WHAT IS ... AN AMOEBA (2)?

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(0) What is "WHAT IS..."? The *Notices of the American Mathematical Society* recently started a new column called the *"WHAT IS ..."* column with the intention to explain notions from highly active research areas that people hear about in lectures leading them to ask, "What is that thing?".[1] The installments of this column should together yield a dictionary of important modern mathematical terms.

The inaugural installment of the column was Oleg Viro's "WHAT IS ... an amoeba?" [Vir02] about amoebae in convex geometry and complex analysis. These amoebae are images of a plane complex algebraic curve under the logarithm function, and were introduced by Gelfand, Kapranov and Zelevinsky in 1994. Viro mentions in his footnote:

> "I was told that in mathematical logic amoebas were known for more than twenty years. However, they belong to an entirely different kind of mathematical microbes, and have never bitten me, so I cannot tell you about them. [Vir02, p. 916, fn. 1]"

This brief note is to be understood as a second entry in the *"WHAT IS ..."* dictionary under the lemma "Amoeba".

(1) History of the Amoeba order. It has been more than thirty years ago that the order that would become known as the amoeba order was first used in set theory:

In their famous paper [MarSol70] in which Tony Martin and Bob Solovay introduced Martin's Axiom MA, they also introduced two orders, the amoeba order $\mathbb{A}$ and the amoeba meager order $\mathbb{UM}$ in order to show that Martin's Axiom implies that the additivity of the ideal of Lebesgue null sets and the ideal of meager sets (countable unions of nowhere dense sets) is large.[2] In their paper, Martin and Solovay didn't name the orders $\mathbb{A}$ and $\mathbb{UM}$—the name "amoeba" appeared in print for the first time in [Tru77].

(2) Definition(s). For each $\varepsilon \leq 1$, let

$$A_\varepsilon := \{P \subseteq [0,1] \times [0,1]\,;\, P \text{ is open and } \mu(P) < \varepsilon\},$$

where μ is Lebesgue measure on the Euclidean unit square. The amoeba order $\mathbb{A}$ is the set of pairs $\langle P, \varepsilon\rangle$ where $P \in A_\varepsilon$. We order the elements of the amoeba order by

$$\langle P, \varepsilon\rangle \leq \langle Q, \varepsilon^*\rangle \iff P \supseteq Q \text{ and } \varepsilon \leq \varepsilon^*.$$

The name "amoeba" stems from the following feature of this order: Suppose you have a Lebesgue measure zero subset Z of the Euclidean unit square and an element $\langle Q, \varepsilon^*\rangle$ of the amoeba order. Then there is an element $\langle P, \varepsilon\rangle \leq \langle Q, \varepsilon^*\rangle$ such that P contains Z:

Let $\mu(Q) = \eta < \varepsilon^*$. Since Z has measure zero, there is an open set $R \supseteq Z$ with $\mu(R) < \varepsilon^* - \eta$. Let $P := Q \cup R$ and $\varepsilon := \varepsilon^*$. Then $\mu(P) \leq \mu(Q) + \mu(R) < \eta + \varepsilon^* - \eta = \varepsilon$. This can be interpreted as an amoeba Q reaching out with its pseudopodium R to consume the set Z as seen in Figure 1.

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[1] Some of this description of the scope of the *"WHAT IS ...?"* column stems from personal communication with Allyn Jackson. A basic mission statement can be found in [Jac02].

[2] The additivity of an ideal $\mathcal{I}$ is the minimal number of sets in the ideal whose union is a set not in the ideal: it is well-known that the ideal of Lebesgue null sets is countably additive, i.e., every countable union of null sets is again null. On the other hand, it is independent from the standard axioms of set theory whether there is an uncountable cardinality κ such that unions of κ many null sets are null. The mentioned result of Martin and Solovay using the amoeba order $\mathbb{A}$ was that under assumption of MA, every uncountable cardinal κ strictly less than the cardinality of the set of real numbers has that property.

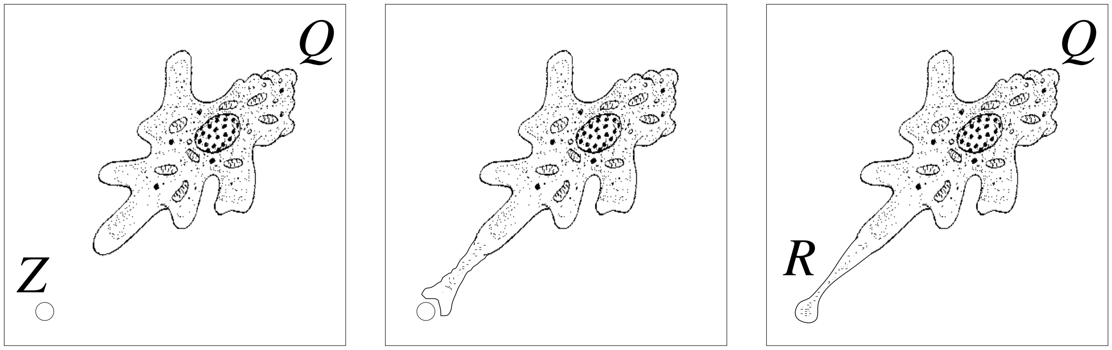

FIGURE 1. Amoeba Q feeding on the small set Z by extending its pseudopodium R

There are several variant definitions of the Amoeba order: We define $\mathbb{A}_\eta$ to be the suborder of $\mathbb{A}$ consisting of those elements $\langle P, \varepsilon\rangle$ such that $\varepsilon \leq \eta$; furthermore, we define $\mathbb{A}^*_\eta$ to be the set A_η ordered by inclusion.[3] All of these variants are equivalent in a certain technical sense (see below).

In addition to the classical Amoeba order(s), there are also the Amoeba meager order and the Amoeba Sacks order. For these, let us work on Cantor space $2^{\mathbb{N}}$. Cantor space is a topological space homeomorphic to the Cantor subset of the unit interval. If T is a binary branching tree, we can see the branches through T (denoted by $[T]$) as subset of Cantor space. On Cantor space, every closed set is the set of branches through a binary branching tree and *vice versa*. We call a binary branching tree **perfect** if each node has a (not necessarily immediate) successor which is a branching node. It is easy to see that perfect trees correspond to perfect subsets of Cantor space (*i.e.*, closed sets without isolated points).

If T is a binary branching tree, we can recursively assign depths to the nodes of the tree, starting with 0 at the root, and then denote by $T\restriction n$ the tree restricted to nodes of depth less than n. Amoeba Sacks forcing $\mathbb{AS}$ is the set of pairs $\langle n, T\rangle$ such that T is a perfect binary tree and n is a natural number, ordered by

$$\langle n, T\rangle \leq \langle m, S\rangle :\iff n \geq m \;\&\; T \subseteq S \;\&\; T\restriction m = S\restriction m.$$

The Amoeba meager order denoted by $\mathbb{UM}$ is a suborder of $\mathbb{AS}$ consisting of only those pairs $\langle n, T\rangle$ where $[T]$ is a nowhere dense subset of $2^{\mathbb{N}}$.

(3) The Amoeba order and forcing. The Amoeba order and its relatives are used in applications of the metamathematical technique of forcing. Forcing has been invented by Paul Cohen in 1963 and was prominently used to show that neither the Continuum Hypothesis ($2^{\aleph_0} = \aleph_1$) nor the Axiom of Choice follow from the standard (Zermelo-Fraenkel) axioms of set theory.

Forcing is a set theoretical model construction that takes a model M of the axioms of set theory and a partial order $\mathbb{P} = \langle P, \leq\rangle$ and produces an extension $M[G]$ of M by adding a generic subset G of $\mathbb{P}$. In most cases, G can be seen as an object having many properties of the elements of $\mathbb{P}$ at once. The key point of any forcing proof is the prudent choice of the right partial order $\mathbb{P}$.[4]

Very important partial orders that have been used in forcing are random forcing $\mathbb{B}$ (the quotient algebra of the Borel sets modulo the ideal of Lebesgue null sets; random forcing has been used by Solovay to show that the construction of a non-Lebesgue measurable set necessarily uses the Axiom of Choice), Cohen forcing $\mathbb{C}$ (the set of finite binary strings ordered by extension; Cohen forcing was used by Cohen in his pivotal 1963 result) and Sacks forcing $\mathbb{S}$ (the set of perfect binary trees ordered by inclusion; Sacks forcing was used by Sacks to construct models with exactly two constructibility degrees).

These three orders generate three kinds of generic objects (always depending on the model M we start with): random reals (over M), Cohen reals (over M), and Sacks reals (over M), and are closely associated

[3]The original version of the Amoeba order in [MarSol70] was $\mathbb{A}^*_\eta$. Truss introduced $\mathbb{A}$ in his [Tru77].

[4]In this brief note, there is no chance to give more details of this construction. The reader who is not familiar with forcing should consider the operation $M \mapsto M[G]$ as a combinatorial black box. More details can be found in textbooks of set theory, *e.g.*, [Kun80].

to three ideals, the ideal of Lebesgue null sets, the ideal of meager sets and the ideal of Marczewski null sets (A is called Marczewski null if for every perfect set P there is a perfect subset $Q \subseteq P$ such that $A \cap Q = \emptyset$).

This is a general feature of a large class of forcing constructions used in applications of set theory in the theory of the real numbers: we have a **characteristic triple** $\langle \mathbb{P}, \mathcal{I}_{\mathbb{P}}, \mathcal{G}_{\mathbb{P}} \rangle$ consisting of a partial order $\mathbb{P}$, a (term describing an) ideal $\mathcal{I}_{\mathbb{P}}$ associated to the partial order, and a (term describing the) set of $\mathbb{P}$-generic objects $\mathcal{G}_{\mathbb{P}}$. As soon as we fix a model M of set theory to start from (the so-called *ground model*) and an extension $M[G]$, we can look at the ideal $\mathcal{I}_{\mathbb{P}}^M$ in the smaller model M, the ideal $\mathcal{I}_{\mathbb{P}}^{M[G]}$ in the larger model $M[G]$, and the set $\mathcal{G}_{\mathbb{P}}^{M;M[G]}$ of elements of $M[G]$ that are $\mathbb{P}$-generic over M.

(4) Forcing equivalence. The creation of generic objects over a ground model M is the main reason to deal with forcing. Therefore, we say that two partial orders $\mathbb{P}$ and $\mathbb{Q}$ are **forcing equivalent** if they produce the same generic objects up to bijections definable in the ground model.

One way to formalize this is the following: If $\mathbb{P} = \langle P, \leq \rangle$ is a partial order, the set of cones $C_p := \{q \in P\,;\, q \leq p\}$ forms a topology base. Let τ be the topology on P generated by the topology base of the cones. As usual, we call a subset $X \subseteq P$ **regular** if it is equal to the interior of its closure (in τ). The set of regular open subsets of P forms a complete Boolean algebra denoted by r.o.($\mathbb{P}$). We now say that $\mathbb{P}$ and $\mathbb{Q}$ are **forcing equivalent** if r.o.($\mathbb{P}$) = r.o.($\mathbb{Q}$).[5]

Truss and Kutyłowski have shown that the different versions of the Amoeba order $\mathbb{A}$, $\mathbb{A}_\eta$ and $\mathbb{A}_\eta^*$ are forcing equivalent.[6]

(5) Being an amoeba of $\mathbb{P}$. A closer look at the characteristic triples of the mentioned important forcing notions yields one of the central properties of the Amoeba algebra:

Let $M[H]$ be an extension of a group model H constructed with the Amoeba algebra. Then Martin and Solovay have proved[7] that in $M[H]$, the set of random reals over M has Lebesgue measure 1; in other words, $\mathbb{R} \backslash \mathcal{G}_{\mathbb{B}}^{M;M[H]} \in \mathcal{I}_{\mathbb{B}}^{M[H]}$.

We can use this fact as a motivation for the following abstract definition. Let $\mathbb{P}$ and $\mathbb{Q}$ be partial orders with characteristic triples $\langle \mathbb{P}, \mathcal{I}_{\mathbb{P}}, \mathcal{G}_{\mathbb{P}} \rangle$ and $\langle \mathbb{Q}, \mathcal{I}_{\mathbb{Q}}, \mathcal{G}_{\mathbb{Q}} \rangle$ We then call a partial order $\mathbb{Q}$ **an amoeba of** $\mathbb{P}$ if in every extension $M[H]$ constructed with $\mathbb{Q}$, we have

$$\mathbb{R} \backslash \mathcal{G}_{\mathbb{P}}^{M;M[H]} \in \mathcal{I}_{\mathbb{P}}.$$

In this sense, the amoeba order $\mathbb{A}$ is an amoeba over $\mathbb{B}$, the Amoeba meager order $\mathbb{UM}$ is an amoeba over $\mathbb{C}$ and the Amoeba Sacks order introduced by Shelah is an amoeba over $\mathbb{S}$. There are also orders that are their own amoebae: Mathias forcing (which is connected to the Ramsey property of infinitary combinatorics) is an amoeba of itself.

(6) The Amoeba product. In some cases, building an amoeba for a partial order $\mathbb{P}$ can be seen as a special case of a special product on partial orders introduced by Skřivánek [Skř88]: the **amoeba product**.

For partial orders $\mathbb{P} = \langle P, \leq \rangle$ and $\mathbb{Q} = \langle Q, \leq \rangle$ with a monotone map $\pi : \mathbb{P} \to \mathbb{Q}$ we define $\mathbb{P} \otimes_\pi \mathbb{Q} := \langle P \otimes_\pi Q, \leq \rangle$, where $P \otimes_\pi Q := \{\langle p, q \rangle\,;\, \pi(p) < q\}$ and $\langle p, q \rangle \leq \langle p^*, q^* \rangle$ if and only if $p \geq p^*$ and $q \leq q^*$.

Now let $\mathbb{O}$ be the partial order of open subsets of $[0,1] \times [0,1]$ ordered by inclusion and $\mathbb{I}$ be the unit interval $[0,1]$ with the usual order. Take Lebesgue measure μ as a monotone map from $\mathbb{O}$ to $\mathbb{I}$. Then $\mathbb{A} = \mathbb{O} \otimes_\mu \mathbb{I}$.

Similarly, we can see the Amoeba meager and the Amoeba Sacks orders as Amoeba products of certain infinite tree orders with a finite tree order.

(7) Applications of the Amoeba order(s). Amoeba orders have been used in many applications of set theory. They have been used in the context of Martin's Axiom by Carlson, Laver, Martin, and Solovay, in the investigation of cardinal invariants of the continuum by Bagaria, Brendle, and Judah, in the investigation of regularity properties of sets of real numbers by Brendle and Shelah, in infinitary combinatorics by Džamonja, Shelah, and Zapletal.

[5] *Cf.* [Kun80, VII, § 7] for a detailed discussion of why this gives the desired consequences for the generic objects.
[6] *Cf.* [Tru88, Theorem 2.1 & 2.2].
[7] *Cf.* [MarSol70, p. 172].

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