UNIVERSITEIT VAN AMSTERDAM
INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Axiomatische Verzamelingentheorie

2005/2006; 2nd Semester
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Homework Set # 14 (the last) *Deadline:* May 25th, 2006

Exercise 37 (total of eight points).

For cardinals κ and λ, write $[\kappa;\lambda]$ for $\bigcup\{\alpha^\lambda\,;\,\alpha<\kappa\}$.
Let κ be a limit cardinal and $\lambda \geq \mathrm{cf}(\kappa)$. Prove that

$$\kappa^\lambda = [\kappa;\lambda]^{\mathrm{cf}(\kappa)}.$$

Exercise 38 (total of eight points).

Define the **gimel function** $\gimel(\kappa) := \kappa^{\mathrm{cf}(\kappa)}$. Assume CH (*i.e.*, $2^{\aleph_0} = \aleph_1$) and "for all singular cardinals λ, we have $\gimel(\lambda) = \lambda^+$".
Compute ${\aleph_\omega}^{\aleph_0}$ (1 point), ${\aleph_{\omega+n}}^{\aleph_0}$ (2 points), ${\aleph_{\omega+\omega}}^{\aleph_0}$ (1 point), ${\aleph_{\omega_1+1}}^{\aleph_1}$ (2 points).
What are the best upper and lower bounds for ${\aleph_{\omega+\omega}}^{\aleph_1}$ that you can give under these assumptions (don't prove that they are optimal, just argue why they are upper and lower bounds; 2 points).

Exercise 39 (total of eight points).

A cardinal κ is called **inaccessible** if it is regular and for all $\lambda < \kappa$, we have that $2^\lambda < \kappa$. Prove that if κ is an inaccessible cardinal, then $\mathbf{V}_\kappa \models \mathsf{Repl}$ where Repl stands for the axiom scheme of replacement. Point out exactly where the two properties of κ (regularity and strong limit) are needed.

Important: Please make sure that you are very precise about what it means that $\mathbf{V}_\kappa \models \mathsf{Repl}$ (you have to relativize all formulas to $\mathbf{V}_\kappa$). Only a properly written solution that does not fall into the metamathematical pitfalls here will get full credit.