Forcing and Independence Proofs:
Martin's Axiom

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Structure

1. Formulation of Martin's Axiom.
2. Martin's Axiom and CH.
3. An equivalent to MA in terms of BAs.

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1. Formulation of Martin's Axiom. — Definitions
2. Martin's Axiom and CH. — DSL, Lemma III.3.7
3. An equivalent to MA in terms of BAs. — $i : \mathbb{P} \to \mathbb{B}$

Formulation of Martin's Axiom: Definitions

Definition

- A *forcing poset* is a triple $(\mathbb{P}, \leqslant, \mathbb{1})$ such that $\leqslant$ is a preorder on $\mathbb{P}$ and $\mathbb{1} \in \mathbb{P}$ is a largest element $(\forall p \in \mathbb{P}\ p \leqslant \mathbb{1})$.

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- $p, q \in \mathbb{P}$ are *incompatible* $(p \perp q)$ iff they have no common extension $(\neg \exists r \in \mathbb{P}\ (r \leqslant p \wedge r \leqslant q))$. An *antichain* is a subset $A \subseteq \mathbb{P}$ whose elements are pairwise incompatible. $\mathbb{P}$ has the *countable chain condition* iff every antichain in $\mathbb{P}$ is countable.

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- $D \subseteq \mathbb{P}$ is *dense* in $\mathbb{P}$ iff $\forall p \in \mathbb{P}\ \exists q \in D\ q \leqslant p$.

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- $D \subseteq \mathbb{P}$ is *dense* in $\mathbb{P}$ iff $\forall p \in \mathbb{P}\ \exists q \in D\ q \leqslant p$.
- $G \subseteq \mathbb{P}$ is a *filter* on $\mathbb{P}$ iff
  - $\mathbb{1} \in G$.
  - $\forall p, q \in G\ \exists r \in G\ (r \leqslant p \wedge r \leqslant q)$.
  - $\forall p, q \in \mathbb{P}\ (q \leqslant p \wedge q \in G \to p \in G)$.

Example

For any I, J: $\mathsf{Fn}(I, J)$ is the set of all *finite partial functions* from I to J; that is $p \in \mathsf{Fn}(I, J)$ iff $p \in [I \times J]^{<\omega}$ and p is the graph of a function. We make $\mathsf{Fn}(I, J)$ into a forcing poset by letting $\leqslant$ be $\supseteq$ and $\mathbb{1} = \varnothing$.

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For $\mathbb{P} = \mathsf{Fn}(\omega, \omega)$,

- $\{(0,0)\}, \{(1,0)\}, \{(0,1)\} \in \mathbb{P}$;
- $\{(0,0)\} \not\perp \{(1,0)\}$ since $\{(0,0),(1,0)\}$ is a common extension;
- $\{(0,0)\} \perp \{(0,1)\}$;
- $D = \{p \in \mathbb{P} : \ \exists k \in \mathbb{N}\ |\mathsf{dom}(p)| = 2k\}$ is dense in $\mathbb{P}$;
- $G = \{p \in \mathbb{P} : \ 1 \notin \mathsf{dom}(p)\}$ is a filter on $\mathbb{P}$.

Definition

- $MA_{\mathbb{P}}(\kappa)$ is the statement that whenever $\mathcal{D}$ is a family of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| \leqslant \kappa$, there exists a filter G on $\mathbb{P}$ such that $G \cap D \neq \varnothing$ for all $D \in \mathcal{D}$.
- $MA(\kappa)$ is the statement that $MA_{\mathbb{P}}(\kappa)$ holds for all ccc posets $\mathbb{P}$.
- MA is the statement $\forall \kappa < \mathfrak{c}\ MA(\kappa)$.

Lemma (III.3.13)

$MA(\kappa)$ *fails for* $\kappa \geqslant \mathfrak{c}$.

Lemma (III.3.14)

$MA(\kappa)$ *holds for* $\kappa = \aleph_0$.

Definition

A family of sets $\mathcal{A}$ forms a *delta system* with *root* R iff $X \cap Y = R$ whenever $X, Y \in \mathcal{A}$ with $X \neq Y$.

Lemma (Delta System)

Let κ be an uncountable regular cardinal, and let $\mathcal{A}$ be a family of finite sets with $|\mathcal{A}| = \kappa$. Then there is a $\mathcal{B} \in [\mathcal{A}]^{\kappa}$ such that $\mathcal{B}$ forms a delta system.

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Fn(I, J) *has the ccc iff* $I = \varnothing$ *or* J *is countable.*

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- CH $\rightarrow$ MA.
- ZFC + MA + $\neg$ CH is consistent. (Proof uses iterated forcing.)
- By identifying certain small cardinals with $\mathfrak{c}$, MA puts restrictions on what $\mathfrak{c}$ can be. E.g., if MA holds then $\mathfrak{c}$ is regular.

Theorem (Equivalence)

For any infinite cardinal κ, the following are equivalent:

1. $MA(\kappa)$.
2. $MA_{\mathbb{B}}(\kappa)$ *holds for all complete ccc Boolean algebras* $\mathbb{B}$.

Definition

A *Boolean algebra* is a structure $(\mathbb{B}, \neg, \vee, \wedge, \mathbb{0}, \mathbb{1})$ such that

- $\leqslant$ is a partial order
- For every $a, b \in \mathbb{B}$, $a \wedge b$ and $a \vee b$ exist
- distributivity of $\wedge$ and $\vee$
- For all $b \in \mathbb{B}, \mathbb{0} \leqslant b \leqslant \mathbb{1}$
- For all $b \in \mathbb{B}$ there is a complement $\neg b$ ($b \wedge \neg b = 0$, $b \vee \neg b = 1$)

$\mathbb{B}$ is *complete* if for every $S \subseteq \mathbb{B}$, inf(S) and sup(S) exist.

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If $\mathbb{B}$ is a Boolean algebra, then $\mathbb{B}$ and $\mathbb{B} \setminus \{\mathbb{0}\}$ are forcing posets.

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If $\mathbb{B}$ is an atomless BA, then $\mathbb{B} \backslash \{\mathbb{0}\}$ is an atomless forcing poset.

Proof of the Equivalence: Idea

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Lemma (2)

Let $i : \mathbb{Q} \to \mathbb{P}$ be a dense embedding. Then $MA_{\mathbb{P}}(\kappa)$ implies $MA_{\mathbb{Q}}(\kappa)$.

Proof of Lemma 1: Definitions

Definition

Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing posets and $i : \mathbb{Q} \to \mathbb{P}$. Then i is a *dense embedding* iff:

1. $i(\mathbb{1}_{\mathbb{Q}}) = \mathbb{1}_{\mathbb{P}}$.
2. $\forall q_1, q_2 \in \mathbb{Q}\ [q_1 \leqslant_{\mathbb{Q}} q_2 \to i(q_1) \leqslant_{\mathbb{P}} i(q_2)]$.
3. $\forall q_1, q_2 \in \mathbb{Q}\ [q_1 \perp_{\mathbb{Q}} q_2 \leftrightarrow i(q_1) \perp_{\mathbb{P}} i(q_2)]$.
4. $i(\mathbb{Q})$ is a dense subset of $\mathbb{P}$.

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Example

Let $\mathbb{P} = \mathsf{Fn}(\omega, \omega)$.

- Let $\mathbb{T} = \{p \in \mathbb{P} : \mathsf{dom}(p) \in \omega\}$. Then $i : \mathbb{T} \to \mathbb{P}$ with $i(p) = p$ is a dense embedding.

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- Let $\mathbb{T} = (\mathbb{N}, \geqslant)$. Then there cannot be an embedding $i : \mathbb{P} \to \mathbb{T}$; and there cannot be a *dense* embedding $\mathbb{T} \to \mathbb{P}$.

Definition

If $\mathbb{P}$ is a forcing poset, the *poset topology* on $\mathbb{P}$ is defined by $\mathcal{T}_{\mathbb{P}} = \{U \subseteq \mathbb{P} : \forall s \in U(s \downarrow \subseteq U)\}$.

Recall: $s \downarrow = \{x \in \mathbb{P} : x \leqslant_{\mathbb{P}} s\}$.

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ro(X) is always complete. int(cl(U)) $= \{x : \forall y, y \leqslant x \; \exists z, z \leqslant y \; z \in U\}$

Example

Consider $(\mathbb{P}, \mathcal{T}_{\mathbb{P}})$ with $\mathbb{P} =$ Fn(ω, ω).

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- For $U = \{p : p(1) = 0\}$:
  $\mathrm{cl}(U) = \{p : p(1) = 0 \vee 1 \notin \mathrm{dom}(p)\}$.
  $\mathrm{int}(\mathrm{cl}(U)) = U$.
  So $U \in \mathrm{ro}(\mathbb{P})$.
- For $U = \{p : 1 \in \mathrm{ran}(p)\}$:
  $\mathrm{cl}(U) = \mathbb{P}$.
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Proof of Lemma 1: Definitions

Definition

Let X be a non-empty topological space. Then its *regular open algebra*, $\text{ro}(X)$, is the set of all $U \subseteq X$ that are both open and *regular* $(U = \text{int}(\text{cl}(U)))$. The $\leqslant, \wedge, \mathbb{0}, \mathbb{1}$ are $\subseteq, \cap, \varnothing, X$, respectively. $U \vee V = \text{int}(\text{cl}(U \cup V))$ and $\neg U = \text{int}(X \backslash U)$.

$\text{ro}(X)$ is always complete. $\text{int}(\text{cl}(U)) = \{x : \forall y, y \leqslant x \ \exists z, z \leqslant y \ z \in U\}$

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  So $U \notin \text{ro}(\mathbb{P})$.

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Lemma (1)

For every forcing poset $\mathbb{P}$*, there is a complete BA* $\mathbb{B}$ *and a dense embedding* $i : \mathbb{P} \to \mathbb{B} \setminus \{\mathbb{0}\}$.

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Give $\mathbb{P}$ the poset topology $\mathcal{T}_{\mathbb{P}}$. Let $\mathbb{B} = \mathsf{ro}(\mathbb{P}, \mathcal{T}_{\mathbb{P}})$, and let $i(p) = \mathsf{int}(\mathsf{cl}(p\downarrow))$.

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4. $i(\mathbb{Q})$ is a dense subset of $\mathbb{P}$.

Proof of Lemma 2

Lemma

The statement:
'Whenever $\mathcal{D}$ is a family of maximal antichains in $\mathbb{P}$ with $|\mathcal{D}| \leqslant \kappa$, there exists a linked family [filter] A in $\mathbb{P}$ such that $D \cap A \neq \emptyset$ for all $D \in \mathcal{D}$'
is equivalent to $MA_{\mathbb{P}}(\kappa)$.

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Lemma

If $i : \mathbb{P} \to \mathbb{Q}$ is a dense embedding, then for all maximal antichains $A \subseteq \mathbb{P}$, $i(A)$ is a maximal antichain in $\mathbb{Q}$.

Proof of the Equivalence

Theorem (Equivalence)

For any infinite cardinal κ, the following are equivalent:

1. $MA(\kappa)$.
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$MA_{\mathbb{P}}(\kappa)$ follows from $MA_{\mathbb{B}}(\kappa)$ by Lemma 2.

Thank you!
Questions?