Relative Cons. Proof:

$$\mathrm{Con}(ZFC) \to \mathrm{Con}(ZFC + CH)$$

if $M \models ZFC$ then ... $N \models ZFC + CH$

① If you assume ZFC + Large Cardinals:

$$\exists M \models ZFC \rightsquigarrow \exists N \models ZFC + CH$$

② Relativization: φ^M

prop. class

If I say "$\mathcal{M} \models ZFC + CH$" this actually mean

$\boxed{ZFC \vdash \varphi^{\mathcal{M}}}$ for each φ of ZFC + CH

$\Rightarrow$ Finitary proof of $\mathrm{Con}(ZFC) \to \mathrm{Con}(ZFC + CH)$

Bec: suppose $ZFC + CH \vdash \perp$

Then $\varphi_1 \dots \varphi_n \vdash \perp$

Then $ZFC \vdash \varphi_1^{\mathcal{M}} \dots \varphi_n^{\mathcal{M}}$

Then $ZFC \vdash \perp^{\mathcal{M}}$

Then $ZFC \vdash \perp$ $\square$

model theory

Tarski Sat. Relation

⚠ Difference between φ^M and $M \models \ulcorner \varphi \urcorner$

set

encoded by $n \in \mathbb{N}$ (sound)

- It is ok to say $ZFC \vdash \exists \varphi \, (\mathcal{M} \models \varphi \wedge \mathcal{N} \not\models \varphi)$

a formal object.

set set

- It is not ok to say $ZFC \vdash \cancel{\exists \varphi \, (\varphi^{\mathcal{M}} \wedge \neg \varphi^{\mathcal{N}})}$

Instead: there is a φ such that $ZFC \vdash \varphi^{\mathcal{M}} \wedge \neg \varphi^{\mathcal{N}}$

<u>Relativization</u>: syntactic: $\varphi \rightsquigarrow \varphi^{\mathcal{M}}$

prop. class

- $\exists x \, (\ldots) \rightsquigarrow \exists x \in \mathcal{M} \, (\ldots)^{\mathcal{M}}$
- $\forall x \, (\ldots) \rightsquigarrow \forall x \in \mathcal{M} \, (\ldots)^{\mathcal{M}}$
- The rest does not change

<u>Thm</u>: for every φ, $ZFC \vdash \ldots \varphi^{\mathcal{M}} \ldots$ ← Like Subset-Schema.

$\Rightarrow$ ∞-many theorems!

<u>Note</u>: if M is a set

$ZFC \vdash M \models \ulcorner \varphi \urcorner \leftrightarrow \varphi^{M}$ for a concrete φ

$ZFC \vdash \ulcorner \varphi^{\mathcal{M}} \urcorner$

$$\boxed{\text{for all } \varphi \quad \cancel{ZFC \vdash Tr(\ulcorner \varphi \urcorner) \leftrightarrow \varphi}}$$

$ZFC \nvdash \exists M$ s.t. φ^{M} for all φ in ZFC (Gödel)

① Formal $\models$: $ZFC \nvdash \exists M \ \forall \ulcorner\varphi\urcorner \in ZFC \ (M \models \ulcorner\varphi\urcorner)$

② Relativization φ^M : $PA \vdash$ (for all φ in ZFC, $ZFC \vdash \varphi^M$ (Schema of ∞. many theorems))

even less

~~Subset Axiom: " $\forall \phi \ \forall x \ \exists y \ \forall z \ \ldots \ \phi \ \ldots$ "~~

- ~~If we had a truth predicate then we could say $ZFC \vdash \forall \ulcorner\varphi\urcorner \in ZFC : True(\ulcorner\varphi\urcorner, \ulcorner M\urcorner)$~~

$ZFC \vdash$ " $M \models ZFC$ "