Perfect Set Theorems at singular cardinals

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RIMS Set Theory Workshop 2023, 27. October 2023

Joint work in progress with Vincenzo Dimonte (Udine) and Martina Iannella (Vienna).

Remember that a set of real numbers has the *Perfect Set Property (PSP)* if it is either countable or contains a non-empty closed subset without isolated points.

Cantor showed that every closed set of reals has the PSP and Suslin extend this implication to all analytic sets.

While Gödel showed that the existence of a co-analytic set of reals without the PSP is consistent, seminal results of Shelah–Woodin and Martin–Steel showed that the existence of large cardinals implies that every set of reals in $L(\mathbb{R})$ has the PSP.

There has recently been an interest to develop a *generalized descriptive set theory* that allows the study of definable objects of higher cardinalities.

While it is known that several key results of the classical theory cannot be directly generalized to all higher cardinalities, several settings with rich structure theories for definable sets were isolated.

The work presented in this talk contributes to the study of one of these settings that originates in Hugh Woodin's work on large cardinal assumptions close to the *Kunen Inconsistency*.

Definition

A non-trivial elementary embedding $j : L(V_{\lambda+1}) \longrightarrow L(V_{\lambda+1})$ for some ordinal λ is an *I0-embedding* if $\mathrm{crit}(j) < \lambda$ holds.

Given a non-empty set X and an infinite cardinal μ, we equip the set ${}^{\mu}X$ of all functions from μ to X with the topology whose basic open sets consists of all functions that extend a given function $s : \xi \longrightarrow X$ with $\xi < \mu$.

In addition, we equip the set $\mathcal{P}(\nu)$ of all subsets of an infinite cardinal ν with the topology whose basic open sets consist of all subsets of ν whose intersection with a given ordinal $\eta < \nu$ is equal to a fixed subset of η.

Finally, we say that a map $\iota : X \longrightarrow Y$ between topological spaces is a *perfect embedding* if it induces a homeomorphism between X and the subspace $\mathrm{ran}(\iota)$ of Y.

Theorem (Cramer, Shi & Woodin)

If $j : L(V_{\lambda+1}) \longrightarrow L(V_{\lambda+1})$ is an I0-embedding and X is a subset of $\mathcal{P}(\lambda)$ of cardinality greater than λ that is an element of $L(V_{\lambda+1})$, then there is a perfect embedding $\iota : {}^{\omega}\lambda \longrightarrow \mathcal{P}(\lambda)$ with $\mathrm{ran}(\iota) \subseteq X$.

Question

Do weaker large cardinal assumptions suffice to derive the above conclusion for smaller classes of definable subsets of $\mathcal{P}(\lambda)$?

Definition

A class X is *definable* by a formula $\varphi(v_0, \ldots, v_n)$ and parameters $z_0, \ldots, z_{n-1}$ if

$$X = \{y \mid \varphi(y, z_0, \ldots, z_{n-1})\}.$$

Definition

- A formula in the language of set theory is a Σ_0-*formula* if it is contained in the smallest collection of $\mathcal{L}_\in$-formulas that contains all atomic formulas and is closed under negation, disjunction and bounded quantification.
- Given $n < \omega$, an $\mathcal{L}_\in$-formula is a Σ_{n+1}-*formula* if it is of the form $\exists x \, \neg\varphi(x)$ for some Σ_n-formula φ.

Theorem (L.–Müller)

If λ is a limit of measurable cardinals and X is a subset of $\mathcal{P}(\lambda)$ of cardinality greater than λ that is definable by a Σ_1-formula with parameters in $V_\lambda \cup \{\lambda\}$, then there is a perfect embedding $\iota : {}^{\mathrm{cof}(\lambda)}\lambda \longrightarrow \mathcal{P}(\lambda)$ with $\mathrm{ran}(\iota) \subseteq X$.

Theorem (L.–Müller)

Let λ be a singular strong limit cardinal with the property that for every subset of $\mathcal{P}(\lambda)$ of cardinality greater than λ that is definable by a Σ_1-formula with parameters in $H_\kappa \cup \{\kappa\}$ contains the range of a perfect embedding of ${}^{\mathrm{cof}(\lambda)}\lambda$ into $\mathcal{P}(\lambda)$. Then there is an inner model with a sequence of measurable cardinals of length $\mathrm{cof}(\lambda)$.

Note that, if λ is an infinite cardinal, then $L(V_{\lambda+1})$ contains all subsets of $\mathcal{P}(\lambda)$ that are definable by a Σ_1-formula with parameters in H_{λ^+}, because H_{λ^+} is contained in $L(V_{\lambda+1})$ and all Σ_1-formulas are absolute between V and H_{λ^+}.

Question

Can we derive a stronger Perfect Set Theorem at limits of measurable cardinals?

What happens if we allow other simple parameters, like V_λ or a cofinal ω-sequence in λ, in our Σ_1-definitions?

Theorem (Dimonte–Iannella–L.)

If $\vec{\lambda}$ is a strictly increasing sequence of measurable cardinals with limit λ, then the following statements hold in an inner model M containing $\vec{\lambda}$:

- The sequence $\vec{\lambda}$ consists of measurable cardinals.
- If $\vec{\nu}$ is a strictly increasing ω-sequence of regular cardinals with limit λ, then there is a subset of $\mathcal{P}(\lambda)$ that does not contain the range of a perfect embedding of ${}^{\omega}\lambda$ into $\mathcal{P}(\lambda)$ and is definable by a Σ_1-formula with parameters in $H_{\aleph_1} \cup \{\vec{\nu}\}$.

Definition

An elementary embedding $j : V \longrightarrow M$ with critical sequence $\langle \lambda_n \mid n < \omega \rangle$ is an *I2-embedding* if $V_\lambda \subseteq M$, where $\lambda = \sup_{n<\omega} \lambda_n$.

Theorem (Dimonte–Iannella–L.)

Let $j : V \longrightarrow M$ be an I2-embedding with critical sequence $\vec{\lambda} = \langle \lambda_n \mid n < \omega \rangle$ and set $\lambda = \sup_{n<\omega} \lambda_n$. If X is a subset of $\mathcal{P}(\lambda)$ of cardinality greater than λ that is definable by a Σ_1-formula with parameters in $V_\lambda \cup \{V_\lambda, \vec{\lambda}\}$, then there is a perfect embedding $\iota : {}^\omega\lambda \longrightarrow \mathcal{P}(\lambda)$ with $\mathrm{ran}(\iota) \subseteq X$.

Theorem (Dimonte–Iannella–L.)

If $j : V \longrightarrow M$ is an I2-embedding with critical sequence $\vec{\lambda} = \langle \lambda_n \mid n < \omega \rangle$ and $\lambda = \sup_{n<\omega} \lambda_n$, then the following statements hold in an inner model:

- There is an I2-embedding whose critical sequence has supremum λ.
- There is a subset of $\mathcal{P}(\lambda)$ that does not contain the range of a perfect embedding of ${}^{\omega}\lambda$ into $\mathcal{P}(\lambda)$ and is definable by a Σ_1-formula with parameters in $\mathcal{P}(\lambda)$.

The results discussed above suggest the possibility of studying large cardinal assumptions inducing singular cardinals λ of countable cofinality through the provable regularity properties of simply definable subsets of $\mathcal{P}(\lambda)$.

More specifically, they suggest we can assign definable subsets of $V_{\lambda+1}$ to these axioms in a way that $\mathbf{ZFC}$ proves that all subsets of $\mathcal{P}(\lambda)$ of cardinality greater than λ that are definable by Σ_1-formulas using parameters from the given subset have the Perfect Set Property.

Note that, since this approach is based on provable implications and not consistency strength, it is less affected by the current technical limitations of inner model theory and therefore provides a new angle to study strong large cardinal axioms.

Descriptive properties of l2-embeddings

In the following, we outline the proof of our main result:

Theorem (Dimonte–Iannella–L.)

Let $j : V \longrightarrow M$ be an I2-embedding with critical sequence $\vec{\lambda} = \langle \lambda_n \mid n < \omega \rangle$ and set $\lambda = \sup_{n<\omega} \lambda_n$.

If X is a subset of $\mathcal{P}(\lambda)$ of cardinality greater than λ that is definable by a Σ_1-formula with parameters in $V_\lambda \cup \{V_\lambda, \vec{\lambda}\}$, then there is a perfect embedding $\iota : {}^\omega\lambda \longrightarrow \mathcal{P}(\lambda)$ with $\mathrm{ran}(\iota) \subseteq X$.

We will in fact show that the above conclusion holds for a larger class of parameters that we will now define.

Classical results of Martin show that I2-embeddings $j : V \longrightarrow M$ are ω-*iterable*, i.e. there exists a commuting system

$$\langle\langle M^j_\alpha \mid \alpha \leq \omega\rangle, \langle j : M^j_\alpha \longrightarrow M^j_\beta \mid \alpha \leq \beta \leq \omega\rangle\rangle$$

of inner models and elementary embeddings with:

- $M^j_0 = V$ and $j_{0,1} = j$.
- If $n < \omega$, then $j_{n+1,n+2} = \bigcup\{j_{n,n+1}(j_{n,n+1} \upharpoonright V_\alpha) \mid \alpha \in \mathrm{Ord}\}$.
- $\langle M^j_\omega, \langle j_{n,\omega} \mid n < \omega\rangle\rangle$ is a direct limit of

$$\langle\langle M^j_n \mid n < \omega\rangle, \langle j_{m,n} : M^j_m \longrightarrow M^j_n \mid m \leq n < \omega\rangle\rangle.$$

If $\vec{\lambda} = \langle \lambda_n \mid n < \omega\rangle$ is the critical sequence of j and $\lambda = \sup_{n<\omega} \lambda_n$, then:

- Given $m \leq n < \omega$, we then have $V_\lambda \subseteq M^j_\omega \subseteq M^j_n \subseteq M^j_m$, $\mathrm{crit}(j_{n,n+1}) = \lambda_n = j_{m,n}(\lambda_m)$, $j_{m,n}(\lambda) = \lambda$ and $j_{n,\omega}(\lambda_n) = \lambda$.
- $j_{0,\omega}(\lambda^+) = \lambda^+$ and $(2^\lambda)^{M^j_\omega} < \lambda^+$.
- $\vec{\lambda}$ is Prikry-generic over M^j_ω and hence $(2^\lambda)^{M^j_\omega[\vec{\lambda}]} < \lambda^+$.

Theorem (Laver)

Let $j : V \longrightarrow M$ be an I2-embedding with critical sequence $\langle \lambda_n \mid n < \omega \rangle$ and set $\lambda = \sup_{n<\omega} \lambda$.

If $d \in V_\lambda$ and $r : d \longrightarrow \mathrm{Ord}$ is a function, then the function

$$j_{0,\omega} \circ r : d \longrightarrow \mathrm{Ord}$$

is an element of M^j_ω.

Using Laver's result, we will be able to prove a strengthening of the above Perfect Set Theorem.

Theorem (Dimonte–Iannella–L.)

Let $j : V \longrightarrow M$ be an I2-embedding with critical sequence $\vec{\lambda} = \langle \lambda_n \mid n < \omega \rangle$ and let N be an inner model of $\mathbf{ZFC}$ with $M^j_\omega \cup \{\vec{\lambda}\} \subseteq N$. Set $\lambda = \sup_{n<\omega} \lambda_n$.

If X is a subset of ${}^\omega\lambda$ with $|X| > (2^\lambda)^N$ that is definable by a Σ_1-formula with parameters in $V^N_{\lambda+1}$, then there is a perfect embedding $\iota : {}^\omega\lambda \longrightarrow C(\vec{\lambda})$ with $\mathrm{ran}(\iota) \subseteq X$.

- A subset of ${}^{\omega}\lambda$ is definable by a Σ_1-formula with parameters in $V^N_{\lambda+1}$ iff it is definable over V_λ by a Σ^1_2-formula with parameters in $V^N_{\lambda+1}$.

- A subset of ${}^{\omega}\lambda \times {}^{\omega}\lambda$ that is definable over V_λ by a Σ^1_1-formula with parameters in $V^N_{\lambda+1}$ can be represented as the projection $p[T]$ of the set $[T]$ of all cofinal branches through a subtree $T \in N$ of $({}^{<\omega}V_\lambda)^3$.

- We can build a *Shoenfield tree* for the Σ^1_2-subset of ${}^{\omega}\lambda$ defined by T.

  Let S^V_T denote the Shoenfield tree of T in V and let S^N_T denote the Shoenfield tree of T in N.

- Then $S^N_T \subseteq S^V_T$ and we can use Laver's theorem to find an embedding of S^V_T into S^N_T that is the identity on the first coordinate.

- We then know that $p[S^N_T]^V = p[S^V_T]^V$.

Lemma

Let $\vec{\lambda} = \langle \lambda_n \mid n < \omega \rangle$ be a strictly increasing sequence of infinite cardinals with limit λ and let $T \subseteq {}^{<\omega}a \times {}^{<\omega}b$ be a tree such that $p[T]$ does not contain the range of a perfect embedding of ${}^{\omega}\lambda$ into ${}^{\omega}a$.

If N is an inner model with $V_\lambda \cup \{T, \vec{\lambda}\} \subseteq N$, then $p[T]^V \subseteq N$.

- Assume that $p[S_T^V]^V$ has cardinality greater than $(2^\lambda)^N$.
- Then $p[S_T^N]^V = p[S_T^V]^V \nsubseteq N$.
- The lemma shows that $p[S_T^V]^V$ contains the range of a perfect embedding of ${}^{\omega}\lambda$ into itself.

Open questions

Question

Assume that $j : V_{\lambda+1} \longrightarrow V_{\lambda+1}$ is an I1-embedding and X is a subset of $\mathcal{P}(\lambda)$ of cardinality greater than λ that is definable in H_{λ^+}.

Does X contain the range of a perfect embedding of ${}^{\omega}\lambda$ into $\mathcal{P}(\lambda)$?

Question

Let $\vec{\lambda} = \langle \lambda_n \mid n < \omega \rangle$ be a strictly increasing sequence of cardinals with limit λ such that each λ_n is $<\lambda$-supercompact.

If X is a subset of $\mathcal{P}(\lambda)$ of cardinality greater than λ that is definable by a Σ_1-formula with parameters in $V_\lambda \cup \{\vec{\lambda}\}$, does X contain the range of a perfect embedding of ${}^{\omega}\lambda$ into $\mathcal{P}(\lambda)$?

Thank you for listening!