UNIVERSITEIT VAN AMSTERDAM
INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Axiomatische Verzamelingentheorie

2005/2006; 2nd Semester
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Homework Set # 13 *Deadline:* May 18th, 2006

Exercise 34 (total of seven points).

Recursively define

$$\begin{aligned} \beth_0 &:= \omega \\ \beth_{\alpha+1} &:= 2^{\beth_\alpha} \\ \beth_\lambda &:= \bigcup_{\xi<\lambda} \beth_\xi. \end{aligned}$$

As in the lecture, let GCH be the statement "$\forall\alpha \in \mathrm{Ord}(2^{\aleph_\alpha} = \aleph_{\alpha+1})$". Show that GCH is equivalent to the statement "for all α, $\aleph_\alpha = \beth_\alpha$" (2 points).
We call a cardinal κ a **beth fixed point** if $\kappa = \beth_\kappa$. Prove that there is a beth fixed point κ such that $\mathrm{cf}(\kappa) = \aleph_1$ (5 points).

Exercise 35 (total of eight points).

Use the axiom of choice to prove that every successor cardinal is regular (4 points). Prove that there is a proper class of singular cardinals (4 points).

Exercise 36 (total of seven points).

Let $\mathbf{M} \subseteq \mathbf{N}$ be transitive models of set theory with the same $\in$-relation, $\mathbf{M} \models \mathsf{ZF}$ and $\mathbf{N} \models \mathsf{ZFC}$. Let $\alpha \in \mathbf{M}$ be an ordinal in both $\mathbf{M}$ and $\mathbf{N}$ such that

- $\mathbf{M} \models$ "α is the least uncountable cardinal" and
- $\mathbf{M} \models \mathrm{cf}(\alpha) = \aleph_0$.

Show that $\mathbf{N} \models$"α is countable" (7 points).