Cantor (1).

Georg Cantor
(1845-1918)
studied in Zürich, Berlin, Göttingen
Professor in Halle

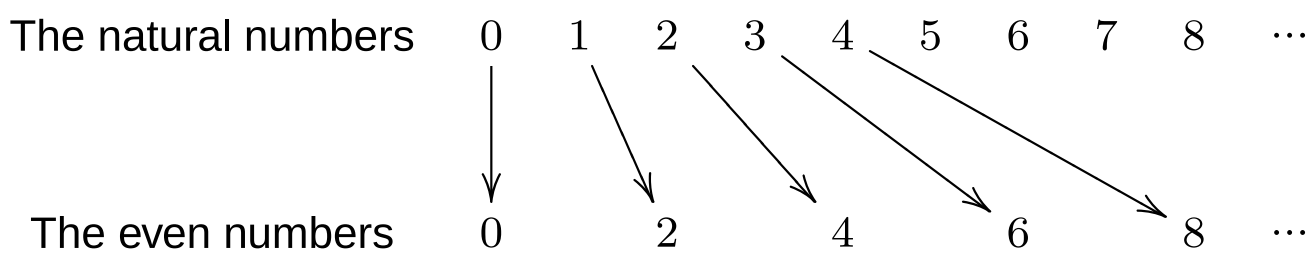
- Work in analysis leads to the notion of cardinality (1874): most real numbers are transcendental.
- Correspondence with Dedekind (1831-1916): bijection between the line and the plane.
- Perfect sets and iterations of operations lead to a notion of ordinal number (1880).

Cantor (2).

Georg Cantor (1845-1918)

- 1877. Leopold Kronecker (1823-1891) tried to prevent publication of Cantor's work.
- Cantor is supported by Dedekind and Felix Klein.
- 1884: Cantor suffers from a severe depression.
- 1888-1891: Cantor is the leading force in the foundation of the *Deutsche Mathematiker-Vereinigung*.
- Development of the foundations of set theory: 1895-1899.

Cardinality (1).



- There is a 1-1 correspondence (bijection) between $\mathbb{N}$ and the even numbers.
- There is a bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$.
- There is a bijection between $\mathbb{Q}$ and $\mathbb{N}$.
- There is no bijection between the set of infinite 0-1 sequences and $\mathbb{N}$.
- There is no bijection between $\mathbb{R}$ and $\mathbb{N}$.

Cardinality (2).

Theorem (Cantor). There is no bijection between the set of infinite 0-1 sequences and $\mathbb{N}$.

Theorem (Cantor). There is a bijection between the real line and the real plane.

Proof. Let's just do it for the set of infi nite 0-1 sequences and the set of pairs of infi nite 0-1 sequences:
If x is an infi nite 0-1 sequence, then let

$$x_0(n) := x(2n), \text{ and}$$

$$x_1(n) := x(2n+1).$$

Let $F(x) := \langle x_0, x_1 \rangle$. F is a bijection. q.e.d.

Cantor to Dedekind (1877): "*Ich sehe es, aber ich glaube es nicht!*"

Transfiniteness (1).

If $X \subseteq \mathbb{R}$ is a set of reals, we call $x \in X$ isolated in X if no sequence of elements of X converges to x.

Cantor's goal: Given any set X, give a construction of a nonempty subset that doesn't contain any isolated points.

Idea: Let X^{isol} be the set of all points isolated in X, and define $X' := X \backslash X^{\mathrm{isol}}$.

Problem: It could happen that $x \in X'$ was the limit of a sequence of points isolated in X. So it wasn't isolated in X, but is now isolated in X'.

Solution: Iterate the procedure: $X_0 := X$ and $X_{n+1} := (X_n)'$.

Transfiniteness (2).

$X' := X \backslash X^{\text{isol}}$; $X_0 := X$ and $X_{n+1} := (X_n)'$.

Question: Is $\bigcap_{n\in\mathbb{N}} X_n$ a set without isolated points?

Answer: In general, no!

So, you could set $X_\infty := \bigcap_{n\in\mathbb{N}} X_n$, and then $X_{\infty+1} := (X_\infty)'$; in general, $X_{\infty+n+1} := (X_{\infty+n})'$.

The indices used in transfinite iterations like this are called ordinals.

Sets (1).

The notion of cardinality needs a general notion of function as a special relation between sets. In order to make the notion of an ordinal precise, we also need sets.

What is a set?

> *Eine Menge ist eine Zusammenfassung bestimmter, wohlunterschiedener Dinge unserer Anschauung oder unseres Denkens zu einem Ganzen.* (Cantor 1895)

Sets (2).

Eine Menge ist eine Zusammenfassung bestimmter, wohlunterschiedener Dinge unserer Anschauung oder unseres Denkens zu einem Ganzen. (Cantor 1895)

Example. Call a linear ordering $\leq$ on a set X a wellorder if any nonempty set $A \subseteq X$ has a $\leq$-least element.

Question. Can we define a wellorder on the set $\mathbb{R}$ of real numbers?

Answer (Zermelo 1908). Yes! The proof uses the following statement about sets: "Whenever I is an index set and for each $i \in I$, the set X_i is nonempty, then the set C of functions $f : I \to \bigcup X_i$ such that for all i, we have $f(i) \in X_i$ is nonempty as well."

$\rightsquigarrow$ Problems in the Foundations of Mathematics (next week)

Syllogistics versus Propositional Logic.

Deficiencies of Syllogistics:

Not expressible:

Every X is a Y and a Z. *Ergo...* Every X is a Y.

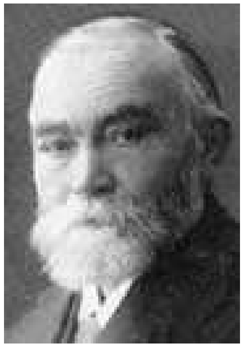
Deficiencies of Propositional Logic:

- $X\mathrm{a}Y$ can be represented as $Y \to X$.
- $X\mathrm{e}Y$ can be represented as $Y \to \neg X$.

Not expressible:

$X\mathrm{i}Y$ and $X\mathrm{o}Y$.

Frege.



Gottlob Frege
1848 - 1925

- Studied in Jena and Göttingen.
- Professor in Jena.
- *Begriffsschrift* (1879).
- *Grundgesetze der Arithmetik* (1893/1903).

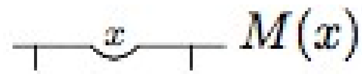
“Every good mathematician is at least half a philosopher, and every good philosopher is at least half a mathematician. (G. Frege)”

Frege's logical framework.

"Everything is M"

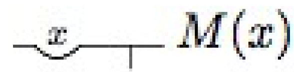
$\forall x\, M(x)$

"Something is M"



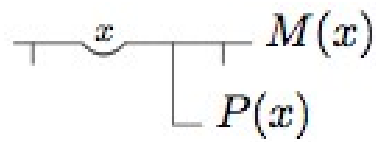
$\exists x\, M(x) \equiv \neg \forall x\, \neg M(x)$

"Nothing is M"



$\forall x\, \neg M(x)$

"Some P is an M"



$$\exists x\, (P(x) \wedge M(x))$$
$$\equiv \neg \forall x\, (P(x) \rightarrow \neg M(x))$$

Second order logic allowing for quantification over properties.

Frege's importance.

- Notion of a formal system.
- Formal notion of proof in a formal system.
- Analysis of number-theoretic properties in terms of second-order properties.
  $\rightsquigarrow$ Russell's Paradox
  (*Grundlagekrise der Mathematik*)

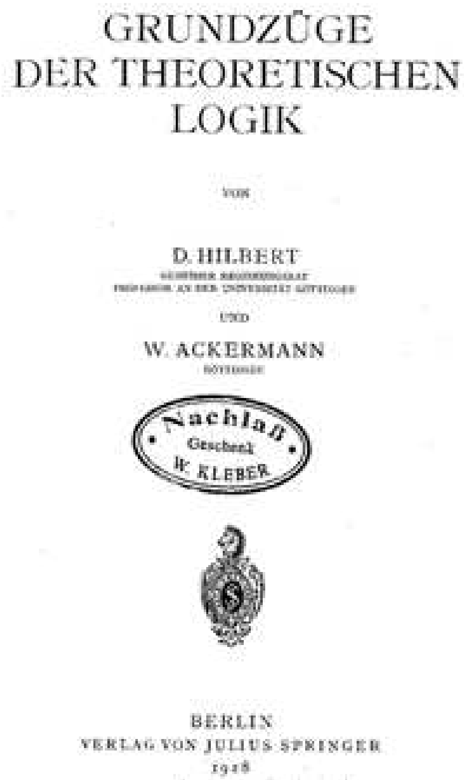
Hilbert (1).

David Hilbert (1862-1943)
Student of Lindemann
1886-1895 Königsberg
1895-1930 Göttingen

1899: *Grundlagen der Geometrie*

"*Man muss jederzeit an Stelle von 'Punkten', 'Geraden', 'Ebenen' 'Tische', 'Stühle', 'Bierseidel' sagen können.*"
"It has to be possible to say 'tables', 'chairs' and 'beer mugs' instead of 'points', 'lines' and 'planes' at any time."

Hilbert (2).


GRUNDZÜGE
DER THEORETISCHEN
LOGIK

VON

D. HILBERT

UND

W. ACKERMANN

BERLIN
VERLAG VON JULIUS SPRINGER
1928

1928: **Hilbert-Ackermann**
Grundzüge der Theoretischen Logik

Wilhelm Ackermann (1896-1962)

First order logic (1).

A first-order language $\mathcal{L}$ is a set $\{\dot{\mathrm{f}}_i\, ;\, i \in I\} \cup \{\dot{\mathrm{R}}_j\, ;\, j \in J\}$ of function symbols and relation symbols together with a signature $\sigma : I \cup J \to \mathbb{N}$.

- $\sigma(\dot{\mathrm{f}}_i) = n$ is interpreted as "$\dot{\mathrm{f}}_i$ represents an n-ary function".
- $\sigma(\dot{\mathrm{R}}_i) = n$ is interpreted as "$\dot{\mathrm{R}}_i$ represents an n-ary relation".

In addition to the symbols from $\mathcal{L}$, we shall be using the logical symbols $\forall$, $\exists$, $\wedge$, $\vee$, $\to$, $\neg$, $\leftrightarrow$, equality $=$, and a set of variables Var.

First order logic (2).

We fi x a fi rst-order language $\mathcal{L} = \{\dot{\mathrm{f}}_i\ ;\ i \in I\} \cup \{\dot{\mathrm{R}}_j\ ;\ j \in J\}$ and a signature $\sigma : I \cup J \to \mathbb{N}$.

Definition of an $\mathcal{L}$-term.

- Every variable is an $\mathcal{L}$-term.
- If $\sigma(\dot{\mathrm{f}}_i) = n$, and $t_1, ..., t_n$ are $\mathcal{L}$-terms, then $\dot{\mathrm{f}}_i(t_1, ..., t_n)$ is an $\mathcal{L}$-term.
- Nothing else is an $\mathcal{L}$-term.

Example. Let $\mathcal{L} = \{\dot{\times}\}$ be a first order language with a binary function symbol.

- $\dot{\times}(x, x)$ is an $\mathcal{L}$-term (normally written as $x \dot{\times} x$, or x^2).
- $\dot{\times}(\dot{\times}(x, x), x)$ is an $\mathcal{L}$-term (normally written as $(x \dot{\times} x) \dot{\times} x$, or x^3).

First order logic (3).

Definition of an $\mathcal{L}$-formula.

- If t and t^* are $\mathcal{L}$-terms, then $t = t^*$ is an $\mathcal{L}$-formula.
- If $\sigma(\dot{\mathrm{R}}_i) = n$, and $t_1, ..., t_n$ are $\mathcal{L}$-terms, then $\dot{\mathrm{R}}_i(t_1, ..., t_n)$ is an $\mathcal{L}$-formula.
- If φ and ψ are $\mathcal{L}$-formulae and x is a variable, then $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$, $\varphi \leftrightarrow \psi$, $\forall x\,(\varphi)$ and $\exists x\,(\varphi)$ are $\mathcal{L}$-formulae.
- Nothing else is an $\mathcal{L}$-formula.

An $\mathcal{L}$-formula without free variables is called an $\mathcal{L}$-sentence.

Semantics (1).

We fix a first-order language $\mathcal{L} = \{\dot{\mathrm{f}}_i \,;\, i \in I\} \cup \{\dot{\mathrm{R}}_j \,;\, j \in J\}$ and a signature $\sigma : I \cup J \to \mathbb{N}$.

A tuple $\mathbf{X} = \langle X, \langle f_i \,;\, i \in I\rangle, \langle R_j \,;\, j \in J\rangle\rangle$ is called an $\mathcal{L}$-structure if f_i is an $\sigma(\dot{\mathrm{f}}_i)$-ary function on X and R_i is an $\sigma(\dot{\mathrm{R}}_i)$-ary relation on X.

An X-interpretation is a function $\iota : \mathrm{Var} \to X$.

If ι is an X-interpretation and $\mathbf{X}$ is an $\mathcal{L}$ then ι extends to a function $\hat{\iota}$ on the set of all $\mathcal{L}$-terms.

If $\mathbf{X}$ is an $\mathcal{L}$-structure and ι is an X-interpretation, we define a semantics for all $\mathcal{L}$-formulae by recursion.

Semantics (2).

If $\mathbf{X}$ is an $\mathcal{L}$-structure and ι is an X-interpretation, we define a semantics for all $\mathcal{L}$-formulae by recursion.

- $\mathbf{X}, \iota \models t = t^*$ if and only if $\hat{\iota}(t) = \hat{\iota}(t^*)$.
- $\mathbf{X}, \iota \models \dot{\mathrm{R}}_j(t_1, ..., t_n)$ if and only if $R(\hat{\iota}(t_1), ..., \hat{\iota}(t_n))$.
- $\mathbf{X}, \iota \models \varphi \wedge \psi$ if and only if $\mathbf{X}, \iota \models \varphi$ and $\mathbf{X}, \iota \models \psi$.
- $\mathbf{X}, \iota \models \neg\varphi$ if and only if it is not the case that $\mathbf{X}, \iota \models \varphi$.
- $\mathbf{X}, \iota \models \forall x\,(\varphi)$ if and only if for all X-interpretations ι^* with $\iota \sim_x \iota^*$, we have $\mathbf{X}, \iota^* \models \varphi$.

- $\mathbf{X} \models \varphi$ if and only if for all X-interpretations ι, we have $\mathbf{X}, \iota \models \varphi$.

Object Language $\leftrightarrow$ Metalanguage.

Semantics (3).

Object Language $\leftrightarrow$ Metalanguage.

Let $\mathbf{X}$ be an $\mathcal{L}$-structure. The theory of $\mathbf{X}$, $\mathrm{Th}(\mathbf{X})$, is the set of all $\mathcal{L}$-sentences φ such that $\mathbf{X} \models \varphi$.

Under the assumption that the *tertium non datur* holds for the metalanguage, the theory of $\mathbf{X}$ is always complete:

For every sentence φ, we either have $\varphi \in \mathrm{Th}(\mathbf{X})$ or $\neg\varphi \in \mathrm{Th}(\mathbf{X})$.

Deduction (1).

Let Φ be a set of $\mathcal{L}$-sentences. A Φ-proof is a finite sequence $\langle \varphi_1, ..., \varphi_n \rangle$ of $\mathcal{L}$-formulae such that for all i, one of the following holds:

- $\varphi_i \equiv t = t$ for some $\mathcal{L}$-term t,
- $\varphi_i \in \Phi$, or
- there are $j, k < i$ such that φ_j and φ_k are the premisses and φ_i is the conclusion in one of the rows of the following table.

Premisses		Conclusion
$\varphi \wedge \psi$		φ
$\varphi \wedge \psi$		ψ
φ	ψ	$\varphi \wedge \psi$
φ	$\neg\varphi$	ψ
$\varphi \to \psi$	$\neg\varphi \to \psi$	ψ
$\forall x\,(\varphi)$		$\varphi\frac{s}{x}$
$\varphi\frac{y}{x}$		$\forall x\,(\varphi))$
$t = t^*$	$\varphi\frac{t}{x}$	$\varphi\frac{t^*}{x}$

Deduction (2).

If Φ is a set of $\mathcal{L}$-sentences and φ is an $\mathcal{L}$-formula, we write $\Phi \vdash \varphi$ if there is a Φ-proof in which φ occurs.

We call a set Φ of sentences a theory if whenever $\Phi \vdash \varphi$, then $\varphi \in \Phi$ ("Φ is deductively closed").

Example. Let $\mathcal{L} = \{\leq\}$ be the language of partial orders. Let $\Phi_{\mathrm{p.o.}}$ be the axioms of partial orders, and let Φ be the deductive closure of $\Phi_{\mathrm{p.o.}}$. Φ is not a complete theory, as the sentence $\forall x \forall y (x \leq y \vee y \leq x)$ is not an element of Φ, but neither is its negation.

Completeness.

Kurt Gödel (1906-1978)

Semantic entailment. We write $\Phi \models \varphi$ for "whenever $\mathbf{X} \models \Phi$, then $\mathbf{X} \models \varphi$".

Gödel Completeness Theorem (1929).

$\Phi \vdash \varphi$ if and only if $\Phi \models \varphi$.

"there is a Φ-proof of φ" "for all $\mathbf{X} \models \Phi$, we have $\mathbf{X} \models \varphi$"

$\Phi \nvdash \varphi$ if and only if $\Phi \not\models \varphi$.

"no Φ-proof contains φ" "there is some $\mathbf{X} \models \Phi \wedge \neg\varphi$"

Applications (1).

The Model Existence Theorem.

If Φ is consistent (*i.e.*, $\Phi \nvdash \bot$), then there is a model $\mathbf{X} \models \Phi$.

The Compactness Theorem.

Let Φ be a set of sentences. If every finite subset of Φ has a model, then Φ has a model.

Proof. If Φ doesn't have a model, then it is inconsistent by the **Model Existence Theorem**.
So, $\Phi \vdash \bot$, *i.e.*, there is a Φ-proof P of $\bot$.
But P is a finite object, so it contains only finitely many elements of Φ. Let Φ_0 be the set of elements occurring in P. Clearly, P is a Φ_0-proof of $\bot$, so Φ_0 is inconsistent. Therefore Φ_0 cannot have a model. q.e.d.

Applications (2).

The Compactness Theorem. Let Φ be a set of sentences. If every fi nite subset of Φ has a model, then Φ has a model.

Corollary 1. Let Φ be a set of sentences that has arbitrary large finite models. Then Φ has an infinite model.

Proof. Let $\psi_{\geq n}$ be the formula stating "there are at least n different objects". Let $\Psi := \{\psi_{\geq n}\,;\, n \in \mathbb{N}\}$. The premiss of the theorem says that every fi nite subset of $\Phi \cup \Psi$ has a model. By compactness, $\Phi \cup \Psi$ has a model. But this must be infi nite. q.e.d.

Let $\mathcal{L} := \{\leq\}$ be the first order language with one binary relation symbol. Let $\Phi_{\text{p.o.}}$ be the axioms of partial orders.

Corollary 2. There is no sentence σ such that for all partial orders $\mathbf{P}$, we have

$$\mathbf{P} \text{ is finite if and only if } \mathbf{P} \models \sigma.$$

[If σ is like this, then **Corollary 1** can be applied to $\Phi_{\text{p.o.}} \cup \{\sigma\}$.]