Capita Selecta: Set Theory

Lecture II

M set of moves

$M^{<\omega}$ positions

M^{ω} plays/runs

$A \subseteq M^{\omega}$ payoff set

$G(A)$ game w/ payoff A

[$G(T; A)$ game w/ payoff A & rules T]

σ strategy

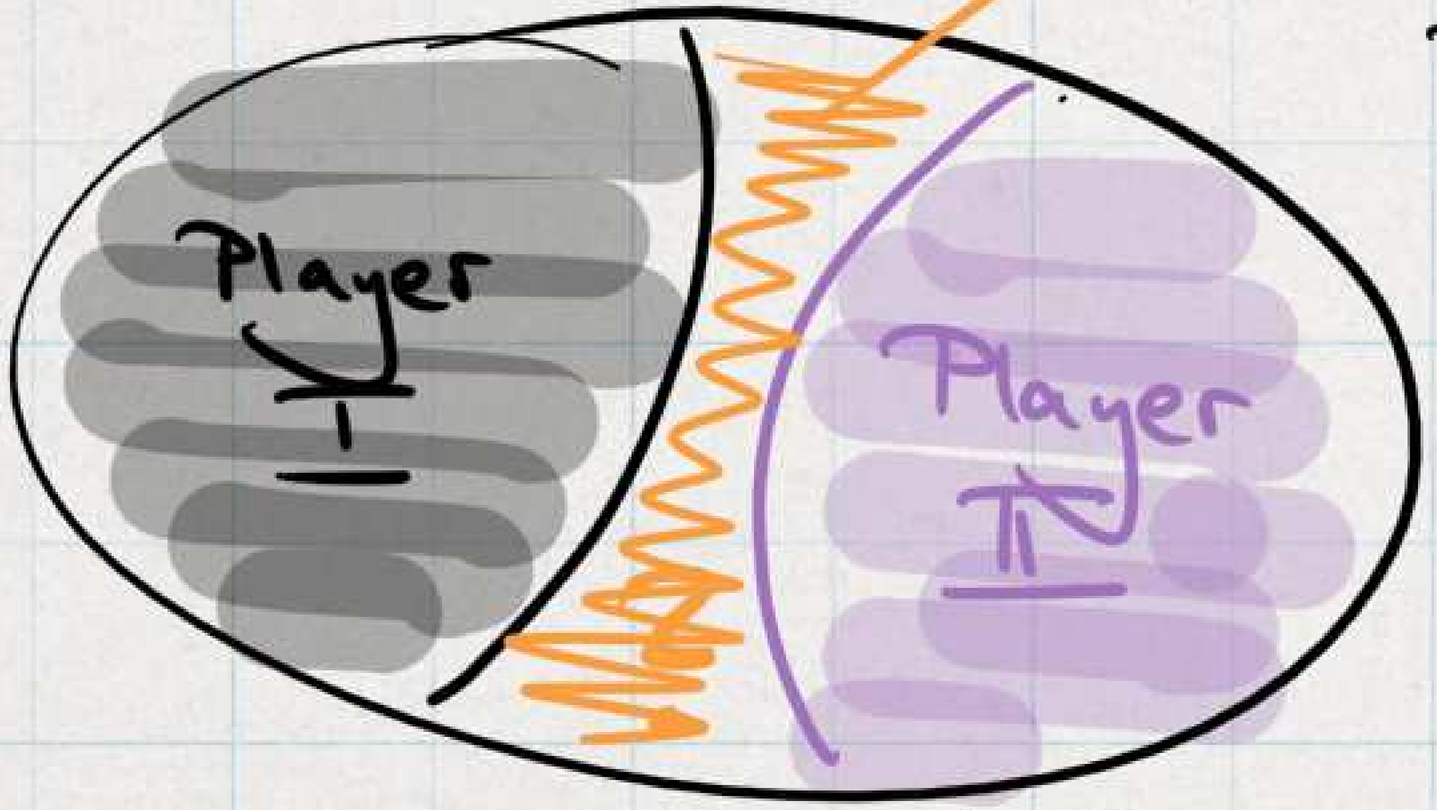
$\sigma * \tau$

σ winning for I if $\forall \tau \ \sigma * \tau \in A$

τ winning for II if $\forall \sigma \ \sigma * \tau \notin A$

$\sigma : M^{<\omega} \longrightarrow M$

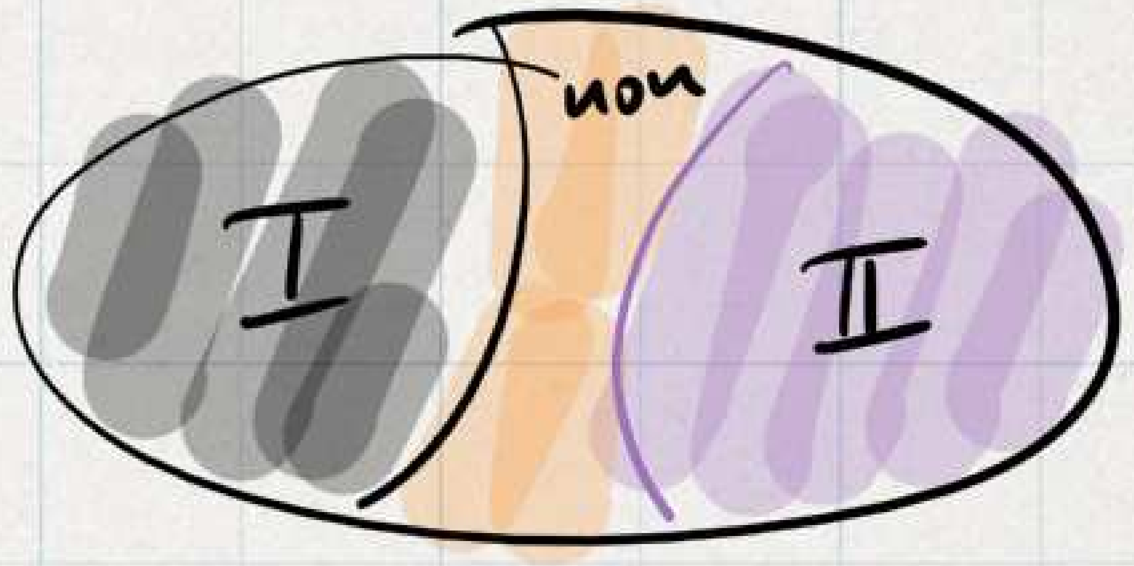
non-determined



Power set of M^{ω}

It is impossible to have a w.s. for I and for II

Def. A is determined if either player I or II has a w.s. in $G(A)$.



Ulam

Can we characterise the set of A s.t. I has a w.s. or the set of A s.t. II has a w.s.

Modern Can we characterise the set of determined sets.

TODAY These are non-trivial questions.

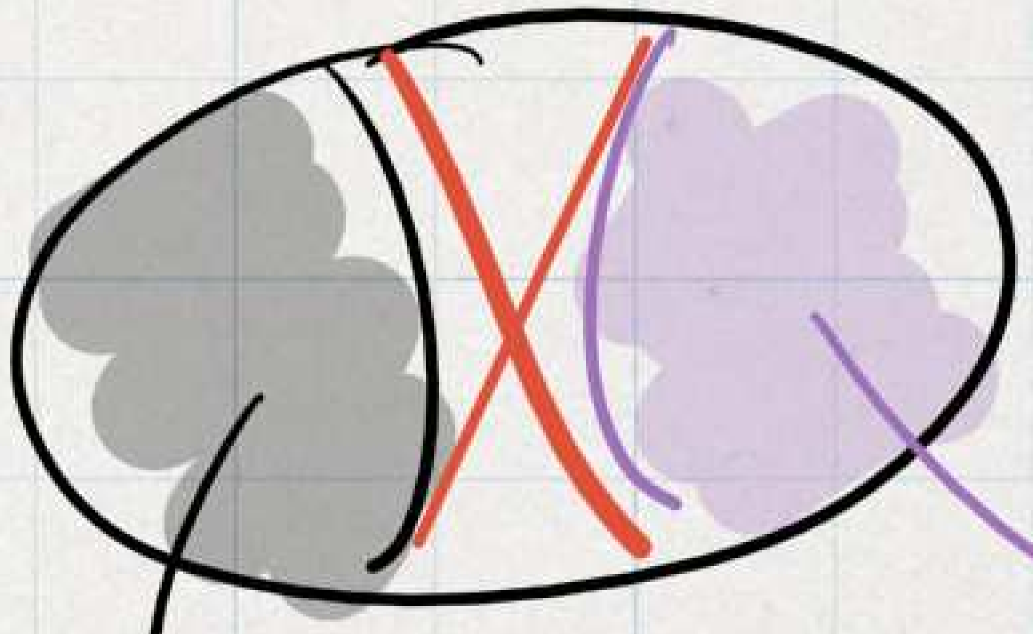
[In particular: $AC \implies \neq \emptyset$.]

What about the choice of M?

Completely trivial case : $|M| = 1$.

$M = \{m\}$

$|M^\omega| = 1$, viz. the constant function



$\mathcal{B}(M^\omega) = \{\emptyset, M^\omega\}$

$\emptyset$ is won by player II

M^ω is won by player I

If $|M| \geqslant 2$, let $m_0 \neq m_1 \in M$

$x \longmapsto z_x \qquad z_x(n) := m_{x(n)}$

This is inj. from 2^ω into M^ω.

So $|M^\omega| \geq 2^{\aleph_0}$.

So $k(M^\omega) \geqslant 2^{(2^{\aleph_0})}$

Ulam Can we give suff./nec. criteria for

I wins $G(A)$

II wins $G(A)$

Theorem If $|M| \geq 2$ and A is countable, then II has a w.s. in $G(A)$.

Proof. Suppose $A = \{a_i ; i \in \mathbb{N}\}$

So, player II in his i-th move plays an element $m \in M$ s.t. $m \neq a_i(2i+1)$.

Possible since $|M| \geq 2$.

τ

This wins against every str. σ for player I:

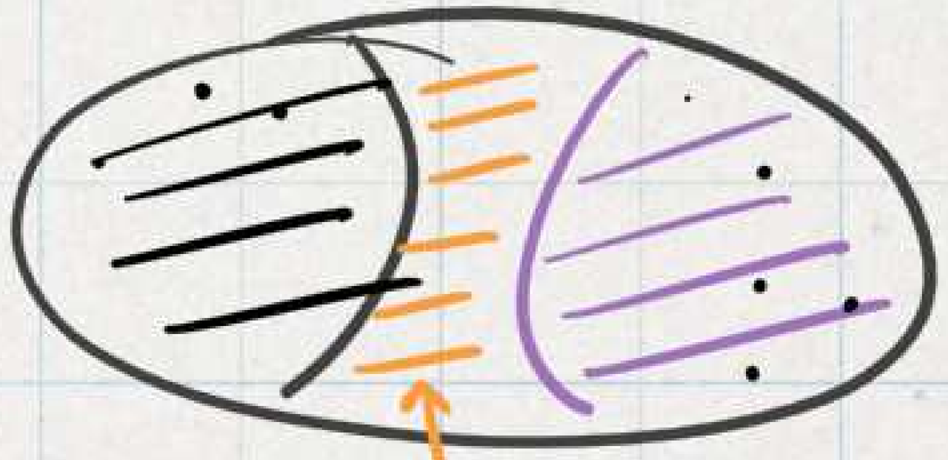
Suppose $\sigma * \tau \in A$, find i s.t. $\sigma * \tau = a_i$.

But then $\sigma * \tau(2i+1) = \tau(\sigma * \tau \upharpoonright 2i+1)$

$\neq a_i(2i+1)$.

q.e.d.

Similarly, if $M^\omega \setminus A$ is countable, then player I has a w.s.



if $\neq \emptyset$, then sets must be uncountable + complement uncountable.

$\mathcal{P}(M^\omega)$

NOTATION

If $x \in M^\omega$, then

$x_{\mathrm{II}}, x_{\mathrm{I}} \in M^\omega$

$$x_{\mathrm{I}}(n) := x(2n)$$

$$x_{\mathrm{II}}(n) := x(2n+1)$$

$x \in M^\omega$

$$\sigma_x(p) := x\left(\frac{lh(p)}{2}\right) \text{ if } lh(p) \text{ is even}$$

$$\tau_x(p) := x\left(\frac{lh(p)-1}{2}\right) \text{ if } lh(p) \text{ is odd}$$

BLINDFOLDED STRATEGIES

<u>EXAMPLE</u>

$$(\sigma * \tau_x)_{\underline{II}} = x$$

$$(\sigma_x * \tau)_{\underline{I}} = x.$$

[Easy induction proofs.]

This tells us how many strategies there are:
If $|M| = 2$, then there are precisely $2^{\aleph_0}$ strategies.

[$2^{\aleph_0} \leq |\text{Strategies}|$: each blindfolded str. is a str.]

$|\text{Strategies}| \leq 2^{\aleph_0}$: $\sigma : \underset{\text{ctble}}{M^{<\omega}} \longrightarrow \underset{|M|=2.}{M}$

If M is bigger, there could be more strategies.

<u>Fact</u> If σ is a str. then σ is winning for $\underline{\mathrm{I}}$

iff

σ wins against each blindfolded strategy

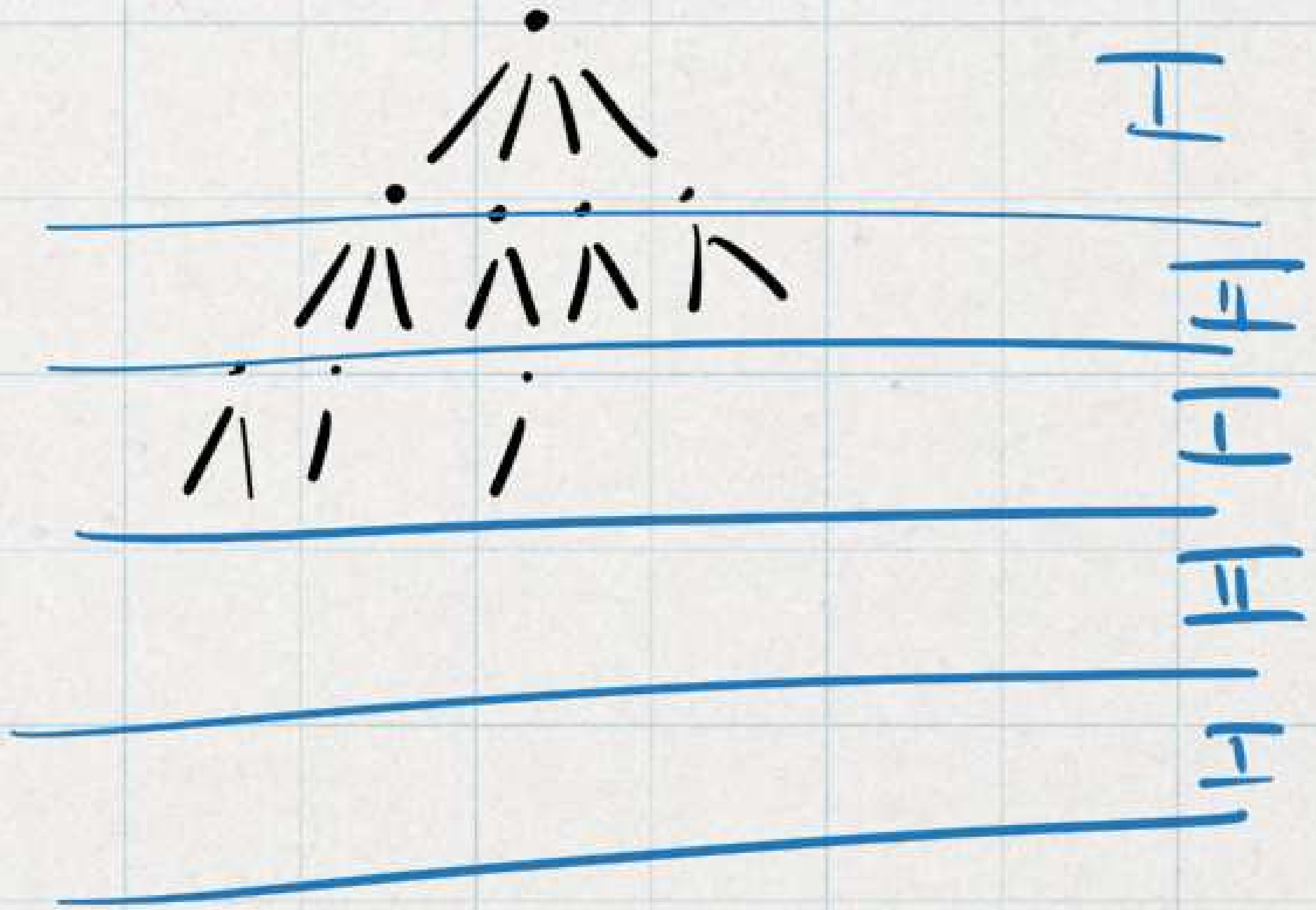
<u>Proof.</u> "$\Longrightarrow$" OK.

"$\Longleftarrow$" Suppose not. Let τ be a counterstr. that wins against σ.

$\sigma * \tau \notin A$. Define $x := \sigma * \tau$. Then $\sigma * \tau_y = \sigma * \tau \notin A$.

Define $y := x_{\mathrm{II}}$. So σ loses against τ_y. q.e.d.

Necessary criteria.



We re-think our notion of strategy.

$$\sigma : M^{<\omega} \longrightarrow M$$

Think of a strategy as a tree

$$T \subseteq M^{<\omega}.$$

Def T^{I}_{σ} STRATEGIC I-tree for σ

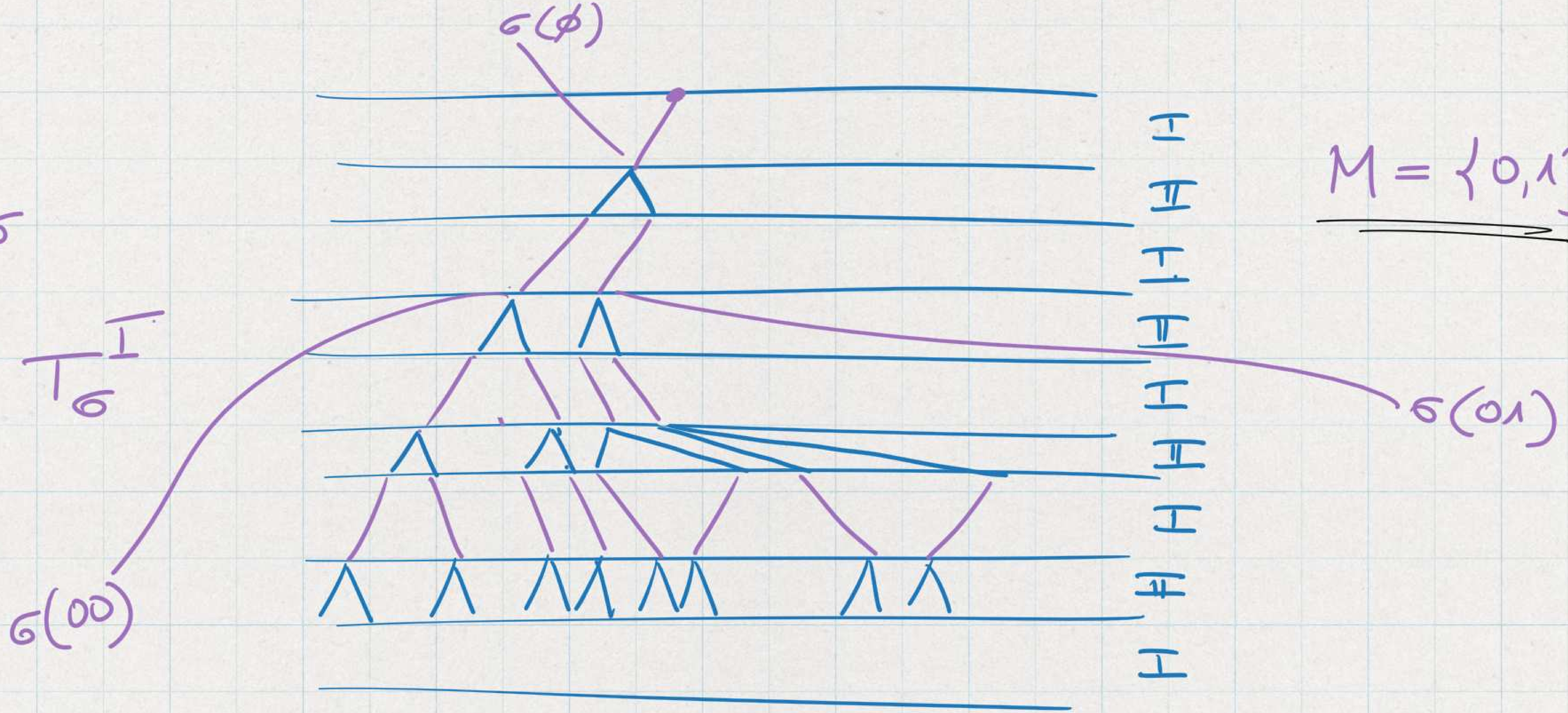
$$p \in T^{\mathrm{I}}_{\sigma} :\Longleftrightarrow \text{f.a. } 2n < lh(p) \quad p(2n) = \sigma(p \upharpoonright 2n)$$

$$p \in T^{\mathrm{II}}_{\sigma} :\Longleftrightarrow \text{f.a. } 2n+1 < lh(p) \quad p(2n+1) = \sigma(p \upharpoonright 2n+1).$$

$M = \{0,1\}$

σ

T_σ^{I}



If σ, τ strategies

$$\{\sigma * \tau\} = [T_\sigma^{\mathrm{I}}] \cap [T_\tau^{\mathrm{II}}]$$

Proof. Obviously by construction $\sigma * \tau \in [T_\sigma^{\mathrm{I}}] \cap [T_\tau^{\mathrm{II}}]$.

Suppose $z \in [T_\sigma^{\mathrm{I}}] \cap [T_\tau^{\mathrm{II}}]$, $z \neq \sigma * \tau$.

Let i be least s.t. $z(i) \neq (\sigma * \tau)(i)$.

$$z \upharpoonright i = (\sigma * \tau) \upharpoonright i.$$

W.l.o.g., i even $z \in [T_\sigma^{\mathrm{I}}] \implies z(i) = \sigma(z \upharpoonright i)$

$$= \sigma(\sigma * \tau \upharpoonright i)$$

$$= \sigma * \tau(i).$$

WITHOUT LOSS OF GENERALITY

So, this means that a strategy σ is winning for I iff

$$[T^{\mathrm{I}}_{\sigma}] \subseteq A.$$

Similarly, τ is winning for II iff

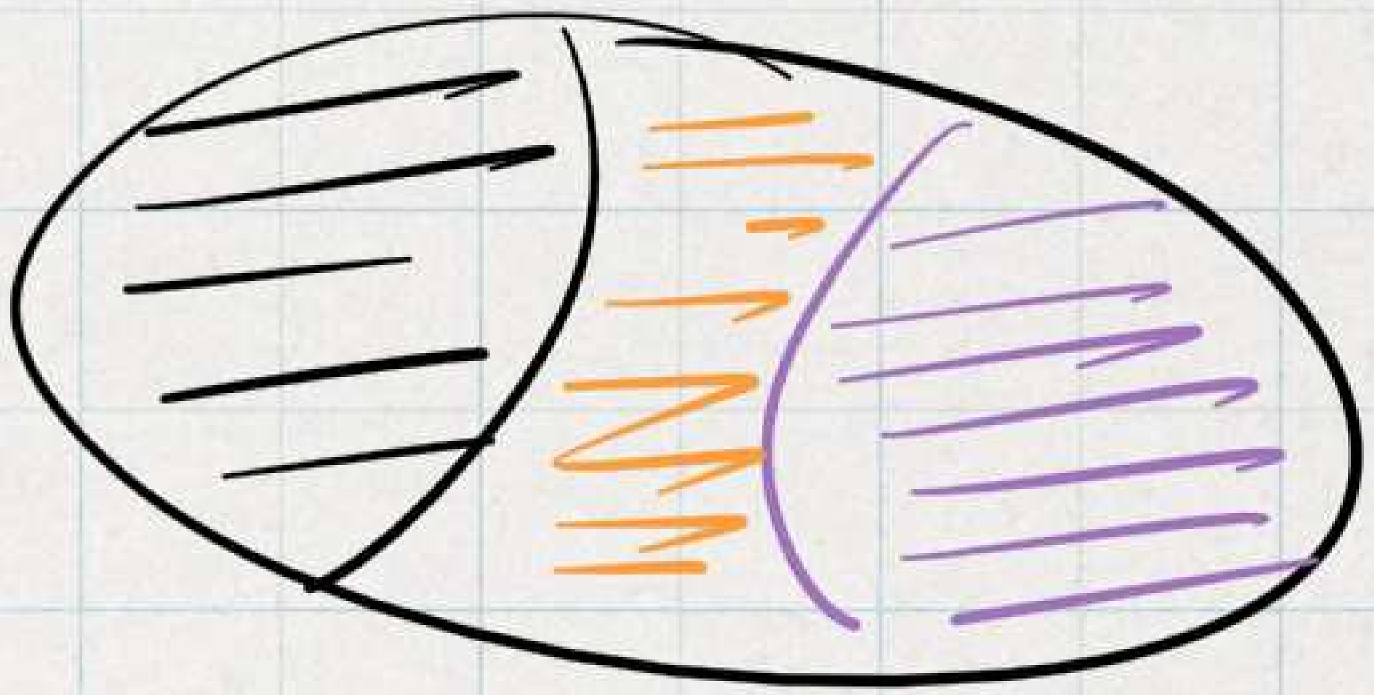
$$[T^{\mathrm{II}}_{\tau}] \cap A = \emptyset \qquad [\,[T^{\mathrm{II}}_{\tau}] \subseteq M^{\omega} \setminus A\,]$$

Clearly, $|[T^{\mathrm{I}}_{\sigma}]| = |[T^{\mathrm{II}}_{\tau}]| = 2^{\aleph_0}$.

[Map $x \longmapsto \sigma * \tau_x \in [T^{\mathrm{I}}_{\sigma}]$; this is an injection of 2^{ω} into $[T^{\mathrm{I}}_{\sigma}]$.]

Corollary If I has w.s. in $G(A)$, then $|A| \geq 2^{\aleph_0}$.

If II has a w.s. in $G(A)$, then $|M^\omega \setminus A| \geq 2^{\aleph_0}$.



? Does orange exist?

We'll now show that AC $\Longrightarrow$ there are non-determined sets.

[Certainly know to Banach 1930; published Gale-Stewart 1953]

Theorem AC $\Longrightarrow$ existence of non-determined set

Proof. By diagonalisation: construct A s.t. $\forall\sigma\,\forall\tau$

$$[T_\sigma^{\mathrm{I}}] \not\subseteq A \longrightarrow \text{I doesn't win}$$

$$[T_\tau^{\mathrm{II}}] \not\subseteq M^\omega \setminus A \longrightarrow \text{II doesn't win}$$

First question: how many strategies (equivalently, how many strategic trees) are there

$\longrightarrow 2^{\aleph_0}$.

We list all strategic trees in order type $2^{\aleph_0}$:

$$\{T_\alpha \, ; \, \alpha < 2^{\aleph_0}\}$$

[first use of AC to wellorder $2^{\aleph_0}$]

For each α, $|[T_\alpha]| = 2^{\aleph_0}$.

We define sets A, B s.t. $A \cap B = \emptyset$ in stages:

$\{A_\alpha \, ; \, \alpha < 2^{\aleph_0}\}$

$$A = \bigcup_{\alpha < 2^{\aleph_0}} A_\alpha \qquad B = \bigcup_{\alpha < 2^{\aleph_0}} B_\alpha .$$

For each α, $[T_\alpha] \not\subseteq A$ and $[T_\alpha] \not\subseteq M^\omega \setminus A$.

s.t. $\quad |A_\alpha| = |\alpha| = |B_\alpha|$.

and $\quad \underline{A_{\alpha+1} \cap [T_\alpha]} \neq \emptyset$ and $\underline{B_{\alpha+1} \cap [T_\alpha]} \neq \emptyset$.

$A_0 := \emptyset \qquad B_0 := \emptyset$

If λ is a limit ordinal and for $\alpha < \lambda$ A_α, B_α are defined with the desired properties, then we let

$$A_\lambda := \bigcup_{\alpha<\lambda} A_\alpha \qquad B_\lambda := \bigcup_{\alpha<\lambda} B_\alpha .$$

Suppose A_α, B_α are defined. with $\alpha < 2^{\aleph_0}$

$$|A_\alpha| = |\alpha| = |B_\alpha| \qquad |\alpha| < 2^{\aleph_0}.$$

Use AC to pick $a_\alpha \neq b_\alpha \in [T_\alpha] \setminus (A_\alpha \cup B_\alpha) \Longrightarrow \underbrace{[T_\alpha]}_{2^{\aleph_0}} \setminus \underbrace{(A_\alpha \cup B_\alpha)}_{> \, 2\cdot|\alpha|} \neq \emptyset$

$$A_{\alpha+1} := A_\alpha \cup \{a_\alpha\}$$

$$B_{\alpha+1} := B_\alpha \cup \{b_\alpha\}$$

Picking from nonempty subsets of M^ω.

$\Longrightarrow$ It's enough to have a wellordering of M^ω.

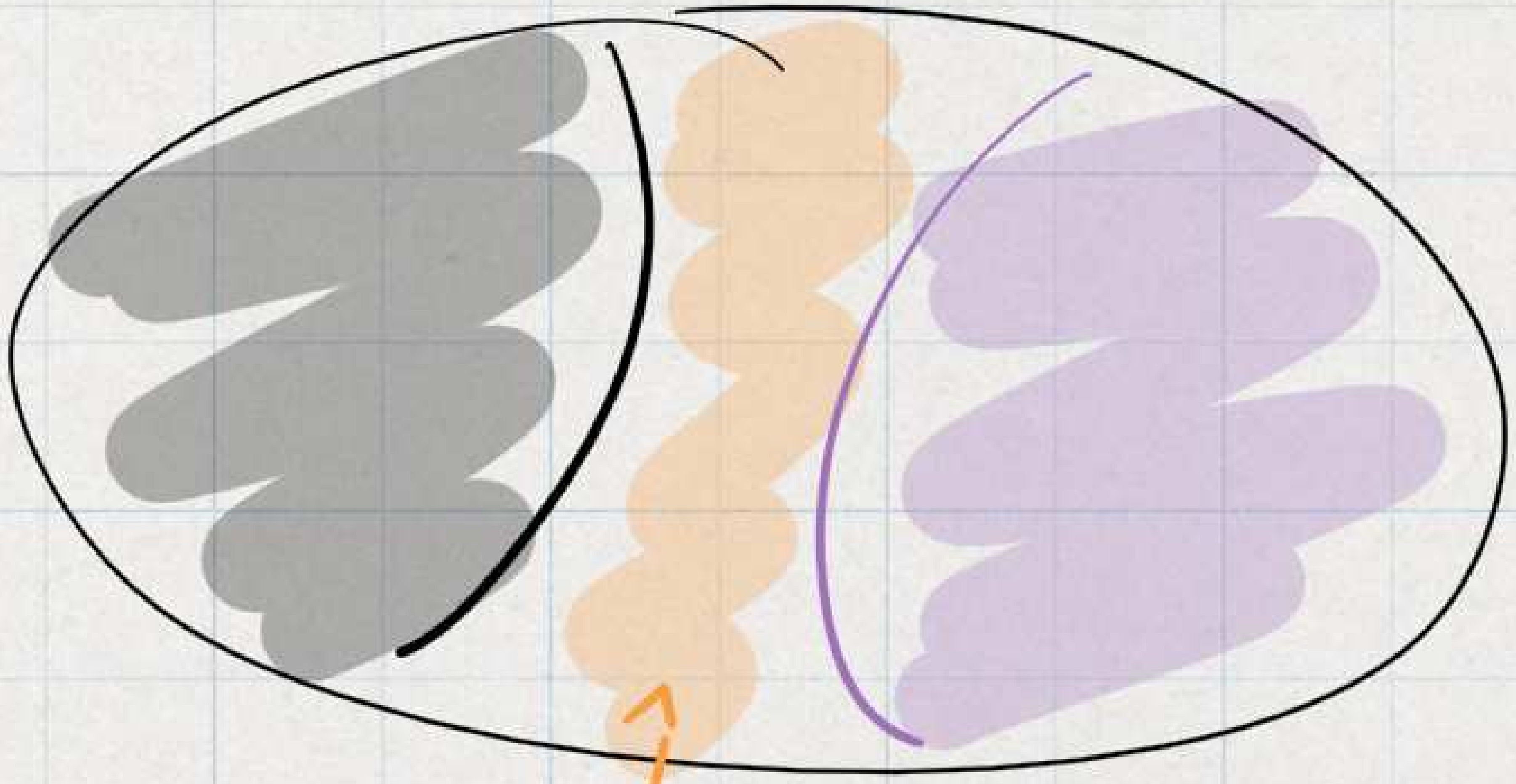
q.e.d.

The Axiom of Determinacy

MYCIELSKI proposed

AD_M

For every $A \subseteq M^\omega$, A is determined.



$AC \implies \neg AD_2$

$AD_2 \implies \neg AC$

If $|M| \geq 2$ and M^ω can be wellordered, then orange exists.