0 \\ \hline v & F & H^{-1} \\ \hline 0 & (H^t)^{-1} & 0 \end{array}\right),$$

where $v = H^{-1} u$ and $F_{ij} = b_{ij} + b_{ji}$. We calculate

$$\begin{aligned} \nabla_{Y_i^-} T &= b_{ij} \frac{\partial}{\partial x_j} + H^{ij} \nabla_{\frac{\partial}{\partial x_{j+n}}} T \\ &= -v_i T + -H^{ij} \frac{\partial}{\partial x_{j+n}} \\ &\quad + \left(b_{ij} - F_{ji} + \frac{1}{2} H^{ik} H^{jl} \left(T(G_{kl}) + \frac{\partial}{\partial x_{k+n}}(u_l) - \frac{\partial}{\partial x_{l+n}}(u_k)\right)\right) \frac{\partial}{\partial x_j}. \end{aligned}$$

Hence, $\nabla_{Y_i^-}T = -Y_i^-$ is equivalent to

$$2b_{ij} = F_{ji} - \frac{1}{2}H^{ik}H^{jl}\left(T(G_{kl}) + \frac{\partial}{\partial x_{k+n}}(u_l) - \frac{\partial}{\partial x_{l+n}}(u_k)\right).$$

which gives

$$2(b_{ij} + b_{ji}) = 2F_{ji} - H^{ik}H^{jl}T(G_{kl}).$$

This implies that also G does not depend on t, but together with (8.12) it also gives a formula for b_{ij}, namely

$$b_{ij} = -\frac{1}{2}H^{ik}H^{jl}\left(u_k u_l + G_{kl} + \frac{1}{2}\left(\frac{\partial}{\partial x_{k+n}}(u_l) - \frac{\partial}{\partial x_{l+n}}(u_k)\right)\right). \qquad (8.14)$$

Finally, we evaluate the integrability of $E^- := \mathrm{span}(Y_i^-)_{i=1}^n$. We write this condition as

$$\nabla_{[Y_i^-, Y_j^-]}T = -[Y_i^-, Y_j^-]$$

and obtain after a lengthy but straightforward calculation that this is equivalent to

$$0 = \Lambda_{ij}\left[b_{iq}\left(\frac{\partial}{\partial x_q}(b_{jp}) - H_{lr}b_{rp}\frac{\partial}{\partial x_q}(H^{jl})\right) + H^{iq}\left(\frac{\partial}{\partial x_{q+n}}(b_{jp}) - H_{lr}b_{rp}\frac{\partial}{\partial x_{q+n}}(H^{jl})\right)\right],$$

in which Λ_{ij} denotes the skew symmetrization with respect to the indices i and j. Although we do not find the general solution of this equation we will construct solutions with $b_{ij} \equiv 0$. We make the following ansatz. We assume that

$$\begin{aligned} \frac{\partial}{\partial x_{i+n}}(u_j) &= 0, \text{ and set} \\ G_{ij} &:= -u_i \cdot u_j. \end{aligned}$$

This implies that $b_{ij} = 0$ which gives that $Y_i^- = H^{ij}\left(u_j \cdot T + \frac{\partial}{\partial x_{j+n}}\right)$ with $H_{ij} = \frac{1}{2}\partial_j u_i$, and ensures that $E^- = \mathrm{span}(Y_i^-)$ is the (-1)-eigen-space of ∇T and integrable. In fact, we get for the Levi-Civita connection of this metric:

$$\begin{aligned} \nabla_T T &= 0 \\ \nabla_T \partial_i &= \partial_i \\ \nabla_T \partial_{i+n} &= -u_i T - \partial_{i+n}, \text{ i.e. } \nabla_T Y_i^- = -Y_i^- \\ \nabla_{\partial_i}\partial_j &= H^{kl}\partial_i(H_{lj})\partial_k \\ \nabla_{\partial_{i+n}}\partial_{j+n} &= 2u_i u_j T + u_i\partial_{j+n} + u_j\partial_{i+n} \\ \nabla_{\partial_i}\partial_{j+n} &= -H_{ij}T - u_j\partial_i, \end{aligned}$$

which implies

$$\begin{aligned} \nabla_{\partial_i}Y_j^- &= \delta_{ij}T + H_{kl}\partial_i(H^{jk})Y_l^- \\ \nabla_{\partial_{i+n}}Y_j^- &= u_i Y_j^- \\ \nabla_{Y_i^-}Y_j^- &= 0 \end{aligned}$$

Now we check that the curvature of the metric is not constant. Calculating the curvature and denoting Y_i^+ by ∂_i, we get

$$\begin{array}{rcl} R(T,Y_i^{\pm}) & : & \left\{ \begin{array}{ccc} T & \mapsto & Y_i^{\pm} \\ Y_i^{\mp} & \mapsto & T \end{array} \right. \\ R(Y_i^{\pm},Y_j^{\pm}) & : & \quad Y_k^{\mp} \quad \mapsto \quad \delta_{jk} Y_i^{\pm} - \delta_{ik} Y_j^{\pm} \\ R(Y_i^{\pm},Y_j^{\mp}) & : & \left\{ \begin{array}{ccc} Y_k\pm & \mapsto & -\delta_{jk} Y_i^{\pm} - 2\delta_{ij} Y_k^{\pm} \\ Y_k\mp & \mapsto & \delta_{ik} Y_i^{\mp} + 2\delta_{ij} Y_k^{\mp} \end{array} \right. \end{array}$$

and the remaining terms being zero. The last terms show that (M,g) does not have constant sectional curvature. This can also be seen by calculating the derivatives of the curvature which are zero apart from one term:

$$\begin{array}{rcl} \left(\nabla_{\partial_i} R\right)(T, Y_j^-, \partial_k, Y_l^-) & = & -\Big(\underbrace{R(\partial_i, Y_j^-, \partial_k, Y_l^-)}_{=-\delta_{kj}\delta_{il} - 2\delta_{ij}\delta_{kl}} + \underbrace{R(T, Y_j^-, \partial_k, \delta_{il} T)}_{=-\delta_{il}\delta_{kj}} \Big) \\ & = & 2\left(\delta_{kj}\delta_{il} + \delta_{ij}\delta_{kl}\right). \end{array}$$

Hence, (M,g) is not locally symmetric. Note that the curvature R of this metric is of the form

$$R = R_1 - 2\omega \otimes J,$$

where R_1 is the curvature of a space of constant curvature, J the para-complex structure and $w = g(.,J.)$ is the para-Kähler form. Altogether we have proven:

Proposition 8.3. *Let $(t, x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n})$ be coordinates on $\mathbb{R}^{2n+1}$ and consider the metric g given by*

$$g \;=\; \left(\begin{array}{c|c|c} -1 & 0 & u^t \\ \hline 0 & 0 & H^t \\ \hline u & H & G \end{array} \right)$$

in which

- *$u = (u_1, \ldots, u_n)$ is a diffeomorphism of $\mathbb{R}^n$, depending on $x_1, \ldots, x_n$,*
- *$H = \frac{1}{2}\left(\frac{\partial}{\partial x_j}(u_i)\right)_{i,j=1}^n$ is its non-degenerate Jacobian, and*
- *G is the symmetric matrix given by $G_{ij} = -u_i u_j$,*

i.e.

$$g = -dt^2 + \sum_{i=1}^n u_i dx_i dt + \frac{1}{2}\sum_{i,j=1}^n \frac{\partial}{\partial x_j}(u_i) dx_i dx_{j+n} - \sum_{i,j=1}^n u_i u_j dx_{i+n} dx_{j+n}. \tag{8.15}$$

Then the manifold $(\mathbb{R}^{2n+1}, g)$ is para-Sasakian, not locally symmetric, and its curvature is given by the following formulas

$$\begin{array}{rcl} R|_{T^\perp \times T^\perp \times T^\perp} & = & \left(R^1(J.,J.) - 2\omega \otimes J\right)|_{T^\perp \times T^\perp \times T^\perp}, \ \textit{and} \\ R(T,.) & = & R^1(T,.), \end{array}$$

where R^1 is the curvature tensor of a space of constant curvature 1 in dimension $2n+1$, J the para-complex structure and $w = g(J.,.)$ is the para-Kähler form. In particular, the space-like cone over $(\mathbb{R}^{2n+1}, g)$ is para-Kähler and non-flat, i.e. its holonomy representation is non-trivial and decomposes into two totally isotropic invariant subspaces.

Remark 8.1.

1. It is obvious that the Abelian group $\mathbb{R}^{n+1}$ acts isometrically on $(\mathbb{R}^{2n+1}, g)$ via
$$\mathbb{R}^{n+1} \ni (c, c_1, \dots, c_n) : \begin{pmatrix} t \\ x_i \\ x_{i+n} \end{pmatrix} \mapsto \begin{pmatrix} t+c \\ x_i \\ x_{i+n} + c_i \end{pmatrix}$$
As these isometries also fix the para-Sasaki vector field $T = \frac{\partial}{\partial t}$, they are automorphisms of the para-Sasaki structure (g, T). Hence, we can consider a lattice $\Gamma \subset \mathbb{R}^{n+1}$ and compactify $(\mathbb{R}^{2n+1}, g)$ along these directions in order to obtain a para-Sasakian structure on
$$\mathbb{R}^{2n+1}/\Gamma = T^{n+1} \times \mathbb{R}^n,$$
where T^{n+1} denotes the $(n+1)$-torus. We do not know under which conditions on the u_i's there are more automorphisms, and if one can find enough in order to compactify the manifold by this method.

2. The manifolds obtained in this way are curvature homogeneous.

More examples of para-Kähler cones are given in [10] by conical special para-Kähler manifolds defined by a holomorphic prepotential of homogeneity 2. Further results on the holonomy of para-Kähler manifolds can be found in [8].

Para-3-Sasakian manifolds and para-hyper-Kähler cones

Now we study the case when the holonomy algebra $\widehat{\mathfrak{h}}$ of the cone $\widehat{M}$ preserves two complementary isotropic subspaces $T^{\pm}$ and a skew-symmetric complex structure J such that $JT^+ = T^-$. Let us provide the definitions needed to formulate a result analogous to Theorem 8.1 in this case.

Definition 8.4.

1. Let V be a real finite dimensional vector space. A *para-hyper-complex structure* on V is a triple $(J_1, J_2, J_3 = J_1J_2)$, where (J_1, J_2) is a pair of anticommuting para-complex structures on V.

2. Let M be a smooth manifold and $\mathcal{V}$ be a distribution on M. An *almost para-hyper-complex structure* on $\mathcal{V}$ is a triple $J_\alpha \in \Gamma(End\,\mathcal{V})$, $\alpha = 1, 2, 3$, such that, for all $p \in M$, $(J_1, J_2, J_3)_p$ is a para-hyper-complex structure on $\mathcal{V}_p$. It is called *integrable* if the J_α are integrable.

3. A *para-hyper-Kähler manifold* is a pseudo-Riemannian manifold (M, g) endowed with a parallel para-hyper-complex structure (J_1, J_2, J_3) consisting of skew-symmetric endomorphisms $J_\alpha \in \Gamma(End\,TM)$.

4. A *para-3-Sasakian manifold* is a pseudo-Riemannian manifold (M, g) of signature $(n+1, n)$ endowed with three orthogonal unit Killing vector fields (T_1, T_2, T_3) such that

(i) the vector fields T_1, T_2 are time-like and define para-Sasakian structures (g, T_1), (g, T_2), see Definition 8.3,

(ii) T_3 is space-like and defines a (pseudo-)Sasakian structure (g, T_3),

(iii) $\nabla_{T_2} T_1 = T_3$,

(iv) the vector fields satisfy the following $\mathfrak{sl}_2(\mathbb{R})$ commutation relations:

$$[T_1, T_2] = -2T_3, \quad [T_1, T_3] = -2T_2, \quad [T_2, T_3] = 2T_1 \quad \text{and}$$

(v) the tensors ∇T_α define a para-hyper-complex structure on $E := \mathrm{span}\{T_1, T_2, T_3\}^\perp$. (Here we are using that the conditions (ii-iii) imply $\nabla_E T_\alpha \subset E$.)

The assumption that $\widehat{\mathfrak{h}}$ preserves two complementary isotropic subspaces $T^\pm \subset T_p\widehat{M}$ and a skew-symmetric complex structure $J \in End\, T_p\widehat{M}$ with $JT^+ = T^-$ can be now reformulated by saying that $\widehat{M}$ locally admits a para-hyper-Kähler structure $(\widehat{g}, \widehat{J}_1, \widehat{J}_2, \widehat{J}_3 = \widehat{J}_1\widehat{J}_2)$, where $\widehat{J}_1|_{T^\pm} = \pm Id$ and $(\widehat{J}_3)_p = J$. The corresponding geometry of the base manifold (M, g) is para-3-Sasakian:

Theorem 8.2. *Let (M, g) be a pseudo-Riemannian manifold. There is a one-to-one correspondence between para-3-Sasakian structures (M, g, T_1, T_2, T_3) on (M, g) and para-hyper-Kähler structures $(\widehat{M}, \widehat{g}, \widehat{J}_1, \widehat{J}_2, \widehat{J}_3 = \widehat{J}_1\widehat{J}_2)$ on the cone $(\widehat{M}, \widehat{g})$. The correspondence is given by $T_\alpha \mapsto \widehat{J}_\alpha := \widehat{\nabla} T_\alpha$.*

Proof. By Theorem 8.1 the para-Sasakian structures (g, T_1) and (g, T_2) induce two para-Kähler structures $(\widehat{g}, \widehat{J}_1)$ and $(\widehat{g}, \widehat{J}_2)$ on the space-like cone $(\widehat{M}, \widehat{g})$. Similarly, the pseudo-Sasakian structure (g, T_3) induces a pseudo-Kähler structure $(\widehat{g}, \widehat{J}_3)$ on $\widehat{M}$. It suffices to show that $\widehat{J}_1\widehat{J}_2 = -\widehat{J}_2\widehat{J}_1 = \widehat{J}_3$. We recall that the vector fields T_α (considered as vector fields on $\widehat{M}$) are related to $T_0 := r\partial_r$ by $T_\alpha = \widehat{J}_\alpha T_0$. Using $\widehat{J}_\alpha = \widehat{\nabla} T_\alpha$, we show that the conditions (iii-iv) in Definition 8.4 imply that the structures $\widehat{J}_\alpha$ preserve the four-dimensional distribution

$$H := \mathrm{span}\{T_i | i = 0, 1, 2, 3\}$$

and act as the standard para-hyper-complex structure on H. In fact, first it is clear that $\widehat{J}_\alpha$ acts in the standard way on the plane P_α spanned by T_0 and T_α. Second the relations (iii-iv) easily imply that $\widehat{J_\alpha}$ preserves the plane $P'_\alpha = P_\alpha^\perp \cap H$. Since $\widehat{J}_\alpha^2 = \pm Id$, the action of $\widehat{J_\alpha}$ on P'_α is completely determined by:

$$\begin{aligned}
\widehat{J}_1 T_2 &= \widehat{\nabla}_{T_2} T_1 = \nabla_{T_2} T_1 \overset{(iii)}{=} T_3 \\
\widehat{J}_2 T_1 &= \nabla_{T_1} T_2 \overset{(iv)}{=} -2T_3 + \nabla_{T_2} T_1 = -T_3 \\
\widehat{J}_3 T_1 &= \nabla_{T_1} T_3 \overset{(iv)}{=} -2T_2 + \nabla_{T_3} T_1 = -2T_2 - g(\nabla_{T_3} T_1, T_2) T_2 \\
&= -2T_2 + g(T_3, \nabla_{T_2} T_1) T_2 - T_2.
\end{aligned}$$

This shows that the endomorphisms $\widehat{J}_\alpha$ act as the standard para-hyper-complex structure on H. Finally, the condition (v) in Definition 8.4 shows that the $\widehat{J}_\alpha$ act also as a para-hyper-complex structure on $E = H^\perp$. □

9 Lorentzian cones

Theorem 9.1. *1. Let (M, g) be a Lorentzian manifold of signature $(+, \cdots, +, -)$ or a negative definite Riemannian manifold and $(\widehat{M} = \mathbb{R}^+ \times M, \widehat{g})$ the cone over M equipped with the Lorentzian metric $\widehat{g} = dr^2 + r^2 g$ (of signature $(+, \cdots, +, -)$ or $(+, -, \cdots, -)$). Suppose that the cone $(\widehat{M}, \widehat{g})$ admits a parallel distribution of isotropic lines. If M is simply connected, then $(\widehat{M}, \widehat{g})$ admits a non-zero parallel light-like vector field.*

2. *Let (M, g) be a negative definite Riemannian manifold and $(\widehat{M}, \widehat{g})$ the cone over M equipped with the Lorentzian metric of signature $(+, -, \cdots, -)$. Suppose that the cone $(\widehat{M}, \widehat{g})$ admits a non-zero parallel light-like vector field, then each point $x \in M$ has a neighbourhood of the form*

$$M_0 = (a, b) \times N, \quad a \in \mathbb{R} \cup \{-\infty\},\ b \in \mathbb{R} \cup \{+\infty\},\ a < b,$$

and for the metric $g|_{M_0}$ we have

$$g|_{M_0} = -ds^2 + e^{-2s} g_N,$$

where (N, g_N) is a negative definite Riemannian manifold. If the holonomy algebra $\mathfrak{hol}(\widehat{M_0}, \widehat{g|_{M_0}})$ of the manifold $(\widehat{M_0}, \widehat{g|_{M_0}})$ is indecomposable, then

$$\mathfrak{hol}(\widehat{M_0}, \widehat{g|_{M_0}}) \cong \mathfrak{hol}(N, g_N) \ltimes \mathbb{R}^{\dim N}.$$

If the manifold (M, g) is complete, then $M_0 = M$, $(a, b) = \mathbb{R}$ and (N, g_N) is complete.

3. *Let (M, g) be a Lorentzian manifold and $(\widehat{M}, \widehat{g})$ the cone over M equipped with the Lorentzian metric. Suppose that the cone $(\widehat{M}, \widehat{g})$ admits a non-zero parallel light-like vector field, then there exist disjoint open subspaces $\{W_i\}_{i \in I} \subset M$ such that the open subspace $\cup_{i \in I} W_i \subset M$ is dense. Any point x of each W_i has an open neighbourhood of the form*

$$U_i = (a, b) \times N \subset W_i, \quad a \in \mathbb{R} \cup \{-\infty\},\ b \in \mathbb{R} \cup \{+\infty\},\ a < b,$$

and for the metric $g|_{U_i}$ we have

$$g|_{U_i} = -ds^2 + e^{-2s} g_N,$$

where (N, g_N) is a Riemannian manifold. If the holonomy algebra $\mathfrak{hol}(\widehat{U_i}, \widehat{g|_{U_i}})$ of the manifold $(\widehat{U_i}, \widehat{g|_{U_i}})$ is indecomposable, then

$$\mathfrak{hol}(\widehat{U_i}, \widehat{g|_{U_i}}) \cong \mathfrak{hol}(N, g_N) \ltimes \mathbb{R}^{\dim N}.$$

If the manifold (M, g) is complete, then $(a, b) = \mathbb{R}$ and $U_i = W_i$.

Proof. 1. Suppose that $(\widehat{M}, \widehat{g})$ admits a parallel distribution of isotropic lines. Then there exists on $\widehat{M}$ a nowhere vanishing recurrent light-like vector field p_1. We have the decomposition

$$p_1 = \alpha \partial_r + Z,$$

where α is a function on $\widehat{M}$ and $Z \in TM \subset T\widehat{M}$. Consider the open subset $U = \{x \in \widehat{M} | \alpha(x) \neq 0\}$. We claim that the subset U is dense in $\widehat{M}$. Indeed, suppose that $\alpha = 0$ on an open subset $V \subset \widehat{M}$, then $p_1 = Z$ on V. Let $Y \in TM$. We have

$$\widehat{\nabla}_Y p_1 = \widehat{\nabla}_Y Z = \nabla_Y Z - rg(Y, Z)\partial_r.$$

Since p_1 is recurrent,

$$\widehat{\nabla}_Y p_1 = \beta(Y) p_1 = \beta(Y) Z,$$

where β is a 1-form on $\widehat{M}$. Hence, $g(Y, Z) = 0$ on V for all $Y \in TM$. Thus, $Z = 0$ and $p_1 = 0$ on V.

Let $y \in U$. We have $\widehat{R}_y(X,Y)\partial_r = 0$ for all $X, Y \in T\widehat{M}$. Hence, $\widehat{R}_y(X,Y)p_1 = \widehat{R}_y(X,Y)Z$. On the other hand, $\widehat{R}_y(X,Y)$ takes values in the holonomy algebra $\mathfrak{hol}_y$ and $\mathfrak{hol}_y$ preserves the line $\mathbb{R}p_{1y}$. Hence, $\widehat{R}_y(X,Y)p_{1y} = C(X,Y)p_{1y} = C(X,Y)(\alpha\partial_r + Z)_y$, where $C(X,Y) \in \mathbb{R}$. Thus, $C(X,Y) = 0$ and $\widehat{R}(X,Y)Z = 0$ on U. Since $U \subset \widehat{M}$ is dense, $\widehat{R}(X,Y)Z = 0$ on $\widehat{M}$ and $\widehat{R}(X,Y)p_1 = 0$ on $\widehat{M}$.

Let $x \in U$ and let $p_x = p_{1x}$. Consider any curve $\gamma(t)$, $t \in [a,b]$ such that $\gamma(a) = x$ and denote by $\tau_\gamma : T_x\widehat{M} \to T_{\gamma(b)}\widehat{M}$ the parallel displacement along γ. For any $X, Y \in T_{\gamma(b)}\widehat{M}$ we have

$$\widehat{R}(X,Y)\tau_\gamma(p_x) = \widehat{R}(X,Y)(cp_{1\gamma(b)}) = 0,$$

where $c \in \mathbb{R}$. From this and the Ambrose-Singer theorem it follows that $\mathfrak{hol}_x$ annihilates the vector p_x. Since $\widehat{M} = \mathbb{R}^+ \times M$ is simply connected, we get a parallel light-like vector field p on $\widehat{M}$. Claim 1 of the theorem is proved.

Now suppose that we have a light-like parallel vector field p on $\widehat{M}$. Consider the decomposition

$$p = \alpha\partial_r + Z,$$

where α is a function on $\widehat{M}$ and $Z \in TM \subset T\widehat{M}$. Note that

$$\widehat{g}(Z,Z) = -\alpha^2, \tag{9.1}$$

and Z is nowhere vanishing. As above we can prove that the open subset $U = \{x \in \widehat{M} | \alpha(x) \neq 0\}$ is dense in $\widehat{M}$.

Lemma 9.1. *Let $Y \in TM \subset T\widehat{M}$. We have*

1. $\partial_r\alpha = 0$, $Y\alpha = rg(Y,Z)$.
2. $\widehat{\nabla}_Y Z = -\frac{\alpha}{r}Y$.
3. $\widehat{\nabla}_{\partial_r} Z = \partial_r Z + \frac{1}{r}Z = 0$, *i.e.* $Z = \frac{1}{r}\tilde{Z}$, *where* $\tilde{Z}$ *is a vector field on* M.
4. $\tilde{Z}\alpha = -\alpha^2$.

Proof. Claims 1-3 follow from the fact that $\widehat{\nabla} p = 0$. Claim 4 follows from (9.1) and Claim 1 of the lemma. □

From Claim 1 of Lemma 9.1 it follows that α can be considered as a function on M and α is constant in the directions orthogonal to the vector field $\tilde{Z}$.

Let $x \in M$, $\alpha(x) \neq 0$ and let $\gamma(t)$ be the curve of the vector field $\tilde{Z}$ passing through the point x. From Claim 1 of Lemma 9.1 it follows that along $\gamma(t)$ we have

$$\alpha = \frac{1}{t+c},$$

where $c \in \mathbb{R}$ is a constant.

2. Suppose that (M, g) is a negative definite Riemannian manifold. In this case the vector field $\tilde{Z}$ is nowhere light-like. From Lemma 9.1 it follows that the gradient of the function α is equal to the vector field $\tilde{Z}$. Hence each point $x \in M$ has an open neighborhood M_0 diffeomorphic to the product $(a, b) \times N$, where N is a manifold diffeomorphic to the level sets of the function $\alpha|_{M_0}$. Note also that the level sets of the function $\alpha|_{M_0}$ are orthogonal to the vector field $\tilde{Z}$. Consequently the metric $g|_{M_0}$ must have the following form

$$g = -ds^2 + g_1,$$

where g_1 is a family depending on the parameter s of Riemannian metrics on the level sets of the function $\alpha|_{M_0}$, and

$$\partial_s = \frac{\tilde{Z}}{\alpha}.$$

From Lemma 9.1 it follows that the function $\alpha|_{M_0}$ satisfies the following differential equation

$$\partial_s \alpha = -\alpha.$$

Hence,

$$\alpha(s) = c_1 e^{-s},$$

where $c_1 \in \mathbb{R}$ is a constant. Changing s, we can assume that $c_1 = \pm 1$. Both cases are similar and we suppose that $c_1 = 1$. Note that $(a, b) = -\ln(\inf \alpha|_{M_0}, \sup \alpha|_{M_0})$.

Let $Y_1, Y_2 \in TM$ be vector fields orthogonal to $\tilde{Z}$ and such that $[Y_1, \partial_s] = [Y_2, \partial_s] = 0$. From Lemma 9.1 it follows that $\nabla_{Y_1} \partial_s = -Y_1$. From the Koszul formula it follows that $2g(\nabla_{Y_1} \partial_s, Y_2) = \partial_s g(Y_1, Y_2)$. Thus we have

$$-2g_1(Y_1, Y_2) = \partial_s g_1(Y_1, Y_2).$$

This means that

$$g_1 = e^{-2s} g_N,$$

where the metric g_N does not depend on s.

Thus we get the decompositions

$$\widehat{M_0} = \mathbb{R}^+ \times (a, b) \times N$$

and

$$\widehat{g|_{M_0}} = dr^2 + r^2(-ds^2 + e^{-2s}g_N).$$

Define the manifold

$$M_1 = \mathbb{R}^+ \times \mathbb{R} \times N$$

and extend the metric $\widehat{g|_{M_0}}$ to the metric g_1 on M_1.

Consider the diffeomorphism

$$\mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \times \mathbb{R}$$

given by

$$(x, y) \mapsto \left(\sqrt{2xy}, \ln\sqrt{\frac{2x}{y}}\right).$$

The inverse diffeomorphism has the form

$$(r, s) \mapsto \left(\frac{r}{2}e^s, re^{-s}\right).$$

We have

$$\begin{aligned} \partial_x &= e^{-s}\partial_r + \tfrac{e^{-s}}{r}\partial_s, \\ \partial_y &= \tfrac{e^s}{2}\partial_r - \tfrac{e^s}{2r}\partial_s. \end{aligned}$$

We get the decomposition

$$M_1 = \mathbb{R}^+ \times \mathbb{R}^+ \times N,$$

and the metric g_1 has the form

$$g_1 = 2dxdy + y^2 g_N.$$

Obviously there exist two intervals $(a_1, b_1), (a_2, b_2) \subset \mathbb{R}^+$ such that $1 \in (a_2, b_2)$ and for $M_2 = (a_1, b_1) \times (a_2, b_2) \times N$ we have $M_2 \subset \widehat{M_0} \subset M_1$. Let $g_2 = \widehat{g}|_{M_2}$.

Applying Theorem 4.2 in [16] to the Lorentzian situation it follows that

$$\mathfrak{hol}(M_1, g_1) = \mathfrak{hol}(M_2, g_2) \cong \mathfrak{hol}(N, g_N) \ltimes \mathbb{R}^{\dim N},$$

where $\mathfrak{hol}(N, g_N)$ is the holonomy algebra of the Riemannian manifold (N, g_N). Thus, $\mathfrak{hol}(\widehat{M_0}, \widehat{g|_{M_0}}) \cong \mathfrak{hol}(N, g_N) \ltimes \mathbb{R}^{\dim N}$.

If the manifold (M, g) is complete, then the global decomposition follows from Proposition 7.1. From Proposition 2.5 it follows that $(a, b) = \mathbb{R}$ and that (N, g_N) is complete. Claim 2 of the theorem is proved.

3. Suppose that (M, g) is a Lorentzian manifold. Consider the open subset $U_1 = \{x \in M | \alpha(x) \neq 0\} \subset M$. Obviously, U_1 is dense in M. Let $\cup_{i \in I} W_i = U_1$ be the representation of the open subset $U_1 \subset M$ as a union of disjoint connected open subsets. At each $x \in W_i$ we have $g_x(\tilde{Z}, \tilde{Z}) \neq 0$. Hence for each W_i we can use that arguments of the proof of Claim 2 of the theorem.

Suppose that (M, g) is complete. As in proof of Claim 2 of the theorem we can show that $U_i = W_i$ for each $i \in I$. From Claim 2 of Lemma 9.1 it follows that the vector field $\frac{\tilde{Z}}{\alpha}$ is a geodesic vector field on U_1. Let $x \in U_1$ and let $\gamma(s)$ be the geodesic such that $\gamma(0) = x$ and $\dot{\gamma}(s) = \frac{\tilde{Z}(\gamma(s))}{\alpha(\gamma(s))}$ if $\gamma(s) \in U_1$. Along the set $\{\gamma(s) | \gamma(s) \in U_1\}$ we have $\alpha(\gamma(s)) = e^{-s}$. Hence, $\gamma(s)$ is defined for all $s \in \mathbb{R}$, $\gamma(\mathbb{R}) \subset U_1$ and $\alpha(\gamma(\mathbb{R})) = \mathbb{R}^+$, i.e. $(a, b) = \mathbb{R}$. $\square$

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D. V. Alekseevsky
The University of Edinburgh and Maxwell Institute for Mathematical Sciences, JCMB, The Kings buildings, Edinburgh, EH9 3JZ, UK, `D.Aleksee@ed.ac.uk`

V. Cortés and T. Leistner
Department Mathematik und Zentrum für Mathematische Physik, Universität Hamburg, Bundesstraße 55, D-20146 Hamburg, Germany, `cortes@math.uni-hamburg.de`, `leistner@math.uni-hamburg.de`

A. S. Galaev
Masaryk University, Faculty of Science, Department of Mathematics, Janáčkovo nám. 2a, 60200 Brno, Czech Republic, `galaev@math.muni.cz`