Unbeatable Strategies

"Stochastic Dynamics in Economics and Finance"

HIM Trimester program, Bonn, Germany

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June 10, 2013

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0 Introduction

This course presents an overview of the rich mathematical theory of *two-person, perfect-information, zero-sum games*, of finite or infinite length. The systematic study of mind games such as chess, checkers, Go etc. goes back to the middle ages, although it was Ernst Zermelo in 1913 who started the modern investigation of *strategies* in such mind games, using modern set-theoretic techniques and setting the stage for further analysis. Various mathematicians, including Dénes König and László Kalmár, followed in Zermelo's footsteps by correcting, improving and extending Zermelo's original theory in the 1930s and 1940s. Roughly at the same time (but largely unnoticed by the mathematical community) Polish mathematicians such as Stefan Banach and Stanisław Mazur used infinite games to solve problems in pure topology and set theory. In the 50s, 60s and 70s, the theory of infinite games gained increasing popularity after Jan Mycielski and Hugo Steinhaus introduced the *Axiom of Determinacy* (a statement contradicting the Axiom of Choice) and Tony Martin proved *Borel determinacy*. Following this, infinite games became a key topic of research in pure set theory, yielding mathematical advances that seem unimaginable without the use of infinite games.

In this course we will cover the history of finite and infinite games from its beginning. In Part I, we present the basic theory of games following Zermelo, König, Kalmár, Gale and Stewart. Here, we will distinguish between three concepts: *finite games*, i.e., games in which the maximum number of moves is pre-determined by a natural number N; *finite-unbounded games*, i.e., those in which "winning" or "losing" is determined at a finite stage but the maximum number of moves is not bounded; and *infinite games*, i.e., those which go on forever, so to say, and where "winning" or "losing" is only determined at the "limit". The three concepts are closely related, and we will see how the latter concept can conveniently capture the former two.

In Part II, we will see more advanced applications of the theory of (infinite) games to analysis, topology and set theory.

It is interesting to note that the progress from the finite to the infinite represents a gradual paradigm shift: from using mathematical tools to study real-life games (which are deemed interesting in themselves), to "constructing" or "inventing" games in order to study other mathematical objects (deemed interesting in themselves). The games change from being the subject of research to being tools in the study of other subjects.

Prerequisites

In this course we assume familiarity with basic mathematical concepts and structures, in particular *infinite sets*, power sets, the set of all functions from one set to another, etc. We assume that the readers know terms such as "surjection", "injection", "transitive relation" and so on. We also assume familiarity with the notions of *cardinality* of a set, writing $|A|$ to denote the cardinality of A. Familiarity with abstract set theory will not be assumed, although some knowledge of *ordinals and cardinals* and the *Axiom of Choice* would be useful. Meta-mathematical results (i.e., results about provability/unprovability of statements in formal ZFC set theory and similar facts) will be mentioned in passing but will not form an important part of the course. A basic knowledge of topology will be assumed, although most concepts will be defined.

Notation used in this course

$\mathbb{N}$ denotes the set of all natural numbers $\{0, 1, 2, 3, \dots\}$. $\mathbb{N}^n$ is the n-Cartesian product of $\mathbb{N}$, i.e., $\mathbb{N} \times \cdots \times \mathbb{N}$ repeated n times. We will denote elements of $\mathbb{N}^n$ by $\langle x_0, x_1, \dots, x_{n-1}\rangle$ where x_i is a natural number, and call these *finite sequences* (of natural numbers). For technical reasons we will frequently identify a finite sequence with a function $f : \{0, \dots, n-1\} \longrightarrow \mathbb{N}$, and if f is such a function we may write $f(m)$ to denote the m-th element of the sequence f. We will usually use letters s, t etc. for finite sequences. The *empty sequence* is denoted by $\langle\rangle$. The set of *all finite sequences* is denoted by $\mathbb{N}^*$, i.e.,

$$\mathbb{N}^* = \bigcup_{n\in\mathbb{N}} \mathbb{N}^n,$$

and for $s \in \mathbb{N}^*$, $|s|$ is the *length of* s, i.e., $|s| = n$ such that $s \in \mathbb{N}^n$.

Generalizing this to an infinite Cartesian product, we let

$$\mathbb{N}^{\mathbb{N}} := \{f : \mathbb{N} \longrightarrow \mathbb{N}\}$$

be the set of all functions from $\mathbb{N}$ to $\mathbb{N}$. We can view such functions as *infinite sequences of natural numbers*, and for that reason we shall sometimes use a notation like $\langle 0, 0, 0, \dots\rangle$, or $\langle x_0, x_1, \dots\rangle$ to denote such infinite sequences. We will usually use the letters x, y etc. for elements of $\mathbb{N}^{\mathbb{N}}$.

If $x \in \mathbb{N}^{\mathbb{N}}$ and $n \in \mathbb{N}$, then $x{\restriction}n$ denotes the *initial segment of* x *of length* n:

$$x{\restriction}n := \langle x(0), x(1), \dots, x(n-1)\rangle$$

If $s \in \mathbb{N}^*$ and $x \in \mathbb{N}^{\mathbb{N}}$, then we say that s *is an initial segment of* x, denoted by

$$s \lhd x$$

if $s(m) = f(m)$ for all $m < |s|$ (equivalently, if $x{\restriction}|s| = s$). For two finite sequences $s, t \in \mathbb{N}^*$, the *concatenation of* s *and* t, denoted by $s^\frown t$, is defined by

$$s^\frown t(i) := \begin{cases} s(i) & \text{if } i < |s| \\ t(i - |s|) & \text{if } |s| \le i < |t| \end{cases}$$

In other words, if $|s| = n$ and $|t| = m$ then $s^\frown t = \langle s(0), \dots, s(n-1), t(0), \dots, t(m-1)\rangle$. For $s \in \mathbb{N}^*$ and $x \in \mathbb{N}^{\mathbb{N}}$ the concatenation $s^\frown x$ is defined analogously.

Any piece of notation not included in the list above will be defined as we go along.

Acknowledgments

I would like to extend my sincere gratitude to Prof. Igor Evstingeev for inviting me to give a series of lectures at the trimester programme "Stochastic Dynamics in Economics and Finance" at the Hausdorff Institute for Mathematics in Bonn.

The material and presentation in this course is largely based on the inspiring and motivating lectures of Prof. Benedikt Löwe, from whom I also learned most of the theory covered here.

These lecture notes are adapted from an earlier version of a course that I taught at Sofia University, Bulgaria, in the summer of 2010. I would like to thank Prof. Alexandra Soskova for hosting me in Sofia and giving me the opportunity to teach the course.

Part I
Finite, finite-unbounded and infinite games: from Zermelo to Martin

1 Finite games

1.1 Our basic setting

We start by considering real-life, finite games. We will present the general setting, derive a mathematical formalism and then show how it applies to some concrete examples such as chess. Technically, the finite case is very straightforward, but the formalism developed in this section should help understanding the more complex situations with infinite games that will be considered later.

The games we will consider are typically called *two-player, perfect information, zero sum* games. Let us unravel the definitions:

- "Two-player" refers to the following setting: there are two players, called *Player I* and *Player II*, who are playing a turn-based mind game against each other. By convention, Player I is a male player whereas Player II is female. Player I always starts the game by making some move, after which Player II makes a move, and then they both take turns in playing the game.
- "Perfect information" means that at every stage of the game, both Players I and II have complete access to and knowledge of the way the game has been played so far.
- "Zero-sum" means that exactly one of the two players wins the game, there are no "draws" and no "mutual benefits". In other words

  Player I wins a game if and only if
  Player II loses the game, and vice versa.

The reader might be surprised by the "zero-sum" condition, as most two-player games, such as chess, checkers etc., do not seem to satisfy it because of the existence of draws. Indeed, in our mathematical formalism draws will not exist, so a particular game must always end with a win for one of the two players. Therefore, when modeling a game like chess, we must decide that a draw signifies a win for one of the two players (for example, a draw in chess is a win for Black). At the end of this section, we will see why this is justified and does not, in fact, limit the class of games we can model.

Our framework includes such famous and popular mind games as chess, checkers, Go, Nine Men's Morris, tic-tac-toe, etc. But at the same time, many important games are left out. Specifically, we will not be considering games where:

1. Chance events are involved, e.g., throwing a die, dealing cards, turning a roulette etc.
2. Players take turns *simultaneously* (or so quickly following one another that they have no chance to react to the opponent's move), e.g. "Rock-Paper-Scissors".
3. Certain moves are known to one player but hidden from the other, e.g. "Stratego".

A crucial distinction involves the *length* of a game, i.e., the number of moves that can be made. In real life, a game must end after a finite number of moves. But do we know in advance what that number is? We will use the following terminology:

- A game is *finite* if there exists a pre-determined number N such that a game can never take more than N moves.
- A game is *finite-unbounded* if it is finite in principle, but the number of moves after which it ends is potentially unbounded.
- A game is *infinite* if there it goes on for countably-infinite many moves.

In the first section, we consider only finite games. Note that, even in this finite setting, we do *not* restrict the number of possible options that a certain player has at each given stage, in particular, there can be an infinite number of options. So we can imagine such games being played "on an infinite board", "with infinitely many pieces", and so on. This is not as far-fetched an assumption as it may sound: while actual board games, of course, are finite, it is easy to think of real-life examples of games played on an "infinite board": an infinitary version of Gomoku ("five in a row"), which can be simulated on a computer; infinite versions of Nim; any other game in which players may pick arbitrarily large natural numbers, etc.

1.2 Chess

Let us start by a (still informal) discussion of the game of chess, which, as Zermelo put it in 1913, is the "most well-known of all games of this kind". As already mentioned, we alter the rules of chess so that a draw is considered a win for Black (and we will later see why this is justified). It is clear that, with this condition, chess fulfills all our requirements: there are two players—White and Black in this case—White starts and then the players alternate in taking turns. At each stage, the players have perfect information about all preceding moves, as the board is fully visible to both players. At the end, either White wins, or Black wins, or there is a draw—which again means that Black wins. So chess is a two-player, perfect information, zero-sum game.

Is chess a finite game (in the sense of our definition above)? The answer depends somewhat on the exact rules we choose to adopt. In chess, there is usually the so-called "threefold repetition rule", i.e., "if a position on the board is repeated three times, with the same player having to go, then the game is called a draw". The number of positions in chess is finite: there are 64 squares, each can be occupied by at most one piece, there are 32 pieces, so there are at most 64^{33} unique positions. Thus no game of chess can take longer than $3 \cdot 64^{33}$ moves.

Notice that we could easily get a much smaller estimate if we took into account how many pieces of each kind there are in chess, that some pieces are identical and do not need distinguishing between each other, that many combinations of pieces on the board are not even legal, and so on. But in this course we are not interested in questions of real-life complexity, and for our purposes any finite number is as good as another.

How can we model or formalize the game of chess? Obviously, there are many ways. The most natural one, perhaps, is to use *algebraic chess notation.* Each game of chess can then be written down as a sequence of moves. Below is an example of a short game of chess (*scholar's mate*):

White:	e4		Qh5		Bc4		Qxf7#
Black:		e5		Nc6		Nf6	

An alternative way would be to assign a natural number between 0 and 64^{33} to each unique position of the pieces on the board, and to write the *positions*, rather than the *moves*, in a table analogous to the one above:

White:	x_0		x_1		x_2		x_3	
Black:		y_0		y_1		y_2		$\dots$

So x_0 is a number $\leq 64^{33}$ which encodes the position of the pieces on the board after the first move by White. y_0 is again a number that encodes the position after the first move by Black, etc. Whether we use this notation or the "algebraic notation", we require that each step in the game corresponds to a legal move according to the rules of chess, and when the game ends, there is a clear algorithm for determining who the winner is. Using the first formalism, this is incorporated into the notation (a "#" means "check-mate"), whereas in the second one, certain numbers n correspond to a winning or losing position.

One could think of many other ways of encoding a game of chess. Regardless which method we use, each completed game of chess is encoded as a *finite sequence* of natural numbers of length at most $2 \cdot 3 \cdot 64^{33}$. In other words, each game is an elements of $\mathbb{N}^n$, for some $n \leq 2 \cdot 3 \cdot 64^{33}$. Let LEGAL be the set of those finite sequences that correspond to a sequence of legal moves according to the rules of chess (keeping in mind the particular encoding we have chosen). Now let $\mathsf{WIN} \subseteq \mathsf{LEGAL}$ be the subset of all such sequences which encode a winning game for White. Clearly $\mathsf{LEGAL} \setminus \mathsf{WIN}$ is the set of legal games that correspond to a win by Black. Thus, once the formalism of encoding moves of chess into natural numbers has been agreed upon, the game called "chess" is completely determined by the sets LEGAL and WIN.

1.3 General finite games

After an informal introduction to the mathematization of chess, we give a general definition of finite games as a natural abstraction from the particular case handled above.

Definition 1.1. (Two-person, perfect-information, zero-sum, finite game.) Let N be a natural number (the *length* of the game), and A an arbitrary subset of $\mathbb{N}^{2N}$. The game $G_N(A)$ is played as follows:

- There are two players, Player I and Player II, which take turns in picking one natural number at each step of the game.
- At each turn i, we denote Player I's choice by x_i and Player II's choice by y_i.
- After N turns have been played the game looks as follows:

I:	x_0		x_1		$\dots$		x_{N-1}	
II:		y_0		y_1		$\dots$		y_{N-1}

  The sequence $s := \langle x_0, y_0, x_1, y_1, \dots, x_{N-1}, y_{N-1}\rangle$ is called a *play of the game* $G_N(A)$.

- Player I *wins the game* $G_N(A)$ if $s \in A$, and Player II wins if $s \notin A$. The set A is called the *pay-off set* for Player I or the *set of winning conditions* for Player I.

If we look at this definition we immediately notice two aspects in which it differs from the informal discussion of chess in the last section. Firstly, we are only considering sequences of length *exactly* $2N$ as valid plays, rather than sequences of length less then or equal to $2N$. Secondly, instead of restricting the possible plays to some given set (like we defined LEGAL before) and only considering those sequences, we allow *any* sequence of natural numbers of length $2N$ to be considered a valid play.

As it turns out, this change of formalism does not restrict the class of games we can model. The first issue is easily fixed simply by assigning one particular natural number (say 0) to represent the state in which "the game has been completed". For example, suppose a particular game of chess only took 20 moves, but our model requires the game to be N moves long (for $N = 3 \cdot 64^{33}$, say). Then we simply fill in 0's for all the moves after the 20-th until the N-th. It is clear that this allows us to model the same class of games.

For the second point, let us think about the following situation: suppose in a game of chess, a player makes an illegal move, i.e., a move that is not allowed by the rules of chess. One could then do one of two things: tell the player that the move was illegal and request that it be re-played, or (in a rather strict environment) disqualify the player immediately, thus making him or her lose that particular game. In our mathematical formalism, we choose the second option: so instead of stipulating that only certain moves are allowed, we allow *all* possible moves to be played but make sure that any player who makes an illegal move immediately loses the game. That information can be encoded in the pay-off set $A \subseteq \mathbb{N}^{2N}$. It is clear that the essence of the game remains the same, so, in fact, Definition 1.1 is sufficient to mathematically model games such as chess.

The reason we chose this formalism, rather than the one involving LEGAL, is purely technical: it is much easier to work with one set A rather than a combination of two sets.

At this point, it is a useful exercise to think of some concrete games where Players I and II pick natural numbers and the winning conditions are given by simple informal rules, and to try to write down the game according to Definition 1.1 (it is not relevant whether the game is interesting or trivial in practice).

Example 1.2. Consider games with the following (silly?) rules:

1. Players I and II make 10 moves each; the first person to play a 5 loses; if no 5's have been played, II wins.

   $N = 10$, $A = \{s \in \mathbb{N}^{20} \mid \exists n \leq 10\, [s(2n+1) = 5 \ \wedge \ (\forall j < 2n+1 (s(j) \neq 5))]\}$.

2. Players I and II make 100 moves each; the first person to play a number that has already been played before, loses; if no numbers have been repeated, II wins.

   $N = 100$, $A = \{s \in \mathbb{N}^{200} \mid \exists n \leq 100\, [\exists i < 2n+1 (s(2n+1) = s(i)) \ \wedge \ \forall i \neq j < 2n+1\, (s(i) \neq s(j))]\}$.

3. Players I and II make 100 moves each; player I starts by playing any number x_0; every number must be strictly larger than the one played in the previous move; when 100 moves have been played, I wins if and only if the sum of all the numbers played is a multiple of x_0.

   $N = 100$, $A = \{s \in N^{200} \mid \exists n \leq 100\, [s(2n+1) \leq s(2n) \ \wedge \ \forall j < 2n\, (s(j+1) > s(j))] \ \wedge \ \exists m\, (\sum_{n=0}^{200} s(n) = s(0) \cdot m)\}$.

The readers can experiment further with games of this kind.

1.4 Strategies

So far, we have only discussed a convenient mathematical abstraction of finite games, but we have not seen anything of mathematical importance yet. The main concept in the study of games is that of a *strategy.* Informally, a strategy for a player is a method of determining the next move based on the preceding sequence of moves (remember that both players have complete information of the preceding sequence of moves). Formally, we introduce the following definition:

Definition 1.3. Let $G_N(A)$ be a finite game of length N. A *strategy for Player I* is a function

$$\sigma : \{s \in \bigcup_{n<2N} \mathbb{N}^n \ \mid \ |s| \text{ is even } \} \longrightarrow \mathbb{N}$$

A *strategy for Player II* is a function

$$\tau : \{s \in \bigcup_{n<2N} \mathbb{N}^n \ \mid \ |s| \text{ is odd } \} \longrightarrow \mathbb{N}$$

So a strategy for Player I is a function assigning a natural number to any even sequence of natural numbers, i.e., assigning the next move to any sequence of preceding moves; the same holds for Player II. Note that it is Player I's turn to move if and only if the sequence of preceding moves is even, and Player II's turn if and only if it is odd.

Given a strategy σ for Player I, we can look at any sequence $t = \langle y_0, \ldots, y_{N-1}\rangle$ of moves by Player II, and consider the play of a game $G_N(A)$ which arises when Player I uses strategy σ and II plays the moves given by t. We denote this play by $\sigma * t$. By symmetry, if τ is a strategy for Player II and $s = \langle x_0, \ldots, x_{N-1}\rangle$ is the sequence of I's moves, we denote the result by $s * \tau$. Formally:

Definition 1.4.

1. Let σ be a strategy for player I in the game $G_N(A)$. For any $t = \langle y_0, \ldots, y_{N-1}\rangle$ we define
$$\sigma * t := \langle x_0, y_0, x_1, y_1, \ldots, x_{N-1}, y_{N-1}\rangle$$
where the x_i are given by the following inductive definition for $i < N$:
   - $x_0 := \sigma(\langle\rangle)$
   - $x_{i+1} := \sigma(\langle x_0, y_0, x_1, y_1, \ldots, x_i, y_i\rangle)$

2. Let τ be a strategy for player II in the game $G_N(A)$. For any $s = \langle x_0, \ldots, x_{N-1}\rangle$ we define
$$s * \tau := \langle x_0, y_0, x_1, y_1, \ldots, x_{N-1}, y_{N-1}\rangle$$
where the y_i are given by the following inductive definition for $i < N$:
   - $y_0 := \sigma(\langle x_0\rangle)$
   - $y_{i+1} := \sigma(\langle x_0, y_0, x_1, y_1, \ldots, x_i, y_i, x_{i+1}\rangle)$

Definition 1.5. Let $G_N(A)$ be a game and σ a strategy for Player I. We denote by

$$\mathsf{Plays}_N(\sigma) := \{\sigma * t \mid t \in \mathbb{N}^N\}$$

the set of all possible plays in the game $G_N(A)$ in which I plays according to σ. Similarly,

$$\mathsf{Plays}_N(\tau) := \{s * \tau \mid s \in \mathbb{N}^N\}$$

denotes the set of all possible plays in which II plays according to τ.

Now we introduce what may be called the most important concept in game theory:

Definition 1.6. Let $G_N(A)$ be a finite game.

1. A strategy σ is a *winning strategy for Player I* if for any $t \in \mathbb{N}^N$, $\sigma * t \in A$.
2. A strategy τ is a *winning strategy for Player II* if for any $s \in \mathbb{N}^N$, $s * \tau \notin A$.

Lemma 1.7. *For any $G_N(A)$, Players I and II cannot both have winning strategies.*

Proof. Suppose both I and II have winning strategies σ and τ. Let $\sigma * \tau$ be the result of the game where I plays according to σ and II according to τ. Then on one hand $\sigma * \tau \in A$ but on the other $\sigma * \tau \notin A$, contradiction. □

Again, it is a useful exercise to find winning strategies for I and II in simple games such as the ones in Example 1.2 (see also Exercise 2).

1.5 Determinacy of finite games

The following question naturally arises: is it always the case that either Player I or Player II has a winning strategy in a given finite game $G_N(A)$? We refer to this as the *determinacy* of a game.

Definition 1.8. A game $G_N(A)$ is called *determined* if either Player I or Player II has a winning strategy.

Theorem 1.9. *Every finite game $G_N(A)$ is determined.*

Proof. Let us analyze the concept of a winning strategy once more. On close inspection it becomes clear that Player I has a winning strategy in the game $G_N(A)$ if and only if the following holds:

- $\exists x_0 \forall y_0 \exists x_1 \forall y_1 \dots \exists x_{N-1} \forall y_{N-1} \; (\langle x_0, y_0, x_1, y_1, \dots, x_{N-1}, y_{N-1} \rangle \in A)$

So suppose I does *not* have a winning strategy. Then

- $\neg(\exists x_0 \forall y_0 \exists x_1 \forall y_1 \dots \exists x_{N-1} \forall y_{N-1} \; (\langle x_0, y_0, x_1, y_1, \dots, x_{N-1}, y_{N-1} \rangle \in A))$

By elementary rules of logic, this implies

- $\forall x_0 \neg(\forall y_0 \exists x_1 \forall y_1 \dots \exists x_{N-1} \forall y_{N-1} \; (\langle x_0, y_0, x_1, y_1, \dots, x_{N-1}, y_{N-1} \rangle \in A))$

Continuing in this fashion ($2N$ times) we derive the following true statements in sequence

- $\forall x_0 \exists y_0 \neg(\exists x_1 \forall y_1 \dots \exists x_{N-1} \forall y_{N-1} \; (\langle x_0, y_0, x_1, y_1, \dots, x_{N-1}, y_{N-1} \rangle \in A))$
- $\forall x_0 \exists y_0 \forall x_1 \neg(\forall y_1 \dots \exists x_{N-1} \forall y_{N-1} \; (\langle x_0, y_0, x_1, y_1, \dots, x_{N-1}, y_{N-1} \rangle \in A))$
- $\forall x_0 \exists y_0 \forall x_1 \exists y_1 \neg(\dots \exists x_{N-1} \forall y_{N-1} \; (\langle x_0, y_0, x_1, y_1, \dots, x_{N-1}, y_{N-1} \rangle \in A))$

  ...
- $\forall x_0 \exists y_0 \forall x_1 \exists y_1 \dots \forall x_{N-1} \exists y_{N-1} \; (\langle x_0, y_0, x_1, y_1, \dots, x_{N-1}, y_{N-1} \rangle \notin A)$

Now it is easy to see that the last statement holds if and only if Player II has a winning strategy in $G_N(A)$. □

The above theorem is frequently credited to Zermelo, though in fact, the games considered by Zermelo were somewhat more complicated and it is not very clear what he, in fact, proved about them.[1]

1.6 Back to real-life games

So what does the above have to say about actual, real-life games? Certainly, games that are zero-sum (no draws) are determined by Theorem 1.9. For chess, for example, it follows that either White has a winning strategy or Black has a strategy to win or draw. But what does this have to say about *real* chess, where a draw is an option? The easiest trick is the following: simply define two different games, call them "white-chess" and "black-chess", which are played exactly as chess but in the first case, a draw is considered a win for White and in the second case, a win for Black. Both games are

[1] See [SW01] for a discussion of this point and a translation of Zermelo's paper.

finite, zero-sum, therefore determined. If White has a winning strategy in Black-chess, then White has a strategy to win actual chess, by definition. Likewise, if Black has a winning strategy in White-chess, then Black can win chess. Both of these cannot happen at the same time (why?), however, it can happen that White has a winning strategy in White-chess and Black has one in Black-chess. For real chess, this means that White and Black have "drawing strategies", i.e., strategies such that, if they follow them, the game will result in a draw. Table 1 illustrates this:

	White wins White-chess	Black wins White-chess
White wins Black-chess	White wins chess	Impossible
Black wins Black-chess	Draw	Black wins chess

Table 1: White-chess, Black-chess and real chess.

The corollary of this (again, typically attributed to Zermelo) is:

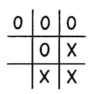
Corollary 1.10. In chess, either White has a winning strategy, or Black has a winning strategy, or both have a drawing strategy.

Of course, the above corollary merely tells us a mathematical fact, namely that *there is* such or such a strategy, and obviously does not tell us *which one it is*, which is much harder to determine. To accomplish this, one would need to parse through the tree of all possible games of chess, a feat which would involve such enormous computational power that it is practically not feasible (although there are easier games than chess for which this has been achieved—most notably checkers, by the team led by Jonathan Schaeffer in 2007). Moreover, a tree-parsing method is only possible in games with a finite number of possible moves but we have not taken any such restrictions in our setting, so Theorem 1.9 applies equally well to finite games with infinite possibilities of moves.

1.7 Exercises

1. Describe at least two ways of formalizing the game "tic-tac-toe" (noughts and crosses). What is the length of the game? What kind of winning, losing or drawing strategies do the players have?

2. Who has a winning strategy in the games from Example 1.2? Describe the winning strategy informally.



2 Finite-unbounded games

2.1 Basic definitions

Our next paradigm is a partial extension from the finite to the infinite. We stick with the real-life intuition that games end after finitely many moves, but we depart somewhat from real life in considering games that can, at least in theory, continue *ad infinitum* and remain undecided. In the next chapter, we will let go entirely of the finiteness-concept, and see that, in a sense, things will get easier. But to motivate this transition to purely infinite games, it is instructive and interesting to consider this intermediate stage, namely the class of finite-unbounded games, which also formed the main subject of concern for mathematicians in the early 20th century such as Zermelo, König, Kalmár and von Neumann (although the setting they used was not precisely the same we are presenting). It is also interesting to note that, although the games we are describing are finite in a sense, it is not possible to give a correct formalization without mentioning *infinite plays* or *infinite sequences of numbers*.

Let us start again by looking at chess (equating draw with a win for Black), but *without* the "threefold repetition rule". It is now possible that a game of chess goes on forever, with neither player winning or losing at any stage. We would like to stress that this is not the same as a draw. Although some authors (Zermelo, Kalmár) did talk of an infinite run of a game as a draw, we would like to explicitly distinguish between the two concepts. For us a "draw", such as a stalemate, is something that is determined and known at a finite stage of the game, and we handle that by equating it to a win by Black. On the other hand, a game that does not stop is conceptually different, since, at any finite stage, it may be impossible to tell the outcome of the game. In chess, there are some rules which specifically take care of this—for example, if there are only two Kings left on the board, the game is decided and called a draw, since it is clear that neither of the players can win. But there are other situations (e.g., "perpetual check") in which the game can go on forever with no official rule saying that the game has ended with a draw, other than the "threefold repetition rule" which we explicitly abandoned.

The formalization of such games, and the related concepts such as winning strategies etc., are now more involved for several reasons:

1. We cannot simply say, as we did in the previous chapter, that "when a game has been completed, extend it with 0's to form a game of length N", since now we do not have such an N.

2. Consequently, it is not sufficient to encode the game with one set A.

3. There are two different goals a player can have in mind, namely

   (a) winning the game, and

   (b) prolonging the game *ad infinitum*.

We now give a formal definition of a finite-unbounded game analogously to Definition 1.1. Recall the notation from the introduction: $\mathbb{N}^*$ stands for the set of all finite sequences of natural numbers (but without a bound on their length) and $\mathbb{N}^{\mathbb{N}}$ stands for the set of infinite sequences of natural numbers.

Definition 2.1. (Two-person, perfect-information, zero-sum, finite-unbounded game.) Let A_{I} and A_{II} be disjoint subsets of $\mathbb{N}^*$. The game $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}})$ is played as follows:

- There are two players, Player I and Player II, who take turns in picking natural numbers at each step of the game.
- At each turn i, we denote Player I's choice by x_i and Player II's choice by y_i.

I:	x_0		x_1		x_2		$\dots$	
II:		y_0		y_1		y_2		$\dots$

- The game is said to be *finished* or *decided* if $\langle x_0, y_0, \dots, x_{n-1}, y_{n-1} \rangle \in A_{\mathrm{I}} \cup A_{\mathrm{II}}$ for some value of n.
- Player I *wins the game* $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}})$ iff for some n, $\langle x_0, y_0, \dots, x_n, y_n \rangle \in A_{\mathrm{I}}$, and Player II *wins the game* $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}})$ iff, for some n, $\langle x_0, y_0, \dots, x_n, y_n \rangle \in A_{\mathrm{II}}$. The game is *undecided* iff $\langle x_0, y_0, \dots, x_n, y_n \rangle \notin A_{\mathrm{I}} \cup A_{\mathrm{II}}$ for any $n \in \mathbb{N}$.
- A_{I} is called the *pay-off set* for Player I or the *set of winning conditions* for Player I, and A_{II} the *pay-off set* or *set of winning conditions* for Player II.

For purely technical reasons, it will be convenient to assume that the pay-off sets A_{I} and A_{II} are closed under extensions. The intuition here is that, after a game has been completed, the players can still carry on playing the game, but the outcome will remain the same (it would be possible to avoid this convention, but that would complicate the definitions and proofs unnecessarily.)

Convention 2.2. (Recall the notation $s \lhd t$ for "s is an initial segment of t"). If $s \in A_{\mathrm{I}}$ and $s \lhd t$ then $t \in A_{\mathrm{I}}$. If $s \in A_{\mathrm{II}}$ and $s \lhd t$ then $t \in A_{\mathrm{II}}$.

It is easy to define strategies for finite-unbounded games in an analogous way as we did before. Since we do not have an upper bound on the length of the games, we make sure that strategies are defined on *all* sequences of natural numbers.

Definition 2.3. Let $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}})$ be a finite-unbounded game. A *strategy for Player I* is a function

$$\sigma : \{ s \in \mathbb{N}^* \mid |s| \text{ is even } \} \longrightarrow \mathbb{N}$$

A *strategy for Player II* is a function

$$\tau : \{ s \in \mathbb{N}^* \mid |s| \text{ is odd } \} \longrightarrow \mathbb{N}$$

The notation $\sigma * t$ and $s * \tau$, for $s, t \in \mathbb{N}^*$, is the same as before. Also, the sets $\mathsf{Plays}(\sigma)$ and $\mathsf{Plays}(\tau)$ of all (finite) plays of the game according to σ or τ, respectively, remains the same. The problem arises when trying to define *winning strategies*. In the finite case, it was easy—every play of the game was either a win for I or a win for II. Now, we have the additional possibility of an undecided game. With it comes the following scenario: even if a player may not have the strategy to win, he or she might still have a strategy to prevent the opponent from winning. So we have the distinct concepts of a *winning* strategy and a *non-losing* strategy. The latter can be defined without reference to infinite sequences, but the former cannot.

Definition 2.4. Let $G_{<\infty}(A_{\rm I}, A_{\rm II})$ be a finite-unbounded game.

1. (a) A strategy ∂ is a *non-losing strategy for Player I* iff for any $t \in \mathbb{N}^*$, $\sigma * t \notin A_{\rm II}$.
   (b) A strategy ρ is a *non-losing strategy for Player II* iff for any $s \in \mathbb{N}^*$, $s * \rho \notin A_{\rm I}$.
2. (a) A strategy σ is a *winning strategy for Player I* iff
   $$\forall y \in \mathbb{N}^{\mathbb{N}} \, \exists n \, (\sigma * (y \restriction n) \in A_{\rm I}).$$
   (b) A strategy τ is a *winning strategy for Player II* iff
   $$\forall x \in \mathbb{N}^{\mathbb{N}} \, \exists n \, ((x \restriction n) * \tau \in A_{\rm II}).$$

In words: a non-losing strategy guarantees that, after a finite play of the game following this strategy, the game is either undecided or a win; a winning strategy guarantees that, if the strategy is followed, then, no matter what the opponent plays, at some stage the game will be decided and result in a win (although we do not say anything about how soon this will happen). Notice that this latter concept cannot be formulated without reference to $\mathbb{N}^{\mathbb{N}}$.

As in Lemma 1.7, it is clear that I and II cannot both have winning strategies. Moreover, if I has a winning strategy, II cannot have a non-losing strategy, and vice versa (exercise). However, it is perfectly possible for both to have non-losing strategies: in that case, the result of a play where both use these strategies will always be an infinite, undecided run of the game.

Again, it is instructive to consider some games with simple, informally-defined rules.

Example 2.5. Consider finite-unbounded games with the following rules:

1. The first person to play a 5 loses.
   $A_{\rm I} = \{s \in \mathbb{N}^* \mid \exists n \, [s(2n+1) = 5 \, \wedge \, (\forall j < 2n+1(s(j) \neq 5))]\}.$
   $A_{\rm II} = \{s \in \mathbb{N}^* \mid \exists n \, [s(2n) = 5 \, \wedge \, (\forall j < 2n(s(j) \neq 5))]\}.$
2. The first person to play a number that has already been played before, loses.
   $A_{\rm I} = \{s \in \mathbb{N}^* \mid \exists n \, [\exists i < 2n+1(s(2n+1) = s(i)) \wedge \forall i \neq j < 2n+1 \, (s(i) \neq s(j))]\}.$
   $A_{\rm II} = \{s \in \mathbb{N}^* \mid \exists n \, [\exists i < 2n(s(2n) = s(i)) \, \wedge \, \forall i \neq j < 2n \, (s(i) \neq s(j))]\}.$

It is obvious that in the above games, neither I nor II has a winning strategy, but both have non-losing strategies (which one)? On the other hand, if the rules would say "the first to play a 5 wins", then I would have a winning strategy. How about the game with the rule "the second one to play a 5 wins"?

2.2 Determinacy of finite-unbounded games

How can we properly define the important concept of *determinacy*? Clearly, the definition "either I or II has a winning strategy" will not suffice, because, as we have seen in the above examples, it is not true even for very simple games. However, what we want in this setting is the following:

Definition 2.6. A game $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}})$ is called *determined* if either I has a winning strategy, or II has a winning strategy, or both I and II have non-losing strategies.

We will show that finite-unbounded games are, indeed, determined in the above sense. For this, we prove the following theorem (which we credit jointly to Zermelo/König/Kalmár or to Gale-Stewart, since it is not entirely clear who deserves the full credit for it).

Theorem 2.7 (Zermelo/König/Kalm'ar; Gale-Stewart)**.** *Let $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}})$ be a finite-unbounded game. If I does not have a winning strategy, then II has a non-losing strategy. If II does not have a winning strategy, then I has a non-losing strategy.*

Before proceeding with the proof, let's give an intuitive motivation. Suppose that, in some simple board-game, we know that Player I does not have a winning strategy. Will that always remain the case as the game progresses? In other words, after n moves have been played, will it still be the case that Player I has no winning strategy in the game *from the n-th move onwards*? Surely, this doesn't seem right. After all, Player II might make a mistake. She might play badly, meaning that even though I had no winning strategy to begin with, he might acquire one following a mistake made by Player II. But, surely, this should not happen if Player II plays optimally, in some sense? In fact, we will show that there is such an optimal strategy, i.e., the strategy of "not making any mistakes", and that this is precisely the non-losing strategy we need to prove the theorem. First we have to define what we mean by a certain *position* in a game.

Definition 2.8. Let $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}})$ be a finite-unbounded game. If $s \in \mathbb{N}^*$ is a sequence of even length, then let $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}}; s)$ denote the game in which Player I starts by playing x_0, Player II continues with y_0, etc., and Player I wins the game iff $s^\frown \langle x_0, y_0, \dots x_n, y_n\rangle \in A_{\mathrm{I}}$ for some n, and Player II wins the game iff $s^\frown \langle x_0, y_0, \dots x_n, y_n\rangle \in A_{\mathrm{II}}$ for some n

In other words, $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}}; s)$ refers to the game $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}})$ but instead of starting at the initial position, starting at position s, i.e., te position in which the first moves played are exactly $s(0), s(1), \dots, s(n-1)$ (where $n = |s|$). The reason we only consider sequences of even length is because it corresponds to a certain number of complete moves having been made (and it is again I's turn to move, as in the beginning of the game).

Lemma 2.9. *The game $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}}; s)$ is the same as the game $G_{<\infty}(A_{\mathrm{I}}/s, A_{\mathrm{II}}/s)$ where*

$$A_{\mathrm{I}}/s := \{t \in \mathbb{N}^* \mid s^\frown t \in A_{\mathrm{I}}\}$$
$$A_{\mathrm{II}}/s := \{t \in \mathbb{N}^* \mid s^\frown t \in A_{\mathrm{II}}\}$$

Proof. Exercise 1. □

Because of this lemma we do not need to introduce new terminology when talking about games at certain positions, but can simply refer to a different game. For example, I has a winning strategy in $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}}; s)$ iff he has one in $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}}; s)$, and the same for Player II.

Proof of Theorem 2.7. The conceptual idea for both directions is the same, but we will give both proofs since the details are slightly different.

Suppose Player I does not have a winning strategy in $G_{<\infty}(A_{\rm I}, A_{\rm II})$. We will define ρ in such a way that for any s, Player I does not have a winning strategy in $G_{<\infty}(A_{\rm I}, A_{\rm II}; s*\rho)$, by induction on the length of s.

The base case is $s = \langle\rangle$. By definition $s * \rho$ is also the empty sequence, i.e., the initial position of the game, so Player I does not have a winning strategy in $G(A)$ by assumption.

Assume that ρ is defined on all s of length $\leq n$, and Player I does not have a winning strategy in $G_{<\infty}(A_{\rm I}, A_{\rm II}; s*\rho)$. We will show that the same holds for sequences of length $n+1$. Fix an s with $|s| = n$.

Claim. *For all x_0 there is a y_0 such that Player I still does not have a winning strategy in $G_{<\infty}(A_{\rm I}, A_{\rm II}; (s*\rho)^\frown\langle x_0, y_0\rangle)$.*

Proof of Claim. If not, then there exists some x_0 such that for all y_0 Player I has a winning strategy, say σ_{x_0,y_0}, in the game $G_{<\infty}(A_{\rm I}, A_{\rm II}; (s*\rho)^\frown\langle x_0, y_0\rangle)$. But then Player I already had a winning strategy in the game $G_{<\infty}(A_{\rm I}, A_{\rm II}; s*\rho)$, namely the following one: *play x_0, and after II replies with y_0, continue following strategy σ_{x_0,y_0}*. This contradicts the inductive hypothesis. □ (Claim)

Now we extend ρ by defining, for every x_0, $\rho((s*\rho)^\frown\langle x_0\rangle) := y_0$, for the particular y_0 given to us by the Claim. It is clear that now I has no winning strategy in $G_{<\infty}(A_{\rm I}, A_{\rm II}; (s^\frown\langle x_0\rangle)*\rho)$ for any x_0, so we have extended the induction with one more step.

It remains to prove that ρ is a non-losing strategy for II. But suppose, towards contradiction, that it weren't: then for some s, $s*\rho \in A_{\rm I}$. But then I has a trivial winning strategy in the game $G_{<\infty}(A_{\rm I}, A_{\rm II}; (s*\rho))$, namely the trivial (empty) strategy—contradiction.

The other direction is very similar, with the roles of I and II reversed. Suppose now II has no winning strategy in $G_{<\infty}(A_{\rm I}, A_{\rm II})$. We define ∂ by induction. The base case is again the empty play; assume ∂ is defined for all t and II has no winning strategy in $G_{<\infty}(A_{\rm I}, A_{\rm II}; (\partial * t))$.

Claim. *There exists an x_0 such that, for any y_0, Player II still does not have a winning strategy in $G_{<\infty}(A_{\rm I}, A_{\rm II}; (\partial * t)^\frown\langle x_0, y_0\rangle)$.*

Proof of Claim. If not, then for all x_0 there is y_0 such that Player II has a winning strategy, say τ_{x_0,y_0}, in $G_{<\infty}(A_{\rm I}, A_{\rm II}; (\partial*t)^\frown\langle x_0, y_0\rangle)$. But then II already had a winning strategy in $G_{<\infty}(A_{\rm I}, A_{\rm II}; (\partial * t))$, namely the following one: *if I plays x_0, respond by playing y_0 as given above, and after that continue following τ_{x_0,y_0}*. This contradicts the inductive hypothesis. □ (Claim)

Now extend ∂ by defining $\partial(\partial * s) := x_0$. Clearly II has no winning strategy in $G_{<\infty}(A_{\rm I}, A_{\rm II}; (\partial * (t^\frown\langle y_0\rangle))$ for any y_0, so we have extended the induction with one more step. The rest is obvious. □

Corollary 2.10. *Finite-unbounded games are determined (in the sense of Definition 2.6).*

2.3 Upper bound on the number of moves

Except for the determinacy of finite-unbounded games which we have proved, Zermelo, König, Kalmár and von Neumann were also concerned with the following question: assuming a player does have a winning strategy, is there one (uniform) $N \in \mathbb{N}$ such that this player can win in at most N moves, regardless of the moves of the opponent? Notice that the definition of a *winning strategy* (say, for I), says "$\forall y \in \mathbb{N}^{\mathbb{N}}\, \exists n\, (\sigma * (y \restriction n) \in A_{\mathrm{I}})$". But n may, in general, depend on y. Under which conditions is there a *uniform* n, i.e., a fixed n such that $\forall y \in \mathbb{N}^{\mathbb{N}}\, (\sigma * (y \restriction n) \in A_{\mathrm{I}})$?

Zermelo in [Zer12] approached this question and claimed to prove that, assuming a finite number of possible positions in a game, there is such a uniform n. König filled a gap (or carelessness?) in Zermelo's argument, in the paper [Kön27] where he also presented his famous result nowadays called *König's Lemma*: every finitely branching, infinite tree contains an infinite branch.

Theorem 2.11 (Zermelo/König)**.** *Let $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}})$ be a finite-unbounded game. If Player I has a winning strategy σ, and if, at every move, Player II has at most finitely many options to play (otherwise she loses immediately), then there exists $N \in \mathbb{N}$ such that, if I follows σ, he will win in at most N moves. Likewise for Player II.*

Before proving this Theorem, let us see why the condition of finitely many options is necessary. Consider the board game depicted in Figure 1, with the following rule: there is a chip on field 0, Player I starts, and each player can move the chip one step forward. The first player unable to make a move loses. We leave it up to the reader to check that in this finite-unbounded game, Player I has a winning strategy, but for any $N \in \mathbb{N}$, Player II can make sure that she does not lose in less than N moves.

Proof of Theorem 2.11. The proof is a direct reformulation of the proof of König's Lemma. We will just do the case for Player I, as the other case is identical. Fix a game $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}}))$ and a winning strategy σ for Player I. Let's introduce the following terminology:

- A position $\sigma * t$ of the game is a *win-in-n* for Player I if, assuming he follows strategy σ in the game $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}}); \sigma * t)$, he will win in at most n moves (compare this to *mate-in-n* chess problems).

The goal is to prove that the initial position $\langle\rangle$ is already a win-in-n for some $n \in \mathbb{N}$. We proceed inductively. Towards contradiction, assume that the initial position is not a win-in-n, and let x_0 be I's first move. For every move y_0 by Player II, let's again introduce the following terminology:

- y_0 is *non-optimal* if there exists an n such that $\langle x_0, y_0\rangle$ is a win-in-n-position for Player I, and
- y_0 is *optimal* if that's not the case.

Now notice that, by assumption, II has at most finitely many "legal" moves. According to our formalism this means that for all y_0 except for finitely many, $\langle x_0, y_0\rangle$ is already a loss for II, trivially implying that such y_0 is non-optimal (it is a win-in-0-position for I). So there remain at most k possible values, say $y_0^0, \dots, y_0^k$. We claim that at least one of them is optimal. Assume, towards contradiction, that all of them were non-optimal. Then, for each $i \leq k$, there is an n_k such that $\langle x_0, y_0^i\rangle$ is a win-in-n_k-position

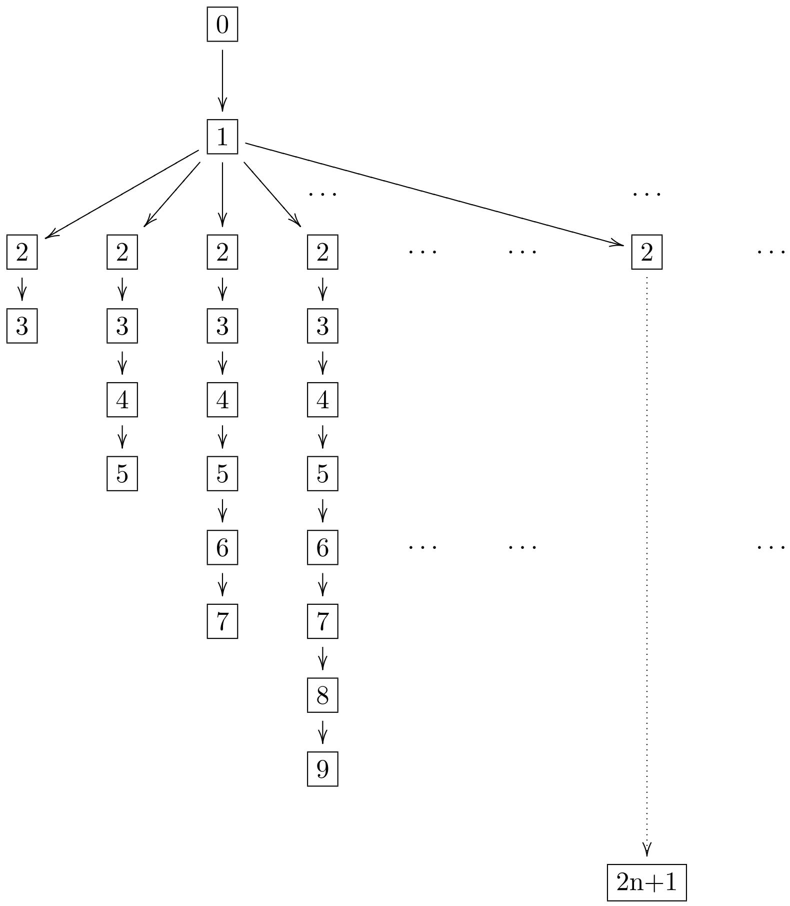


Figure 1: An infinite board-game

for I. But then, letting $N := \max\{n_i \mid i \leq k\}$, the initial position was a win-in-N for I, contradicting the assumption.

So, fix one y_0 which is optional for II, and continue inductively. Let x_1 be I's next move according to σ, and consider all possible next moves y_1. Again, only finitely many are "legal", and among those, by the same argument as above, at least one must be optimal (otherwise it would contradict the optimality of y_0 from the last step). So, again, fix such a y_1 and continue inductively.

We can clearly continue the procedure in this fashion, constructing an infinite sequence $y := \langle y_0, y_1, y_2, \dots \rangle \in \mathbb{N}^{\mathbb{N}}$, such that every y_i is optimal.

But σ is a winning strategy! By definition, there must be some $m \in \mathbb{N}^{\mathbb{N}}$ such that $\sigma * (y \restriction m)$ is a win for I. Then $\sigma * (y \restriction m)$ is a win-in-0-position for I, contradicting the assumption that all y_i were optimal. This completes the proof. $\square$

Although we gave a direct proof along the lines of König's original argument (largely for historical reason), we could easily have given a short proof by alluding to König's Lemma, see Exercise 4.

2.4 Exercises

1. Prove Lemma 2.9.

2. Prove that if I has a winning strategy then II cannot have a non-losing strategy, and vice versa.

3. Consider the games in Example 2.5. Who has winning and/or non-losing strategies in these games? Describe this strategy (informally).

4. König's Lemma says the following: if a tree contains infinitely many nodes but every node is only finitely branching, then there exists an infinite path through the tree. Prove Theorem 2.11 in the following way: let T be the tree of finite sequences t such that $\sigma * t \notin A_{\mathrm{I}}$, ordered by end-extension. Assume that there is no N such that σ is winning in at most N moves, and, using König's Lemma, conclude that σ cannot be a winning strategy.

3 Infinite games

3.1 Basic definitions

In the previous section we considered a version of finite games with no bound on the possible number of moves. We saw that, although the games we considered were finite in a sense, a formal study involves the mentioning of infinite sequences, i.e., elements of $\mathbb{N}^{\mathbb{N}}$. On the other hand, we were bound by the criterion that winning/losing must happen at a finite stage. In this section, we let go entirely of this finiteness restriction, and simply consider infinite games. This concept takes some time getting used to, but in the long-run, the readers will see that most concepts become cleaner and easier. Moreover, we will see how the finite-unbounded paradigm is easily captured in the paradigm of infinite games. Also, we no longer need to worry about the sets A_{I} and A_{II}, and we may go back to the convenient formalism used in the first chapter, namely, that a game is completely determined by its payoff set A. As from now on we will only consider infinite games, we use the simple notation $G(A)$.

Definition 3.1. (Two-person, perfect-information, zero-sum, infinite game.) Let A be an arbitrary subset of $\mathbb{N}^{\mathbb{N}}$. The game $G(A)$ is played as follows:

- There are two players, Player I and Player II, who take turns in picking natural numbers at each step of the game.
- At each turn i, we denote Player I's choice by x_i and Player II's choice by y_i.

I:	x_0		x_1		x_2		$\dots$	
II:		y_0		y_1		y_2		$\dots$

- Let $z := \langle x_0, y_0, x_1, y_1, x_2, y_2, \dots \rangle \in \mathbb{N}^{\mathbb{N}}$ be an infinite sequence, called a *play of the game* $G(A)$. Player I wins if and only if $z \in A$, otherwise II wins. A is called the *pay-off set* for Player I or the *set of winning conditions* for Player I.

Strategies are defined exactly as in the finite-unbounded case. The same holds for $\sigma * t$ and $s * \tau$, where s and t are finite. However, now it is more important to have a notation for the result of playing a strategy agains an *infinite* sequences of moves by the opponent.

Definition 3.2.

1. Let σ be a strategy for player I. For any $y = \langle y_0, y_1, \dots \rangle$ define

$$\sigma * y := \langle x_0, y_0, x_1, y_1, \dots \rangle$$

   where the x_i are given by the following inductive definition:

   - $x_0 := \sigma(\langle\rangle)$
   - $x_{i+1} := \sigma(\langle x_0, y_0, x_1, y_1, \dots, x_i, y_i \rangle)$

2. Let τ be a strategy for player II. For any $x = \langle x_0, x_1, \dots \rangle$ define

$$s * \tau := \langle x_0, y_0, x_1, y_1, \dots \rangle$$

   where the y_i are given by the following inductive definition:

- $y_0 := \sigma(\langle x_0 \rangle)$
- $y_{i+1} := \sigma(\langle x_0, y_0, x_1, y_1, \ldots, x_i, y_i, x_{i+1} \rangle)$

Also, it seems more interesting to define the following $\mathsf{Plays}(\sigma) := \{\sigma * x \mid x \in \mathbb{N}^{\mathbb{N}}\}$ and $\mathsf{Plays}(\tau) := \{y * \tau \mid y \in \mathbb{N}^{\mathbb{N}}\}$. Unlike the previous case, these two sets are subsets of $\mathbb{N}^{\mathbb{N}}$.

It is now straightforward to define winning strategies.

Definition 3.3. Let $A \subseteq \mathbb{N}^{\mathbb{N}}$ be a given pay-off set and $G(A)$ an infinite game.

1. A strategy σ is a *winning strategy for Player I* in $G(A)$ if $\forall y \in \mathbb{N}^{\mathbb{N}} \, (\sigma * y \in A)$.
2. A strategy τ is a *winning strategy for Player II* in $G(A)$ if $\forall x \in \mathbb{N}^{\mathbb{N}} \, (x * \tau \notin A)$.

There are no "non-losing" strategies in this context, as not losing is always equivalent to winning. There are no "undecided games", since all games are decided at the limit anyway. Notice how much closer this formalism is to the case of finite games. Indeed, we may consider infinite games to be games of fixed length, namely length ω (the first uncountable ordinal number).

Lemma 3.4. *For any $A \subseteq \mathbb{N}^{\mathbb{N}}$, Players I and II cannot both have winning strategies.*

Proof. Obvious. □

Let's see what kind of interesting games we can model.

Example 3.5.

1. Player I wins iff infinitely many 5's have been played.

   $A := \{x \in \mathbb{N}^{\mathbb{N}} \mid \forall n \exists m \geq n \, (x(m) = 5)\}$.

   Who has a winning strategy in this game?

2. I and II pick numbers x_i and y_i. Let z be the result of the infinite game. Player I wins if and only if the infinite sum of the $\frac{1}{x_i+1}$'s and $\frac{1}{y_i+1}$'s, for all i, is convergent.

   $A := \{z \in \mathbb{N}^{\mathbb{N}} \mid \sum_{n=0}^{\infty} \frac{1}{z(n)+1} < \infty\}$.

   Who has a winning strategy in this game?

3. Now, the game is as above, but with the additional assumption that, at every step, II's move must be at least as large as the preceding move by I.

   $A := \{z \in \mathbb{N}^{\mathbb{N}} \mid \forall n \, (z(2n+1) \geq z(2n)) \, \wedge \, \sum_{n=0}^{\infty} \frac{1}{z(n)+1} < \infty\}$.

   Now who has a winning strategy?

3.2 Cardinality arguments

Before getting into the determinacy of infinite games, let us prove some easy results that might give the flavour of what is to come. As will become increasingly clear, the study of infinite games $G(A)$ is essentially related to the complexity of the sets A themselves, among other things, the cardinality of A.

Lemma 3.6. *Let $A \subseteq \mathbb{N}^{\mathbb{N}}$ be a countable set. Then II has a winning strategy in $G(A)$.*

Proof. Let $\{a_0, a_1, a_2, \dots\}$ enumerate A. Let τ be the strategy that says "at your i-th move, play any natural number different from $a_i(2i+1)$ (this is the $(2i+1)$-st digit of the i-th element of A)". Let z be the result of this strategy against anything played by Player I, and write $z := \langle x_0, y_0, x_1, y_1, \dots \rangle$. By construction, for each i:

$$z(2i+1) = y_i \neq a_i(2i+1)$$

Hence, for each i, $z \neq a_i$. $\square$

We continue with some more "cardinality arguments". Recall that $\mathbb{N}^{\mathbb{N}}$ has the cardinality of the continuum $2^{\aleph_0}$.

Lemma 3.7. *Let A be a set with $|A| < 2^{\aleph_0}$. Then Player I cannot have a winning strategy in $G(A)$.*

Proof. Assume, towards contradiction, that σ is a winning strategy for Player I. It is easy to see that for all $y_1, y_2 \in \mathbb{N}^{\mathbb{N}}$, if $y_1 \neq y_2$ then $\sigma * y_1 \neq \sigma * y_2$, so there is an injection from $\mathbb{N}^{\mathbb{N}}$ to the set $\mathsf{Plays}(\sigma)$. But since σ is winning, $\mathsf{Plays}(\sigma) \subseteq A$, contradicting $|A| < 2^{\aleph_0}$. $\square$

Note that if the Continuum Hypothesis is true, i.e., if $2^{\aleph_0}$ is the smallest uncountable cardinality, then Lemma 3.7 follows from Lemma 3.6 and Lemma 3.4. So Lemma 3.7 has relevance only if the Continuum Hypothesis is false.

Obviously these two theorems also hold with the roles of I and II reversed, i.e., if $\mathbb{N}^{\mathbb{N}} \setminus A$ is countable then I has a winning strategy, and if $|\mathbb{N}^{\mathbb{N}} \setminus A| < 2^{\aleph_0}$ then II cannot have a winning strategy.

3.3 Determinacy of infinite games

We come to the main point of the theory. Since, as we mentioned, every infinite game results in a win for exactly one of the players (and there no undecided games), the only sensible definition of determinacy is the generalization of the concept from finite games.

Definition 3.8. A game $G(A)$ is *determined* if either Player I or Player II has a winning strategy.

Here we see the first essential difference between infinite games and their finitary version.

Theorem 3.9 ([MS62])**.** *Not every infinite game is determined.*

The proof is by transfinite induction on ordinals $\alpha < 2^{\aleph_0}$. Since we do not assume familiarity with ordinals in this course, we supply a black box result which encompasses exactly what we need for the proof.

Lemma 3.10. *For every set X, there exists a well-ordered set $(I, \leq)$, which we call the* index set for X*, such that*

1. $|I| = |X|$, *and*
2. *For every $\alpha \in I$, the set $\{\beta \in I \mid \beta < \alpha\}$ has cardinality strictly less than $|I| = |X|$.*

Proof. (Ordinal and cardinal theory.) By the Axiom of Choice every set X can be well-ordered, hence there is an ordinal α order-isomorphic to it. Let κ be $|\alpha| = |X|$, i.e., the least ordinal in bijection with α. Since κ is a cardinal, clearly $|\kappa| = \kappa = |\alpha| = |X|$, and for any $\gamma < \kappa$, the set $\{\beta < \kappa \mid \beta < \gamma\} = \beta$ has cardinality $< \kappa$. So $(\kappa, \in)$ is the desired index set. □

Those unfamiliar with ordinals can treat this lemma as a black box result and ignore its proof. Intuitively, one can compare the situation with that of a countable set X, in which case the index set is simply $(\mathbb{N}, \leq)$.

Proof of Theorem 3.9. We start by counting the possible number of strategies. A strategy is a function from a subset of $\mathbb{N}^*$ to $\mathbb{N}$. But $\mathbb{N}^*$ is countable, so, there are as many strategies as functions from a countable set to a countable set, namely $2^{\aleph_0}$.

Let $\mathsf{Strat}(\mathrm{I})$ be the set of all possible strategies of Player I and $\mathsf{Strat}(\mathrm{II})$ the set of strategies of Player II. Applying Lemma 3.10, let I be an index set, of cardinality $2^{\aleph_0}$, which is in bijection with $\mathsf{Strat}(\mathrm{I})$ and $\mathsf{Strat}(\mathrm{II})$. Now we can use I to "enumerate" the strategies, as follows:

$$\mathsf{Strat}(\mathrm{I}) = \{\sigma_\alpha \mid \alpha \in I\}$$
$$\mathsf{Strat}(\mathrm{II}) = \{\tau_\alpha \mid \alpha \in I\}$$

We will produce two subsets of $\mathbb{N}^{\mathbb{N}}$: $A = \{a_\alpha \mid \alpha \in I\}$ and $B = \{b_\alpha \mid \alpha \in I\}$, by induction on $(I, \leq)$.

- **Base case:** Let $0 \in I$ stand for the $\leq$-least member of I. Arbitrarily pick any $a_0 \in \mathsf{Plays}(\tau_0)$. Now, $\mathsf{Plays}(\sigma_0)$ clearly contains more than one element, so we can pick $b_0 \in \mathsf{Plays}(\sigma_0)$ such that $b_0 \neq a_0$.

- **Induction step:** Let $\alpha \in I$ and suppose that for all $\beta < \alpha$, a_β and b_β have already been chosen. We will chose a_α and b_α.

  Note that since $\{b_\beta \mid \beta < \alpha\}$ is in bijection with $\{\beta \in I \mid \beta < \alpha\}$, it has cardinality strictly less than $2^{\aleph_0}$ (by Lemma 3.10 (2)). On the other hand, we already saw that $\mathsf{Plays}(\tau_\alpha)$ has cardinality $2^{\aleph_0}$. Therefore there is at least one element in $\mathsf{Plays}(\tau_\alpha)$ which is not in $\{b_\beta \mid \beta < \alpha\}$. Pick any one of these and call it a_α.

  Now do the same for the collection $\{a_\beta \mid \beta < \alpha\} \cup \{a_\alpha\}$. This still has cardinality less than $2^{\aleph_0}$ whereas $\mathsf{Plays}(\sigma_\alpha)$ has cardinality $2^{\aleph_0}$, so we can pick a b_α in $\mathsf{Plays}(\sigma_\alpha)$ which is not a member of $\{a_\beta \mid \beta < \alpha\} \cup \{a_\alpha\}$.

This completes the inductive definition of A and B. Now we claim the following:

Claim 1. $A \cap B = \varnothing$.

Proof. Take any $a \in A$. By construction, there is some $\alpha \in I$ such that $a = a_\alpha$. Now, recall that at "stage α" of the inductive procedure, we made sure that a_α is not equal to b_β for any $\beta < \alpha$. On the other hand, at each "stage γ" for $\gamma \geq \alpha$, we made sure that b_γ is not equal to a_α. Hence a_α is not equal to any $b \in B$. □(Claim 1)

Claim 2. *$G(A)$ is not determined.*

Proof. First, assume that I has a winning strategy σ in $G(A)$. Then $\mathsf{Plays}(\sigma) \subseteq A$. But there is an $\alpha \in I$ such that $\sigma = \sigma_\alpha$. At "stage α" of the inductive procedure we picked a $b_\alpha \in \mathsf{Plays}(\sigma_\alpha)$. But by Claim 1, b_α cannot be in A—contradiction

Now assume II has a winning strategy τ in $G(A)$. Then $\mathsf{Plays}(\tau) \cap A = \varnothing$. Again, $\tau = \tau_\alpha$ for some α, but at "stage α" we picked $a_\alpha \in \mathsf{Plays}(\tau_\alpha)$—contradiction. □(Claim 2)

By a similar argument, $G(B)$ is not determined either. □

The above prove is entirely non-constructive, i.e., it shows that an undetermined infinite game must exist, but gives no information as to what this game really is. It also uses the Axiom of Choice in an essential way (we will come back to this in Section 4.4). Therefore, it is similar to the proof that, e.g., there exists a non-Lebesgue-measurable set (Vitali set), a set without the property of Baire etc.

Although this is a delimitative result, showing that we cannot hope to prove that *all* games are determined, we will go on to show that this does not undermine the whole enterprise of infinite game theory. For example, the games from Example 2.5, despite having infinitary rules, were determined. Also, consider the following: if $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}})$ is a finite-unbounded game, define $A = \{x \in \mathbb{N}^{\mathbb{N}} \mid \exists s \in A_{\mathrm{I}}\ (s \lhd x)\}$, i.e., A is the set of all infinite extensions of completed finite games that are won by I. The *Gale-Stewart theorem*, which we present in the next section, is precisely the statement that such games are determined.

Although the Gale-Stewart theorem can be formulated and proved directly using Corollary 2.10, it is best understood in the language of a topology on the space $\mathbb{N}^{\mathbb{N}}$. In fact, the connections between infinite games and topology are numerous, surprising and very fruitful. The bottom line is that there is an intrinsic relation between determinacy of games $G(A)$ and the complexity of the set A, where *complexity* is measured topologically.

3.4 Exercises

1. Describe the winning strategies of the games in Example 3.5.

2. Let $z \in \mathbb{N}^{\mathbb{N}}$ be an infinite sequence. Describe informally the game $G(A)$ where $A = \{z\}$. Who has a winning strategy in this game? How many moves does that player need to make sure he or she has won the game?

3. For every set $A \subseteq \mathbb{N}^{\mathbb{N}}$ and every $n \in \mathbb{N}$, define

$$\langle n \rangle ^\frown \overline{A} := \{\langle n \rangle ^\frown x \mid x \notin A\}$$

(a) Prove, or at least argue informally, that for every $A \subseteq \mathbb{N}^{\mathbb{N}}$, Player II has a winning strategy in $G(A)$ if and only if for every n, Player I has a winning strategy in $G(\langle n \rangle ^\frown \overline{A})$.

(b) Similarly, prove that for every $A \subseteq \mathbb{N}^{\mathbb{N}}$, Player I has a winning strategy in $G(A)$ if and only if there is some n such that Player II has a winning strategy in $G(\langle n \rangle ^\frown \overline{A})$.

4.* Adapt the proof of Theorem 3.9 to prove that the property of "being determined" is not closed under complements, i.e., that there is a set A such that $G(A)$ is determined but $G(\mathbb{N}^{\mathbb{N}} \setminus A)$ is not.

4 Topology on $\mathbb{N}^\mathbb{N}$, Gale-Stewart, and Borel determinacy

4.1 Topology on the Baire space

The following is called the *standard topology* on $\mathbb{N}^\mathbb{N}$.

Definition 4.1. For every $s \in \mathbb{N}^*$, let

$$O(s) := \{x \in \mathbb{N}^\mathbb{N} \mid s \lhd x\},$$

i.e., $O(s)$ is the set of all infinite sequences of natural numbers of which s is an initial segment. The *standard topology* on $\mathbb{N}^\mathbb{N}$ is generated by the basic open sets $\{O(s) \mid s \in \mathbb{N}^*\}$. The space $\mathbb{N}^\mathbb{N}$ endowed with this topology is called the *Baire space*.

This is equivalent to the *product topology* (*Tychonoff topology*) generated by countably many copies of $\mathbb{N}$ with the discrete topology. It is also equivalent to the topology inherited from the following metric:

$$d(x,y) := \begin{cases} 0 & \text{if } x = y \\ 1/2^n & \text{where } n \text{ is least s.t. } x(n) \neq y(n) \end{cases}$$

The following basic properties of the topology are easy to see, but useful to keep in mind:

Definition 4.2. Let $s, t \in \mathbb{N}^*$. We say that s and t are *compatible*, notation $s||t$, if either $s \lhd t$ or $t \lhd s$ (or $s = t$). Otherwise s and t are called *incompatible*, denoted by $s \perp t$.

Fact 4.3.

1. *$s \lhd t$ if and only if $O(t) \subseteq O(s)$,*
2. *$s||t$ if and only if $O(s) \subseteq O(t)$ or $O(t) \subseteq O(s)$,*
3. *$s \perp t$ if and only if $O(s) \cap O(t) = \varnothing$,*
4. *$O(s) \cap O(t)$ is either $\varnothing$ or basic open,*
5. *for any $x \in \mathbb{N}^\mathbb{N}$ and $n \in N$, $O(x \restriction n)$ is the open ball around x with radius $\epsilon = 1/2^n$.*

The Baire space has many similarities with the real line $\mathbb{R}$—in fact, so many that pure set theorists prefer to study the Baire space instead of $\mathbb{R}$ and call elements $x \in \mathbb{N}^\mathbb{N}$ *real numbers*. The Baire space is homeomorphic to the space of irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ with the standard topology. But there are also some differences: the Baire space is *totally disconnected*, as follows from the following lemma.

Lemma 4.4. *For every s, $O(s)$ is clopen.*

Proof. Show that $\mathbb{N}^\mathbb{N} \setminus O(s) = \bigcup\{O(t) \mid |t| = |s| \text{ and } t \neq s\}$. □

It is also useful to keep in mind what convergence and continuity are in this topology:

Fact 4.5.

1. *Let* $\{x_n\}_{n\in\mathbb{N}}$ *be an infinite sequence of elements of* $\mathbb{N}^\mathbb{N}$*. Then* $x = \lim_{n\to\infty} x_n$ *iff the following holds*
$$\forall s \lhd x\, \exists N\, \forall n \geq N\, (s \lhd x_n).$$
In words: if we fix a finite initial segment of x*, eventually every* x_n *will extend it. We will usually just write "*$x_n \to x$*".*

2. *A function* $f : \mathbb{N}^\mathbb{N} \longrightarrow \mathbb{N}^\mathbb{N}$ *is continuous iff for each* $x \in \mathbb{N}^\mathbb{N}$ *we have*
$$\forall s \lhd f(x)\, \exists t \lhd x\, \forall y\, (t \lhd y \to s \lhd f(y))$$
In words: if we fix a finite initial segment s *of* $f(x)$*, then we can find a finite initial segment* t *of* x*, such that if* y *extends* t *then* $f(y)$ *extends* s*.*

4.2 The Gale-Stewart theorem, and relation to finite-unbounded games

If A is an open set, then $A = \bigcup\{O(s) \mid s \in E\}$ for some subset $E \subseteq \mathbb{N}^*$. Therefore, if any *infinite* sequence x is in A, then there must be a *finite* initial segment $s \lhd x$, such that *any other* y extending s will also be in A. In other words: *membership of* x *in* A *is secured at a finite stage.* Games $G(A)$ for such sets A are determined, as proved by Gale-Stewart in [GS53].

Theorem 4.6 (Gale-Stewart)**.** *If* A *is open or closed, then* $G(A)$ *is determined.*

Proof. Suppose A is open. Assume I does not have a winning strategy. As in the proof of Theorem 2.7, say that $G(A;s)$ is the game $G(A)$ played starting from position s, i.e., $G(A;s) := G(A/s)$ where $A/s := \{x \in \mathbb{N}^\mathbb{N} \mid s^\frown x \in A\}$. Using the exact same argument as in the proof of Theorem 2.7, we can construt a stragegy ρ for Player II, such that, for all $t \in \mathbb{N}^*$, I does not have a winning strategy in the game $G(A; (t * \rho))$.

But then, we claim that ρ must in fact be a winning strategy for Player II. Suppose, towards contradiction, that $x \in \mathbb{N}^\mathbb{N}$ is such that $x * \rho \in A$. Since A is open, there is an initial segment $s \lhd (x * \rho)$ (wlog. $|s|$ is even) such that all $y \in O(s)$ are in A. But that means that I has a winning strategy in $G(A;s)$, contradicting the definition of ρ.

Similarly, if A is closed, then assume II does not have a winning strategy, and by the analogous argument produce a strategy ∂ for I such that II does not have a winning strategy in $G(A; \partial * t)$, for any t. Then ∂ is winning for I by the same argument as above, using that $\mathbb{N}^\mathbb{N} \setminus A$ is open. □

The Gale-Stewart theorem is, essentially, a restatement of Corollary 2.10 which said that finite-unbounded games are determined. Let us now see precisely how the finite-unbounded paradigm can be translated into the infinite one and vice versa. Recall that when we handled finite games in Chapter 1, we required that all games have a fixed length N, even if, actually, the game ended much earlier. Here we can do the same, but instead of a fixed length N, we used the fixed uncountable length ω. What used to be an "undecided game" can now be called a "draw"; and "draws", as we have seen in Chapter 1, can be handled by defining two games, one in which the draw is a win for I and another in which it is a win for II. This is exactly what we now use.

Let $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}})$ be a finite-unbounded game. Let $\tilde{A}_{\mathrm{I}} := \bigcup\{O(s) \mid s \in A_{\mathrm{I}}\}$ and $\tilde{A_{\mathrm{II}}} := \bigcup\{O(s) \mid s \in A_{\mathrm{II}}\}$. So every $x \in \tilde{A}_{\mathrm{I}}$ is an infinite extension of a finite play which is a win for I in $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}})$, and every $x \in \tilde{A}_{\mathrm{I}}$ is an infinite extension of a finite play which is a win for II in $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}})$. Clearly $\tilde{A}_{\mathrm{I}}$ and $\tilde{A_{\mathrm{II}}}$ are disjoint, but there may be $x \in \mathbb{N}^{\mathbb{N}}$ which are neither in $\tilde{A}_{\mathrm{I}}$ nor in $\tilde{A_{\mathrm{II}}}$, namely, the undecided plays of $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}})$. If we stipulate that such undecided plays are a win for Player II, we get the infinite game $G(\tilde{A}_{\mathrm{I}})$. On the other hand, if such undecided plays are wins for I, we get the infinite game $G(\mathbb{N}^{\mathbb{N}} \setminus \tilde{A_{\mathrm{II}}})$.

Lemma 4.7. *Let $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}})$ be a finite-unbounded game and $\tilde{A}_{\mathrm{I}}$ and $\tilde{A_{\mathrm{II}}}$ as above. Then:*

1. *I has a winning strategy in $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}})$ iff I has a winning strategy in $G(\tilde{A}_{\mathrm{I}})$.*
2. *I has a non-losing strategy in $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}})$ iff I has a winning strategy in $G(\mathbb{N}^{\mathbb{N}} \setminus \tilde{A_{\mathrm{II}}})$.*
3. *II has a winning strategy in $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}})$ iff II has a winning strategy in $G(\mathbb{N}^{\mathbb{N}} \setminus \tilde{A}_{\mathrm{I}})$.*
4. *II has a non-losing strategy in $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}})$ iff II has a winning strategy in $G(\tilde{A}_{\mathrm{I}})$.*

Proof. We leave the details as an exercise to the reader. It is a straightforward matter of translating the definitions of winning/non-losing strategies. □

Since $\tilde{A}_{\mathrm{I}}$ and $\tilde{A_{\mathrm{II}}}$ are open sets, all the infinite games in the lemma are determined. Therefore, the determinacy of finite-unbounded games (Corollary 2.10) follows from the Gale-stewart theorem, using the same trick we used to analyse chess (cf. Table 1).

	I wins $G(\mathbb{N}^{\mathbb{N}} \setminus \tilde{A_{\mathrm{II}}})$	II wins $G(\mathbb{N}^{\mathbb{N}} \setminus \tilde{A_{\mathrm{II}}})$
I wins $G(\tilde{A}_{\mathrm{I}})$	I wins $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}})$	Impossible
II wins $G(\tilde{A}_{\mathrm{I}})$	$G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}})$ is undecided (both I and II have non-losing strategies)	II wins $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}})$

Table 2: Finite-unbounded games vs. infinite games

In the other direction, a given infinite game $G(A)$ for open A can be translated to a finite-unbounded game, as follows. Since A is a union of basic open sets $O(s)$, we can find $A_{\mathrm{I}} \subseteq \mathbb{N}^*$ such that $A = \bigcup\{O(s) \mid s \in A_{\mathrm{I}}\}$. On other hand, let $A_{\mathrm{II}} := \{s \in \mathbb{N}^* \mid O(s) \subseteq (\mathbb{N}^{\mathbb{N}} \setminus A)\}$. Notice that $\bigcup A_{\mathrm{II}}$ is the *topological interior* of the closed set $\mathbb{N}^{\mathbb{N}} \setminus A$.

Lemma 4.8.

1. *I has a winning strategy in $G(A)$ iff I has a winning strategy in $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}})$.*

2. *II has a winning strategy in $G(A)$ iff II has a non-losing strategy in $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}})$.*

Proof. Again, a direct consequence of the definitions. □

Similarly, if A was closed, we could apply the same trick to $\mathbb{N}^{\mathbb{N}} \setminus A$. The bottom line is that the Gale-stewart theorem follows from Theorem 2.7.

4.3 Beyond open and closed

Although games $G(A)$ for open and closed A represent a direct abstraction from finite-unbounded games, they are pretty simple from a topological point of view. Once we are in the infinitary context, why not look at more complex pay-off sets? The games from Example 2.5 are not open or closed, but still easily seen to be determined. So the natural question is: what about the determinacy of games with more complex pay-off sets? For example, what about F_σ and G_δ sets? In 1955 Philip Wolfe [Wol55] proved the determinacy of F_σ and G_δ sets, in 1964 (almost ten years later!) Morton Davis [Dav64] proved determinacy of the next level in the Borel hierarchy. Finally, in 1975 (again about ten years later!) Tony Martin [Mar75] proved *Borel determinacy*, a result that had been expected for a while.

Theorem 4.9 (Tony Martin)**.** *If $A \subseteq \mathbb{N}^{\mathbb{N}}$ is Borel then $G(A)$ is determined.*

Unfortunately, it is far beyond the scope of our course to provide a proof of this theorem. Interested readers can find a clear exposition in [Kec95, pages 140–146].

While the Borel sets are sufficient for many applications in topology and analysis, there are some natural mathematical operations that transcend them, for example, the *analytic* sets (continuous images of Borel sets), then the *coanalytic sets* (complements of analytic sets), and in general, the *projective sets* (obtained by an iterated application of the operations of taking complements and continuous images). In general, determinacy is usually studied in the following context:

Definition 4.10. Let $\mathbf{\Gamma} \subseteq \mathscr{P}(\mathbb{N}^{\mathbb{N}})$ be a collection of subsets of $\mathbb{N}^{\mathbb{N}}$. We call $\mathbf{\Gamma}$ a *boldface point-class*[2] iff it is:

1. closed under continuous pre-images (i.e., $A \in \mathbf{\Gamma} \rightarrow f^{-1}[A] \in \mathbf{\Gamma}$), and

2. closed under intersections with closed sets (i.e., $A \in \mathbf{\Gamma}$ and C closed $\rightarrow A \cap C \in \mathbf{\Gamma}$).

Closed, F_σ, G_δ, and Borel sets $\mathcal{B}$, are all examples of boldface pointclasses, as well as the analytic, coanalytic sets and further classes of the projective hierarchy. $\mathscr{P}(\mathbb{N}^{\mathbb{N}})$ is a trivial boldface pointclass.

Definition 4.11. For a boldface pointclass $\mathbf{\Gamma}$, "$\mathsf{Det}(\mathbf{\Gamma})$" abbreviates the statement: "for every $A \in \mathbf{\Gamma}$, $G(A)$ is determined."

So the Gale-Stewart theorem says Det(open) and Det(closed), and Martin's Theorem 4.9 says $\mathsf{Det}(\mathcal{B})$. On the other hand, Theorem 3.9 says $\neg\mathsf{Det}(\mathscr{P}(\mathbb{N}^{\mathbb{N}}))$.

It turns out that if we focus on pointclasses $\mathbf{\Gamma}$ that extend the Borel sets, but are still below $\mathscr{P}(\mathbb{N}^{\mathbb{N}})$, then typically the statement $\mathsf{Det}(\mathbf{\Gamma})$ is independent of the basic axioms

[2]The name comes from the fact that such classes were traditionally denoted by boldface letters. Although this funny blend of syntax and semantics might theoretically lead to problems, in practice this does not occur, because it is always clear from context which $\mathbf{\Gamma}$ one has in mind.

of set theory, i.e., it is not possible to prove or refute the statement based on the axioms alone. Nevertheless, as long as $\mathsf{Det}(\mathbf{\Gamma})$ is not outright contradictory, we can take it as an axiom and look at its consequences.

4.4 Axiom of Determinacy and the Axiom of Choice

In this section presents a "meta-mathematical" reflection concerning determinacy and the Axiom of Choice AC. It is of no direct relevance for the rest of our course, but still represents an interesting point of view.

Mycielski and Steinhaus [MS62] pondered on the determinacy of all infinite games and proposed the following axiom:

Definition 4.12. The Axiom of Determinacy (abbreviated by AD) is the statement

All infinite games are determined.

Of course, Mycielski and Steinhaus were aware of Theorem 3.9, but thought that AD provided an interesting alternative view to mathematics. Formally, this mathematics is encoded in the axiomatic theory $\mathsf{ZF}+\mathsf{AD}$, i.e., Zermelo-Fraenkel set theory, *without* the Axiom of Choice, together with the Axiom of Determinacy. As we will see in Part II, determinacy has many desirable consequences in the fields of analysis and topology, and mathematics in which AC has been replaced by AD implies, in particular, that all those consequences hold, and typical *pathological* examples of sets with bad behavior do not exist.

The immediate worry of a mathematician is that, if we banish AC altogether, many familiar facts may not hold any more (for example, to prove that a countable union of countable sets is countable requires Choice). Fortunately, AD itself implies a weak version of AC.

Definition 4.13. The *Axiom of Countable Choices over* $\mathcal{X}$, denoted by $\mathsf{AC}_{\mathsf{ctbl}}(\mathcal{X})$, is the following statement:

Every countable collection $\{X_n \mid n \in \mathbb{N}\}$ *of*
non-empty subsets $X_i \subseteq \mathcal{X}$ *has a choice function.*

Lemma 4.14. AD *implies* $\mathsf{AC}_{\mathsf{ctbl}}(\mathbb{N}^{\mathbb{N}})$.

Using standard coding arguments, it is easy to replace $\mathbb{N}^{\mathbb{N}}$ in the above statement by $\mathbb{R}$, $\mathscr{P}(\mathbb{N})$, $2^{\mathbb{N}}$ etc.

Proof. Let $\{X_n \mid n \in \mathbb{N}\}$ be a collection of non-empty subsets of $\mathbb{N}^{\mathbb{N}}$. Consider the following infinite game: Players I and II choose x_i and y_i as usual; let $y := \langle y_0, y_1, y_2, \dots\rangle$ be the sequence of II's moves; Player II wins iff $y \in X_n$. Formally, this game is given by the pay-off set

$$A := \{z \in \mathbb{N}^{\mathbb{N}} \mid (z)_{\text{odd}} \notin X_{z(0)}\},$$

where $(z)_{\text{odd}}$ is defined by $(z)_{\text{odd}}(n) = z(2n+1)$. Note that only the first move of Player I matter for the game—his other moves are irrelevant.

By AD this game is determined. Assume first that I has a winning strategy σ in $G(A)$. Let $n := \sigma(\langle\rangle)$ be the first move. No matter which sequence $y \in \mathbb{N}^{\mathbb{N}}$ Player II will play, I will win the game, implying that $y \notin X_n$. But II can play anything, so this means that $X_n = \varnothing$. But we have assumed that this was not the case.

Therefore, Player II must have a winning strategy τ. But then, define f as follows: for any n, let $f(n)$ be the sequence of II's moves, played according to τ, if I plays the sequence $\langle n, 0, 0, 0, \dots \rangle$. Since τ is a winning strategy, $f(n)$ is an element of X_n. This holds for every n, so f is a choice function for $\{X_n \mid n \in \mathbb{N}\}$. □

This countable choice principle is sufficient for many mathematical results (e.g., the countable union of countable sets is countable), although obviously not for "there exists a well-order of $\mathbb{N}^{\mathbb{N}}$".

We end Part I by mentioning some meta-mathematical considerations (without too many details, and assuming some knowledge of abstract set theory on the reader's part) relating determinacy with *large cardinals*. In set theory, it has become customary (since the latter half of the 20th century) to consider additional mathematical axioms, postulating the existence of cardinals with very strong combinatorial properties. The existence of such cardinals cannot be proved from the standard set theory axioms ZFC, however, their introduction is seen as a natural generalization of the principles leading to mathematical axioms. The process is similar to postulating that *infinite sets exist*, although this does not follow from the other set theoretic axioms. More about large cardinals can be found in [Kan03].

Following the introduction of AD and the concept of infinite games in abstract mathematics, a prominent research programme among set theorists became the following: prove determinacy for larger pointclasses $\mathbf{\Gamma}$ assuming large cardinals. For example:

Theorem 4.15 (Tony Martin [Mar70])**.** *If there exists a* measurable cardinal *then $G(A)$ is determined for all analytic A.*

Further progress followed from the pivotal work of Martin, Steel and Woodin [MS89].

Theorem 4.16 (Martin-Steel-Woodin)**.**

1. *If there exist n Woodin cardinals and a measurable cardinal above them, then $G(A)$ is determined for every $\mathbf{\Pi}^1_{n+1}$ set A.*
2. *If there are infinitely many Woodin cardinals, then $G(A)$ is determined for every projective A.*
3. *If there are infinitely many Woodin cardinals and a measurable above them, then there exists a class of sets, called $L(\mathbb{R})$, which satisfies* ZF + AD *(i.e., all the set-theoretic axioms without the Axiom of Choice, but with the Axiom of Determinacy).*

The third part, in particular, implies that the Axiom of Determinacy is, in fact, consistent *to the extent that it does not contradict* AC, i.e., it holds in the natural model of sets where the Axiom of Choice fails ($L(\mathbb{R})$ can be seen as the "constructive" or "definable" component of the mathematical universe).

We shall let these considerations rest, and in Part II look at the various interesting consequences of determinacy of infinite games.

4.5 Exercises

1. Prove that the metric d defined in Section 4.1 satisfies the triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$.

2. Prove that the Baire space is Hausdorff, i.e., that for any two $x \neq y$ there are two disjoint open neighbourhoods of x and y.

3. A topological space is called *totally separated* if for every two $x \neq y$ there are open sets U and V such that $x \in U$, $y \in V$ and $U \cup V$ equals the whole space. Prove that the Baire space is totally separated.

4. Prove the statements in Fact 4.3 and Fact 4.5.

5. Give a detailed proof of Lemma 4.7 and Lemma 4.8.

6.* Consider the modification of infinite games, where Players I and II choose not natural numbers at each stage, but objects from some set X. Let AD_X be the postulate that such games are determined, and let $\mathsf{AD}_{\text{everything}}$ be the postulate $\forall X\, (\mathsf{AD}(X))$. Show that, based on ZF alone (i.e., without Choice), $\mathsf{AD}_{\text{everything}}$ is false.

   Hint: first prove that $\mathsf{AD}_{\text{everything}}$ implies full AC, by a similar argument as in Lemma 4.14.

   Remark: in fact, already the statement $\mathsf{AD}(\omega_1)$, where ω_1 denotes the first uncountable ordinal, is false based on ZF alone.

Part II

Applications of infinite games to analysis, topology and set theory

5 Introduction

We have now arrived at the stage promised in the introduction, namely where we can construct and use infinite games as tools in the study of various mathematical objects. We will be interested in various problems related to the study of the continuum. By the *continuum* we will usually mean the Baire space $\mathbb{N}^{\mathbb{N}}$ with the standard topology defined in Section 4.1, but most results (with the exception of those in Section 9) can easily be adapted to work for $\mathbb{R}$ or $\mathbb{R}^n$. When we treat the case of Lebesgue measure, we will explicitly work with $\mathbb{R}$.

In each situation, we are going to define a specific infinite game $G(A)$, and show that *if* $G(A)$ is determined *then* A satisfies some desirable property (e.g., is Lebesgue measurable). Such results can be viewed in one of two ways:

- **View 1:** Assuming $\mathsf{ZF} + \mathsf{AD}$, all sets A satisfy the desirable property (e.g., are Lebesgue measurable), or
- **View 2:** Assuming only $\mathsf{Det}(\boldsymbol{\Gamma})$ for a fixed boldface pointclass $\boldsymbol{\Gamma}$, all sets A in $\boldsymbol{\Gamma}$ have this desirable property (e.g., are Lebesgue measurable).

Proving the first result is easier technically, but is problematic because of the contradiction with the Axiom of Choice. We will usually focus on the second point of view, however, this will require that we check whether the coding we use in the games preserves membership in $\boldsymbol{\Gamma}$, which requires a little extra work (but usually not much). In this short section we explain how this is done.

The point is that the games we need are not exactly the infinite games that fall under Definition 3.1. For example, the games may require the players to play other mathematical objects than natural numbers, and the winning condition may be rather complicated. To deal with this problem we must introduce the technique of *continuous coding* which allows us to treat complex rules as standard games falling under Definition 3.1. As motivation let us consider the following example (to which we will return in section 7.2 in more detail):

Definition 5.1. Let $A \subseteq \mathbb{N}^{\mathbb{N}}$ be a set, and consider the following game, called the *Banach-Mazur game* and denoted by $G^{**}(A)$: Players I and II alternate in taking turns, but instead of natural numbers, they play *non-empty sequences of natural numbers* $s_i, t_i \in \mathbb{N}^*$:

I:	s_0		s_1		$\dots$	
II:		t_0		t_1		$\dots$

Then let $z := s_0{}^\frown t_0{}^\frown s_1{}^\frown t_1{}^\frown \dots$ and say that Player I wins if and only if $z \in A$.

This game, at least on first sight, does not appear to be an infinite game according to our definition. However, as you may recall from our treatment of chess in the first chapter, the same was true there as well. What we did in order to formalize various games in a uniform way was to *code* the positions, moves etc. of the game as natural numbers. Clearly the same can be done here, too: since $\mathbb{N}^*$ is countable, we can fix a bijection $\varphi : \mathbb{N} \longrightarrow \mathbb{N}^* \setminus \{\langle\rangle\}$ and formulate the Banach-Mazur game as a game with natural numbers. But then, we must change the pay-off set: if A was the pay-off set of the Banach-Mazur game, let

$$A^{**} := \{z \in \mathbb{N}^{\mathbb{N}} \mid \varphi(z(0))^\frown\varphi(z(1))^\frown\varphi(z(2))^\frown \cdots \in A\}$$

It is easy to see that Player I wins $G^{**}(A)$ if and only if he wins $G(A^{**})$, and the same holds for Player II. Hence the two games are equivalent, and we have succeeded in formalizing the Banach-Mazur game as a game on natural numbers as in Definition 3.1.

For reasons of legibility, it will be much more convenient to talk about such games as if the moves are really some other objects, rather than their codes by natural numbers. For example, if σ is a strategy in the Banach-Mazur game, we will assume that σ is a function which takes $\langle s_0, t_0, \dots, s_n, t_n\rangle$ as parameters, rather then the coded version $\big\langle\varphi^{-1}(s_0), \varphi^{-1}(t_0), \dots, \varphi^{-1}(s_n), \varphi^{-1}(t_n)\big\rangle$.

This is sufficient if we want to assume the Axiom of Determinacy: since all games are determined, in particular $G(A^{**})$ is, and therefore, $G^{**}(A)$ is. But if we only assume $\mathsf{Det}(\mathbf{\Gamma})$ for some pointclass $\mathbf{\Gamma}$, and we have a set $A \in \mathbf{\Gamma}$, we can only conclude that $G^{**}(A)$ is determined *if we can prove that* $A^{**} \in \mathbf{\Gamma}$.

But now, recall that our pointclass $\mathbf{\Gamma}$ was not just any collection of sets, but one closed under continuous pre-images and intersections with closed sets. In the example above, we can use the following lemma.

Lemma 5.2. *The function* $f : \mathbb{N}^{\mathbb{N}} \longrightarrow \mathbb{N}^{\mathbb{N}}$ *given by*

$$f(z) := \varphi(z(0))^\frown\varphi(z(1))^\frown\varphi(z(2))^\frown \dots$$

is continuous.

Proof. Recalling Fact 4.5, fix z and let $s \lhd f(z)$. Let n be least such that

$$s \lhd \varphi(z(0))^\frown \dots ^\frown\varphi(z(n)).$$

Let $t := \langle z(0), z(1), \dots, z(n)\rangle \lhd z$. Now, for any other y with $t \lhd y$ we have

$$f(y) = \varphi(z(0))^\frown \dots ^\frown\varphi(z(n))^\frown\varphi(y(n+1))^\frown \dots$$

Therefore in any case $s \lhd f(y)$ holds, completing the proof. □

Since $A^{**} = f^{-1}[A]$ and $\mathbf{\Gamma}$ is closed under continuous preimages, $A \in \mathbf{\Gamma}$ implies that also $A^{**} \in \mathbf{\Gamma}$. In general, we will usually have a situation similar to the one above, and we will need a combination of continuous functions and closed sets in order to reduce a complex game to a standard game. In a few cases we may need additional closure properties of $\mathbf{\Gamma}$ (this will be made explicit).

Each of the following sections can be read independently, since in each one we study different properties. The sections are ordered (roughly) in order of increasing sophistication of the game-theoretic arguments involved.

6 The perfect set property

6.1 Trees

When studying the Baire space, it is convenient to view closed sets as sets of infinite branches through trees. The idea is that since infinite sequences $x \in \mathbb{N}^\mathbb{N}$ can be finitely approximated by their initial segments $s \lhd x$, a set $A \subseteq \mathbb{N}^\mathbb{N}$ can be approximated by the set of all initial segments of all its members. Although x is completely determined by the set of all its initial segments $\{s \in \mathbb{N}^* \mid s \lhd x\}$, this does not apply to subsets of $\mathbb{N}^\mathbb{N}$. One simple reason is that there are $2^{(2^{\aleph_0})}$ possible subsets of $\mathbb{N}^\mathbb{N}$ whereas there are only $2^{\aleph_0}$ possible sets of finite sequences (because $\mathbb{N}^*$ is countable). Hence, the transition from $A \subseteq \mathbb{N}^\mathbb{N}$ to the set of finite approximations of all its members can, in general, involve loss of information. But we will see that this is not the case for *closed* A, and that in fact there is a one-to-one correspondence between trees and closed sets.

Definition 6.1. A *tree* T is any collection of finite sequences closed under initial segments, i.e., $T \subseteq \mathbb{N}^*$ such that if $t \in T$ and $s \lhd t$ then $s \in T$.

It is easy to see why such objects are called "trees": the finite sequences are like nodes that can branch off in different directions, with the infinite sequences forming branches through the trees.

Definition 6.2. Let T be a tree.

1. A $t \in T$ is called a *node of* T. The *successors of* t are all nodes $s \in T$ such that $t \lhd s$ and $|s| = |t| + 1$.
2. A node $t \in T$ is called *terminal* if it has no successors.
3. A node $t \in T$ is called *splitting* if it has more than one successor, and *non-splitting* if it has exactly one successor.

Definition 6.3. For a tree T, a *branch through* T is any $x \in \mathbb{N}^\mathbb{N}$ such that $\forall n\,(x{\restriction}n \in T)$. The set of branches through T is denoted by $[T]$.

Lemma 6.4.

1. *For any tree T, $[T]$ is a closed set.*
2. *For any closed set C, there is a tree T_C such that $C = [T_C]$.*

Proof.

1. Pick any convergent sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[T]$ and let x be its limit point. Then for every $s \lhd x$, there is N such that $\forall n \geq N\,(s \lhd x_n)$. Since $x_n \in [T]$ for all such n, by definition we have $s \in T$. Since this holds for all $s \lhd x$, again by definition $x \in [T]$. So $[T]$ is closed.
2. Let $T_C := \{x{\restriction}n \mid x \in C, n \in \mathbb{N}\}$. Clearly, if $x \in C$ then by definition $x \in [T_C]$. Conversely, if $x \notin C$ then, since the complement of C is open, there is $s \lhd x$ such that $O(s) \cap C = \varnothing$. But then, no infinite extension y of s is in C, so s cannot be a branch through T_C by definition. Therefore $x \notin [T_C]$. □

The above proof actually gives an explicit one-to-one correspondence between closed sets and trees, given by $C \mapsto T_C$ one way and $T \mapsto [T]$ in the other. In fact, it is not hard to see that the operation $A \mapsto [T(A)]$ is the *topological closure* of the set A.

6.2 Perfect sets

The questions in this section are motivated by an early attempt of Georg Cantor to solve the Continuum Hypothesis.

Definition 6.5. A tree T is called a *perfect tree* if every node $t \in T$ has at least two incompatible extensions in T, i.e., $\exists s_1, s_2 \in T$ such that $t \lhd s_1$ and $t \lhd s_2$ and $s_1 \perp s_2$. A closed set C is called a *perfect set* if it is the set of branches $[T]$ for a perfect tree T.

Perfect sets also have a topological characterization: recall that for a set X, $x \in X$ is called an *isolated point* of X if there is an open neighbourhood O of x such that $O \cap C = \{x\}$.

Lemma 6.6. *A closed set C is perfect if and only if it contains no isolated points.*

Proof. Exercise 7. □

Lemma 6.7. *If T is a perfect tree then $[T]$ has the cardinality of the continuum $2^{\aleph_0}$.*

Proof. Let $2^{\mathbb{N}}$ and 2^* denote the set of all infinite, respectively finite, sequences of 0's and 1's. We know that $|2^{\mathbb{N}}| = 2^{\aleph_0}$. By induction, define a function φ from 2^* to the splitting nodes of T:

- $\varphi(\langle\rangle) :=$ least splitting node of T,
- If $\varphi(s)$ has been defined for $s \in 2^*$, then $\varphi(s)$ is a splitting node of T, hence there are different n and m such that $\varphi(s)^\frown\langle n\rangle$ and $\varphi(s)^\frown\langle m\rangle$ are both in T. Now let $\varphi(s^\frown\langle 0\rangle)$ be the least splitting node of T extending $\varphi(s)^\frown\langle n\rangle$ and $\varphi(s^\frown\langle 1\rangle)$ be the least splitting node of T extending $\varphi(s)^\frown\langle m\rangle$.

We can now lift the function φ to $\hat{\varphi} : 2^{\mathbb{N}} \longrightarrow [T]$ by setting

$$\hat{\varphi}(x) := \text{ the unique } z \in [T] \text{ such that } \forall s \lhd x\ (\varphi(s) \lhd z)$$

It only remains to verify that $\hat{\varphi}$ is injective. But if $x, y \in 2^{\mathbb{N}}$ and $x \neq y$ then there is a least n such that $x{\restriction}n \neq y{\restriction}n$. But φ was inductively defined in such a way that $\varphi(x{\restriction}n) \neq \varphi(y{\restriction}n)$. Since $\varphi(x{\restriction}n) \lhd \hat{\varphi}(x)$ and $\varphi(t{\restriction}n) \lhd \hat{\varphi}(y)$, it follows that $\hat{\varphi}(x) \neq \hat{\varphi}(y)$.

Therefore there is an injection from $2^{\mathbb{N}}$ to $[T]$ and hence $[T]$ has cardinality $2^{\aleph_0}$. □

This theorem is much more intuitive then may first seem. It simply says that if a set contains a "copy" of the full binary tree, then it must have the cardinality of the full binary tree, namely $2^{\aleph_0}$.

Now recall that the Continuum Hypothesis is the statement that every uncountable set has cardinality $2^{\aleph_0}$. In Cantor's time, one of the methods for proving this hypothesis that mathematicians considered went along the following lines of reasoning: if a subset of the reals, or of $\mathbb{N}^{\mathbb{N}}$, is uncountable, then there must be an explicit reason for it to be so. The only reason that seems explicit is that the set would contain a copy of the full binary tree, i.e., a perfect set. From this, the following dichotomy can be deduced (as usual, we present it in the setting of $\mathbb{N}^{\mathbb{N}}$ but analogous results hold for the real numbers).

Definition 6.8. A set $A \subseteq \mathbb{N}^{\mathbb{N}}$ has the *perfect set property*, abbreviated by PSP, if it is either countable or contains a perfect set (i.e., $[T] \subseteq A$ for some perfect tree T).

The ideas was that, if *every* set satisfied the perfect set property, then the Continuum Hypothesis would hold (at least, among subsets of the real numbers). The perfect set property is easily falsified by a diagonal construction using the Axiom of Choice, similarly to our proof of Theorem 3.9 (see also Exercise 8). However, in this section we will show that it is reasonable for PSP to hold for sets in some limited pointclass, and the way we do that is by showing that PSP follows from determinacy.

6.3 The $*$-game

The contents that follow are due to Morton Davis [Dav64], although we reformulate it in the context of $\mathbb{N}^{\mathbb{N}}$ instead of $2^{\mathbb{N}}$ which was used by Davis.

Definition 6.9. Let $A \subseteq \mathbb{N}^{\mathbb{N}}$ be a set. The game $G^*(A)$ is played as follows:

- Player I picks non-empty sequences of natural numbers, and Player II plays natural numbers.

I :	s_0		s_1		s_2		$\dots$
II :		n_1		n_2		$\dots$	

- Player I wins $G^*(A)$ if and only if
    1. $\forall i \geq 1$: $s_i(0) \neq n_i$, and
    2. $x := s_0{}^\frown s_1{}^\frown s_2{}^\frown \cdots \in A$.

In this game, the roles of I and II are not symmetric. The intuition is that Player I attempts, in the limit, to form an infinite sequence in A. Meanwhile, Player II chooses numbers n_i such that, in the next move, I may not play any sequence s_i starting with n_i (but has full freedom to do anything else). I wins the game if he can overcome the challenges set by II and produce an infinite sequence in A. II wins if she can choose numbers in such a way as to prevent I from reaching his objective.

Before studying the consequences of the determinacy of $G^*(A)$ we must show that this game can be coded into a standard game, with moves in $\mathbb{N}$. Fix a bijection $\varphi : \mathbb{N} \longrightarrow (\mathbb{N}^* \setminus \langle\rangle)$. Then the $*$-game can be reformulated as a standard game with the pay-off set given by

$$\begin{aligned} A^* = \{z \in \mathbb{N}^{\mathbb{N}} \mid \forall n \geq 1\ (\varphi(z(2n))(0) \neq z(2n-1)) \\ \wedge\ \ \varphi(z(0))^\frown \varphi(z(2))^\frown \varphi(z(4))^\frown \cdots \in A\} \end{aligned}$$

Lemma 6.10. *The function $f : \mathbb{N}^{\mathbb{N}} \longrightarrow \mathbb{N}^{\mathbb{N}}$ given by $f(z) := \varphi(z(0))^\frown \varphi(z(2))^\frown \varphi(z(4))^\frown \dots$ is continuous.*

Proof. Similar to Lemma 5.2. □

Lemma 6.11. *The set $C := \{z \in \mathbb{N}^{\mathbb{N}} \mid \forall n \geq 1\ (\varphi(z(2n))(0) \neq z(2n-1))\}$ is closed.*

Proof. $C = \bigcap_{n \geq 1} C_n$ where

$$C_n := \{z \mid \varphi(z(2n))(0) \neq z(2n-1)\}$$

so it remains to show that each C_n is closed. But we can write $C_n = [T_n]$ where $T_n := \{t \in \mathbb{N}^* \mid \text{if } |t| > 2n \text{ then } \varphi(t(2n))(0) \neq t(2n-1)\}$. □

Now $A^* = C \cap f^{-1}[A]$, and since $\mathbf{\Gamma}$ is closed under continuous pre-images and intersections with closed sets, it follows that whenever $A \in \mathbf{\Gamma}$, $A^* \in \mathbf{\Gamma}$.

Theorem 6.12 (Morton Davis, 1964)**.** *Let $A \subseteq \mathbb{N}^{\mathbb{N}}$ be a set.*

1. If Player I has a winning strategy in $G^*(A)$ then A contains a perfect set.

2. If Player II has a winning strategy in $G^*(A)$ then A is countable.

Proof.

1. Let σ be a winning strategy for Player I in the game $G^*(A)$. Although we are not talking about standard games, we can still use the notation $\sigma * y$ for an infinite run of the game in which II plays y and I according to σ, and similarly $\sigma * t$ for a finite position of the game. Let $\mathsf{Plays}^*(\sigma) := \{\sigma * y \mid y \in \mathbb{N}^{\mathbb{N}}\}$ be as before and additionally let
$$T_\sigma := \{s \in \mathbb{N}^* \mid s \lhd (\sigma * t) \text{ for some } t\}$$
be the tree of the closed set $\mathsf{Plays}^*(\sigma)$. Since σ is winning for I, clearly $\mathsf{Plays}^*(\sigma) \subseteq A$. So it remains to show that T_σ is a perfect tree.

   Pick any $t \in T_\sigma$ and consider the least move i such that $t \lhd s_0{}^\frown \dots {}^\frown s_i$. Now II can play n_{i+1} in her next move, after which I, assuming he follows σ, must play an s_{i+1} such that $s_{i+1}(0) \neq n_{i+1}$. Let $m_{i+1} := s_{i+1}(0)$. Instead of playing n_{i+1}, Player II could also have played m_{i+1} in which case I would have been forced to play a t_{i+1} such that $t_{i+1}(0) \neq m_{i+1}$. But then $t_{i+1} \neq s_{i+1}$, and both the sequence $s_0{}^\frown \dots {}^\frown s_i{}^\frown t_{i+1}$ and the sequence $s_0{}^\frown \dots {}^\frown s_i{}^\frown s_{i+1}$ are incompatible extensions of t according to σ, and hence are members of T_σ. So T_σ is perfect.

2. Now fix a winning strategy τ for II. Suppose p is a partial play according to τ, and such that it is Player I's turn to move, i.e., $p = \langle s_0, n_1, s_1, \dots, s_{i-1}, n_i \rangle$. Then we write $p^* := s_0{}^\frown \dots {}^\frown s_{i-1}$. For such p and $x \in \mathbb{N}^{\mathbb{N}}$ we say:

   - p is *compatible with* x if there exists an s_i such that $s_i(0) \neq n_i$ and $p^{*\frown} s_i \lhd x$. Note that this holds if and only if $p^* \lhd x$ and n_i (II's last move) doesn't "lie on x". Intuitively, this simply says that at position p, Player I still has a chance to produce x as the infinite play.
   - p *rejects* x if it is compatible with x and maximally so, i.e., if for all s_i with $s_i(0) \neq n_i$, we have $p^\frown \langle s_i, \tau(p^\frown \langle s_i \rangle)\rangle$ is *not* compatible with x any longer. In other words, at position p Player I still has a chance to extend the game in the direction of x, but for just one more move, because, no matter which s_i he plays, Player II will reply with n_{i+1} according to her strategy τ, after which I will not have a chance to produce x any more.

   Claim 1. *For every $x \in A$, there is a p which rejects it.*

   Proof. Fix an $x \in A$ and towards contradiction, suppose there is no p which rejects it. Then at every stage of the game, Player I can play an s_i such that $s_0{}^\frown \dots {}^\frown s_i \lhd x$ and such that Player II's strategy τ can do nothing to stop him. That means there is a sequence y played by I such that $y * \tau = x \in A$, contradicting the fact that τ is a winning strategy for II. □ (Claim 1)

Claim 2. *Every p rejects at most one x.*

Proof. Suppose p rejects x and y and $x \neq y$. By definition, p is compatible with both x and y, so Player I can play some s_i with $s_i(0) \neq n_i$ and $p^* {}^\frown s_i \lhd x$ and $p^* {}^\frown s_i \lhd y$. But then, he can also play s_i to be *maximal* in this sense, i.e., such that any further extension $p^* {}^\frown s_i {}^\frown \langle n \rangle$ cannot be an initial segment of both x and y (this is always possible since there is an n such that $x(n) \neq y(n)$).

Then consider $n_{i+1} := \tau(p^\frown \langle s_i \rangle)$. Clearly n_{i+1} cannot lie on both x and y, so $p^\frown \langle s_i, n_{i+1} \rangle$ can still be extended by Player I to be compatible with either x or y. Therefore, p does not reject both x and y. □ (Claim 2)

If we now define $K_p := \{x \in \mathbb{N}^{\mathbb{N}} \mid p \text{ rejects } x\}$ we see that by Claim 1, $A \subseteq \bigcup_p K_p$, by Claim 2 each K_p is a singleton, and moreover there are only countably many p's. Hence A is contained in a countable union of singletons, so it is countable. □

Corollary 6.13. $\mathsf{Det}(\mathbf{\Gamma})$ *implies that all sets in* $\mathbf{\Gamma}$ *have the perfect set property.*

Proof. Let A be a set in $\mathbf{\Gamma}$. Since A^* is also in $\mathbf{\Gamma}$, $G^*(A) = G(A^*)$ is determined. If Player I has a winning strategy in $G^*(A)$ then A contains a perfect tree, and if Player II has a winning strategy, then A is countable. □

6.4 Exercises

1. For $x \in \mathbb{N}^{\mathbb{N}}$ let $A_{\leq x} := \{y \in \mathbb{N}^{\mathbb{N}} \mid \forall n\ (y(n) \leq x(n))\}$ and $T_{\leq x} := \{s \in \mathbb{N}^* \mid \forall n\ (s(n) \leq x(n))\}$. Show that $T_{\leq x}$ is a tree, that $[T_{\leq x}] = A$, and conclude that $A_{\leq x}$ is closed.
2. Repeat the previous exercise for the sets $A_{\geq x}$ and $T_{\geq x}$ defined analogously but with "$\leq$" replaced by "$\geq$".
3. Prove that for any $n, m \in \mathbb{N}$, the set $A_{n \mapsto m} := \{x \in \mathbb{N}^{\mathbb{N}} \mid x(n) = m\}$ is clopen.
4. Let C_n be a closed set for every n. Show that the set $C := \{x \in \mathbb{N}^{\mathbb{N}} \mid \forall n\ (x \in C_n)\}$ is closed. Come up with an example showing that this does not hold for the set $A := \{x \in \mathbb{N}^{\mathbb{N}} \mid \exists n\ (x \in C_n)\}$.
5. Conclude from Exercises 8 and 9 (or prove directly) that for infinite sequences $\langle n_0, n_1, \ldots \rangle$ and $\langle m_0, m_1, \ldots \rangle$ the set $A_{\vec{n} \mapsto \vec{m}} := \{x \in \mathbb{N}^{\mathbb{N}} \mid \forall i\ (x(n_i) = m_i)\}$ is closed.
6. A tree is called *pruned* if every node has a successor, i.e., $\forall s \in T\ \exists t \in T$ such that $s \lhd t$. Show that every tree T can be turned into a pruned tree $\mathrm{pr}(T)$ in such a way that $[\mathrm{pr}(T)] = [T]$.
7. Prove Lemma 6.6.
8. Use AC to show directly that there is a set that does not satisfy the Perfect Set Property (hint: use Theorem 3.9).

9.* A set K in a topological space is called *compact* if every infinite cover of K by open sets has a finite subcover, i.e., if for every J and $K \subseteq \bigcup\{O_j \mid j \in J\}$ with each O_j open, there exists a finite subset $I \subseteq J$ such that $K \subseteq \bigcup\{O_j \mid j \in I\}$. Show that in the Baire space, a closed set K is compact if and only if in the corresponding tree T_K, every node is finitely splitting.

7 Baire property and Lebesgue measure

7.1 Lebesgue measure

Here we will focus on the actual real number line $\mathbb{R}$, and assume some familiarity with standard measure theory. The standard Lebesgue measure on $\mathbb{R}$ will be denoted by μ. Also, we will assume here that $\boldsymbol{\Gamma}$ is, additionally, closed under finite unions, intersections and complements, and contains the F_σ sets (the $\boldsymbol{\Delta}^1_n$ sets, projective sets etc. satisfy this property). Notice that $\mathbb{N}^\mathbb{N}$ is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$, so any pointclass $\boldsymbol{\Gamma}$ generates a pointclass on $\mathbb{R} \setminus \mathbb{Q}$, and we can also extend it to $\mathbb{R}$ by stipulating for $A \subseteq \mathbb{R}$ that $A \in \boldsymbol{\Gamma}$ iff $A \setminus \mathbb{Q} \in \boldsymbol{\Gamma}$.

The result of this section was first proved by Mycielski-Świerczkowski [MŚ64], but we present a proof due to Harrington. We will define the *covering game* G_μ, for which we first need to fix some setup. Note that it is sufficient to prove that every subset of $[0, 1]$ in $\boldsymbol{\Gamma}$ is measurable.

Definition 7.1.

1. Fix an enumeration $\{I_n \mid n \in \mathbb{N}\}$ of all possible finite unions of open intervals in $[0, 1]$ with rational endpoints (e.g., I_n is of the form $(q_0, q_1) \cup \dots \cup (q_k, q_{k+1})$ for some k, $q_i \in \mathbb{Q}$, etc.) This is possible since $\mathbb{Q}$ is countable.
2. For $x \in 2^\mathbb{N}$, let $a : 2^\mathbb{N} \longrightarrow [0, 1]$ be the function given by

$$a(x) := \sum_{n=0}^{\infty} \frac{x(n)}{2^{n+1}}$$

It is not hard to see that a, as a function from the Baire space to $[0, 1]$, is continuous and that its range is all of $[0, 1]$ (but a is not injective, e.g., both $\langle 1, 0, 0, 0, \dots \rangle$ and $\langle 0, 1, 1, 1, \dots \rangle$ map into $\frac{1}{2}$ —think of x as the binary expansion of $a(x)$).

For every $\epsilon > 0$, we define a game $G_\mu(A, \epsilon)$.

Definition 7.2. Let A be a subset of $[0, 1]$ and $\epsilon > 0$. The game $G_\mu(A, \epsilon)$ is defined as follows:

- At each turn, Player I picks 0 or 1, and Player II picks natural numbers.

I :	x_0		x_1		x_2		$\dots$
II :		y_0		y_1		y_2	

- At every move n, Player II must make sure that

$$\mu(I_{y_n}) < \frac{\epsilon}{2^{2(n+1)}},$$

(otherwise she loses).

- Player I wins iff $a(x) \in A \setminus \bigcup_{n=0}^{\infty} I_{y_n}$.

So the idea here is that Player I attempts to play a real number in $A \subseteq [0,1]$, essentially by using the infinite binary expansion of that real, while Player II attempts to "cover" that real with a countable union of the I_n's, but of an increasingly smaller measure.

Showing that this game can be formulated as a game within the same pointclass is a bit more involved.

Lemma 7.3. *Given A in $\mathbf{\Gamma}$ and ϵ, there exists a set $A^{\mu,\epsilon} \subseteq \mathbb{N}^{\mathbb{N}}$ which is in $\mathbf{\Gamma}$ and such that $G_\mu(A,\epsilon) = G(A^{\mu,\epsilon})$.*

Proof. First of all, clearly the functions $f(z)(n) := z(2n)$ and $g(z)(n) := z(2n+1)$ are continuous. Then, if z is the result of the game in the standard sense, Player I should win $G_{\mu,\epsilon}(A)$ iff

1. $\forall n(f(z)(n) \in \{0,1\})$,
2. $a(f(z)) \in A$, and
3. $a(f(z)) \notin \bigcup_{n=1}^{\infty} I_{g(z)(n)}$,

   or if
4. $\exists n$ such that $\mu(I_{g(z)(n)}) \geq \frac{\epsilon}{2^{2(n+1)}}$

So define the following sets:

1. $C_1 := \{z \mid \forall n(z(n) \in \{0,1\})\}$,
2. $C_2 := \{(a,y) \in \mathbb{R} \times \mathbb{N}^{\mathbb{N}} \mid \forall n(a \notin I_{y(n)})\}$,
3. $C_3 := \{z \mid \exists n(\mu[I_{z(n)}] \geq \frac{\epsilon}{2^{2(n+1)}})\}$.

And let

$$A^{\mu,\epsilon} := \big(f^{-1}[C_1] \ \cap \ (a \circ f)^{-1}[A] \ \cap \ ((a \circ f) \times g)^{-1}[C_2]\big) \ \ \cup \ \ g^{-1}[C_3]$$

Clearly $G_\mu(A,\epsilon) = G(A^{\mu,\epsilon})$. Using reasoning as in Exercise 6.4 (1)–(5), it is easy to see that C_1 is closed and C_3 is open. Concerning C_2, let's write the complement of C_2 in the following form

$$(\mathbb{R} \times \mathbb{N}^{\mathbb{N}}) \setminus C_2 = \bigcup_{n=0}^{\infty} \{(a,y) \mid a \in I_{y(n)}\}$$

$$= \bigcup_{n=0}^{\infty} \{(a,y) \mid \exists m \,(a \in I_m \ \wedge \ y(n) = m)\}$$

$$= \bigcup_{n=0}^{\infty} \bigcup_{m=0}^{\infty} \{(a,y) \mid a \in I_m \ \wedge \ y(n) = m\}$$

$$= \bigcup_{n=0}^{\infty} \bigcup_{m=0}^{\infty} I_m \times \{y \mid y(n) = m\}$$

Since the I_m are finite unions of open intervals, they are all open; also the $\{y \mid y(n) = m\}$ are open by Exercise 6.4 (3). So the product of these two sets is open in the product topology of $\mathbb{R} \times \mathbb{N}^\mathbb{N}$, and hence the countable union is open. Therefore the complement of C_2 is open, hence C_2 is closed.

Now all the functions involved are continuous and it is easy to see that $A^{\mu,\epsilon} \in \mathbf{\Gamma}$ (recall that we assumed $\mathbf{\Gamma}$ to be closed under finite unions and intersections). $\square$

Now let us return to the main result.

Theorem 7.4. *Let $A \subseteq \mathbb{N}^\mathbb{N}$ and ϵ be given. Then*

1. *If Player I has a winning strategy in $G_\mu(A, \epsilon)$ then there is a measurable set Z with $\mu(Z) > 0$ such that $Z \subseteq A$ (i.e., the inner measure of A is > 0).*

2. *If Player II has a winning strategy in $G_\mu(A, \epsilon)$ then there is an open O such that $A \subseteq O$ and $\mu(O) < \epsilon$ (i.e., the* outer measure *of A is $< \epsilon$).*

Proof.

1. Let σ be a winning strategy for I. It is clear that the mapping $y \mapsto \sigma * y$ is continuous. But then, also the mapping $y \mapsto a(f(\sigma * y))$ is continuous (where f is defined as in Lemma 7.3). Let $Z := \{a(f(\sigma * y)) \mid y \in \mathbb{N}^\mathbb{N}\}$. This is an *analytic* set (continuous image of a closed set) which is well-known to be measurable (by a classical result of Suslin from 1917). As we assumed σ to be winning, $Z \subseteq A$. But if $\mu(Z) = 0$ then (again by standard measure-theory) there exists a cover of Z by a sequence of sets $\{I_{y_n} \mid n \in \mathbb{N}\}$ satisfying $\forall n\, (\mu(I_{y_n}) < \frac{\epsilon}{2^{2(n+1)}})$. Then if II plays the sequence $y = \langle y_0, y_1, \dots\rangle$, we will get $a(f(\sigma * y)) \in Z \subseteq \bigcup_{n=0}^\infty I_{y_n}$, contradicting that σ is winning for I.

2. Now suppose II has a winning strategy τ. For each $s \in 2^*$ of length n, define $I_s := I_{(s*\rho)(2n-1)}$, i.e., I_s is the $I_{y_{n-1}}$ where y_{n-1} is the last move of the game in which I played s and II used τ. As τ is winning for II, for every $a \in A$ and every $x \in 2^\mathbb{N}$ such that $a(x) = a$, there must be some n such that $a \in I_{x\restriction n}$. In other words, $a \in \bigcup\{I_s \mid s \lhd x\}$ where x is such that $a(x) = a$. Therefore, in particular,

$$A \subseteq \bigcup_{s \in 2^\mathbb{N}} I_s = \bigcup_{n=1}^\infty \bigcup_{s \in \{0,1\}^n} I_s.$$

Now notice that, since τ was winning, for every s of length $n \geq 1$, $\mu(I_s) < \epsilon/2^{2n}$. Therefore

$$\mu\left(\bigcup_{s \in \{0,1\}^n} I_s\right) < \frac{\epsilon}{2^{2n}} \cdot 2^n = \frac{\epsilon}{2^n}.$$

It follows that

$$\mu\left(\bigcup_{s \in 2^\mathbb{N}} I_s\right) < \sum_{n=1}^\infty \frac{\epsilon}{2^n} = \epsilon.$$

Therefore, indeed, A is contained in an open set of measure $< \epsilon$. $\square$

Now it only remains to use the above dichotomy to show that it implies that every set is measurable.

Corollary 7.5. *Let $X \subseteq [0,1]$ be any set in $\mathbf{\Gamma}$ and assume* $\mathsf{Det}(\mathbf{\Gamma})$*. Then X is measurable.*

Proof. Let δ be the outer measure of X. Then there is a G_δ set B such that $X \subseteq B$ and $\mu(B) = \delta$. Now consider the games $G_\mu(B \setminus X, \epsilon)$, for all ϵ. Notice that our closeure properties on $\mathbf{\Gamma}$ guarantee that $X \setminus B \in \mathbf{\Gamma}$. By Theorem 7.4 each such game is determined. But if, for at least one $\epsilon > 0$, I would have a winning strategy, then there would exist a measurable set $Z \subseteq B \setminus X$ of positive measure, implying that X is contained in a set $B \setminus Z$ of measure strictly less than δ, thus contradicting that the outer measure of X was δ. Therefore, by determinacy, II must have a winning strategy in every game $G_\mu(B \setminus X, \epsilon)$ for every $\epsilon > 0$. But that implies that, for every ϵ, $B \setminus X$ can be covered by an open set of measure $< \epsilon$, therefore $B \setminus X$ itself has measure 0. Since X is equal to a G_δ set modulo a measure-zero set, X itself must be measurable. □

7.2 Baire property

Next, we consider a well-known topological property, frequently seen as a counterpart to Lebesgue-measurability. The game used in this context is the Banach-Mazur game from Definition 5.1.

Recall the following topological definitions:

Definition 7.6. Let $X \subseteq \mathbb{N}^\mathbb{N}$. We say that

1. X is *nowhere dense* if every basic open $O(t)$ contains a basic open $O(s) \subseteq O(t)$ such that $O(s) \cap X = \varnothing$,
2. X is *meager* if it is the union of countably many nowhere dense sets.
3. X has the *Baire property* if it is equal to an open set modulo meager, i.e., if there is an open set O such that $(X \setminus O) \cup (O \setminus X)$ is meager.

Just as with the perfect set property, it is possible to show (using the Axiom of Choice) that there are sets without the Baire property. We will prove that it follows from determinacy (for boldface pointclasses $\mathbf{\Gamma}$).

So, let $G^{**}(A)$ be the Banach-Mazur game from Definition 5.1; recall that we already proved that the coding involved in the game is continuous. Originally, this theorem is due to Banach and Mazur; it can be found in [Oxt57].

Theorem 7.7. *Let $A \subseteq \mathbb{N}^\mathbb{N}$ be a set and $G^{**}(A)$ the Banach-Mazur game.*

1. *If Player II has a winning strategy in $G^{**}(A)$ then A is meager.*
2. *If Player I has a winning strategy in $G^{**}(A)$ then $O(s) \setminus A$ is meager for some basic open $O(s)$.*

Proof.

1. This part of the proof is similar to the proof with the $*$-game in the previous section. Let τ be a winning strategy of Player II. For a position $p := \langle s_0, t_0, \dots, s_n, t_n \rangle$ write $p^* := s_0 {}^\frown t_0 {}^\frown \dots {}^\frown s_n {}^\frown t_n$. For any position p and $x \in \mathbb{N}^\mathbb{N}$ we say that
   - p is *compatible with x* if $p^* \lhd x$.

- p *rejects* x if it is compatible and maximally so, i.e., if for any s_{n+1}, the next position according to τ, i.e., the position $p^\frown \langle s_{n+1}, \tau(p^\frown \langle s_{n+1}\rangle)\rangle$ is not compatible with x.

Claim 1. *For every $x \in A$, there is a p which rejects it.*

Proof. Just as in the proof of Theorem 6.12, if there were no p which rejected x then there is a sequence y of moves by Player I such that $x = y*\tau \in A$ contradicting the assumption that τ is winning for Player II. □ (Claim 1)

Claim 2. *For every p, the set $F_p := \{x \mid p \text{ rejects } x\}$ is nowhere dense.*

Proof. Let $O(s)$ be any basic open set. If $p^* \ntriangleleft s$ then we can extend s to some t such that any $x \in O(t)$ is incompatible with p^*, and hence not in F_p, i.e., $O(t) \cap F_p = \varnothing$. So the only interesting case is if $p^* \triangleleft s$. Now we use the following trick: suppose $p = \langle s_0, \ldots, t_n\rangle$. Let s_{n+1} be the sequence such that $p^{*\frown} s_{n+1} = s$. Now let t_{n+1} be τ's answer, i.e., let $t_{n+1} := \tau(p^\frown \langle s_{n+1}\rangle)$. Then let $t := s^\frown t_{n+1}$. It is clear that $s \triangleleft t$ and hence $O(t) \subseteq O(s)$. We claim that $O(t) \cap F_p = \varnothing$ which is exactly what we need.

Let $x \in O(t)$, i.e., $t \triangleleft x$. But if we look at the definition of rejection, it is clear that p cannot reject x, because for s_{n+1} Player II's response is t_{n+1} and the play $p^{*\frown} \langle s_{n+1}, t_{n+1}\rangle = t$ is compatible with x. Thus $x \notin F_p$. □ (Claim 2)

Now the rest follows: by Claim 1, $A \subseteq \bigcup_p F_p$ which, by Claim 2, is a countable union of nowhere dense sets. Therefore A is meager.

2. Now we assume that Player I has a winning strategy σ in $G^{**}(A)$. Let s be I's first move according to the winning strategy, i.e., $s := \sigma(\langle\rangle)$. Then we claim:

   Claim 3. *Player II has a winning strategy in the game $G^{**}(O(s) \setminus A)$*

   Proof. Here we shall see the first instance of how a player can translate an opponent's winning strategy into his/her own. We will describe this strategy informally:

   Let s_0 be I's first move in the game $G^{**}(O(s) \setminus A)$.

   - **Case 1.** $s \ntriangleleft s_0$. Then play any t_0 such that $s_0{}^\frown t_0$ is incompatible with s. After that, play anything whatsoever. It is clear that the result of this game is some real $x \notin O(s)$, hence also $x \notin O(s) \setminus A$, and therefore is a win for II.
   - **Case 2.** $s \triangleleft s_0$. Then let s_0' be such that $s^\frown s_0' = s_0$. Now Player II does the following trick: to determine her strategy she "plays another game on the side", a so-called *auxiliary game*. This auxiliary game is the original game $G^{**}(A)$ in which Player I plays according to his winning strategy σ. Player II will determine her moves based on the moves of Player I in the auxiliary game.

     The first move in the auxiliary game is $s := \sigma(\langle\rangle)$. Then Player II plays s_0' as the next move of the auxiliary game. To that, in the auxiliary game Player

I responds by playing $t_0 := \sigma(\langle s, s'_0 \rangle)$. Now Player II switches back to the "real" game, and copies that t_0 as her first response to I's "real" move, s_0.

Next, in the "real" game she observes an s_1 being played by Player I. She then copies it as her next move in the auxiliary game, in which I responds according to σ with $t_1 := \sigma(\langle s, s'_0, t_0, s_1 \rangle)$. II copies t_1 on to the real game, and so it goes on. You can observe this in Figure 2, where the first game represents the real game and the second the auxiliary one:

I:	s_0		s_1		$\dots$
II:		t_0		t_1	

I:	$s = \sigma(\langle\rangle)$		$t_0 = \sigma(\langle s, s'_0 \rangle)$		$t_1 = \sigma(\langle s, s'_0, t_0, s_1 \rangle)$	
II:		s'_0		s_1		$\dots$

Figure 2: The "real" and "auxiliary" games

In the final play of the real game, the infinite play

$$x := s_0{}^\frown t_0{}^\frown s_1{}^\frown t_1{}^\frown \dots$$

is produced. But clearly $x = s^\frown s'_0{}^\frown t_0{}^\frown s_1{}^\frown t_1{}^\frown \dots$ and that was a play in the auxiliary game $G^{**}(A)$ in which Player I used his winning strategy σ. That means that $x \in A$. Therefore in the real game, $x \notin O(s) \setminus A$ which means that the strategy which Player II followed was winning for her. And that completes the proof of Claim 3. □ (Claim 3)

Now it follows directly from part 1 of the theorem that $O(s) \setminus A$ is meager, which is exactly what we had to show. □

We are now close to the final result, but not done yet. What we have proven is that for boldface pointclasses $\mathbf{\Gamma}$, if $\mathsf{Det}(\mathbf{\Gamma})$ holds then for every $A \in \mathbf{\Gamma}$, either A is meager or $O(s) \setminus A$ is meager for some basic open $O(s)$, which is a kind of "weak" Baire property. So it only remains to prove the following lemma:

Lemma 7.8. *Let $\mathbf{\Gamma}$ be a boldface pointclass. If for every $A \in \mathbf{\Gamma}$, either A is meager or $O(s) \setminus A$ is meager for some $O(s)$, then every A in $\mathbf{\Gamma}$ satisfies the Baire property.*

Proof. Pick $A \in \mathbf{\Gamma}$. If A is meager we are done because A is equal to the open set $\varnothing$ modulo a meager set, hence has the Baire property. Otherwise, let

$$O := \bigcup \{O(s) \mid O(s) \setminus A \text{ is meager }\}$$

This is an open set, and by definition $O \setminus A$ is a countable union of meager sets, hence it is meager. It remains to show that $A \setminus O$ is also meager. But since $\mathbf{\Gamma}$ is closed under intersections with closed sets, $A \setminus O \in \mathbf{\Gamma}$. So if it is not meager, then there is $O(s)$ such that $O(s) \setminus (A \setminus O)$ is meager. That implies

1. $O(s) \setminus A$ is meager, and
2. $O(s) \cap O$ is meager.

But the first statement implies, by definition, that $O(s) \subseteq O$. Then the second statement states that $O(s)$ is meager, and we know that that is false in the Baire space. So we conclude that $A \setminus O$ is meager and so A has the Baire property. $\square$

Corollary 7.9. $\mathsf{Det}(\mathbf{\Gamma})$ implies that every set in $\mathbf{\Gamma}$ has the Baire property.

8 Flip sets

In this section we investigate another non-constructive object, the so-called "flip sets". For the time being we focus on the space $2^{\mathbb{N}}$ of all infinite sequences of 0's and 1's, rather than the Baire space. As $2^{\mathbb{N}}$ is a closed subspace of $\mathbb{N}^{\mathbb{N}}$, it is also a topological space with the subspace topology inherited from $\mathbb{N}^{\mathbb{N}}$, and the topology behaves exactly the same way (basic open sets are $O(t) := \{x \in 2^{\mathbb{N}} \mid t \lhd x\}$).

Definition 8.1. An $X \subseteq 2^{\mathbb{N}}$ is called a *flip set* if for all $x, y \in 2^{\mathbb{N}}$, if x and y differ by exactly one digit, i.e., $\exists! n(x(n) \neq y(n))$, then

$$x \in X \iff y \notin X$$

Flip sets can be visualized by imagining an infinite sequence of light-switches such that flipping each switch turns the light on or off (in X or not in X). It is clear that if x and y differ by an even number of digits then $x \in X \iff y \in X$ whereas if they differ by an odd number then $x \in X \iff y \notin X$. If x and y differ by an infinite number of digits, we do not know what happens. Flip sets are also called *infinitary XOR gates* or *infinitary XOR functions* in computer science.

Although this gives us a nice description of flip sets, it is not clear whether such sets exist.

Lemma 8.2. *Assuming* AC*, flip sets exist.*

Proof. Let $\sim$ be the equivalent relation on $2^{\mathbb{N}}$ such that $x \sim y$ iff $\{n \mid x(n) \neq y(n)\}$ is finite. For each equivalence class $[x]_\sim$, let $s_{[x]_\sim}$ be some fixed element from that class. Now define

$$x \in X \iff |\{n \mid x(n) \neq s_{[x]_\sim}(n)\}| \text{ is even.}$$

This is a flip set: any x, y which differ by exactly one digit must be in the same equivalence class, hence $s_{[x]_\sim} = s_{[y]_\sim}$. But then, clearly exactly one of x, y must be in X. $\square$

Note: this particular proof comes from [KN12].

Although flip sets exist assuming AC, they can be considered pathological objects. We will show that $\mathsf{Det}(\boldsymbol{\Gamma})$ implies that there are no flip sets in $\boldsymbol{\Gamma}$ (and $\mathsf{ZF} + \mathsf{AD}$ implies that there are no flip sets at all).

We consider a version of the Banach-Mazur game which is exactly as in Definition 5.1 but with I and II playing non-empty sequences of 0's and 1's. For a set $X \subseteq 2^{\mathbb{N}}$, we denote the game by the same symbol $G^{**}(X)$. The fact that this can be coded using a continuous function is analogous to the previous case and we leave the details to the reader.

The way to prove that $\mathsf{Det}(\boldsymbol{\Gamma})$ implies that there are no flip sets in $\boldsymbol{\Gamma}$ is not by a direct application of determinacy, but rather by a sequence of Lemmas which, assuming a flip set exists in $\boldsymbol{\Gamma}$, lead to absurdity.

Lemma 8.3. *Let X be a flip set.*

1. *If I has a winning strategy in $G^{**}(X)$ then he also has a winning strategy in $G^{**}(2^{\mathbb{N}} \setminus X)$.*
2. *If II has a winning strategy in $G^{**}(X)$ then she also has a winning strategy in $G^{**}(2^{\mathbb{N}} \setminus X)$.*

Proof. For 1, assume σ is a winning strategy for Player I in $G^{**}(X)$ then let σ' be as follows:

- The first move $\sigma'(\langle\rangle)$ is any sequence of the same length as $\sigma(\langle\rangle)$ and differs from it by exactly one digit.
- The next moves are played according to σ, pretending that the first move was $\sigma(\langle\rangle)$ and not $\sigma'(\langle\rangle)$.

It is clear that for any sequence y of II's moves, $\sigma * y$ and $\sigma' * y$ differ by exactly one digit. Since $\sigma * y \in X$ and X is a flip set, $\sigma' * y \notin X$, hence σ' is winning for I in $G^{**}(2^{\mathbb{N}} \setminus X)$.

Part 2 is analogous (Player II changes exactly one digit of her first move t_0). □

Lemma 8.4. *Let X be a flip set. If II has a winning strategy in $G^{**}(X)$ then I has a winning strategy in the game $G^{**}(2^{\mathbb{N}} \setminus X)$.*

Proof. Let τ be II's winning strategy in $G^{**}(X)$. We informally describe Player I's strategy in $G^{**}(2^{\mathbb{N}} \setminus X)$: first he plays an arbitrary s. Player II will answer with some t. Now I starts playing an auxiliary version of $G^{**}(X)$ on the side, in which II uses τ. There he plays $s^\frown t$, and let s_0 be τ's answer in the auxiliary game. He copies s_0 as the next move in the real game. Player II will answer with some t_0. I copies t_0 on to the auxiliary game, etc.

$$G^{**}(2^{\mathbb{N}} \setminus X): \quad \begin{array}{r||ccccccc} \text{I:} & s & & s_0 & & s_1 & & \\ \hline \text{II:} & & t & & t_0 & & \ldots & \end{array}$$

$$G^{**}(X): \quad \begin{array}{r||ccccccc} \text{I:} & s^\frown t & & t_0 & & t_1 & & \\ \hline \text{II:} & & s_0 & & s_1 & & \ldots & \end{array}$$

Now if $x = s^\frown t^\frown s_0{}^\frown t_0{}^\frown \ldots$ is the result of the game, it is the same as the result of the auxiliary game which was played according to τ. As τ was winning, it follows that $x \notin X$ and hence the strategy we just defined is winning for I in $G^{**}(2^{\mathbb{N}} \setminus X)$. □

Lemma 8.5. *Let X be a flip set. If I has a winning strategy in $G^{**}(X)$ then II has a winning strategy in $G^{**}(2^{\mathbb{N}} \setminus X)$.*

Proof. This is slightly more involved because of the order of moves. Let σ be winning for I in $G^{**}(X)$. Player II will, again, play two games: the main one $G^{**}(2^{\mathbb{N}} \setminus X)$, and an auxiliary $G^{**}(X)$, using σ. Let Player I's first move in the real game be s_0. Let $s := \sigma(\langle\rangle)$ be I's first move in the auxiliary game. First, consider the case

- $|s_0| < |s|$.

Then in the real game, let II play t_0 such that $|s_0{}^\frown t_0| = |s|$ and $s_0{}^\frown t_0$ differs from s on an even number of digits. Clearly II can always find such t_0. Then let s_1 be I's next move in the real game. Player II copies it to the auxiliary game, in which I replies with some t_1, which II copies on to the real game, etc.

$$G^{**}(2^{\mathbb{N}} \setminus X): \quad \begin{array}{r||ccccccc} \text{I:} & s_0 & & s_1 & & s_2 & & \\ \hline \text{II:} & & t_0 & & t_1 & & \ldots & \end{array}$$

$$G^{**}(X): \quad \begin{array}{r||ccccccc} \text{I:} & s & & & t_1 & & \ldots & \\ \hline \text{II:} & & & s_1 & & s_2 & & \end{array}$$

Let $x := s_0{}^\frown t_0{}^\frown s_1{}^\frown t_1{}^\frown \dots$ be the result of the main game, and $y := s^\frown s_1{}^\frown t_1{}^\frown \dots$ the result of the auxiliary game. Then $y \in X$, and since by construction x and y differ by an even number of digits and X is a flip set, $x \in X$ follows, i.e., the strategy we described is winning for II in $G^{**}(2^{\mathbb{N}} \setminus X)$.

Now consider the case that

- $|s| \leq |s_0|$

This time Player II first plays any t in the auxiliary game such that $|s^\frown t| > |s_0|$, and finds t' to be Player I's reply in the auxiliary game. Now clearly $|s^\frown t^\frown t'| > |s_0|$ and she can play a t_0 in the real game such that $|s_0{}^\frown t_0| = |s^\frown t^\frown t'|$ and $s_0{}^\frown t_0$ and $s^\frown t^\frown t'$ differ on an even number of digits. After that she proceeds as before, i.e., if s_1 is the next move of I in the real game, she copies it into the auxiliary game, etc.

$$G^{**}(2^{\mathbb{N}} \setminus X): \quad \begin{array}{r||ccccccc} \text{I:} & s_0 & & s_1 & & s_2 & & \\ \hline \text{II:} & & t_0 & & t_1 & & \dots \end{array}$$

$$G^{**}(X): \quad \begin{array}{r||ccccccc} \text{I:} & s & & t' & & t_1 & & \dots \\ \hline \text{II:} & & t & & s_1 & & s_2 & \end{array}$$

Now $x = s_0{}^\frown t_0{}^\frown s_1{}^\frown t_1{}^\frown \dots$ and $y = s^\frown t^\frown t'^\frown s_1{}^\frown t_1{}^\frown \dots$ differ by an even number of digits so the result follows. □

Theorem 8.6. *If* $\mathsf{Det}(\mathbf{\Gamma})$ *then there are no flip sets in* $\mathbf{\Gamma}$.

Proof. Suppose, towards contradiction, that there exists a flip set $X \in \mathbf{\Gamma}$. Then

- I has a winning strategy in $G^{**}(X)$
  $\Longrightarrow$ I has a winning strategy in $G^{**}(2^{\mathbb{N}} \setminus X)$
  $\Longrightarrow$ II has a winning strategy in $G^{**}(X)$.
- II has a winning strategy in $G^{**}(X)$
  $\Longrightarrow$ II has a winning strategy in $G^{**}(2^{\mathbb{N}} \setminus X)$
  $\Longrightarrow$ I has a winning strategy in $G^{**}(X)$.

Both situations are clearly absurd, from which we conclude that there cannot be a flip set in $\mathbf{\Gamma}$. □

9 Wadge reducibility

9.1 Wadge game

Our last application of infinite games relates to Wadge reducibility, a large area of research of which we will only present a small part. The study of it started with the work of William Wadge (pronounced "wage") [Wad83].

Since we will deal with complements of sets a lot in this section, it will be convenient to use the notation $\overline{A} := \mathbb{N}^{\mathbb{N}} \setminus A$.

Definition 9.1. Let $A, B \subseteq \mathbb{N}^{\mathbb{N}}$. We say that A is *Wadge reducible to* B, notation $A \leq_W B$, if there is a continuous function $f : \mathbb{N}^{\mathbb{N}} \longrightarrow \mathbb{N}^{\mathbb{N}}$ such that for all x:

$$x \in A \iff f(x) \in B$$

Clearly, $A \leq_W B$ iff $\overline{A} \leq_W \overline{B}$, and it is not hard to see that $\leq_W$ is a pre-order (reflexive and transitive, but in general not antisymmetric). As is usual practice with pre-orders, we say that A *is Wadge equivalent to* B, denoted by $A \equiv_W B$, if $A \leq_W B$ and $B \leq_W A$. Let $[A]_W$ denote the equivalence class of A for every A, and lift the relation onto equivalence classes by $[A]_W \leq_W [B]_W$ iff $A \leq_W B$. The equivalence classes $[A]_W$ are called *Wadge degrees* and the relation $\leq_W$ on the Wadge degrees is a partial order. As usual, the strict Wadge ordering is defined by

$$A <_W B \text{ iff } A \leq_W B \text{ and } B \not\leq_W A.$$

The theory of Wadge reducibility has many applications, ranging from analysis and topology to computer science. We should, however, note that the theory as presented in this section does not directly generalize to connected spaces such as $\mathbb{R}$ and $\mathbb{R}^n$ (see [Ike10, Chapter 5]).

We will analyse the corresponding theory under determinacy assumptions, which entail a very interesting and rich structure of the Wadge degrees. In this section, we will also assume that $\mathbf{\Gamma}$ is closed under finite intersections and complements.

Definition 9.2. Let A, B be sets. The *Wadge game* $G^W(A, B)$ is played as follows: Players I and II choose natural numbers:

I:	x_0		x_1		$\dots$	
II:		y_0		y_1		$\dots$

If $x = \langle x_0, x_1, \dots \rangle$ and $y = \langle y_0, y_1, \dots \rangle$, then Player II wins $G^W(A, B)$ if and only if

$$x \in A \iff y \in B$$

To see that this game can be coded by a pay-off set in the same pointclass, consider the two functions f and g defined by

- $f(x)(n) := x(2n)$ and
- $g(x)(n) := x(2n + 1)$.

It is straightforward to verify that f and g are continuous. Now note that if z is the outcome of the Wadge game, then Player I wins $G^W(A,B)$ if and only if $f(z) \in A \iff g(z) \notin B$. From this it follows that $G^W(A,B)$ is equivalent to the game

$$G((f^{-1}[A] \setminus g^{-1}[B]) \cup (g^{-1}[A] \setminus f^{-1}[B]))$$

and by our closure assumptions on $\mathbf{\Gamma}$, this set is also in $\mathbf{\Gamma}$, so our coding is adequate.

The main result is the following theorem due to William Wadge.

Theorem 9.3 (Wadge, 1972)**.** *Let $A, B \subseteq \mathbb{N}^{\mathbb{N}}$.*

1. *If Player II has a winning strategy in $G^W(A,B)$ then $A \leq_W B$.*
2. *If Player I has a winning strategy in $G^W(A,B)$ then $B \leq_W \overline{A}$.*

Proof. Let τ be a winning strategy of II. For every x played by Player I, by definition of the winning condition, $x \in A \iff g(x * \tau) \in B$. But it is easy to see that the function mapping x to $x * \tau$ is continuous. Similarly, g is continuous, and so the composition of these two functions is a continuous reduction from A to B, so $A \leq_W B$.

Analogously, if σ is a winning strategy of I then for every y we have $f(\sigma * y) \in A \iff y \notin B$, so again we have a continuous reduction from $\overline{B}$ to A, or equivalently from B to $\overline{A}$. □

Therefore, if we limit our attention to sets in $\mathbf{\Gamma}$ and assume $\mathsf{Det}(\mathbf{\Gamma})$, the Wadge order satisfies the property that for all A, B, either $A \leq_W B$ or $B \leq_W \overline{A}$. This immediately has many implications for the order.

Corollary 9.4. *If $A <_W B$ then*

1. *$A \leq_W \overline{B}$,*
2. *$B \not\leq_W \overline{A}$.*

Proof. For 1, suppose $A \not\leq_W \overline{B}$. Then by Theorem 9.3 we have $\overline{B} \leq_W \overline{A}$, and so $B \leq_W A$, contradicting $A <_W B$.

For 2, suppose $B \leq_W \overline{A}$. Then we have $B \leq_W \overline{A} \leq_W \overline{B} \leq_W A$, where the second inequality is because of $A \leq_W B$ and the third one by the assumption that $B \leq_W \overline{A}$. This again contradicts $A <_W B$. □

A set A, or its corresponding Wadge degree $[A]_W$, is called *self-dual* if $A \equiv_W \overline{A}$.

Corollary 9.5. *If A is self-dual, then for any B, either $B \leq_W A$ or $A \leq_W B$.*

Proof. If $A \not\leq_W B$ then $B \leq \overline{A} \leq_W A$. □

9.2 Martin-Monk Theorem

We end this section, and with it our course, with the proof of the Martin-Monk Theorem, illustrating the standard level of argumentation that can be applied to infinite games.

Theorem 9.6 (Martin-Monk)**.** *If $\mathsf{Det}(\mathbf{\Gamma})$ then the relation $<_W$ restricted to sets in $\mathbf{\Gamma}$ is well-founded.*

Proof. We must show that there are no infinite descending $<_W$-chains of sets in $\mathbf{\Gamma}$. So, towards contradiction, suppose that $\{A_n \mid n \in \mathbb{N}\}$ is an infinite collection of sets in $\mathbf{\Gamma}$ which forms an infinite descending $<_W$-chain:

$$\cdots <_W A_3 <_W A_2 <_W A_1 <_W A_0$$

Since for each n, $A_{n+1} <_W A_n$, by Corollary 9.4, both $A_n \not\leq_W A_{n+1}$ and $A_n \not\leq_W \overline{A_{n+1}}$. Therefore by Theorem 9.3 Player II cannot have a winning strategy in the games $G^W(A_n, A_{n+1})$ and $G^W(A_n, \overline{A_{n+1}})$. By determinacy, Player I must then have winning strategies in both games. We will call these strategies σ_n^0 and σ_n^1, respectively.

We now introduce the following abbreviation:

$$G_n^0 := G^W(A_n, A_{n+1})$$

$$G_n^1 := G^W(A_n, \overline{A_{n+1}})$$

Now to any infinite sequence of 0's and 1's, i.e., any $x \in 2^{\mathbb{N}}$, we can associate an infinite sequence of Wadge games

$$\left\langle G_0^{x(0)}, G_1^{x(1)}, G_2^{x(2)}, \ldots \right\rangle$$

played according to I's winning strategies

$$\left\langle \sigma_0^{x(0)}, \sigma_1^{x(1)}, \sigma_2^{x(2)}, \ldots \right\rangle.$$

Now we fix some particular $x \in 2^{\mathbb{N}}$, and Player II is going to play all these infinite Wadge games simultaneously (as a kind of "simultaneous exhibition", against infinitely many "Players I"). In each game $G_n^{x(n)}$ Player I follows his winning strategy $\sigma_n^{x(n)}$, whereas Player II copies I's moves from the next game $G_{n+1}^{x(n+1)}$. To make this possible, she follows the following diagonal procedure:

- In the first game $G_0^{x(0)}$, let $a_0^x(0)$ be the first move of Player I, according to $\sigma_0^{x(0)}$. The superscript x refers to the infinite sequence we fixed at the start and the subscript 0 refers to the 0-th game.
- To play the next move in the first game, Player II needs information from the second game. Let $a_1^x(0)$ be Player I's first move in the game $G_1^{x(1)}$, according to $\sigma_1^{x(1)}$. Player II copies that move on to the first game.
- Next, Player I plays $a_0^x(1)$ in the first game. To reply to that, Player II needs information from the second game. There, $a_1^x(0)$ has been played, and Player II would like to copy information from the next game.
- So let $a_2^x(0)$ be Player I's first move in the game $G_2^{x(2)}$, according to $\sigma_2^{x(2)}$. Player II copies that on to the second game. Now $a_1^x(1)$ is I's next move in the second game, which Player II copies on to the first game.
- Etc.

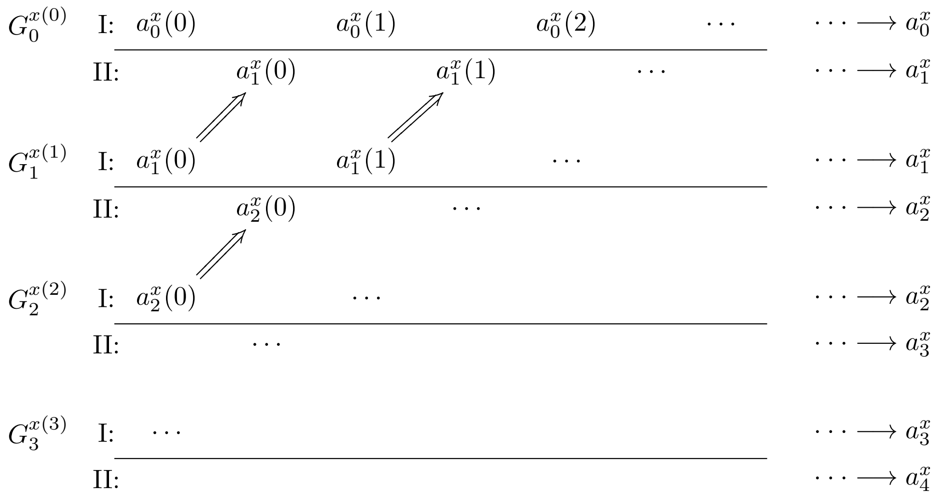


Figure 3: The Martin-Monk theorem: a “simulteneous exhibition” of Player II against infinitely many opponents, in infinitely many games.

All of this is best represented in Figure 3:

Using this procedure the two players are able to fill in the entire table. For each game $G_n^{x(n)}$ let a_n^x be the outcome of Player I’s moves, and a_{n+1}^x be the outcome of Player II’s moves. Note that the same infinite sequence a_{n+1}^x is also the result of I’s moves in the next game, $G_{n+1}^{x(n+1)}$.

Since each game is won by Player I, the definition of the Wadge game implies that for each n:

$$\begin{aligned} x(n) = 0 &\implies (a_n^x \in A_n \ \leftrightarrow\ a_{n+1}^x \notin A_{n+1}) \\ x(n) = 1 &\implies (a_n^x \in A_n \ \leftrightarrow\ a_{n+1}^x \in A_{n+1}) \end{aligned} \qquad (*)$$

Now we compare the procedure described above for different $x, y \in 2^{\mathbb{N}}$.

Claim 1. *If* $\forall m \geq n\ (x(m) = y(m))$ *then* $\forall m \geq n\ (a_m^x = a_m^y)$.

Proof. Simply note that the values of a_m^x and a_m^y depend only on games $G_{m'}^{x(m')}$ and $G_{m'}^{y(m')}$ for $m' \geq m$. Therefore, if $x(m')$ and $y(m')$ are identical, so are the corresponding games and so are a_m^x and a_m^y. □ (Claim 1)

Claim 2. *Let n be such that $x(n) \neq y(n)$ but $\forall m > n\ (x(m) = y(m))$. Then $a_n^x \in A_n \ \leftrightarrow\ a_n^y \notin A_n$.*

Proof.

- **Case 1:** $x(n) = 0$ and $y(n) = 1$. By condition $(*)$ above it follows that

$$\begin{aligned} a_n^x \in A_n &\ \leftrightarrow\ a_{n+1}^x \notin A_{n+1} \\ a_n^y \in A_n &\ \leftrightarrow\ a_{n+1}^y \in A_{n+1}. \end{aligned}$$

Since by Claim 1, $a^x_{n+1} = a^y_{n+1}$, it follows that

$$a^x_n \in A_n \leftrightarrow a^x_{n+1} \notin A_{n+1} \leftrightarrow a^y_{n+1} \notin A_{n+1} \leftrightarrow a^y_n \notin A_n.$$

- **Case 2:** $x(n) = 1$ and $y(n) = 0$. Now $(*)$ implies that

$$\begin{array}{r} a^x_n \in A_n \leftrightarrow a^x_{n+1} \in A_{n+1} \\ a^y_n \in A_n \leftrightarrow a^y_{n+1} \notin A_{n+1}. \end{array}$$

Again by Claim 1 it follows that

$$a^x_n \in A_n \leftrightarrow a^x_{n+1} \in A_{n+1} \leftrightarrow a^y_{n+1} \in A_{n+1} \leftrightarrow a^y_n \notin A_n.$$

□ (Claim 2)

Claim 3. *Let x and y be such that there is a unique n with $x(n) \neq y(n)$. Then $a^x_0 \in A_0 \leftrightarrow a^y_0 \notin A_0$.*

Proof. By Claim 2 we know that $a^x_n \in A_n \leftrightarrow a^y_n \notin A_n$. Since $x(n-1) = y(n-1)$ we again have two cases:

- **Case 1:** $x(n-1) = y(n-1) = 0$. Then by $(*)$

$$\begin{array}{r} a^x_{n-1} \in A_{n-1} \leftrightarrow a^x_n \notin A_n \\ a^y_{n-1} \in A_{n-1} \leftrightarrow a^y_n \notin A_n \end{array}$$

and therefore $a^x_{n-1} \in A_{n-1} \leftrightarrow a^y_{n-1} \notin A_{n-1}$.

- **Case 2:** $x(n-1) = y(n-1) = 1$. Then by $(*)$ we have

$$\begin{array}{r} a^x_{n-1} \in A_{n-1} \leftrightarrow a^x_n \in A_n \\ a^y_{n-1} \in A_{n-1} \leftrightarrow a^y_n \in A_n \end{array}$$

and therefore again $a^x_{n-1} \in A_{n-1} \leftrightarrow a^y_{n-1} \notin A_{n-1}$.

Now we go to the $(n-2)$-th level. Since again $x(n-2) = y(n-2)$ we get, by an application of $(*)$, that $a^x_{n-2} \in A_{n-2} \leftrightarrow a^y_{n-2} \notin A_{n-2}$. We go on like this until we reach the 0-th level, in which case we get $a^x_0 \in A_0 \leftrightarrow a^y_0 \notin A_0$, as required. □ (Claim 3)

Now define

$$X := \{x \in 2^{\mathbb{N}} \mid a^x_0 \in A_0\}$$

It is not hard to see that the map $x \longmapsto a^x_0$ is continuous: if we fix the first n values of a^x_0, we see that they only depend on the first n games $\{G^{x(i)}_i \mid i \leq n\}$. Therefore X is the continuous pre-image of A_0 and therefore $X \in \mathbf{\Gamma}$. But now Claim 3 says that X is a flip set, contradicting Theorem 8.6. □

9.3 Exercises

1. (a) Show that for any set $A \notin \{\varnothing, \mathbb{N}^{\mathbb{N}}\}$, we have both $\varnothing <_W A$ and $\mathbb{N}^{\mathbb{N}} <_W A$.
   (b) Show that $\varnothing \not\leq_W \mathbb{N}^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}} \not\leq_W \varnothing$. Conclude that $[\varnothing]_W = \{\varnothing\}$ and $[\mathbb{N}^{\mathbb{N}}]_W = \{\mathbb{N}^{\mathbb{N}}\}$.
   (c) If $(P, \leq)$ is a partial order, then a subset $A \subseteq P$ is called an *antichain* if $\forall p, q \in A\ (p \not\leq q \wedge q \not\leq p)$. Show that in the partial order $(\mathbf{\Gamma}, \leq_W)$, assuming $\mathsf{Det}(\mathbf{\Gamma})$, antichains have size at most 2.

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