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THE SPECIAL GEOMETRY OF EUCLIDIAN SUPERSYMMETRY: A SURVEY

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ABSTRACT. This is a survey about recent joint work with Christoph Mayer, Thomas Mohaupt and Frank Saueressig on the special geometry of Euclidian supersymmetry. It is based on the second of two lectures given at the II Workshop in Differential Geometry, La Falda, Córdoba, 2005.

INTRODUCTION

The purpose of this note is to present some geometric structures and constructions which arise in the study of supersymmetric field theories on a Euclidian rather than Minkowskian space-time, see [CMMS1, CMMS2].

The text is written for readers with a background in differential geometry. Nevertheless, a rough idea of supersymmetry and the restrictions it imposes on the geometry of the scalar manifold would be helpful for orientation. A short introduction to classical (rigid) supersymmetric field theories and to the notion of special geometry is provided in another contribution to this volume, based on the first of my two talks given at the II Workshop in Differential Geometry. Such theories are usually considered on the d-dimensional Minkowski space. The case $d = 4$ is particularly important since our observed space-time is four-dimensional.

For various physical reasons, it is useful to consider also field theories on Euclidian space, even if our physical space-time metric is not positive definite. In quantum field theory, for instance, one needs to understand instantons, since they contribute to the Feynman path integral. Instantons are particular solutions of the Euler-Lagrange equations of a Euclidian counterpart of the underlying Minkowskian classical field theory.

For the Euclidian 4-space there exists an $N = 2$ super-Poincaré algebra and Euclidian vector multiplets can be defined [CMMS1]. (There exists no $N = 1$ super-Poincaré algebra on the Euclidian 4-space.) The special geometry of the scalar manifold in supersymmetric field theories with Euclidian vector multiplets was determined in [CMMS1] and named *(affine) special para-Kähler geometry*, since the role of the complex structure J in special Kähler geometry is now played by a para-complex structure J, $J^2 = \mathrm{Id}$. Special para-Kähler manifolds are discussed in section 1.

In section 2 we present several maps, which relate various special geometries associated to field theories in five, four and three dimensions. In particular, we discuss two purely geometric constructions of para-hyper-Kähler manifolds, which correspond to the dimensional reduction of a Euclidian or Minkowskian theory in 4 dimensions to a Euclidian theory in 3 dimensions [CMMS2].

1. Special para-Kähler manifolds

1.1. **Definitions.**

Definition 1. *A* para-Kähler manifold *is a pseudo-Riemannian manifold* (M, g) *endowed with a parallel skew-symmetric involution* $J \in \Gamma(EndTM)$.

A special para-Kähler manifold (M, J, g, ∇) *is a para-Kähler manifold* (M, J, g) *endowed with a flat torsion-free connection* ∇ *satisfying*

(i) $\nabla\omega = 0$, *where* $\omega = g(J\cdot, \cdot)$ *is the symplectic form associated to* (M, J, g) *and*
(ii) $(\nabla_X J)Y = (\nabla_Y J)X, \quad \forall X, Y \in \Gamma(TM)$.

From the definition of a para-Kähler manifold it follows that the eigen-distributions $T^{\pm}M$ of J are isotropic, of the same dimension and integrable.

In particular, $\dim M = 2n$ and g is of split signature (n, n).

Definition 2. *A field of involutions on a manifold* M *with integrable eigen-distributions of the same dimension is called a* para-complex structure.

A manifold endowed with a para-complex structure is called a para-complex manifold.

A map $\phi : (M, J) \to (M', J')$ *between para-complex manifolds is called* para-holomorphic *if* $d\phi \circ J = J' \circ d\phi$.

A para-holomorphic function *is a para-holomorphic map* $f : (M, J) \to C$ *with values in the ring of para-complex numbers* $C = \mathbb{R}[e], e^2 = 1$.

For any $p \in M$ there exists an open neighbourhood U and para-holomorphic functions

$$z^i : U \to C, \quad i = 1, \ldots, n = \frac{\dim M}{2},$$

such that the map $(z^1, \ldots, z^n) : U \to C^n = \mathbb{R}^{2n}$ is a diffeomorphism on its image.

Such a system of para-holomorphic functions is called a system of para-holomorphic coordinates.

1.2. **Extrinsic construction of special para-Kähler manifolds.**

Consider the free C-module $V = C^{2n}$ with its global linear para-holomorphic coordinates (z^i, w_i), its standard para-holomorphic symplectic form

$$\Omega = \sum dz^i \wedge dw_i$$

and the standard anti-linear involution $\tau : V \to V$ with the set of fixed points $V^\tau = \mathbb{R}^{2n}$. We define a constant para-Kähler metric by

$$g_V(X, Y) := \mathrm{Re}(e\Omega(X, \tau Y)), \quad X, Y \in V.$$

Definition 3. *Let* (M, J) *be a para-complex manifold of real dimension* $2n$. *A para-holomorphic immersion* $\phi : M \to V = C^{2n}$ *is called* para-Kählerian *(respectively,* Lagrangian*) if* $\phi^* g_V$ *is non-degenerate (respectively, if* $\phi^*\Omega = 0$*).*

It is easy to see that the metric $g = \phi^* g_V$ induced by a para-Kählerian immersion is para-Kählerian. In the following, we will abbreviate para-Kählerian Lagrangian immersion to PKLI.

Lemma 1. *Let $\phi : M \to V$ be a PKLI and $\omega = g(J\cdot, \cdot)$ the corresponding symplectic structure. Then $\omega = 2\sum d\tilde{x}^i \wedge d\tilde{y}_i$, where $\tilde{x}^i = x^i \circ \phi$, $\tilde{y}_i = y_i \circ \phi$, $x^i = Re\, z^i$, $y^i = Re\, w^i$.*

By the lemma $(\tilde{x}^i, \tilde{y}_i)$ defines a system of local coordinates. Therefore, there exists a unique flat and torsion-free connection ∇ on M for which $\tilde{x}^i$ and $\tilde{y}_i$ are affine functions.

Theorem 1. [CMMS1] *Let $\phi : M \to V$ be a PKLI with induced data (J, g, ∇). Then (M, J, g, ∇) is a special para-Kähler manifold.*

Conversely, any simply connected special para-Kähler manifold (M, J, g, ∇) admits a PKLI $\phi : M \to V$ with induced data (J, g, ∇). Moreover, the PLKI ϕ is unique up to an element of $Aff_{Sp(\mathbb{R}^{2n})}(V)$.

Proof of "$\Rightarrow$". Let $\phi : M \to V$ be a PKLI with induced data (J, g, ∇). We have to show that (M, J, g, ∇) is special para-Kähler. We know that (M, J, g) is para-Kähler and that ∇ is flat and torsion-free. By the lemma, the symplectic form ω has constants coefficients with respect to the ∇-affine coordinates $(\tilde{x}^i, \tilde{y}_i)$. Thus $\nabla\omega = 0$. It remains to show that ∇J is symmetric. For a ∇-parallel one-form ξ and vector fields X, Y on M we calculate:

$$\begin{aligned} d(\xi \circ J)(X, Y) \quad &\underset{T^{\nabla}=0}{=} \quad \nabla_X(\xi \circ J)Y - \nabla_Y(\xi \circ J)X \\ &= \quad \xi(\nabla_X(J)Y - \nabla_Y(J)X). \end{aligned}$$

Therefore, it is sufficient to prove that $\xi \circ J$ is closed for $\xi = d\tilde{x}^i$ and $\xi = d\tilde{y}_i$. Let us check this, for example, for $\xi = d\tilde{x}^i$. The function $\tilde{x}^i$ is the real-part of the para-holomorphic function $\tilde{z}^i = z^i \circ \phi$. So $d\tilde{z}^i = d\tilde{x}^i + e d\tilde{x}^i \circ J$. Since $d\tilde{x}^i$ and $d\tilde{z}^i$ are closed, this shows that $d\tilde{x}^i \circ J$ is closed. □

Corollary 1. *Let $F : U \to C$ be a para-holomorphic function defined on a open set $U \subset C^n$ satisfying the non-degeneracy condition det $Im \frac{\partial^2}{\partial z^i \partial z^j} F \neq 0$.*

Then $\phi_F = dF : U \to C^{2n}$,

$$z = (z^1, \ldots, z^n) \mapsto (z, \frac{\partial F}{\partial z^1}(z), \ldots, \frac{\partial F}{\partial z^n}(z)),$$

is a PKLI and hence defines a special para-Kähler manifold M_F.

Conversely, any special para-Kähler manifold is locally of this form.

2. Maps between special geometries from dimensional reduction

Dimensional reduction is a procedure for the construction of a field theory in d space-time dimensions from one in $d + 1$ dimensions.

In the context of special geometry of Euclidian supersymmetry it is natural to ask the following two questions:

(i) Is it possible to construct $N = 2$ supersymmetric field theories with vector multiplets on 4-dimensional Euclidian space from field theories on 5-dimensional Minkowski space?
(ii) Is it possible to construct Euclidian supersymmetric field theories in 3 dimensions out of $N = 2$ supersymmetric field theories with vector multiplets in 4 dimensions?

The first question is given a detailed positive answers in [CMMS1], the second in [CMMS2]. We describe the corresponding geometrical constructions in the remaining two subsections.

2.1. **Dimensional reduction from 5 to 4 dimensions.** The allowed target geometry for the scalar fields in the relevant supersymmetric theories on 5-dimensional Minkowski space is called *(affine) very special*, see [CMMS1].

It is defined by a real cubic polynomial $h(x^1, \ldots, x^n)$ with non-degenerate Hessian $\partial^2 h$ on some domain $U \subset \mathbb{R}^n$.

We found that dimensional reduction of such a Minkowskian theory over time yields a Euclidian $N = 2$ supersymmetric theory with vector multiplets such that the target is special para-Kähler [CMMS1]. This means that we get a map:

$$\{\text{very special manifolds}\} \stackrel{r^{4+1}_{4+0}}{\longrightarrow} \{\text{special para-Kähler manifolds}\},$$

which we call the *para-r-map*.

Theorem 2. [CMMS1] *There exists a map r^{4+1}_{4+0} which associates a special para-Kähler structure on the domain $\widetilde{U} = U + e\mathbb{R}^n \subset C^n$ to any very special manifold $(U, \partial^2 h)$, $U \subset \mathbb{R}^n$. The special para-Kähler structure is defined by the para-holomorphic function*

$$F : \widetilde{U} \to C, \quad F(z^1, \ldots z^n) := \frac{1}{2e} h(z^1, \ldots, z^n),$$

which satisfies det $Im\,\partial^2 F \neq 0$.

This is the para-version of the *r-map*:

$$\{\text{very special manifolds}\} \stackrel{r^{4+1}_{3+1}}{\longrightarrow} \{\text{special pseudo-Kähler manifolds}\},$$

introduced by B. de Wit and A. Van Proeyen in [DV] in the context of supergravity.

2.2. **Dimensional reduction from 4 to 3 dimensions.** We found two ways of constructing Euclidian supersymmetric field theories in 3 dimensions out of $N = 2$ theories with vector multiplets in 4 dimensions [CMMS2].

One can start either with a Minkowskian theory and reduce over time or with a Euclidian theory. This gives us two maps:

$$\{\text{special pseudo-Kähler manifolds}\} \stackrel{c^{3+1}_{3+0}}{\longrightarrow} \{\text{para-hyper-Kähler manifolds}\},$$

$$\{\text{special para-Kähler manifolds}\} \stackrel{c^{4+0}_{3+0}}{\longrightarrow} \{\text{para-hyper-Kähler manifolds}\},$$

which we call the *para-c-maps*. They are para-variants of the *c-map*, worked out by Cecotti, Ferrara and Girardello in [CFG].

Para-hyper-Kähler manifolds.

Definition 4. *A* para-hyper-Kähler manifold *is a pseudo-Riemannian manifold with three pairwise anticommuting parallel skew-symmetric endomorphism fields* J_1, J_2, $J_3 = J_1 J_2$, *such that* $J_1^2 = J_2^2 = -J_3^2 = Id$.

A pseudo-Riemannian manifold is para-hyper-Kähler if and only if its holonomy group

$$\begin{aligned} \mathrm{Hol} \quad &\subset \quad Sp(\mathbb{R}^{2n}) = Id_{\mathbb{R}^2} \otimes Sp(\mathbb{R}^{2n}) \\ &\subset \quad SO(\mathbb{R}^2 \otimes \mathbb{R}^{2n}, \omega_{\mathbb{R}^2} \otimes \omega_{\mathbb{R}^{2n}}) = SO(2n, 2n). \end{aligned}$$

Here $\omega_{\mathbb{R}^{2n}}$ stands for the standard symplectic structure of $\mathbb{R}^{2n}$. In particular, the dimension of any para-hyper-Kähler manifold is divisible by 4.

The para-c-maps. Now I describe, for instance, the para-hyper-Kähler manifold associated to a special para-Kähler manifold (M, J, g, ∇) via the para-c-map c^{4+0}_{3+0}.

Let $N = T^*M$ be the total space of the cotangent bundle $\pi : N \to M$ and consider the decomposition $T_\xi N = \mathcal{H}^\nabla_\xi \oplus T^v_\xi N$, $\xi \in N$, into horizontal and vertical subbundles with respect to the connection ∇.

This defines a canonical identification

$$T_\xi N \cong T_x M \oplus T^*_x M, \quad x = \pi(\xi).$$

With respect to the above identification, we define a pseudo-Riemannian metric g_N on N by

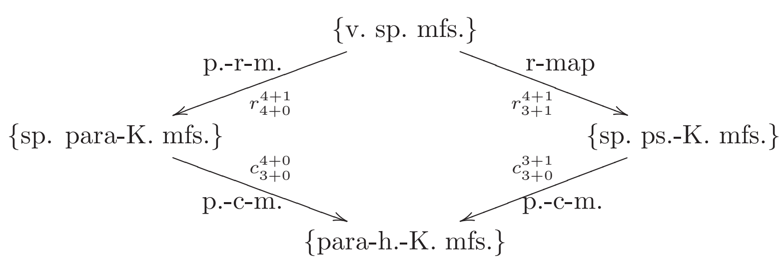
$$g_N := \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}$$

and two involutions J_1, J_2 by

$$J_1 := \begin{pmatrix} J & 0 \\ 0 & J^* \end{pmatrix} \text{ and } J_2 := \begin{pmatrix} 0 & \omega^{-1} \\ \omega & 0 \end{pmatrix}.$$

Theorem 3. [CMMS2] *For any special para-Kähler manifold* (M, J, g, ∇), $(N, g_N, J_1, J_2, J_3 = J_1 J_2)$ *is a para-hyper-Kähler manifold.*

Conclusion. The maps between special geometries induced by dimensional reduction are summarized in the following diagram:



The diagram is essentially commutative:

Theorem 4. [CMMS2] *For any very special manifold* $L = (U, \partial^2 h)$ *the para-hyper-Kähler manifolds* $c^{4+0}_{3+0} \circ r^{4+1}_{4+0}(L)$ *and* $c^{3+1}_{3+0} \circ r^{4+1}_{3+1}(L)$ *are canonically isometric.*

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