Forcing and Independence Proofs 2023: Assignment 1

Part A: object language and meta-language

1. For each of the following statements, determine whether they are made in the formal language of set theory $\mathcal{L}_\in$, or in a meta-language (in which we talk *about* set theory).[1]
   (a) Every convergent sequence in $\mathbb{R}$ is bounded from above.
   (b) $\mathsf{ZFC} \vdash$ "Every convergent sequence in $\mathbb{R}$ is bounded from above".
   (c) ZFC contains infinitely many axioms.
   (d) If ZFC is consistent, then $\mathsf{ZFC} + \mathsf{CH}$ is also consistent
   (e) The addition operation on the ordinals is not commutative.
   (f) Every normal function on the ordinals has a fixed point.
   (g) Ord (the class of all ordinals) is not a set.
   (h) There are classes which are not sets.

2. Consider the following informally stated assertion:

   "For every proper class A and every set X, there exists an injective function $f : X \to A$."

   (a) Write down the above statement formally. You may use the abbreviations "f is a function", "$\mathrm{dom}(f)$" and "$\mathrm{ran}(f)$" without writing them out in detail.
   (b) Is this a statement in the formal language or the meta-language?

3. Find the mistake in the argument below.

   Theorem. ZFC *is inconsistent.*

   Proof. Let $\{\theta_n : n < \omega\}$ be an enumeration of all formulas of $\mathcal{L}_\in$ with exactly one free variable. Let $\psi(x)$ be the formula "$x \in \omega \wedge \neg\theta_x(x)$". Since ψ is a formula of $\mathcal{L}_\in$ with exactly one free variable, it appears in the enumeration, so there exists an $e \in \omega$ such that $\psi = \theta_e$. But then $\mathsf{ZFC} \vdash \theta_e(e) \leftrightarrow \psi(e) \leftrightarrow \neg\theta_e(e)$. □

[1] Note that any statement in the meta-language can *in principle* be formalized and considered a statement in the object language as well. In this exercise, you are asked to consider the standard or most natural meaning.

Part B: relativization, set-models and class-models of set theory

1. (a) Let $F : V \to V$ be a bijective class function. Define $E \subseteq V \times V$ by:

$$xEy \;:\Leftrightarrow\; x \in F(y).$$

We claim that (V, E) is a model of $\mathsf{ZFC} - \mathsf{Foundation}$. Choose any two axioms of $\mathsf{ZFC} - \mathsf{Foundation}$, and prove that they hold in (V, E).

(b) Use the previous claim to show that

$$\mathrm{Con}(\mathsf{ZFC}) \quad \to \quad \mathrm{Con}(\mathsf{ZFC} - \mathsf{Foundation} + \text{“}\exists x\,(x = \{x\})\text{”})$$

Hint: use $F(0) := 1$ and ....

2. (a) Let M and N be sets, and suppose that $(M, \in)$ is an *elementary submodel* of $(N, \in)$ (in the usual model-theoretic sense for set-models, in the language with a binary relation symbol $\in$). We say that an element $c \in N$ is *uniquely definable in* N if there exists a formula $\phi(x)$ such that

$$N \models \forall x\,(\phi(x) \leftrightarrow x = c).$$

Prove that if $c \in N$ is uniquely definable in N, then $c \in M$.

(b) Conclude from this that if $(M, \in)$ is elementary in $(H_{\omega_2}, \in)$, then $\omega_1 \in M$.

(c) Prove that if $(M, \in)$ is elementary in $(V_\omega, \in)$, then $M = V_\omega$.

Hint: Prove, by $\in$-induction, that every $x \in V_\omega$ is uniquely definable in V_ω.