UNIVERSITEIT VAN AMSTERDAM
INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Core Logic

2007/2008; 1st Semester
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Homework Set # 9 *Deadline:* November 14th, 2007

Exercise 30 (9 points).

Translated into the language of Boolean algebras, **Celarent** became

For all a, b, and c, if $ba = \mathbf{0}$ and $c(\mathbf{1} - b) = \mathbf{0}$, then $ca = \mathbf{0}$.

Rephrase **Baroco**, **Darapti**, and **Felapton** in a similar way in the language of Boolean algebras (1 point each). Find out whether these statements are true in Boolean algebras. If they are, prove it from the axioms of Boolean algebras. If not, give a counterexample. (2 points each).

Exercise 31 (3 points).

Let $\mathbf{B} = \langle B, 0, 1, +, \cdot, - \rangle$ be a Boolean algebra. Define an operation $\star$ by $x \star y := -(x + y)$ (the `NOR` or Sheffer operation). Give formulas φ_{mult}, φ_{add}, φ_{comp} in the language just containing $\star$, $=$ and parentheses such that

$$\begin{aligned} \varphi_{\mathrm{mult}}(x, y, z) &\equiv x \cdot y = z \\ \varphi_{\mathrm{add}}(x, y, z) &\equiv x + y = z \\ \varphi_{\mathrm{comp}}(x, z) &\equiv -x = z \end{aligned}$$

(1 point each). (In other words, the $\star$-language is expressive enough to define the language of Boolean algebras.)

Exercise 32 (6 points).

A structure $\langle R, +, \cdot, 0, 1 \rangle$ is called a **ring** if $+$ is a commutative and associative binary operation on R, $\cdot$ is an associative binary operation on R, $\cdot$ distributes over $+$ (*i.e.*, $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$), 0 is the neutral element of $+$ (*i.e.*, $0 + a = a + 0 = a$) and 1 is the neutral element of $\cdot$ (*i.e.*, $a \cdot 1 = 1 \cdot a = a$).
Examples of rings are: the integers $\mathbb{Z}$, the rationals $\mathbb{Q}$, the reals $\mathbb{R}$.

Let $\mathbf{B} = \langle B, 0, 1, \vee, \wedge, - \rangle$ be a Boolean algebra. For $X, Y \in B$, define

$$X + Y := (X \wedge -Y) \vee (-X \wedge Y), \text{ and}$$

$$X \cdot Y := X \wedge Y.$$

We write $\mathrm{R}(\mathbf{B}) := \langle B, +, \cdot, 0, 1 \rangle$.

(1) Prove that $\mathrm{R}(\mathbf{B})$ is a ring (3 points).
(2) Give an example of a ring R such that R is not isomorphic to any $\mathrm{R}(\mathbf{B})$ (with a proof; 3 points).

Exercise 33 (4 points).

In a small Irish village, there are three children, Sean, Mairi, and Breandan, all of which like icecream. Sean likes Mairi and Breandan. Mairi likes Breandan, but not Sean; and Breandan doesn't like either of the other two.

Consider the set $V := \{\mathsf{S}, \mathsf{M}, \mathsf{B}, \mathsf{I}\}$ where S represents Sean, M represents Mairi, B represents Breandan, and I represents icecream. Consider the following set L of two-element subsets of V:

$$\{x, y\} \in L \text{ if and only if either } x \text{ likes } y \text{ or } y \text{ likes } x.$$

Show that $\langle V, L, \in \rangle$ is a plane.