Theorem: $\mathrm{Con}(ZF) \rightarrow \mathrm{Con}(ZFC + GCH)$.

This really means that for every φ in $ZFC + GCH$,

$$ZF \vdash \varphi^L$$

$L :=$ proper class model of $ZFC + GCH$.

First: Recall, downwards Löwenheim-Skolem:

- $X \subseteq M$ M inf. model,

  Then there is $N \preceq M$ (elementary)

  s.t. $X \subseteq N$ and $|N| = \max(\aleph_0, |X|)$

- $N :=$ Skolem Hull of X within M.

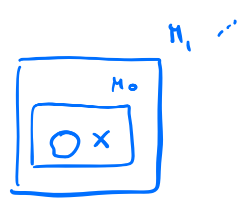
3 Reflection Theorems:

What about Set Theory? For any finite fragment

$$ZFC^* = \{\varphi_1 \ldots \varphi_n\} \subseteq ZFC \qquad (M \preceq_{\varphi_1 \ldots \varphi_n} V)$$

① $\forall X \; \exists M$ (set.) $X \in M$, $M \models ZFC^*$ and $|M| = \max(\aleph_0, |X|)$

[But: M is not transitive.]



② $\forall x \; \exists M$ $x \in M$, $M \models ZFC^*$ and M trans. $\bigcup_n M_n$

[But: M not small; e.g. $M = V_\alpha$]

③ $\forall X$ transitive, $\exists M$, $X \subseteq M$, $M \models ZFC^*$, M transitive

and $|M| = \max(\aleph_0, |x|)$.

[M = Mostowski Collapse of the M from ①]

Follows from ③ with $X = \emptyset$: $\exists$ ctbl. trans. $M \models ZFC^*$.

<u>Def:</u> $X \subseteq M$ ← set is <u>definable</u> over M if there is φ and $a_1 \dots a_n \in M$ such that

$$X = \{ x \in M \mid M \models \varphi(x, a_1 \dots a_n) \}$$

$$D(M) = \{ X \mid X \text{ is definable over } M \}$$

" $\exists \varphi \dots M \models \varphi(\dots)$ " is ok inside ZF, Because M is a <u>set</u>, so we can use formal $M \models \ulcorner \varphi \urcorner$ notion from model theory.

$$L_0 = \emptyset$$
$$L_{\alpha+1} = D(L_\alpha)$$
$$L_\lambda = \bigcup_{\alpha < \lambda} L_\alpha$$
$$L := \bigcup_{\alpha \in Ord} L_\alpha \quad (\text{proper class})$$

Let's compare L & V.

$L_0 = V_0 = \emptyset$

$L_n = V_n$ for all $n < \omega$ (because fin. subsets are def.)

$L_\omega = V_\omega$

$L_{\omega+1} \neq V_{\omega+1}$ in fact $|V_{\omega+1}| = 2^\omega$

$|L_{\omega+1}| = \omega$

Similar Argument: $|L_\alpha| = |\alpha| \quad \forall \alpha \geq \omega$

- $\alpha < \omega_1$: $|L_\alpha| = |\alpha| = \omega$

$\vdots$

$|L_{\omega_1}| = |\bigcup_{\alpha<\omega_1} L_\alpha| = \omega_1$

$\vdots$

$|L_{\omega_1+1}| = \omega_1$

Theorem: (ZF) $L \models ZF$. $[ZF \vdash \varphi^L$ for all φ in ZF$]$

Proof: Mostly not difficult.

Compr. Ax. needs reflection. $\square$

Theorem (ZF): $L \models AC$.

Proof: $AC \cong \forall x$ (x can be wellordered)

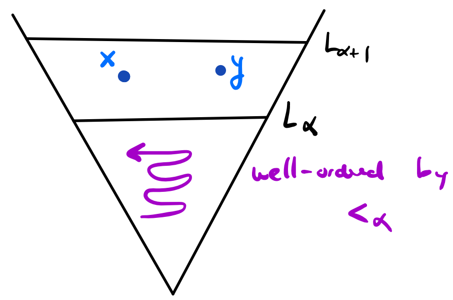
In fact $L \models$ Global Choice, i.e. there is a class $<_L$ which is def. on all of L and is a wellorder.

Inductively wellorder L_α by $<_\alpha$.

At succ. steps, if $x, y \in L_{\alpha+1}$:

① $x \in L_\alpha$, $y \notin L_\alpha$ then $x <_{\alpha+1} y$

(old before new)



② $x, y \in L_\alpha$, then $x <_{\alpha+1} y \leftrightarrow x <_\alpha y$

($<_{\alpha+1}$ extends $<_\alpha$)

③ $x, y \notin L_\alpha$: $x <_{\alpha+1} y$ if

e.g. code φ by nat. number $\ulcorner \varphi \urcorner \in \mathbb{N}$.

the least φ and $<_\alpha^{lex}$-least $a_1 \dots a_n$ defining x

are $<_\alpha^{lex}$-less than

the least ψ and $<_\alpha^{lex}$-least $a_1 \dots a_n$ defining y.

$x = \{ z \dots \mid L_\alpha \models \varphi(z, a_1 \dots a_n) \}$ $\qquad \langle \ulcorner \varphi \urcorner, a_1 \dots a_n \rangle$ which

$y = \{ \qquad \psi \quad b_1 \dots b_k \}$ $\qquad \langle \ulcorner \psi \urcorner, b_1, \dots b_k \rangle$ comes first ?

This way we well-order all of L. $\square$

Now look at GCH:

Main idea: The definition of L ($\alpha \mapsto L_\alpha$) is **absolute** between all transitive models of set theory.

So: $\alpha \in M$ then $L_\alpha \in M$ etc.

Definition: Axiom of Constructibility : $\forall x \, \exists \alpha \, (x \in L_\alpha)$

"$V = L$"

__Thm__: $L \models (V = L)$

__Proof__: Since "L is absolute" it follows trivially.

could be any finite fragment.

__Thm__: L is the minimal trans. class-model of ZFC^*.

__Proof__: Take any class $M \models ZFC^*$, then $\forall \alpha \in M$, so

$\forall \alpha$: $L_\alpha \in M$. So $L \subseteq M$.

__Thm__: If $M \models ZFC^* + (V=L)$, M transitive prop. class, then $M = L$.

__Proof__: $L \subseteq M$ from above.

$M \models \forall x \exists \alpha (x \in L_\alpha)$.

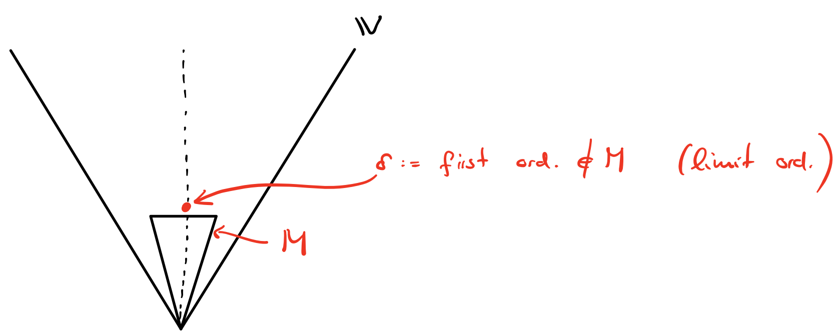
$\therefore \forall x \in M \; \exists \alpha \in M \; (x \in L_\alpha)^M$

$\therefore \forall x \in M \; \exists \alpha \; x \in L_\alpha$

$\therefore M \subseteq L$ $\square$

We also need __set-versions.__

__Def__: M trans. set-model. Then $o(M)$ = height of M := least ordinal $\delta \notin M$. __Equiv__: $\delta = \mathrm{Ord} \cap M$.



__Thm (analogue of above)__: If M trans. set and $M \models ZFC^* + V=L$, then $M = L_\delta$ for $\delta = o(M)$. (*)

Proof: copy-paste:

Proof: Since $\alpha < \delta$ is in $M \Rightarrow L_\alpha \in M \Rightarrow L_\delta \subseteq M$

$M \models \forall x \, \exists \alpha \, (x \in L_\alpha)$.

$\therefore \forall x \in M \, \boxed{\exists \alpha \in M} \, (x \in L_\alpha)^M$

$\therefore \forall x \in M \, \exists \alpha < \delta \, x \in L_\alpha$

$\therefore M \subseteq L_\delta$ $\square$

Idea: L is so minimal that the only submodels of $ZFC^* + V=L$ are either L_δ (if set) or all of L (if class).

Theorem: $V=L \rightarrow GCH$

Proof only CH: we show $\boxed{P(\omega) \subseteq L_{\omega_1}}$.

This is enough:

$$2^{\aleph_0} = |P(\omega)| \leq |L_{\omega_1}| = \omega_1$$

So: take any $x \subseteq \omega$. Let $a = \{x\} \cup \omega$ (to make it transitive).

Then a is trans. and $|a| = \omega$.

Use Reflection 3: Let M be ctbl, trans, $a \subseteq M$

and $M \models ZFC^* + V=L$

But by (*) $M = L_\delta$ and $\delta = o(M)$. But M ctbl,

So $o(M) < \omega_1$.

So $x \in a \in L_\delta \subseteq L_{\omega_1}$. $\Rightarrow$ $x \in L_{\omega_1}$.

<u>So</u>: for any $x \subseteq \omega$, $x \in L_{\omega_1}$ so $\mathcal{P}(\omega) \subseteq L_{\omega_1}$. $\square$.

For GCH: replace $\omega \rightsquigarrow \kappa$
$\omega_1 \rightsquigarrow \kappa^+$

<u>Idea:</u>

"Skolem Hull" of x inside L, can only be some L_δ, $\delta < \omega_1$

"Condensation Lemma"