Infinite Games
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University of Cambridge
Prof. Dr. B. Löwe

Shoenfield's Theorem

Theorem (Shoenfield). Every $\mathbf{\Pi}^1_1$ set is $\aleph_1$-Suslin.

Proof. Let A be $\mathbf{\Pi}^1_1$, then $\omega^\omega\backslash A$ is $\mathbf{\Sigma}^1_1$, so let T be a tree such that

$$\begin{aligned} x \in A &\iff x \notin \mathrm{p}[T] \\ &\iff T_x \text{ is wellfounded} \\ &\iff \text{there is an order preserving map } f : (T_x, \supsetneqq) \to (\aleph_1, <), \end{aligned}$$

where $T_x := \{s\,;\,(s, x{\restriction}|s|) \in T\}$.

Fix a bijection $i \mapsto s_i$ from $\omega \to \omega^{<\omega}$ such that if $s_i \subsetneqq s_j$, then $i < j$. We let $T_t := \{s_i\,;\,|s_i| \leq |t|,\ i \leq |t|$, and $(s_i, t{\restriction}|s_i|) \in T\}$ and observe that $T_x = \bigcup_{n\in\omega} T_{x\restriction n}$.

Let S be any tree on ω. We say that a function $g : \omega \to \omega_1$ is an *order preserving code for* S if for all i and j, if $s_i, s_j \in S$ and $s_i \supsetneqq s_j$, then $g(i) < g(j)$. Similarly, if $u \in {\omega_1}^{<\omega}$, then we say that u is a *partial order preserving code for* S if for all $i, j < |u|$, if $s_i, s_j \in S$ and $s_i \supsetneqq s_j$, then $u(i) < u(j)$. Then the following tree is called the *Shoenfield tree*:

$$\widehat{T} := \{(u, s)\,;\,u \text{ is a partial order preserving code for } T_s\}.$$

We claim that $A = \mathrm{p}[\widehat{T}]$:

"$\subseteq$": If $x \in A$, then let $f : T_x \to \omega_1$ be an order preserving map and define

$$g(i) := \begin{cases} f(s_i) & \text{if } s_i \in T_x \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Then g is an order preserving code for T_x, and thus every $g{\restriction}n$ is a partial order preserving code for $T_{x\restriction n}$. Thus $(g, x) \in [\widehat{T}]$.

"$\supseteq$": If $x \in \mathrm{p}[\widehat{T}]$, find $g \in {\omega_1}^\omega$ such that $(g, x) \in [\widehat{T}]$; this means that for each n, $g{\restriction}n$ is a partial order preserving code for $T_{x\restriction n}$. If g is not an order preserving code for T_x, then there are $s_i, s_j \in T_x$ where $s_i \supseteq s_j$, but $g(i) \geq g(j)$. Find n large enough such that $s_i, s_j \in T_{x\restriction n}$: this is a contradiction to the fact that $g{\restriction}n$ is a partial order preserving code for $T_{x\restriction n}$. Thus g is an order preserving code for T_x, and therefore T_x is wellfounded whence $x \in A$. q.e.d.