Topics in Set Theory
Lent Term 2019
Part III of the Mathematical Tripos
University of Cambridge
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Example Sheet #1

Course webpage: `https://www.math.uni-hamburg.de/home/loewe/Lent2019/TST_L19.html`

Example Classes.
#1. Monday 4 February, 3:30–5pm, MR5.
#2. Monday 18 February, 3:30–5pm, MR5.
#3. Monday 4 March, 3:30–5pm, MR5.
#4. TBD; probably Thursday 14 March.
You hand in your work at the beginning of the Example Class.

1. Let $(M, \in) \models \mathsf{ZFC}$ and refer to the natural numbers of M by the usual symbols 0, 1, etc. Consider the (non-transitive) set $A := M \backslash \{1\}$. In class, we saw that $(A, \in) \models \neg\mathsf{Extensionality}$. Check the other axioms of ZFC for their validity in $(A, \in)$.

2. We called a formula Δ_0 if it is in the closure of the quantifier-free formulas under the operations $\varphi \mapsto \neg\varphi$, $(\varphi, \psi) \mapsto \varphi \wedge \psi$, $(\varphi, \psi) \mapsto \varphi \vee \psi$, $(\varphi, \psi) \mapsto \varphi \to \psi$, $\varphi \mapsto \exists x(x \in y \wedge \varphi)$, and $\varphi \mapsto \forall x(x \in y \to \varphi)$. Check whether the following formulas are Δ_0 and give an argument for your answer:

   (a) $\exists x(\forall z(\neg z \in x) \wedge x \in y)$;
   (b) $(x = y) \vee (z \in x)$;
   (c) $\forall x(x \in y \to x \in z)$;
   (d) $\exists x(x \in y \wedge \neg x \in z)$;
   (e) $\exists x(x \in y \wedge \neg(\exists z(z \in y \wedge (z \in x \vee y \in x))))$.

3. Let T be any $\mathcal{L}_\in$-theory. We called a formula Δ_0^T if the theory T proves that it is equivalent to a Δ_0 formula. Show that the following concepts can be expressed by Δ_0^T-formulas for a reasonable choice of T; also, indicate what T you choose and why.

   (a) $z = \{x, y\}$;
   (b) $z = (x, y)$;
   (c) $z = y \times y$;
   (d) z is a function;
   (e) z is a group;
   (f) z is a linear order;
   (g) z is a set with exactly two elements.

4. The *Mirimanoff rank* is defined by $\varrho(x) := \min\{\alpha\,;\, x \in \mathbf{V}_{\alpha+1}\}$. Show that for any ordinals $\alpha \leq \beta$, the following hold:

   (a) $\mathbf{V}_\alpha$ is transitive;

   (b) $\mathbf{V}_\alpha \subseteq \mathbf{V}_\beta$;

   (c) if $x \in y$, then $\varrho(x) < \varrho(y)$;

   (d) $\varrho(x) := \sup\{\varrho(y) + 1\,;\, y \in x\}$;

   (e) $\alpha = \{\gamma \in \mathbf{V}_\alpha\,;\, \gamma \text{ is an ordinal}\}$.

5. Suppose that $\lambda > \omega$ is a limit ordinal. Show that

   (a) $(\mathbf{V}_\lambda, \in) \models \mathsf{Union}$;

   (b) $(\mathbf{V}_\lambda, \in) \models \mathsf{Separation}$;

   (c) $(\mathbf{V}_\lambda, \in) \models \mathsf{PowerSet}$.

6. Give a concrete example of a wellorder $(X, R) \in \mathbf{V}_{\omega+\omega}$ that is not isomorphic to an ordinal $\alpha \in \mathbf{V}_{\omega+\omega}$.

7. We said that a cardinal κ is *regular* if there is no partition $\kappa = \bigcup_{\beta \in I} A_\beta$ with $|I| < \kappa$ and $|A_\beta| < \kappa$ for all $\beta \in I$.

   Let κ be an arbitrary cardinal and define $\mathrm{cf}(\kappa)$ to be the least cardinal λ such that there is a partition $\kappa = \bigcup_{\beta \in I} A_\beta$ with $|I| = \lambda$ and $|A_\beta| < \kappa$ for all $\beta \in I$.

   Show that $\mathrm{cf}(\kappa)$ is a regular cardinal.

8. A cardinal κ is called an *aleph fixed point* if $\aleph_\kappa = \kappa$. Show in ZFC that there is an aleph fixed point κ such that $\mathrm{cf}(\kappa) = \aleph_0$.

9. Show that if κ is inaccessible, then $\mathbf{V}_\kappa = \mathbf{H}_\kappa$. What can you say about the converse?

10. As usual, work inside a model $(M, \in) \models \mathsf{ZFC}$. Suppose that $A \subseteq M$ is transitive and $(A, \in) \models \mathsf{ZFC}$. Suppose that M and A disagree about the value of $\aleph_1$, i.e., there is a countable ordinal α such that $(A, \in) \models$"α is the first uncountable cardinal". Show that there is some $x \subseteq \mathbb{N}$ such that $x \notin A$.

11. As usual, work inside a model $(M, \in) \models \mathsf{ZFC}$. Let $\Phi(x)$ be the formula expressing "x is an inaccessible cardinal", let $\mathsf{IC} := \exists x \Phi(x)$, and let λ be a limit ordinal.

    (a) Show that Φ is absolute between $\mathbf{V}_\lambda$ and M.

    (b) Show that if κ is the least inaccessible cardinal, then $\mathbf{V}_\kappa \models \mathsf{ZFC} + \neg\mathsf{IC}$.

    (c) Give a proof of $\mathsf{ZFC} \nvdash \mathsf{IC}$ that does not use Gödel's Incompleteness Theorem.

    (d) Show that $\mathsf{ZFC} + \mathsf{IC}$ does not prove that there are two inaccessible cardinals.

12. Work in $\mathsf{ZFC} + \mathsf{IC}$ and show that there is a cardinal λ with $\mathrm{cf}(\lambda) = \aleph_0$ such that $\mathbf{V}_\lambda \models \mathsf{ZFC}$.

    (*Hint.* Define λ as a countable union by recursion and use the Tarski-Vaught criterion to show that $\mathbf{V}_\lambda \prec \mathbf{V}_\kappa$ where κ is the inaccessible cardinal.)