UNIVERSITEIT VAN AMSTERDAM
INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Axiomatische Verzamelingentheorie

2005/2006; 2nd Semester
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Homework Set # 12 *Deadline:* May 11th, 2006

Exercise 31 (total of seven points).

In this exercise, you are **not** supposed to use the result $\kappa + \lambda = \kappa \cdot \lambda = \max(\kappa, \lambda)$ that we proved in class. Instead, you are supposed to prove the equalities between cardinals directly by giving a bijection. For example, in order to show $\kappa \cdot \lambda = \lambda \cdot \kappa$, you should give a bijection between $\kappa \times \lambda$ and $\lambda \times \kappa$. As in Exercise 30, let $\mathrm{Fun}(X, Y)$ be the set of functions from X to Y.
For cardinals κ, λ and μ, prove:

(1) $\kappa \cdot \lambda = \lambda \cdot \kappa$ (1 point),
(2) $(\kappa \cdot \lambda) \cdot \mu = \kappa \cdot (\lambda \cdot \mu)$ (2 points),
(3) $(\kappa + \lambda) \cdot \mu = \kappa \cdot \mu + \lambda \cdot \mu$ (2 points),
(4) $\mathrm{Card}(\mathrm{Fun}(\mu, \mathrm{Fun}(\lambda, \kappa))) = \mathrm{Card}(\mathrm{Fun}(\mu \times \kappa, \kappa))$ (2 points).

Exercise 32 (total of nine points).

Let κ be a cardinal. We call a set X κ**-splittable** if there is a family $\{X_\alpha\,;\,\alpha < \kappa\rangle$ such that for all $\alpha < \kappa$, $\mathrm{Card}(X_\alpha) \leq \kappa$, and $X = \bigcup_{\alpha<\kappa} X_\alpha$.

(1) Prove that every nonempty set X is $\mathrm{Card}(X)$-splittable (1 point).
(2) Use the axiom of choice to prove that no cardinal $\kappa > \aleph_0$ is $\aleph_0$-splittable (2 points).
(3) **Without using the axiom of choice (!)**, prove that no cardinal $\kappa > \aleph_1$ is $\aleph_0$-splittable (6 points).

Hint. If $X \subseteq \kappa$ is countable, then $\mathrm{o.t.}(X) < \omega_1$. Use a family witnessing that κ is $\aleph_0$-splittable to define an injection from κ into $\aleph_1 \times \aleph_0$. Derive a contradiction.

Exercise 33 (total of seven points).

Recall the definition of the Gödel β-function: If $\gamma, \delta, \gamma', \delta'$ are ordinals with $\mu := \max(\gamma, \delta)$ and $\mu' := \max(\gamma', \delta')$, we let

$$\langle \gamma, \delta \rangle \prec \langle \gamma', \delta' \rangle \iff (\mu < \mu') \lor (\mu = \mu' \,\&\, \gamma < \gamma') \lor (\mu = \mu' \,\&\, \gamma = \gamma' \,\&\, \delta < \delta').$$

As proved in the lecture, $\prec$ is a wellordering on any set of pairs of ordinals. For fixed $\langle \gamma, \delta \rangle$, we let

$$O_{\gamma,\delta} := \{\langle \xi, \eta \rangle \,;\, \langle \xi, \eta \rangle \prec \langle \gamma, \delta \rangle\},$$

and then $\beta(\gamma, \delta) := \mathrm{o.t.}(\langle O_{\gamma,\delta}, \prec \rangle)$.
Prove that the ordinal operation $\alpha \mapsto \beta(\alpha, 0)$ is normal (*i.e.*, for all $\gamma < \delta$, we have $\beta(\gamma, 0) < \beta(\delta, 0)$ and for limit λ, we have $\beta(\lambda, 0) = \bigcup_{\alpha<\lambda} \beta(\alpha, 0)$) (2 points). Therefore, this operation has arbitrarily large fixed points. Note that in class, we proved (without using that the operation is normal that all infinite cardinals are fixed points of the operation $\alpha \mapsto \beta(\alpha, 0)$.

Is $\omega \cdot 2$ a fixed point of $\alpha \mapsto \beta(\alpha, 0)$ (prove your claim, 2 points)? Are there fixed points of the operation that are not infinite cardinals (prove your claim, 2 points)? Compute $\beta(\omega+2, \omega+1)$ (1 point).