AN END DEGREE FOR DIGRAPHS

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Abstract. In this paper we define a degree for ends of infinite digraphs. The well-definedness of our definition in particular resolves a problem by Zuther. Furthermore, we extend our notion of end degree to also respect, among others, the vertices dominating the end, which we denote as combined end degree. Our main result is a characterisation of the combined end degree in terms of certain sequences of vertices, which we call end-exhausting sequences. This establishes a similar, although more complex relationship as known for the combined end degree and end-defining sequences in undirected graphs.

§1. Introduction

The notion of ends became crucial for analysing the structure of infinite graphs. An *end* of a graph is an equivalence class of one-way infinite paths, where two such paths are called *equivalent* if they are joined by infinitely many disjoint paths. Degree parameters were defined for ends as well, see e.g. [2, 7], where the basic definition is as follows. The *degree* of an end ω of a graph is the supremum of the number of disjoint one-way infinite paths in ω. It is a non-trivial theorem by Halin [6] that the supremum in this definition is actually an attained maximum. End degrees turned out to be useful parameters for infinite graphs, e.g. for characterising a topological notion of infinite cycles [2], or when studying extremal questions regarding the existence of infinite grid-like subgraphs [7].

A different way to describe the degree of an end is by certain sequences of nested finite vertex separators, so-called *defining sequences*. It was shown in [4] that one can characterise the degree of an end together with the number of vertices dominating it, also referred to as *combined end degree*, via the sizes of the separators within defining sequences. Here, a vertex v is said to *dominate* an end ω if there exist infinitely many paths from v to a one-way infinite path in ω which are all disjoint except from v.

In this paper we shall consider directed graphs, briefly denoted as *digraphs*, which are infinite, and we shall define analogous concepts of end degrees as mentioned above for

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undirected graphs. For this we follow a notion of ends of digraphs defined by Zuther [8,9], which is a natural and analog definition to the one for undirected graphs. An *end* of a digraph is defined as an equivalence class of rays and anti-rays, where a (*anti-*)*ray* is an orientation of a one-way infinite path such that each edge is oriented towards (resp. away from) infinity (see Section 2 for a precise definition). Zuther called two rays or anti-rays R_1, R_2 of a digraph D *equivalent* if there exist infinitely pairwise disjoint directed paths from R_1 to R_2 and vice versa. Note that this definition allows that e.g. R_1 is a ray and R_2 is an anti-ray.

The first result of this paper, Theorem 3.1, resolves a problem stated by Zuther [8, Problem 2] and proves that an end of a digraph which contains any finite number of disjoint rays also contains infinitely many disjoint rays. Hence, Theorem 3.1 is an analogous result to the aforementioned theorem by Halin [6], and allows us to define the *in-degree* (resp. *out-degree*) of an end as the maximum number of disjoint rays (resp. anti-rays) in that end.

The natural question arises whether an end with infinite in- and out-degree might admit a system of pairwise disjoint rays and anti-rays witnessing both of these degrees. By Theorem 4.1 we answer this question negatively and construct a digraph with infinite in- and out-degree where each ray intersects each anti-ray.

The main contribution of this paper is the introduction of *end-exhausting sequences*, a concept for ends of digraphs similar to defining sequences for ends of undirected graphs. Similarly, although more complex as for undirected graphs, we define a *combined in-degree* (and *out-degree*) for ends, and prove an equality to a parameter solely based on end-exhausting sequences in the following main result of this paper:

Theorem 1.1. *Let D be a digraph and let ω be an end of D that contains at least one but at most countably many rays. Then the combined in-degree of ω is the same as*

$$\inf\left\{\liminf_{i\in\mathbb{N}} |U_i| \;\middle|\; (U_i)_{i\in\mathbb{N}} \text{ is an } \omega\text{-exhausting sequence}\right\}.$$

Qualitatively, Theorem 1.1 establishes the same duality between end-exhausting sequences and combined in-degrees (or out-degrees) of ends for digraphs as it is known for end-defining sequences and the combined end degree in the undirected case.

The structure of this paper is as follows. After introducing some terminology in Section 2, we prove in Section 3 that the in- and out-degree of an end is well-defined. In Section 4 we construct a digraph containing infinitely many disjoint rays and infinitely many disjoint anti-rays such that each ray intersects each anti-ray. In Section 5 we define end-exhausting sequences and the combined in- and out-degree of ends, followed by the proof of Theorem 1.1. Finally, we briefly discuss in Section 6 how the results from Section 3 and Section 4 can be proved when edge-disjoint rays (and anti-rays) are considered instead of vertex-disjoint ones.

§2. Preliminaries

For general facts and notation regarding graphs we refer the reader to [3], regarding digraphs in particular to [1].

We call a digraph D *weak* if it is weakly connected. For the sake of brevity we call a directed cycle just a *dicycle* and a directed path just a *dipath*. Given a dipath P containing two vertices a, b such that b is reached from a via P, we define aPb as the subdipath of P starting at a and ending in b. Given two vertex sets A, B, we call a dipath P an A–B *dipath* if P starts in A, ends in B and is internally disjoint from $A \cup B$.

We call a weak digraph where each vertex has in- and out-degree 1 except one vertex v which has in-degree (resp. out-degree) 0 and out-degree (resp. in-degree) 1 a *ray* (resp. *anti-ray*). The vertex v is called the *starting vertex* (resp. *end vertex*) of the ray (resp. anti-ray). We say that a ray *starts* in a vertex set A if it has its starting vertex in A. Given a ray R with starting vertex v and some $x \in V(R)$ we denote by Rx the subdipath vRx of R. A *tail* of R is a subray of R. If this tail starts at x, then we denote it by xR. Similarly, for an anti-ray Q with end vertex v and some $x \in V(Q)$, we denote by xQ the subdipath xQv of Q and by Qx the subanti-ray of Q that ends at x, which we will also call a *tail* of Q. We call a weak digraph where each vertex has in- and out-degree 1 a *double ray*. The *tails* of a double ray are its subdigraphs that are rays or anti-rays.

Let Q and R be rays or anti-rays. We write $Q \leqslant R$ if there are infinitely many pairwise disjoint Q–R dipaths and we write $Q \sim R$ if $Q \leqslant R$ and $R \leqslant Q$. Then $\leqslant$ is a preorder on the set of rays and anti-rays in a digraph D and $\sim$ is an equivalence relation on that set. Note that $\leqslant$ is not anti-symmetric as witnessed by a ray and one of its proper tails. The equivalence classes of $\sim$ are the *ends* of D and we can extend the relation $\leqslant$ to the ends: we write $\eta \leqslant \omega$ for ends η and ω if there are $Q \in \eta$ and $R \in \omega$ with $Q \leqslant R$. Note that $\eta \leqslant \omega$ if and only if $Q \leqslant R$ for every $Q \in \eta$ and $R \in \omega$. In particular, we have $\eta \leqslant \omega$ and $\omega \leqslant \eta$ if and only if $\eta = \omega$. So, $\leqslant$ is a partial order on the ends of a digraph.

We call an oriented tree A that contains a vertex x such that each vertex $y \in V(A)$ is reachable from x (resp. reaches x) in A by a dipath, an *out-arborescence* (resp. *in-arborescence*). The vertex x is called the *root* of A. An out-arborescence (resp. in-arborescence) S with root c whose underlying tree is a star is called an *out-star* (resp. *in-star*) with *centre* c.

An undirected tree C is called a *comb* if it is obtained from a system $\mathcal{P}$ of infinitely many pairwise disjoint finite paths and a one-way infinite path R by gluing one end vertex of each path to a vertex on R such that different paths are glued to different vertices on R. The one-way infinite path R naturally exists as a subgraph in C and is called the *spine* of C. All those end vertices of paths in $\mathcal{P}$ that do not lie on R and those that belong to trivial paths (i.e. paths that consist of just one vertex) are called the *teeth* of C. An orientation of a comb is called an *out-comb* (resp. *in-comb*) if the spine of the comb is oriented as a ray (resp. anti-ray) and each path of $\mathcal{P}$ is oriented as a dipath directed away from (resp. towards) R.

We shall need the following analog for digraphs of the Star-Comb Lemma [3, Lemma 8.2.2] for undirected graphs. The proof is very similar to the undirected version, but since it is short, we include it for the sake of completeness.

Lemma 2.1 (Star-Comb Lemma)**.** *Let D be a digraph and let $x \in V(D)$ and $U \subseteq V(D)$ be infinite such that there exists an x–u dipath for every $u \in U$. Then there exists either an out-comb with all its teeth in U or a subdivided infinite out-star with all its leaves in U.*

Proof. By Zorn's lemma, there exists a maximal out-arborescence T containing the vertices of U such that for every vertex t of $V(T)$ there exists an x–t dipath in T and such that every vertex of T without out-neighbour lies in U. Since U is infinite, T is infinite as well. If T has a vertex of infinite out-degree, then it contains a subdivided infinite out-star as out-arborescence with its centre as root and its leaves in U. So let us assume that all vertices of T have finite out-degree. Then there exists a ray in T starting at x, which we denote by R. In order to construct infinitely many pairwise disjoint R–U dipaths in T, let us assume that we have already constructed $P_0, \ldots, P_{n-1}$ such that the starting vertex of P_i lies before that of P_j on R for $i < j$. Let v be the starting vertex of P_{n-1} and let w be its out-neighbour on R. Then the edge vw lies on an x–u dipath in T for some $u \in U$. This dipath contains a maximal R–U dipath P_n, which is disjoint to all P_i with $i < n$. Thus, we obtain infinitely many pairwise disjoint R–U dipaths. Then R together with all dipaths P_i for $i \in \mathbb{N}$ is an out-arborescence with root x, that is an out-comb. $\square$

§3. End degree

Zuther [8, Theorem 2.17], see also Gut et al. [5], proved that every digraph that contains an arbitrarily large finite number of pairwise disjoint rays contains countably infinitely many pairwise disjoint rays. Zuther posed the problem [8, Problem 2] whether this also holds when we ask all rays to lie in a common end. We settle his problem in the positive.

Theorem 3.1. *Let D be a digraph.*

(i) *If an end of D contains n pairwise disjoint rays for all $n \in \mathbb{N}$, then it contains countably infinitely many pairwise disjoint rays.*

(ii) *If an end of D contains n pairwise disjoint anti-rays for all $n \in \mathbb{N}$, then it contains countably infinitely many pairwise disjoint anti-rays.*

Proof. It suffices to prove (i), since (ii) follows by applying (i) to the digraph with all edge directions reversed.

Let ω be an end of a fixed digraph D such that for all $n \in \mathbb{N}$ there are n pairwise disjoint rays in ω and let R be a ray in ω. For all $n \in \mathbb{N}$, we will recursively construct a set $\mathcal{R}^n = \{R_1^n, \ldots, R_n^n\}$ of n pairwise disjoint rays, a set $X^n := \{x_1^n, \ldots, x_n^n\}$ of n vertices, and a set $\mathcal{P}^n$ of $2n$ dipaths, such that the following hold for all $n \geqslant 1$:

(1) $\mathcal{R}^n \subseteq \omega$;
(2) x_i^n lies on R_i^n for all $1 \leqslant i \leqslant n$;
(3) $R_i^{n-1} x_i^{n-1}$ is a proper starting subdipath of $R_i^n x_i^n$ for all $1 \leqslant i \leqslant n-1$;
(4) $\mathcal{P}^n$ contains a dipath from R to $x_i^{n-1} R_i^n x_i^n$ and a dipath from $x_i^{n-1} R_i^n x_i^n$ to R for all $1 \leqslant i \leqslant n-1$ that avoid $\bigcup_{j<n} \mathcal{P}^j$, where x_i^0 denotes the starting vertex of R_i.

Let $\mathcal{R}^1$ be a set consisting of a single ray R_1^1 in ω, let $\mathcal{P}^1$ consist of an R–R_1^1 dipath and an R_1^1–R dipath and let X^1 consist of a vertex of R_1^1 that lies after all vertices of dipaths in $\mathcal{P}^1$ on R_1^1. By these definitions, (1)–(4) are satisfied for $n = 1$. Let us now assume that we have already constructed $\mathcal{R}^n$, X^n and $\mathcal{P}^n$. See Figure 3.1 for a rough overview of the construction step from $n = 2$ to 3.

Let X be the set of vertices on the dipaths $R_i^n x_i^n$ for $1 \leqslant i \leqslant n$ and let $\mathcal{Q} \subseteq \omega$ be a set of $n+1$ pairwise disjoint rays. Let $Q_1, \ldots, Q_{n+1}$ be tails of the elements of $\mathcal{Q}$ that avoid X. For all $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n+1$, let $P_{i,j}^1, \ldots, P_{i,j}^n$ be n pairwise disjoint $x_i^n R_i^n$–Q_j dipaths that avoid $X \smallsetminus X^n$, which exist since all considered rays lie in a common end. For all $1 \leqslant i \leqslant n$ let h_i denote the last vertex on R_i^n that lies on any of the dipaths $P_{i,j}^k$. Now, for all $1 \leqslant \ell \leqslant n+1$, let y_ℓ be a vertex on Q_ℓ that lies after all vertices on dipaths $P_{i,j}^k$ on Q_ℓ and after all vertices on segments $x_i^n R_i^n h_i$.

Let D' be the finite subdigraph of D induced by all dipaths $x_i^n R_i^n z$, where z is a starting vertex of some $P_{i,j}^k$, by all dipaths $P_{i,j}^k$, and by all dipaths $z' Q_j y_j$, where z' is an end vertex of some dipath $P_{i,j}^k$. Let S be a set of fewer than n vertices in D'. Then S avoids at least one ray $x_i^n R_i^n$, at least one Q_j and at least one $P_{i,j}^k$, that is, we find an x_i^n–y_j dipath. Menger's theorem implies that there are n disjoint dipaths from X^n to $\{y_1, \ldots, y_{n+1}\}$. We may assume that the indices are such that we find dipaths P_i from x_i^n to y_i for all $1 \leqslant i \leqslant n$. We set $R_i^{n+1} := R_i^n x_i^n P_i y_i Q_i$ for all $1 \leqslant i \leqslant n$ and choose R_{n+1}^{n+1} as a tail of Q_{n+1} which is disjoint from all R_i^{n+1}. Finally, we set $\mathcal{R}^{n+1} := \{R_i^{n+1} \mid 1 \leqslant i \leqslant n+1\}$. By construction, (1) holds for $\mathcal{R}^{n+1}$. Let $\mathcal{P}^{n+1}$ be a set of dipaths, one from R to R_i^{n+1} and one from R_i^{n+1} to R for all $1 \leqslant i \leqslant n$, such that all these dipaths avoid the vertices in $\bigcup_{j \leqslant n} \mathcal{P}^j$. Note that these dipaths exist as all rays lie in a common end. For $1 \leqslant i \leqslant n+1$, let x_i^{n+1} be a vertex on R_i^{n+1} after all vertices of elements of $\mathcal{P}^{n+1}$. In particular, x_i^{n+1} lies after x_i^n on R_i^{n+1}. Then, (2)–(4) hold by construction.

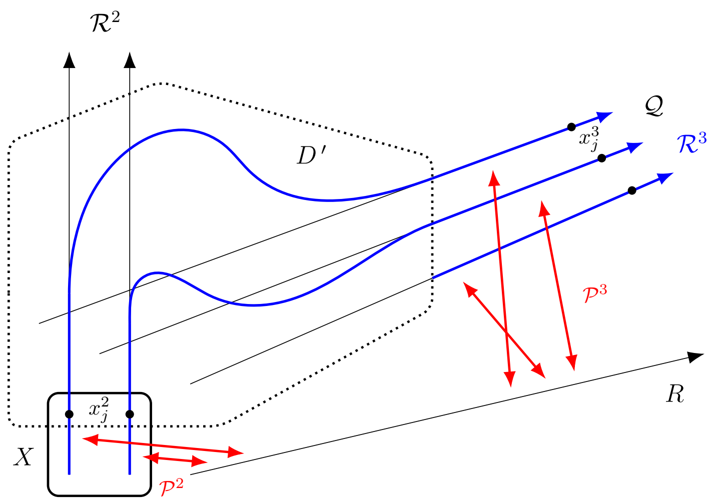


FIGURE 3.1. The step $n = 2$ in the construction for Theorem 3.1.

Thus $\{\bigcup R_i^n x_i^n \mid n \in \mathbb{N}, 1 \leqslant i \leqslant n\}$ is an infinite set of pairwise disjoint rays that are all equivalent to R, and hence lie in ω. ☐

For higher cardinalities, we note the following remark:

Corollary 3.2. *Let D be a digraph, let ω be an end of D and let $\mathcal{R}$ (resp. $\mathcal{A}$) be the set of all sets of disjoint rays (resp. anti-rays) in ω. Then $\sup_{M \in \mathcal{R}} |M| = \max_{M \in \mathcal{R}} |M|$ and $\sup_{M \in \mathcal{A}} |M| = \max_{M \in \mathcal{A}} |M|$.*

Proof. It suffices to prove the corollary for rays as we can obtain the result for anti-rays by applying the result for rays to the digraph with all edge directions reversed. Let $\kappa = \sup_{M \in \mathcal{R}} |M|$. Then we can find a sequence $(M_\alpha)_{\alpha \leqslant \lambda}$ for some $\lambda \leqslant \kappa$ of elements of $\mathcal{R}$ such that $\kappa = \sup_{\alpha \leqslant \lambda} |M_\alpha|$ and $|M_\alpha| < |M_\beta|$ for all $\alpha < \beta$. Due to Theorem 3.1 we may assume that no M_α is a finite set. Now we greedily construct a set $M^* \in \mathcal{R}$ such that $|M^*| = \kappa$, which will complete our proof. For that, we will construct a sequence $(M'_\alpha)_{\alpha \leqslant \lambda}$ with $M'_\alpha \subseteq M'_\beta$ for all $\alpha \leqslant \beta \leqslant \lambda$ and such that $|M'_\lambda| = \kappa$. We start with $M'_1 := M_1$. Then there are only $|M_1| \cdot \aleph_0 = |M_1|$ elements of M_2 that contain vertices from elements of M_1. In particular, we can unite M_1 with $|M_2|$ many elements of M_2 that are disjoint from the rays of M_1 and obtain a set M'_2. We continue transfinitely and follow the same approach for successor ordinals $\alpha + 1$: only $|M_\alpha|$ many rays from $M_{\alpha+1}$ contain vertices from the rays of M'_α. So there are $|M_{\alpha+1}|$ rays in $M_{\alpha+1}$ disjoint from the elements of M'_α that we add to M'_α to obtain $M'_{\alpha+1}$. Say, for any limit ordinal β, we already constructed sets M'_α

for every $\alpha < \beta$. Then we set $M''_\beta := \bigcup_{\alpha<\beta} M'_\alpha$, which is by construction a set of disjoint rays in ω. Since $M''_\beta \in \mathcal{R}$, the set M_β must exist. Furthermore, $|M_\beta| \geqslant |M''_\beta|$ holds. If $|M_\beta| = |M''_\beta|$, we set $M'_\beta := M''_\beta$. Otherwise, we add, as in the successor ordinal case, $|M_\beta|$ many ray from M_β which are disjoint to the elements of M''_β to M''_β in order to form M'_β. With this we can now continue transfinitely, but as our construction stops after at most λ many steps, the supremum is an attained maximum. $\square$

The *in-degree* of an end ω, denoted by $d^-(\omega)$, is the maximum number of pairwise disjoint rays in that end, which is well-defined due to Corollary 3.2. Analogously, we define the *out-degree*, denoted by $d^+(\omega)$, with respect to anti-rays.

§4. Example

In this section, we will discuss an example of a digraph with infinitely many pairwise disjoint rays and infinitely many pairwise disjoint anti-rays such that every ray and every anti-ray share a vertex.

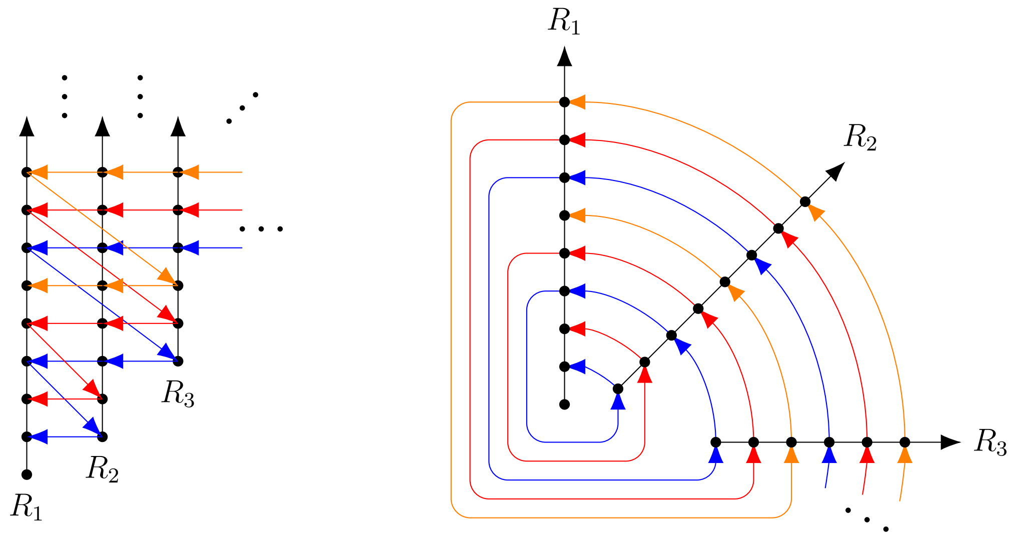
Theorem 4.1. *There exists a digraph D with the following properties:*

(i) *D contains infinitely many pairwise disjoint rays.*
(ii) *D contains infinitely many pairwise disjoint anti-rays.*
(iii) *Every ray and every anti-ray of D share a vertex.*

Proof. For all $i \in \mathbb{N}$ and $i' := (\sum_{j=1}^{i-1} j) + 1$, let $R_i = x^i_{i'} x^i_{i'+1} \dots$ be a ray. Let D be the union of all rays R_i with additional edges $x^{i+1}_k x^i_k$ for all $i \in \mathbb{N}$ and all feasible k and with additional edges from x^1_k with $k = (\sum_{j=1}^{i} j) + k'$ to $x^i_{i''}$ with $i'' = (\sum_{j=1}^{i-1} j) + k'$, for every $k' \in \{1, \dots, i\}$. We call these latter edges *diagonal*. Note that this digraph is planar. See Figure 4.1 (a) for one picture of D and Figure 4.1 (b) for a planar drawing of D.

We note that (i) and (ii) are trivially true. So let Q be a ray and P be an anti-ray in D. Moving on P along the edges in opposite direction away from the end vertex of P, we must meet R_1 after finitely many edges of the form $x^{m+1}_n x^m_n$ or $x^m_n x^m_{n+1}$. This implies that P meets R_1 infinitely many times. Thus, P must also meet R_2 infinitely often, and so on. So P meets each R_i infinitely often. Thus, if Q and some R_i have a common tail, then Q and P must have infinitely many common vertices.

So let us assume that Q has no common tail with any R_i. Note that Q must leave R_i towards R_{i-1}, and R_{i-1} towards R_{i-2}, and so on until it meets R_1. Then Q must use a diagonal edge to some R_k with $k \geqslant i$ when leaving R_1. Afterwards, Q must again traverse all $R_{k'}$ with $k' \leqslant k$. Note that for each ray R_i there are only finitely many diagonal edges that are directed towards that ray. So eventually, Q must use a diagonal edge to some R_k with $k > i$ when leaving R_1. Hence, we obtain that Q meets all R_i infinitely often. Note,

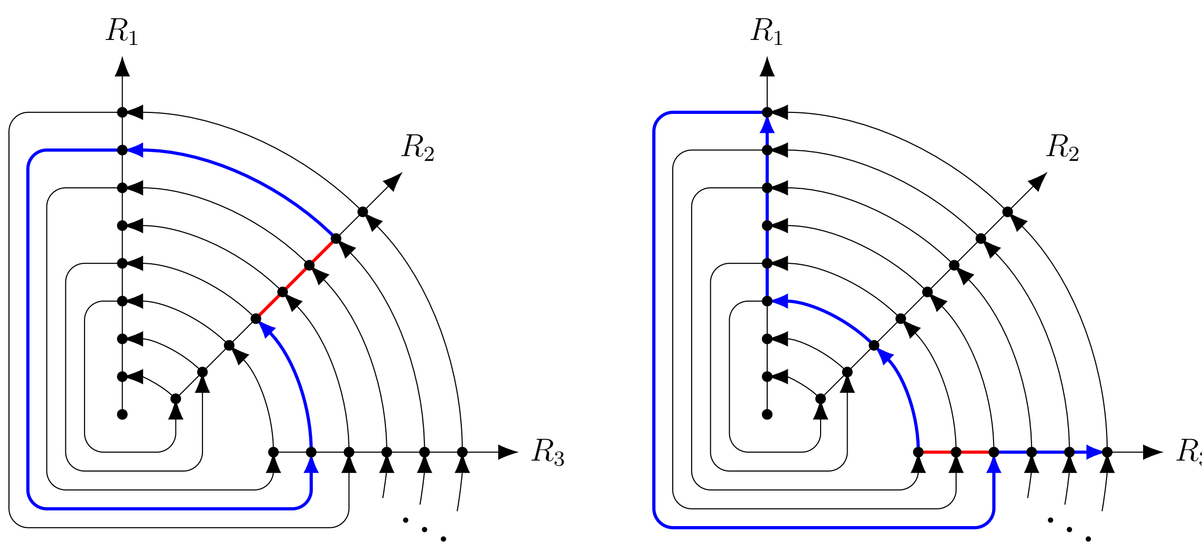


(a) The digraph D.

(b) A planar drawing of the digraph D.

FIGURE 4.1. Two drawings of the digraph D.

furthermore, that due to planarity it is not possible for Q to traverse a ray R_i first through some vertex r_n^i and later through some vertex r_m^i for $m < n$ as the r_n^i–R_1 subdipath of Q together with the R_1–r_m^i subdipath of Q would cause $r_m^i Q$ to intersect $Q r_m^i$ in another vertex than r_m^i, which is impossible, cf. Figure 4.2 (a).



(a) The blue dipath cannot be extended to a ray Q since the finite face bounded by the red dipath and itself cannot be left.

(b) No anti-ray P disjoint to the blue ray Q can enter the finite face bounded by the red dipath and a finite subpath of Q.

FIGURE 4.2. Two cases from the proof of Theorem 4.1.

This implies that there exists $x^j_{\ell_1}$ and $x^j_{\ell_2}$ on some R_j and on Q with $\ell_1 < \ell_2$ such that $x^j_{\ell_1}R_jx^j_{\ell_2}$ does not meet Q and that there exists x^j_ℓ on P such that $\ell < \ell_1$, but there is no $x^j_{\ell*}$ on P with $\ell < \ell^* < \ell_1$. Hence, P must use a diagonal edge $x^1_k x^{j'}_{\ell'}$ after the vertex x^j_ℓ with $\ell' \leqslant \ell$ and $j' \geqslant j$ for some $k \in \mathbb{N}$. Since $\ell < \ell_1$, the dipath $x^j_{\ell_1}Qx^j_{\ell_2}$ must also use a diagonal edge $x^1_{k'}x^{j''}_{\ell''}$ with $k' > k$ for some $j'', \ell'' \in \mathbb{N}$. Now it follows that x^j_ℓ lies in the interior of the face bounded by $x^j_{\ell_1}Qx^j_{\ell_2} \cup x^j_{\ell_1}R_jx^j_{\ell_2}$. Since P intersects R_j again after x^j_ℓ, but uses the diagonal edge $x^1_k x^{j'}_{\ell'}$, we get that P must intersect the subdipath $x^j_{\ell_1}Qx^1_{k'}$ of Q, cf. Figure 4.2 (b), which is a contradiction. □

This leaves the following problem open.

Problem 4.2. *If a digraph D has an end ω such that D contains for all $n \in \mathbb{N}$ a set of n rays and n anti-rays that are pairwise disjoint and lie in ω, does there exist a set of infinitely many rays and infinitely many anti-rays in ω that are pairwise disjoint?*

Note that the proof method used in Theorem 3.1 does not work here since we would need to find two disjoint dipath systems, one for rerouting our initial segments of rays, and one for our endsegments of anti-rays. This, however, is not guaranteed by an application of Menger's theorem as done before.

The previous problem is also motivated by Proposition 4.4 and the following problem by Gut et al. [5], as we shall see below.

Problem 4.3. [5, Problem 1.3] *Is the double ray ubiquitous?**

Proposition 4.4. *For every $n \in \mathbb{N}$, if a digraph D contains a set of n rays and n anti-rays that are all pairwise disjoint and lie in the same end, then there exists a set of n pairwise disjoint double rays in D all of whose tails lie in that end.*

Proof. Let $R_1, \dots, R_n$ be rays and $Q_1, \dots, Q_n$ be anti-rays all in the same end ω of the digraph D and all pairwise disjoint. Let $\mathcal{P}$ be a set of pairwise disjoint dipaths that consists of n many Q_i–R_j dipaths for all $1 \leqslant i, j \leqslant n$. This is possible to choose since all rays and anti-rays lie in ω. For all $1 \leqslant i \leqslant n$, let x_i be a vertex on Q_i such that Q_ix_i contains no vertex from any dipath in $\mathcal{P}$. Let y_i be a vertex on R_i such that y_iR_i contains no vertex from any dipath in $\mathcal{P}$. Let H be the finite digraph on the final subdipaths x_iQ_i, the starting dipaths R_iy_i and the dipaths in $\mathcal{P}$. Then every set of less than n vertices misses one dipath x_iQ_i, one dipath R_jy_j and one Q_i–R_j dipath $P \in \mathcal{P}$. Thus, this vertex set does not separate $X := \{x_i \mid 1 \leqslant i \leqslant n\}$ from $Y := \{y_i \mid 1 \leqslant i \leqslant n\}$. By Menger's theorem, there

*A digraph H is *ubiquitous* if, for any digraph D, the existence of n pairwise disjoint copies of H in D for all $n \in \mathbb{N}$ implies the existence of infinitely many pairwise disjoint copies of H in D.

exist n pairwise disjoint X–Y dipaths in H and hence in D. These dipaths together with the tails $Q_i x_i$ and $y_i R_i$ form n pairwise disjoint double rays all of whose tails lie in ω. □

Let us briefly discuss how Problem 4.2 follows from a positive answer for Problem 4.3 for one-ended digraphs. Assume that the end ω of a one-ended digraph D has for all $n \in \mathbb{N}$ a subset of n rays and n anti-rays that are pairwise disjoint. Then Proposition 4.4 implies that there are n pairwise disjoint double rays in D. So a positive answer for Problem 4.3 gives us infinitely many pairwise disjoint double rays in D which directly implies the existence of a set of infinitely many rays and infinitely many anti-rays in D that are all pairwise disjoint and lie in the unique end ω.

§5. End-exhausting sequences

In this section we define a generalisation of the in-degree of an end, the so-called combined in-degree, and characterise it in terms of certain sequences of vertex sets. Hence, we focus on ends that contain rays. Everything can be done for the out-degree and anti-rays completely analogously, which is why we omit the details for that here.

Let D be a digraph and ω an end of D which contains a ray. Furthermore, let $(U_i)_{i \in \mathbb{N}}$ be a sequence of finite vertex sets of D. We say that the sequence $(U_i)_{i \in \mathbb{N}}$ is *ω-exhausting* if for every ray in ω there exists an $i \in \mathbb{N}$ such that this ray contains a vertex of U_i and if a ray in ω contains a vertex of U_i, then it contains a vertex of U_{i+1}. Note that the vertices of the ray in U_i and in U_{i+1} need not be distinct. Furthermore, note that obviously every countable digraph admits an ω-exhausting sequence for every end ω that contains a ray of the digraph. For uncountable digraphs this is not necessarily true. The following proposition characterises the existence of exhausting sequences and is an analogue of Lemma 5.1 in [4], which characterises the existence of so-called end-defining sequences for ends of undirected graphs.

Proposition 5.1. *Let D be a digraph and let ω be an end of D that contains a ray. Then there exists an ω-exhausting sequence if and only if there exist at most countably many disjoint rays in ω.*

Proof. If there are uncountably many disjoint rays in ω, then clearly there cannot exist an ω-exhausting sequence.

Conversely, let $\mathcal{R} := \{R_i = x_0^i x_1^i \dots \mid i \in \mathbb{N}\}$ be a maximal set of countably many disjoint rays in ω. We set

$$\begin{aligned} V_1 &:= \{x_0^1\} \\ V_2 &:= V_1 \cup \{x_1^1, x_0^2\} \\ V_3 &:= V_2 \cup \{x_2^1, x_1^2, x_0^3\} \\ &\vdots \end{aligned}$$

Let R be a ray in ω. By the maximality of $\mathcal{R}$, there exists some vertex from a ray in $\mathcal{R}$ that lies on R. Let x_j^i be the first such vertex on R. Since x_j^i lies in V_{i+j} and thus in V_k for all $k \geqslant i+j$, we conclude that $(V_i)_{i \in \mathbb{N}}$ is an ω-exhausting sequence. $\square$

Let us call an end ω of a digraph D *countable* if it contains at most countably many disjoint rays. So by the proposition above, every countable end with at least one ray admits an exhausting sequence.

Let X and Y be disjoint sets of ends of D. We say that a vertex set $S \subseteq V(D)$ *separates* X from Y in D if for every $\omega \in Y$ every $R \in \omega$ has a tail Q such that every ray in elements of X that starts at a vertex of Q meets S. In case X (or Y) is a singleton set, we ease the notation and analogously define that the end $\omega_X \in X$ (or X) is separated from Y (or from the end $\omega_Y \in Y$). Note that, if $\eta < \omega$ for ends η and ω, then η is separated from ω by a finite set of vertices.

Similarly, we define for $W \subseteq V(D)$ and a set of ends Y of D that a vertex set $S \subseteq V(D)$ *separates* W from Y in D if for every $\omega \in Y$ every $R \in \omega$ has a tail Q such that every Q–W dipath meets S. As before, we ease the notation and make corresponding definitions in case W or Y are singleton sets. Finally, we accordingly define how a set of vertices and ends is separated from a set of ends.

For an end ω of D, set

$$\omega^- := \{\eta < \omega \mid \eta \text{ end of } D, d^-(\eta) \geqslant 1\}.$$

For a vertex $v \in V(D)$ and a ray R, we call an infinite family of v–R dipaths an *infinite v–R fan* if they pairwise meet only in v.

A vertex v *dominates* an end ω of a digraph D if ω contains a ray and for every ray $R \in \omega$ there is an infinite v–R fan and an R–v dipath. We denote by $\operatorname{dom}(\omega)$ the set of vertices dominating the end ω. Note that looking at infinitely many distinct tails of R, the definition implies the existence of infinitely many distinct R–v dipaths. In contrary to the v–R fan, these dipaths may pairwise intersect in more vertices than just v. Note that, if η

and ω are ends with $\eta < \omega$ and if $v \in V(D)$ dominates η, then either v also dominates ω, or v, and hence η, is separated from ω by the empty set.

For an arbitrary end ω of a digraph D, we define the *combined in-degree* of ω, denoted by $\Delta^-(\omega)$, as

$$d^-(\omega) + \inf\{|S| \mid S \subseteq V(D) \text{ separates } \omega^- \cup \operatorname{dom}(\omega) \text{ from } \omega\}.$$

Analogously, it is possible to define the *combined out-degree* of ω.

Note that $\Delta^-(\omega) = d^-(\omega)$ holds if $d^-(\omega) \geqslant \aleph_0$, since the vertex set of a maximal collection of disjoint rays in ω is a set separating $\omega^- \cup \operatorname{dom}(\omega)$ from ω of size $d^-(\omega)$. Furthermore, note for the case where $d^-(\omega)$ is finite that $\Delta^-(\omega) \leqslant \aleph_0$ as, similarly as before, the vertex set of a maximal collection of disjoint rays in ω is a countable set separating $\omega^- \cup \operatorname{dom}(\omega)$ from ω. The same holds for the out-degree and combined out-degree.

In graphs the combined degree of an end is the maximum number of disjoint rays plus the number of vertices dominating that end. This is known to be equal to the infimum over the sizes of the vertex sets in defining sequences of that end if the end is countable (see [4]). For digraphs, the infimum over the sizes of the sets of exhausting sequences is not the same as the in-degree plus the number of dominating vertices of that end as the following example shows:

Example 5.2. Let D be the digraph as depicted in Figure 5.1. The in-degree of the end ω containing R is 1 and there is no dominating vertex of that end. Intuitively, the ray R^- serves as a dominating vertex that is stretched out as a ray. The combined in-degree of ω is 2 as we can simply let S consist of the bottom left vertex for the definition of the combined in-degree. Furthermore, if U_i is a set consisting of the i-th vertex of R together with its in-neighbour on R^-, then $(U_i)_{i\in\mathbb{N}}$ is an exhausting sequence of ω and there is no exhausting sequence of ω with smaller limit inferior.

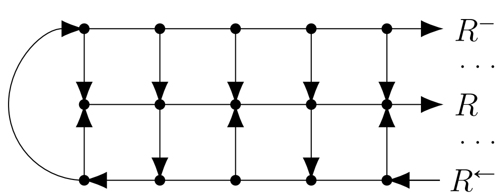


FIGURE 5.1. The end containing R has combined in-degree 2, but contains no two disjoint rays and is not dominated by any vertex.

Our aim in the rest of this section is to show that the combined in-degree can be characterised via exhausting sequences as it is indicated by Example 5.2.

Lemma 5.3. *Let D be a digraph and ω an end of D where $1 \leqslant d^-(\omega) < \infty$. Suppose there exists a finite $S \subseteq V(D)$ separating $\mathrm{dom}(\omega) \cup \omega^-$ from ω. Then there is a sequence $(U_i)_{i\in\mathbb{N}}$ with $U_i \subseteq V(D)$ and $|U_i| = d^-(\omega)$ for all $i \in \mathbb{N}$ such that every ray in ω meets some U_i and if it meets U_i and avoids S then it also meets U_{i+1}.*

Proof. Let U_1 be a vertex set of size $d^-(\omega)$ such that $d^-(\omega)$ many disjoint rays $R_1, \ldots, R_{d^-(\omega)}$ with $R_i = x_0^i x_1^i \ldots$ for all $1 \leqslant i \leqslant d^-(\omega)$ in ω start at U_1. Since S is finite, we may assume that no x_j^i lies in S and that no R_i–$\mathrm{dom}(\omega)$ dipath exists in $D - S$ for any i. We define the U_i recursively for all $i \in \mathbb{N}$. Let us assume that we have already defined U_i. Then there is a smallest vertex set U_{i+1} with $U_i \cap U_{i+1} = \varnothing$ that separates all but finitely many X_ℓ from U_i in $D - S$. Note that this set is finite since there are otherwise infinitely many U_i–X_ℓ dipaths that only share their starting vertex. Hence, there would exist a vertex from $\mathrm{dom}(\omega)$ in U_i, which cannot be. By definition, we have $|U_i| \leqslant |U_{i+1}|$. Then there is a set $\mathcal{P}_i$ of $|U_i|$ pairwise disjoint U_i–U_{i+1} dipaths in $D - S$ by Menger's theorem. Note that every $P \in \mathcal{P}_i$ is disjoint from every $P' \in \mathcal{P}_j$ for $j < i$: otherwise there exists a U_j–U_{i+1} dipath that avoids U_i, which can be extended to a U_j–X_ℓ dipath that avoids U_i as U_{i+1} was chosen smallest. Concatenating the elements of the sets $\mathcal{P}_i$ for all $i \geqslant j$ any $j \in \mathbb{N}$ yields a set of at least $|U_j|$ many disjoint rays, all of which lie in an end $\mu \leqslant \omega$ by construction. But as all the dipaths used to construct the rays avoid S, these rays must be in ω. This shows $|U_j| \leqslant d^-(\omega)$. Since $d^-(\omega) = |U_1| \leqslant |U_j|$, we obtain $|U_j| = d^-(\omega)$.

Now, let $R \in \omega$ be such that R avoids S and let us assume there is an $i \in \mathbb{N}$ such that R contains a vertex v from U_i. Hence, $v \in V(R_j)$ for some $1 \leqslant j \leqslant d^-(\omega)$. Since $R \in \omega$, it contains infinitely many vertices from the rays $R_1, \ldots, R_{d^-(\omega)}$. This implies that R contains vertices from X_n for every $n \in \mathbb{N}$. As U_{i+1} separates some X_m from U_i in $D - S$ and R is disjoint from S, we know that vR intersects U_{i+1}.

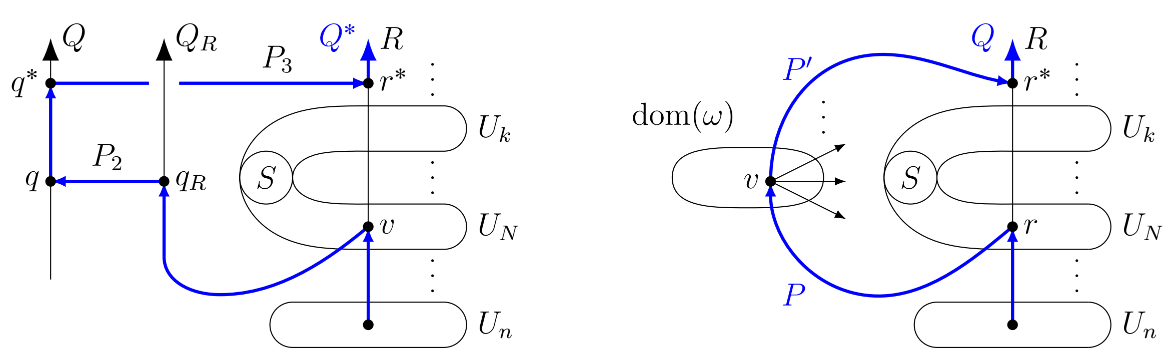
Let us now assume that $R \in \omega$ does not meet any U_i. By the choice of $R_1, \ldots, R_{d^-(\omega)}$, the ray R meets some R_i. Let Q be the ray $x_0^i R_i x R$, where x is a vertex in $R_i \cap R$. Then Q meets any U_j only in $x_0^i R_i x$ and thus it meets only finitely many U_j. Since it meets at least U_1, this contradicts the property that we just proved. Thus, the sequence $(U_i)_{i\in\mathbb{N}}$ satisfies the claim. □

Lemma 5.3 shows that under its assumptions there exists an exhausting sequence all of whose elements have size $\Delta^-(\omega)$: simply take the sequence $(U_i \cup S)_{i\in\mathbb{N}}$. While this just seems to be a part of a special case of Theorem 1.1, it will actually help us in the proof of that theorem.

Lemma 5.4. *Let D be a digraph and let ω be a countable end of D with $d^-(\omega) \geqslant 1$. Let $(U_i)_{i\in\mathbb{N}}$ be an ω-exhausting sequence such that $|U_i| \leqslant k$ for some $k \in \mathbb{N}$ and all $i \in \mathbb{N}$. Then the following hold.*

(i) *For every $\eta \in \omega^-$, either there exists a finite vertex set S that separates η from ω and lies in all but finitely many U_i or $(U_i)_{i\in\mathbb{N}}$ is η-exhausting.*
(ii) *There exists a finite vertex set S that separates* $\mathrm{dom}(\omega)$ *from ω and lies in all but finitely many U_i.*

Proof. Let $\eta \in \omega^-$. Let us assume that there is no finite vertex set that separates η from ω and that lies in all but finitely many U_i. Let us suppose that there is a ray $Q \in \eta$ that avoids all vertex sets U_i. Let S be the set of vertices that are contained in all but finitely many U_i. Hence, S is a finite set, and does not separate η from ω by assumption. So there is a ray $R \in \omega$ such that for every tail T of R there is a ray $Q_T \in \eta$ starting at T and avoiding S. By considering a tail, if necessary, and since $(U_i)_{i\in\mathbb{N}}$ is ω-exhausting, we may assume that R is disjoint from S and the first vertex of R lies in U_n for some $n \geqslant 1$. By assumption, we know that $Q_R \in \eta$ is a ray starting at a vertex v on R and avoiding S. See Figure 5.2 (a) for a figurative sketch of the situation in the proof. Since Q and Q_R are equivalent, there exists a Q_R–Q dipath P_2, starting at q_R and ending at q, that is disjoint to S. Then there exists $N \in \mathbb{N}$ with $N \geqslant n$ such that for all $j > N$ there is no common vertex of U_j and the ray $Q^* := RvQ_Rq_RP_2qQ$. Because of $\eta \leqslant \omega$, there are infinitely many pairwise disjoint qQ–R dipaths. Thus, there exists one such dipath P_3 with first vertex q^* and last vertex $r^* \neq v$ such that neither P_3 nor r^*R contains any vertex from U_k for some $k > N$. Then the digraph $Q^*q^*P_3r^*R$ contains a ray R^* that lies in ω, contains a vertex from U_n but not from U_k. This is a contradiction to $(U_i)_{i\in\mathbb{N}}$ being ω-exhausting. Thus, every ray in η meets some set U_i.



(a) The blue ray Q^* intersects U_n but not U_k. (b) The blue ray Q intersects U_N but not U_k.

FIGURE 5.2. Two cases from the proof of Lemma 5.4.

Let us suppose that there exists a ray $Q \in \eta$ such that Q contains a vertex from U_i but not from U_{i+1}. Let x be on a ray $R \in \omega$ such that xR avoids U_{i+1}, too, which is possible as U_{i+1} is finite. Since $Q \leqslant R$, there exists a Q–xR dipath P with starting vertex y and end vertex z that avoids U_{i+1} such that Qy meets U_i. Then $QyPzR$ contains a ray in ω that contains a vertex from U_i but not from U_{i+1}. This is impossible since $(U_j)_{j\in\mathbb{N}}$ is ω-exhausting. Thus, (i) follows.

Suppose that there is no finite vertex set separating $\mathrm{dom}(\omega)$ from ω that lies in all but finitely many U_i. Let S be the set of vertices that are contained in all but finitely many U_i. Hence, S is a finite set and does not separate some $v \in \mathrm{dom}(\omega)$ from ω by assumption. Again, we may assume that there is a ray $R \in \omega$ that is disjoint from S. Similarly as in the proof of statement (i), we can find an R–v dipath P starting at some $r \in R$ that avoids S. See Figure 5.2 (b) for a picture of the situation in this proof. We may choose r such that Rr contains a vertex from some U_n. Then there exists $N \geqslant n$ such that RrP contains a vertex $u \in U_N$ but $uRrP$ or uP, depending on whether u lies on R or on P, meets U_M for $M \geqslant N$ at most in u. Furthermore, let $k > N$ such that $U_k \cap U_N \subseteq S$. Then $uRrP$ or uP is disjoint from U_k. Let R' be a tail of R that is disjoint from U_k. Since there exists a v–R' fan, there is a v–R' dipath P' with end vertex r' on R' that avoids U_k and such that P' intersects $uRrPv$ only in v. Then $uRrPvP'r'R'$ or $uPvP'r'R'$ is a ray Q in ω that contains a vertex from U_N but avoids U_k, which contradicts that $(U_i)_{i\in\mathbb{N}}$ is ω-exhausting. This shows (ii). $\square$

Now we are able to prove our main result.

Theorem 5.5. *Let D be a digraph and let ω be a countable end of D with $d^-(\omega) \geqslant 1$. Then*

$$K(\omega) := \inf\left\{\liminf_{i\in\mathbb{N}} |U_i| \;\middle|\; (U_i)_{i\in\mathbb{N}} \text{ is an } \omega\text{-exhausting sequence}\right\}$$

is the same as the combined in-degree $\Delta^-(\omega)$.

Proof. Let us define the following:

$$\delta^-(\omega) := \inf\left\{|S| + \sum_{\eta\in B} d^-(\eta) \;\middle|\; \begin{array}{l} A \cup B = \omega^- \cup \{\omega\}, A \cap B = \varnothing, \omega \in B, \\ \qquad S \subseteq V(D) \text{ separates } A \cup \mathrm{dom}(\omega) \text{ from } B \end{array}\right\}.$$

We will actually prove that $K(\omega) = \delta^-(\omega) = \Delta^-(\omega)$. Note that $K(\omega)$ is at most $\aleph_0$ by definition. Since ω is countable by assumption, we have $\Delta^-(\omega) \leqslant \aleph_0$ as noted earlier. Trivially, we have $\delta^-(\omega) \leqslant \Delta^-(\omega)$ and hence $\delta^-(\omega) \leqslant \aleph_0$, too.

In order to prove $K(\omega) \geqslant \delta^-(\omega)$, it suffices to prove the assertion for finite $K(\omega)$. Let $(U_i)_{i\in\mathbb{N}}$ be an ω-exhausting sequence. By thinning out the sequence $(U_i)_{i\in\mathbb{N}}$, we may assume without loss of generality that all sets U_i have the same finite size. Let (A, B) be a partition of $\omega^- \cup \{\omega\}$ such that B consists of those ends η in $\omega^- \cup \{\omega\}$ for which $(U_i)_{i\in\mathbb{N}}$ is an exhausting sequence and such that the (finite) set of all those vertices that lies in all but finitely many U_i does not separate η from ω. By Lemma 5.4 (ii), there exists a finite vertex set S_{dom} that separates $\text{dom}(\omega)$ from ω and that lies in all but finitely many U_i. Since $(U_i)_{i\in\mathbb{N}}$ is an exhausting sequence for every element $\eta \in B$, there exists by Lemma 5.4 (i), for every $\mu \in A \cap \eta^-$, a finite vertex set $S_{\mu,\eta}$ that separates μ from η and lies in all but finitely many U_i. We claim that for every $\mu \in A \setminus \eta^-$ the set $S_{\mu,\omega}$ separates μ from η and lies in all but finitely many U_i. If $S_{\mu,\omega}$ does not separate μ from η, then $S_{\mu,\omega}$ does not separate μ from ω, as $S_{\mu,\omega}$ is not separating η from ω by definition of B, a contradiction. As $K(\omega)$ is finite, there exists a finite vertex set S that separates A from B and that lies in all but finitely many U_i.

If B is infinite or has an element of infinite in-degree, then there exist more than $K(\omega)$ pairwise disjoint rays in elements of B. Since $(U_i)_{i\in\mathbb{N}}$ is η-exhausting for every $\eta \in B$, there exists an $N \in \mathbb{N}$ such that for all $i \geqslant N$ the set U_i contains vertices from more than $K(\omega)$ many of these rays. This contradicts the definition of $K(\omega)$. Hence, B is finite and every element of B has finite in-degree. Thus, the maximum number of pairwise disjoint rays in elements of B is finite and is the same as $\sum_{\eta\in B} d^-(\eta)$. Since $S \cup S_{\text{dom}}$ is finite, by considering tails if necessary, we may assume that there are $\sum_{\eta\in B} d^-(\eta)$ many pairwise disjoint rays in elements of B each of which contains no vertex from $S \cup S_{\text{dom}}$. Since all but finitely many U_i must contain a vertex from each of those rays and $S \cup S_{\text{dom}}$ is contained in all but finitely many U_i, this completes the proof of $K(\omega) \geqslant \delta^-(\omega)$.

In order to prove $K(\omega) \leqslant \delta^-(\omega)$, let us now assume that $\delta^-(\omega)$ is finite, i. e. there are a partition (A, B) of $\omega^- \cup \{\omega\}$ with $\omega \in B$ and some vertex set S separating $A \cup \text{dom}(\omega)$ from B such that $|S| + \sum_{\eta\in B} d^-(\eta)$ is finite. In particular, $\delta^-(\omega)$ being finite implies that B and S are finite. Note that no end η in B is separated from ω by S, as moving η together with all ends in B from which η is not separated from B to A would make the value of $\delta^-(\omega)$ smaller. Let $\eta_1, \ldots, \eta_{|B|}$ be an enumeration of B such that $i \leqslant j$ if $\eta_i \leqslant \eta_j$. Note that $\eta_{|B|} = \omega$. Next we claim that, for every $i \in \{1, \ldots, |B|\}$, there is a finite vertex set S_i that contains S and separates all ends $\mu < \eta_i$ from η_i. Since B is finite, we can separate all ends $\mu < \eta_i$ with $\mu \in B$ with a finite vertex set S'_i from η_i. Hence, we can separate all ends $\mu < \eta_i$ with $\mu \in \omega^-$ from η_i by $S \cup S'_i$. Suppose there were a $\mu < \eta_i$ outside of ω^- which is not separated by $S \cup S'_i$ from η_i. Then, we have $\mu \in \eta_i^- \subseteq \omega^-$, a contradiction. Note that every vertex dominating an end $\eta_j < \omega$ also dominates ω. Therefore, every vertex dominating η_i is separated from η_i by S_i because $\eta_i \in B$ and $S \subseteq S_i$. Thus, Lemma 5.3 implies that there are sequences $(U_i^j)_{i \in \mathbb{N}}$, for every $j \in \{1, \ldots, |B|\}$, with $|U_i^j| = d^-(\eta_j)$ for all $i \in \mathbb{N}$ such that every ray in η_j meets some U_i^j and if it meets U_i^j and avoids S_j then it also meets U_{i+1}^j. Let $R \in \omega$ be a ray. Since S does not separate any η_j from ω, we may assume that there is a dipath from R to every vertex in U_i^j for every $1 \leqslant j \leqslant |B|$ and every $i \in \mathbb{N}$. For every $1 \leqslant j \leqslant |B|$, let $\mathcal{R}^j$ be a maximal set of pairwise disjoint rays in η_j starting at U_1^j.

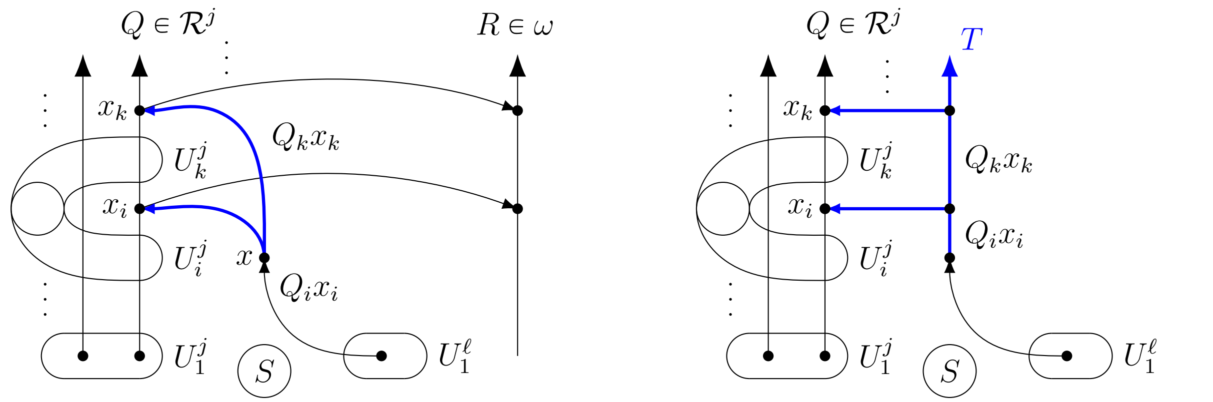
Claim 1. *We may choose the $(U_i^j)_{i \in \mathbb{N}}$ such that there is a dipath from every tail of R to every vertex in U_i^j that avoids S for every $1 \leqslant j \leqslant |B|$ and every $i \in \mathbb{N}$ and such that the elements of all $\mathcal{R}^j$ for $1 \leqslant j \leqslant |B|$ are disjoint from S.*

Proof. Since S does not separate η_j from ω, there exists a ray $Q \in \eta_j$ starting at a vertex on an arbitrary tail of R so that Q is disjoint from S. Hence, Q intersects all but finitely many U_i^j. Let us now fix $i \in \mathbb{N}$ such that Q intersects U_i^j, and let $x \in V(Q) \cap U_i^j$. Then there exists $N \geqslant i$ such that for every $i' \geqslant N$ and every $y \in U_{i'}^j$ there exists an x–y dipath avoiding S. Since S is finite, the claim follows by taking a suitable subsequence of $(U_i^j)_{i \in \mathbb{N}}$ and thus taking the elements of $\mathcal{R}^j$ as suitable tails of the original ones. $\lozenge$

Claim 2. *We may choose the sequences $(U_i^j)_{i \in \mathbb{N}}$ such that every ray in η_j that starts at U_1^ℓ with $\ell > j$ and avoids S, and hence also every ray in η_j that starts at some U_i^ℓ with $\ell > j$ and avoids S, must have a vertex from $\bigcup_{k \leqslant j} U_1^k$.*

Proof. Let us suppose that the claim is false and that j is smallest such that the claim fails for j. In particular, the sets U_1^k with $k < j$ are defined such that they have the desired property. Then there exists for infinitely many $i \geqslant 2$ a ray $Q_i \in \eta_j$ starting in some U_1^ℓ with $\ell > j$ that avoids $\bigcup_{k<j} U_1^k \cup U_i^j \cup S$, as otherwise we could replace the sequence $(U_i^j)_{i \in \mathbb{N}}$ by a suitable final subsequence. Since $Q_i \in \eta_j$, there exists a first vertex x_i on Q_i that lies on some $Q \in \mathcal{R}^j$ such that $x_i Q$ does not meet $\bigcup_{k<j} U_1^k \cup U_i^j$. Note that $\{x_i \mid i \in \mathbb{N}\}$ must be infinite. As U_1^ℓ is finite, infinitely many of the Q_i start at the same vertex of U_1^ℓ. Hence, in the subdigraph induced by the starting dipaths $Q_i x_i$, there exists by Lemma 2.1 either a subdivided infinite out-star with leaves in elements of $\mathcal{R}^j$ or an out-comb with teeth on elements of $\mathcal{R}^j$. See Figure 5.3 for a picture of the two cases.

Let us first suppose that there exists a subdivided infinite out-star with centre x and leaves in elements of $\mathcal{R}^j$. Clearly, there exists an x–R' fan for every ray $R' \in \eta_j$. And since $\eta_j \leqslant \omega$, there is also an x–R'' fan for every ray $R'' \in \omega$. By Claim 1, each U_i^j can be reached from every tail of R via a dipath avoiding S. Since all rays Q_i avoid S, we know



(a) The case when we obtain a blue out-star with centre x.

(b) The case when we obtain a blue out-comb with spine T.

FIGURE 5.3. Two cases from the proof of Claim 2.

that x is reached from every tail of R via a dipath avoiding S. So x is dominating ω, but not separated from it by S, a contradiction to the choice of S.

Thus, we find an out-comb with spine T and teeth in $\{x_i \mid i \geqslant 2\}$ in that subdigraph. We may assume that T has its first vertex in some U_1^ℓ with $\ell > j$. As $\mathcal{R}^j$ is finite, one of its elements contains infinitely many teeth of the out-comb. So T must lie in an element of $\omega^- \cup \{\omega\}$, but as all rays Q_i avoid S, it cannot lie in A, so it lies in some $\eta_k \leqslant \eta_j$. By the minimal choice of j, if $k < j$, then T meets $\bigcup_{i<k} U_1^i$, which is impossible as all the dipaths $Q_i x_i$ avoid that set. Hence, we have $k = j$. By the choice of $(U_i^j)_{i \in \mathbb{N}}$, T must contain a vertex from some U_m^j, and thus from every U_i^j for $i \geqslant m$ as the sequence was chosen according to Lemma 5.3. This shows that infinitely many Q_i contain a vertex from all U_i^j for $i \geqslant m$, which contradicts their choices. Thus, there exists $n \in \mathbb{N}$ such that all rays in η_j starting at some U_1^ℓ with $\ell > j$ that avoid $\bigcup_{k<j} U_1^k \cup S$ contain a vertex from U_n^j. Removing all elements before U_n^j from the sequence $(U_i^j)_{i \in \mathbb{N}}$ yields a sequence for j as desired. $\Diamond$

Note that Claim 2 implies that $S \cup \bigcup_{j<|B|} U_1^j$ separates $\omega^- \cup \mathrm{dom}(\omega)$ from ω. Furthermore, Claim 2 implies that we may have chosen $S_i = S \cup \bigcup_{j<i} U_1^j$, which we assume for the rest of the proof.

We will define a sequence $(V_i)_{i \in \mathbb{N}}$ that is ω-exhausting. We set

$$V_1 := S \cup \bigcup_{1 \leqslant j \leqslant |B|} U_1^j$$

and

$$V_i := S \cup \bigcup_{1 \leqslant j < |B|} U_1^j \cup U_i^{|B|}.$$

Note that

$$V_i = V_1 \setminus U_1^{|B|} \cup U_i^{|B|} = S_{|B|} \cup U_i^{|B|}.$$

Let Q be a ray in ω. By definition of $(U_i^{|B|})_{i\in\mathbb{N}}$, the ray Q contains a vertex from some $U_i^{|B|}$. If it also contains a vertex of $S_{|B|}$, then the ray contains a vertex from all V_j for $j \geqslant i$ by their definition. If Q contains no vertex from $S_{|B|}$, then the definition of the $(U_k^{|B|})_{k\in\mathbb{N}}$ implies that Q contains a vertex from $U_{i+1}^{|B|}$ and inductively from all $U_j^{|B|}$ for $j \geqslant i$. Thus, the sequence $(V_i)_{i\in\mathbb{N}}$ is ω-exhausting. This implies $K(\omega) \leqslant \delta^-(\omega)$.

Since $\delta^-(\omega) \leqslant K(\omega)$ and since the sequence $(V_i)_{i\in\mathbb{N}}$ that we constructed in the proof of $K(\omega) \leqslant \delta^-(\omega)$ is ω-exhausting and has the property $|V_i| \leqslant \delta^-(\omega)$ for all $i \in \mathbb{N}$, we have $|V_i| \leqslant K(\omega)$. Since all V_i have the same size, we have $|V_i| = K(\omega)$ for all $i \in \mathbb{N}$. As $S_{|B|}$ separates $\omega^- \cup \mathrm{dom}(\omega)$ from ω, this implies

$$\Delta^-(\omega) \leqslant |S_{|B|}| + d^-(\omega) = |V_1| = K(\omega).$$

Thus, we have proved

$$\delta^-(\omega) \leqslant \Delta^-(\omega) \leqslant K(\omega) \leqslant \delta^-(\omega),$$

which completes the proof. $\square$

§6. Edge-degrees of ends

In this section, we will consider edge-disjoint rays in ends of digraphs and discuss the corresponding structural results for ends containing an arbitrary or an infinite number of pairwise edge-disjoint rays. The proofs of the results are essentially the same as those given in Section 3 and Section 4, which is why we omit them here. The first result corresponds to Corollary 3.2 for edge-disjoint rays.

Theorem 6.1. *Let D be a digraph, let ω be an end of D and let $\mathcal{R}$ (resp. $\mathcal{A}$) be the set of all sets of edge-disjoint rays (resp. anti-rays) in ω. Then $\sup_{M\in\mathcal{R}} |M| = \max_{M\in\mathcal{R}} |M|$ and $\sup_{M\in\mathcal{A}} |M| = \max_{M\in\mathcal{A}} |M|$.* $\square$

Theorem 6.1 enables us to define the *edge-in-degree* (the *edge-out-degree*) of an end of a digraph as the maximum number of pairwise edge-disjoint rays (anti-rays) in that end.

Similarly as for vertex-disjoint rays and anti-rays as in Theorem 4.1, we obtain a digraph that has infinitely many pairwise edge-disjoint rays and infinitely many pairwise edge-disjoint anti-rays such that every ray and every anti-ray share an edge.

Theorem 6.2. *There exists a digraph D with the following properties:*

(i) *D contains infinitely many pairwise edge-disjoint rays.*

(ii) *D contains infinitely many pairwise edge-disjoint anti-rays.*

(iii) *Every ray and every anti-ray of D share an edge.*

Proof. Let D' be the digraph constructed in the proof of Theorem 4.1. We replace every vertex u by two vertices u^- and u^+ and every edge uv by the edge u^+v^-. We add also all edges of the form u^-u^+ for $u \in V(D')$ in order to obtain the digraph D. Obviously, D satisfies (i) and (ii). Since every vertex of D has either a unique out-neighbour or a unique in-neighbour, edge-disjoint dipaths in D induce disjoint dipaths in D'. Thus, D must satisfy (iii). □

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