CS:ST **Lecture XII** 9 October 2020

Theorem (Shoenfield).
Every $\utilde{\Pi}^1_1$-set is $\aleph_1$-Suslin.

Theorem (Martin).
If there is a measurable cardinal, then all $\utilde{\Pi}^1_1$-sets are determined.

Fundam. Math. 66 (1969/1970), 287-291

Measurable cardinals and analytic games

by

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Introduction. A subset P of ω^ω is *determinate* if, in the sense of [5] the game $G_\omega(P)$ is determined. The assumption that every projective set is determinate implies that every projective set is Lebesgue measurable (see [6]) and leads to a complete solution to the problem of reduction and separation principles in the classical and effective projective hierarchies [1], [4]. Because if these and other consequences it would be pleasant to have a proof that every projective set is determinate. The best available result is that every $F_{\sigma\delta}$ is determinate [2]. It is not provable in Zermelo–Fraenkel set theory that every analytic ($\boldsymbol{\Sigma}^1_1$) set is determinate [5]. [1]

BK-code for T_x: $g : \omega \to \aleph_1$ s.t. if $i, j \in K_x$, then $s_i <_{BK} s_j \iff g(i) < g(j)$

We remember that if $A \in \mathbf{\Pi}^1_1$, then there is a tree T on $\omega \times \omega$ such that

BK

$x \in A$ if and only if $(T_x, \supsetneq)$ is wellfounded
if and only if $(T_x, <_{\mathrm{KB}})$ is wellordered
if and only if there is an order preserving map from $(T_x, <_{\mathrm{KB}})$ to $(\omega_1, <)$
if and only if there is a KB-code for T_x

where $<_{\mathrm{KB}}$ is the Kleene-Brouwer order on $\omega^{<\omega}$. For any $s \in \omega^{<\omega}$, we write $<_s$ for the order induced by the Kleene-Brouwer order on $\omega^{<\omega}$ on K_s, i.e., $i <_s j$ if and only if $s_i <_{\mathrm{KB}} s_j$. Note that since K_s is finite, $(K_s, \leq_s)$ is a (finite) wellorder.

Goal: find a tree $\hat{T}$ on $\omega \times \aleph_1$ s.t. $(x, g) \in [\hat{T}]$ iff g is a BK-code for T_x

So: $x \in p[\hat{T}] \iff x \in A$.

How do we code "finite approximations" to a BK-code in objects of size $\aleph_1$?

$M :=$ the set of partial functions from ω to $\aleph_1$ with finite domain

$|M| = |[\omega]^{<\omega} \times \aleph_1^{<\omega}| = \aleph_1$.

A is $\aleph_1$-Suslin $\overset{?}{\iff}$ A is M-Suslin

Fix a bijection $i \mapsto s_i$ from $\omega \to \omega^{<\omega}$ such that if $s_i \subseteq s_j$, then $i \leq j$. (This implies that $\mathrm{lh}(s_i) \leq i$.) Let $T \subseteq (\omega \times \omega)^{<\omega}$ be a tree, $x \in \omega^\omega$, and $s \in \omega^{<\omega}$. Then we let

$$T_s := \{t \in \omega^{<\omega} ; (s \restriction \mathrm{lh}(t), t) \in T\},$$
$$T_x := \{t \in \omega^{<\omega} ; (x \restriction \mathrm{lh}(t), t) \in T\} = \bigcup_{n \in \mathbb{N}} T_{x \restriction n},$$
$$\boxed{\begin{aligned} K_s &:= \{i \leq \mathrm{lh}(s) ; s_i \in T_s\}, \text{ and} \\ K_x &:= \{i \in \omega ; s_i \in T_x\} = \bigcup_{n \in \mathbb{N}} K_{x \restriction n}. \end{aligned}}$$

We note that T_s is a tree of finite height (every element $t \in T_s$ has length $\leq \mathrm{lh}(s)$) and that K_s is a finite set. We observe that $T_x = \{s_i ; i \in K_x\}$ (but, in general, $T_s \supsetneq \{s_i ; i \in K_s\}$).

Let g be a BK-code for some tree S. Then $u \in M$ is a finite approximation to g if $\mathrm{dom}(u) \subseteq \{i ; s_i \in S\}$ and $u(i) = g(i)$.

Let $v \in \aleph_1^{<\omega}$ and $s \in \omega^{<\omega}$

$v = (v_0, \dots, v_n)$

$s = (s_0, \dots, s_n)$ [fix T tree repr. of $\underset{\sim}{\Sigma}^1_1$ set]

We say v is coherent with s if

- $\forall i \le n \quad \mathrm{dom}(v_i) = K_{s \restriction i}$
- $\forall i \le n \quad v_i : K_{s\restriction i} \longrightarrow \aleph_1$ is order preserving $(K_{s\restriction i}, <_{BK})$ and $(\aleph_1, <)$
- $\forall i \le j \quad v_i \le v_j$.

$\hat{T} := \{(s, v) ; v \text{ is coherent with } s\}$

DEPENDS ON THE CHOICE OF T: T_s, T_x, K_s, K_x

SHOENFIELD tree

Claim $A = p[\hat{T}]$.

[This proves Shoenfield's Thm.]

Pf of Claim "$\subseteq$". Let $x \in A$.

By the equivalences, find $g : \omega \longrightarrow \aleph_1$ BK-code for T_x. $v_i := g \restriction K_{x\restriction i}$

If you define v by

$$v_i := g \upharpoonright K_{x \upharpoonright i},$$

then for each n $(x\upharpoonright n, v\upharpoonright n)$ is coherent by definition and thus

$$(x, v) \in [\hat{T}],$$

so $x \in p[\hat{T}]$.

"$\supseteq$". Let $(x, v) \in [\hat{T}]$.

<u>So</u>: $v_i : K_{x\upharpoonright i} \longrightarrow \aleph_1$.

with $i < j \Longrightarrow v_i \subseteq v_j$

$\hat{v} := \bigcup_{i \in \mathbb{N}} v_i$ is a function with domain

$$\bigcup_{i \in \mathbb{N}} \mathrm{dom}(v_i) = \bigcup_{i \in \mathbb{N}} K_{x \upharpoonright i} = K_x.$$

Together $\hat{v} : K_x \longrightarrow \aleph_1$ order preserving

so $g : n \longmapsto \begin{cases} \hat{v}(n) & \text{if } n \in K_x \\ 0 & \text{o/w} \end{cases}$

is a BK-code for T_x.

Thus $x \in A$. [by our equivalences.]

q.e.d.

Theorem (Martin)

If κ is measurable, then every $\underset{\sim}{\Pi}^1_1$ set is determined.

ROWBOTTOM'S THM If S is countable and $\{f_s;\, s\in S\}$ is a family of finite colouring on κ, then there is a set H [of uncountable card. κ] s.t. H is homogeneous for f_s (f.a. $s\in S$).

κ is measurable $\Longrightarrow$

We prove Martin's thm by showing:
if κ satisfies Rowbottom, then every $\underset{\sim}{\Pi}^1_1$-set is determined.

How does κ even get involved?
Apply Shoenfield's Thm to get that $A\in\underset{\sim}{\Pi}^1_1$ is κ-Suslin and play an auxiliary game on the Shoenfield tree.

$\{u;\ u$ is a partial fn from ω to κ with finite domain$\} =: M$

If $A\in\underset{\sim}{\Pi}^1_1$, get Shoenfield tree on ~~κ~~ M $\hat{T}\subseteq\omega\times M$ s.t. $A=p[\hat{T}]$.

Consider $G(A)$

$$\left.\begin{array}{lllllll} \mathrm{I} & x_0 & & x_2 & & x_4 & \cdots \\ \mathrm{II} & & x_1 & & x_3 & & x_5 \; \cdots \end{array}\right] \longrightarrow x \in \omega^\omega$$

I wins if $x \in A$.

Auxiliary game $G_{aux}(\hat{T})$

$$\begin{array}{lllllllll} \mathrm{I} & x_0, \underline{v_0} & & x_2, \underline{v_1} & & x_4, \underline{v_2} & & x_6, \underline{v_3} & \cdots \\ \mathrm{II} & & x_1 & & x_3 & & x_5 & & \cdots \end{array}$$

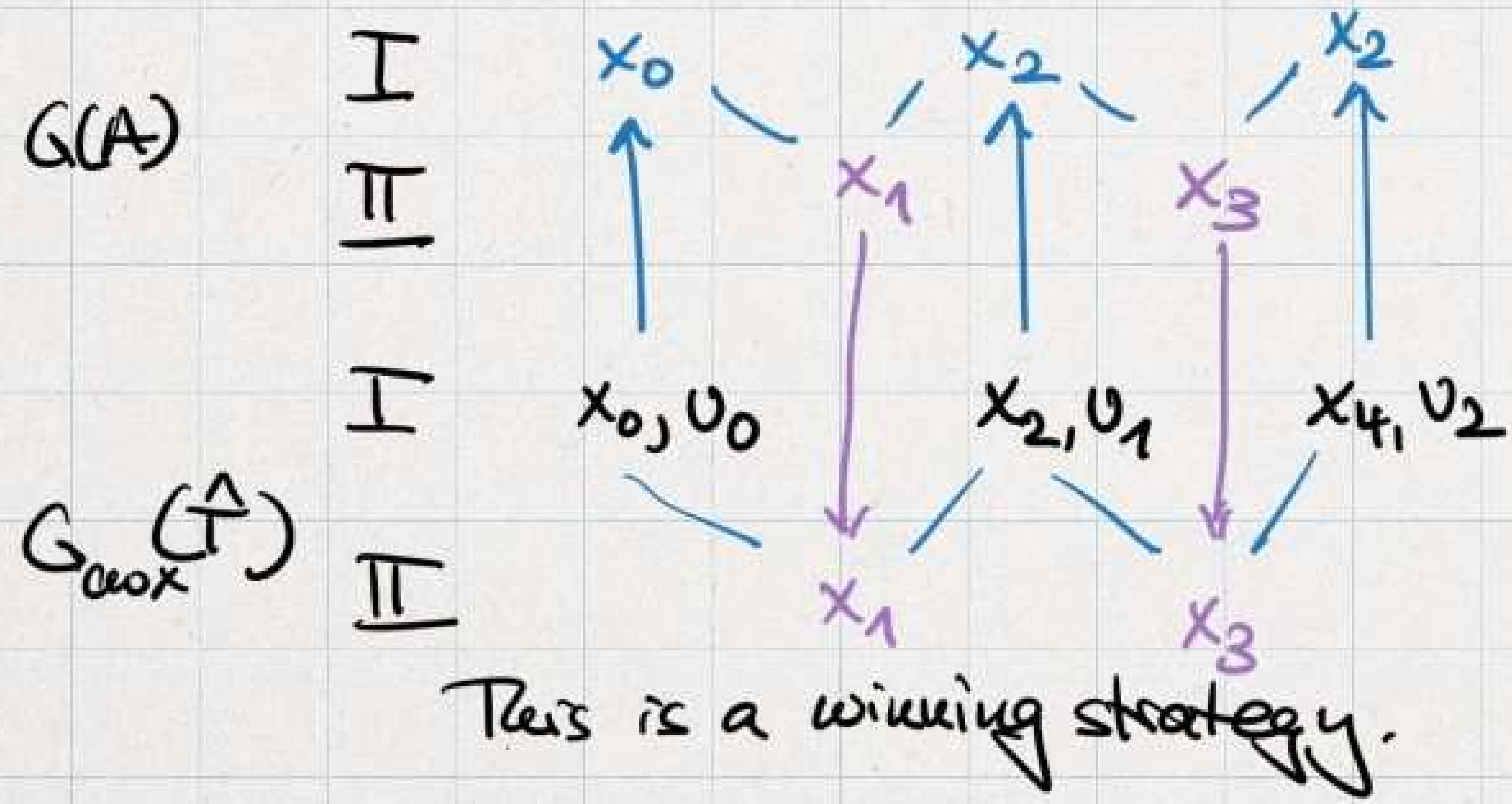
$x = (x_i \,;\, i \in \omega)$

$v = (v_i \,;\, i \in \omega)$

$\underline{\text{I wins}}$ if $\underline{(x, v) \in [\hat{T}]}$.

$\Longrightarrow \quad x \in p[\hat{T}] = A$.

Translation of a winning str. for player I from $G_{aux}(\hat{T})$ to $G(A)$:



This is a winning strategy.

What about player II ?

$G(A)$ $\quad \mathrm{I}$ $\quad x_0$

$\qquad\quad \mathrm{II}$

$G_{aux}(\hat{T})$ $\quad \mathrm{I}$ $\quad x_0, *$

$\qquad\quad \mathrm{II}$ $\quad$?

A winning str. for player II needs input of the form (x_0, v_0) [in the first step] and thus cannot work w/o additional information.

This extra information will be provided by the measurable cardinal!

ROWBOTTOM For S ctble, $\{f_s ; s \in S\}$ finite colouring into κ, there is uncountable H s.t. for each s, H is homogeneous for f_s.

Take a w.s. τ for player II in $G_{aux}(\hat{T})$ use H to define τ_H in $G(A)$ s.t.

τ_H is winning in $G(A)$.

Let $s \in \omega^{<\omega}$. We define f_s:

$$k_s := |K_s|.$$

If $Q \in [\kappa]^{k_s}$, compare

$$(K_s, <_{BK}) \qquad (Q, <)$$

1 0 100 20 21 5 6

$$\alpha_0 < \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < \alpha_5 < \alpha_6$$

$$\alpha_1 \; \alpha_0 \; \alpha_5 \; \alpha_6 \; \alpha_3 \; \alpha_4 \; \alpha_2$$

Let $w : K_s \longrightarrow Q$ be the unique order preserving map w from $(K_s, <_{BK})$ into $(Q, <)$.

$$v_i^{s,Q} := w \restriction K_{s \restriction i}.$$

This is the only way I can define $v_i^{s,Q}$ to get a finite approximation for a BK-code provided that I fix the set Q as range.

If

$$\begin{array}{lcccccc} \mathrm{I} & x_0 & & x_2 & & x_4 & \\ \mathrm{II} & & x_1 & & x_3 & \cdots & x_{2n+1} \end{array} \quad s$$

is a position in $G(A)$ $s = (x_0, \dots, x_{2n+1})$

then

$$\begin{array}{lcccccccc} \mathrm{I} & x_0, v_0^{s,Q} & & x_2, v_1^{s,Q} & & x_4, v_2^{s,Q} & \cdots & \\ \mathrm{II} & & x_1 & & x_3 & & x_5 & x_{2n+1} \end{array}$$

is a position in $G_{aux}(f)$

With fixed s, Q, we write

$s_{*,Q}$ for this uniquely defined position.

$f_s(Q) := \tau(s_{*,Q})$

the answer of the w.s. τ in $G_{\mathrm{aux}}(\hat{T})$ to the position s in $G(A)$ augmented in the unique way assuming range Q.

f_s is a k_s-colouring, so $\{f_s ; s \in \omega^{<\omega}\}$ is a ctble set of finite colouring, so ROWBOTTOM

$\Longrightarrow$ there is an uncountable homogeneous H:

i.e., for every s and every $Q, Q' \in [H]^{k_s}$

$\tau(s_{*,Q}) = f_s(Q) = f_s(Q') = \tau(s_{*,Q'})$.

The τ-answer does not change, as long as $Q \subseteq H$. So, we define

$\boxed{\tau_H}(s) := \tau(s_{*,Q})$ for any $Q \in [H]^{k_s}$.

<u>Claim</u> If τ was winning in $G_{aux}(\hat{T})$, and H is homogeneous for all f_s ($s \in \omega^{<\omega}$) defined via τ, then τ_H is winning in $G(A)$.

<u>Proof.</u> Suppose not: so there is a strategy σ winning against τ_H in $G(A)$!

$\underline{x} := \sigma * \tau_H \in \underline{A}$ any uncountable set of ordinals

$\iff$ there is an order preserving map
$$f : (T_x, <_{BK}) \longrightarrow (\cancel{\omega}_1, <)$$
$\iff$ there is a BK-code for T_x

$\iff$ there is a BK-code g for T_x with $\mathrm{ran}(g) \subseteq H$.

$g : \omega \longrightarrow H$

$\forall i, j \quad s_i, s_j \in T_x \implies i <_{BK} j \iff g(i) < g(j)$.

Define $v_i := g \restriction \underline{k_{x \restriction i}}$.

$$x := \sigma * \tau_H \in A$$

$$g : \omega \longrightarrow H \quad \text{BK-code for } T_x$$

$$\forall i,j \quad s_i, s_j \in T_x \implies$$

$$i <_{BK} j \longleftrightarrow g(i) < g(j)$$

$$v_i := g \upharpoonright K_x \upharpoonright i .$$

$$\begin{tikzcd}
\sigma \quad G(A) \quad \mathrm{I} & x_0 \arrow[dd] & & x_2 \arrow[dd] & & x_4 \arrow[dd] \\
\tau_H \quad \mathrm{II} & & x_1 & & x_3 & \\
G_{aux}(\hat{T}) \quad \mathrm{I} & x_0, v_0 \arrow[dr, "\tau"] & & x_2, v_1 \arrow[dr, "\tau"] & & x_4, v_2 \\
\mathrm{II} & & x_1 \arrow[uu, "\tau_H"'] & & x_3 \arrow[uu, "\tau_H"'] &
\end{tikzcd}$$

If $v_i := g \upharpoonright K_x \upharpoonright i$, then

$$(x_0, v_0, x_1, x_2, v_1, x_3, x_4, v_2, \ldots)$$

is a play of $G_{aux}(\hat{T})$ that follows the strategy τ.

So $(x,v) \notin [\hat{T}]$ because τ was winning.

But by choice of v, $(x,v) \in [\hat{T}]$.

CONTRADICTION!

<u>Summary.</u>

If σ is w.s. for $\underline{\mathrm{I}}$ in $G_{aux}(\hat{T})$,
then the "forgetful" strategy
σ^* is w.s. for $\underline{\mathrm{I}}$ in $G(A)$.

If τ is w.s. for $\underline{\mathrm{II}}$ in $G_{aux}(\hat{T})$
then τ_H is w.s. for $\underline{\mathrm{II}}$ in $G(A)$
for H homogeneous for all of the
fs defined via τ.

Determinacy of $G_{aux}(\hat{T})$ implies
determinacy of $G(A)$.

But $G_{aux}(\hat{T})$ is just $G_{\underline{\omega \times M}}([\hat{T}])$

[Not quite: need to ignore the M-moves of player II.]

This is a closed game, so determined
by G-S.

[Note that this does not require Choice, since
$\omega \times M \sim \omega \times \aleph_1$ is wellorderable.]

q.e.d.