THE GIRTH RAMSEY THEOREM

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Abstract. Given a hypergraph F and a number of colours r, there exists a hypergraph H of the same girth satisfying $H \longrightarrow (F)_r$. Moreover, for every linear hypergraph F there exists a Ramsey hypergraph H that locally looks like a forest of copies of F.

§1. Introduction

1.1. **Colouring vertices.** We commence with the well known fact, due to Erdős and Hajnal [9], that for every $k \geqslant 2$ there exist k-uniform hypergraphs whose girth and chromatic number are simultaneously arbitrarily large (see also [2, 7, 17, 20]).

Recall that the *chromatic number* of a hypergraph H is the least natural number $\chi(H)$ such that there exists a colouring of the vertices of H using $\chi(H)$ colours with the property that no edge of H is monochromatic. This is a Ramsey theoretic invariant of H, for a lower bound estimate of the form $\chi(H) > r$ can equivalently be expressed by the partition relation

$$H \longrightarrow (e)^v_r\,, \tag{1.1}$$

where the superscripted v on the right side means that the objects receiving colours are *vertices* and the letter e enclosed in parentheses indicates that the object we want to find monochromatically is an *edge*.

The absence of cycles can equivalently be described in terms of forests. Let us call a set N of edges a *forest* if there exists an enumeration $N = \{e_1, \dots, e_{|N|}\}$ such that for every $j \in [2, |N|]$ the set $\big(\bigcup_{i<j} e_i\big) \cap e_j$ is either empty or it consists of a single vertex. Now the aforementioned result on hypergraphs with high chromatic number and large girth reformulates as follows.

Theorem 1.1. *For every $k \geqslant 2$ and all $r, n \in \mathbb{N}$ there exists a k-uniform hypergraph H with $H \longrightarrow (e)^v_r$ having the property that any set consisting of at most n edges of H forms a forest.* □

This result is optimal in the sense that for every forest W there is some number of colours r such that every hypergraph H satisfying $H \longrightarrow (e)^v_r$ contains a copy of W.

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From a Ramsey theoretic perspective, it is equally natural to investigate the problem that instead of a monochromatic edge one wishes to enforce a monochromatic copy of a given hypergraph F. For any two hypergraphs F and H we write $\binom{H}{F}$ for the set of all *induced* subhypergraphs of H isomorphic to F. Given $\mathscr{H} \subseteq \binom{H}{F}$ and $r \in \mathbb{N}$ the partition relation

$$\mathscr{H} \longrightarrow (F)^v_r \tag{1.2}$$

is defined to hold if for every colouring of the vertices of H with r colours there exists a monochromatic copy $F_\star \in \mathscr{H}$. The existence of a system $\mathscr{H}$ having this property for given F and r is easily established by starting with a linear $v(F)$-uniform hypergraph whose chromatic number exceeds r, and inserting copies of F into its edges. Pursuing this argument further one arrives at the following result (see [18]).

Theorem 1.2. *For every k-uniform hypergraph F and all $r, n \in \mathbb{N}$ there exists a hypergraph H together with a system of copies $\mathscr{H} \subseteq \binom{H}{F}$ such that*

- *(i)* $\mathscr{H} \longrightarrow (F)^v_r$ *and*
- *(ii) for every $\mathscr{N} \subseteq \mathscr{H}$ with $|\mathscr{N}| \leqslant n$ there exists an enumeration $\mathscr{N} = \{F_1, \dots, F_{|\mathscr{N}|}\}$ with the property that for every $j \in [2, |\mathscr{N}|]$ the sets $\bigcup_{i<j} V(F_i)$ and $V(F_j)$ have at most one vertex in common.* □

Again this result is optimal in the sense that it describes all configurations of copies of F that need to be present in systems $\mathscr{H}$ satisfying $\mathscr{H} \longrightarrow (F)^v_r$ for large r. For a precise statement along these lines, we refer to the recent work of Daskin, Hoshen, Krivelevich, and Zhukovskii [6].

1.2. **Colouring edges.** An entirely new level of difficulty emerges when edges rather than vertices are the entities subject to colouration. Erdős asked more than forty years ago whether for every hypergraph F and every number of colours r there exists a hypergraph H with

$$H \longrightarrow (F)_r\,,$$

where now we are colouring edges* and the desired monochromatic occurrence of F is still supposed to be induced. This problem was first solved in the 2-uniform case [5, 10, 26, 27] and later in full generality [1, 19]. In §3.3 we describe a simple proof of this *induced Ramsey theorem* from [23]. As a matter of fact, the articles [1, 19] show much stronger results allowing subhypergraphs and not only edges to be coloured. Moreover, [19] proves that one can demand H to have certain additional properties, provided that F has these properties as well. For instance, let G be a hypergraph any pair of whose vertices is covered by an edge. Now if F has no subhypergraph isomorphic to G, then we can insist that the Ramsey

*Henceforth all colourings are colourings of edges and attempting to keep the notation simple we refrain from adding a superscripted "e" on the right side of our partition relations.

hypergraph H likewise does not have such a subhypergraph. This result allows to preserve the clique number of F, but not the girth. Let us recall that girth is defined as follows.

Definition 1.3. Given a hypergraph $H = (V, E)$ and an integer $n \geqslant 2$ we say that a cyclic sequence

$$e_1 v_1 \dots e_n v_n \tag{1.3}$$

is an *n-cycle* in H provided

- $(C1)$ the edges $e_1, \dots, e_n \in E$ are distinct;
- $(C2)$ the vertices $v_1, \dots, v_n \in V$ are distinct;
- $(C3)$ and $v_i \in e_i \cap e_{i+1}$ for each $i \in \mathbb{Z}/n\mathbb{Z}$.

Moreover, for an integer $g \geqslant 2$ we write $\mathrm{girth}(H) > g$ if for no $n \in [2, g]$ there is an n-cycle contained in H.

In particular, $\mathrm{girth}(H) > 2$ means that H is *linear* in the sense that no two of its edges intersect in more than one vertex. We can now state the simplest form of the girth Ramsey theorem, which is among the main results of this article.

Theorem 1.4. *Given an integer $g \geqslant 2$, a hypergraph F with $\mathrm{girth}(F) > g$, and a natural number r, there exists a hypergraph H with $\mathrm{girth}(H) > g$ and*

$$H \longrightarrow (F)_r\,. \tag{1.4}$$

Roughly speaking this result gives us quite a lot of local control over the Ramsey hypergraph H of F. Ultimately, one would like to analyse the local structure of Ramsey hypergraphs in the same way as Theorem 1.1 describes the local structure of hypergraphs with large chromatic number. Here we solve this problem for linear hypergraphs F (and therefore, in particular, for all graphs).

Definition 1.5. Let F be a linear hypergraph. We call a set $\mathscr{N}$ of hypergraphs isomorphic to F a *forest of copies of F* if there exists an enumeration $\mathscr{N} = \{F_1, \dots, F_{|\mathscr{N}|}\}$ such that for every $j \in [2, |\mathscr{N}|]$ the set $z_j = \big(\bigcup_{i<j} V(F_i)\big) \cap V(F_j)$ satisfies

- (i) either $|z_j| \leqslant 1$
- (ii) or $z_j \in \big(\bigcup_{i<j} E(F_i)\big) \cap E(F_j)$.

We denote the union of a forest of copies $\mathscr{N}$ by $\bigcup \mathscr{N}$; explicitly, this is the hypergraph with vertex set $\bigcup_{F_\star \in \mathscr{N}} V(F_\star)$ and edge set $\bigcup_{F_\star \in \mathscr{N}} E(F_\star)$. A hypergraph G is said to be a *partial F-forest* if it is an induced subhypergraph of $\bigcup \mathscr{N}$ for some forest $\mathscr{N}$ of copies of F.

It is not difficult to see that every such forest of copies of F needs to be contained in every Ramsey hypergraph of F with sufficiently many colours. Another main result of this work states that, conversely, we can build Ramsey hypergraphs that locally look like forests of copies of F.

Theorem 1.6. *For every linear hypergraph F and all $r, n \in \mathbb{N}$ there exists a linear hypergraph H with $H \longrightarrow (F)_r$ such that every set $X \subseteq V(H)$ whose size it at most n induces a partial F-forest in H.*

An oversimplified way of looking at the construction of H is the following: We start with an extremely large set $\mathscr{H}$ of mutually disjoint copies of F. Then we perform many steps of glueing some copies together along edges, with the aim of obtaining the desired hypergraph H. Now, on the one hand, we need to glue quite a lot in order to ensure the Ramsey property. On the other hand, locally we are not allowed to glue too much because we want to exclude short cycles of copies of F. In any case, H is constructed together with a system of copies $\mathscr{H} \subseteq \binom{H}{F}$ such that $\mathscr{H} \longrightarrow (F)_r$ and we may wonder whether we can insist that all small subsets of $\mathscr{H}$ should be forests of copies of F.

Before answering this question we need to draw attention to a somewhat bizarre difference between the notion of forests of copies of F and the standard forests of edges considered in §1.1. It is well known that if a set of edges forms a forest, then so does each of its subsets. But, as the following counterexample demonstrates, the analogous statement for forests of copies fails (see Figure 1.1).

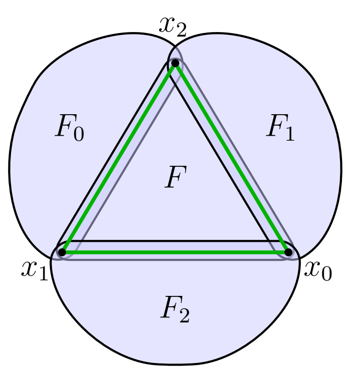


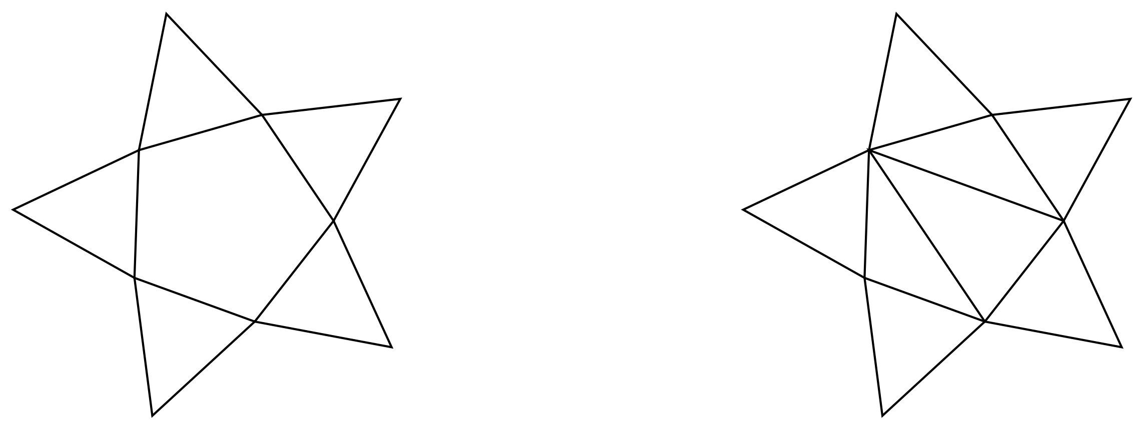
FIGURE 1.1. A subforest $\{F_0, F_1, F_2\}$ that fails to be a forest.

Let F be a graph containing a triangle $x_0x_1x_2$. For every index $i \in \mathbb{Z}/3\mathbb{Z}$ let F_i be a graph isomorphic to F having the edge $x_{i+1}x_{i+2}$ but nothing else in common with F. Suppose that except for these intersections the copies in $\mathscr{N} = \{F, F_0, F_1, F_2\}$ are mutually disjoint. This enumeration exemplifies that $\mathscr{N}$ is a forest of copies. However its subset $\mathscr{N}^- = \mathscr{N} \smallsetminus \{F\}$ fails to be such a forest. For instance, for the enumeration $\mathscr{N}^- = \{F_0, F_1, F_2\}$ the set $z_2 = \big(V(F_0) \cup V(F_1)\big) \cap V(F_2) = \{x_0, x_1\}$ is certainly not in case *(i)* and, as it fails to be an edge of F_0 or F_1, it does not satisfy *(ii)* either. By symmetry a similar problem arises when one enumerates $\mathscr{N}^-$ in any other way.

Summarising this discussion, we have seen that being a forest of copies is not preserved under taking subsets. This phenomenon explains the rôle of $\mathscr{X}$ in the most general version of the girth Ramsey theorem that follows.

Theorem 1.7. *Given a linear hypergraph F and $r, n \in \mathbb{N}$ there exists a linear hypergraph H together with a system of copies $\mathscr{H} \subseteq \binom{H}{F}$ satisfying not only $\mathscr{H} \longrightarrow (F)_r$ but also the following statement: For every $\mathscr{N} \subseteq \mathscr{H}$ with $|\mathscr{N}| \in [2, n]$ there exists a set $\mathscr{X} \subseteq \mathscr{H}$ such that $|\mathscr{X}| \leqslant |\mathscr{N}| - 2$ and $\mathscr{N} \cup \mathscr{X}$ is a forest of copies.*

Let us emphasise again that allowing such a set $\mathscr{X}$ is necessary. For instance, if F is a triangle and r is large, then $\mathscr{H}$ needs to have a subset $\mathscr{N}$ consisting of five triangles arranged "cyclically" (see Figure 1.2a). Now $\mathscr{N}$ itself fails to be a forest of triangles, but by triangulating the pentagon one can create a set $\mathscr{X}$ of three further triangles such that $\mathscr{N} \cup \mathscr{X}$ is a forest of triangles and, hence, unavoidable in $\mathscr{H}$ (see Figure 1.2b).



(a) A cycle of triangles

(b) Adding further triangles creates a forest

FIGURE 1.2. The necessity of $\mathscr{X}$ in Theorem 1.7

In general one needs $|\mathscr{N}| - 2$ triangles for triangulating an $|\mathscr{N}|$-gon and, hence, the bound $|\mathscr{X}| \leqslant |\mathscr{N}| - 2$ is optimal whenever F contains a triangle. If $\mathrm{girth}(F) > g \geqslant 2$, then the upper bound on $|\mathscr{X}|$ can be improved to $|\mathscr{X}| \leqslant (|\mathscr{N}| - 2)/(g - 1)$ (see Theorem 13.12 below).

We conclude this introduction by discussing some partial results towards the girth Ramsey theorem that have been obtained over the years. First, for general k-uniform hypergraphs Theorem 1.7 has been proved by Nešetřil and Rödl [24] for $n = 2$ and their approach yields Theorem 1.4 for $g = 3$ as well.

Most other partial results deal with the case $k = 2$, i.e., with graphs. The main result of [24] asserts that Theorem 1.4 holds for $k = 2$ and $g = 4$ and, as Nešetřil and Rödl point out, by means of a more elaborate version of their argument one can treat every $g \leqslant 7$.

However, the new difficulties arising for $g = 8$ are fairly overwhelming. In general, it seems that even cycles cause more trouble than odd cycles and, in fact, an *odd-girth* version of Theorem 1.4 was obtained in [21].

For $k = 2$ and arbitrary girth Rödl and Ruciński [28] proved probabilistically that Theorem 1.4 holds for $F = C_{g+1}$, thus answering a question of Erdős [8]. Hypergraph extensions of this result follow from the work of Friedgut, Rödl, and Schacht [11], and of Conlon and Gowers [4]. The random graph approach was also used by Haxell, Kohayakawa, and Łuczak [14] in order to determine the smallest number of edges that a Ramsey graph for C_{g+1} can have.

It appears, however, that the usual probabilistic model $G(n, p)$ is not suitable for proving any version of the girth Ramsey theorem for arbitrary graphs or hypergraphs F. We shall thus return to the explicit constructions that were developed in the early days of this area.

Organisation. The next section offers an informal discussion of some aspects of the proof of the girth Ramsey theorem. It ends with an annotated table of contents, that we hope to be helpful for navigating through this article. From a logical point of view, this section can be skipped entirely. The remaining eleven sections, on the other hand, are, with the exception of a small number of subsections, necessary for our argument and the discussion in Section 2 is intended to shed some light on the rôle they will play in due course. These exceptional omittable subsections are typically added at the end of some sections and have intentionally the same title "orientation". They deal with summaries of where we currently are, where we want to go, and how we plan to arrive there.

§2. Overview

It is quite hard to summarise in a few pages how the girth Ramsey theorem is proved, but we would like to use this section for pointing to some problems one naturally encounters when thinking about it, and to some ideas we use for solving them. Throughout this informal discussion, we assume some familiarity with the partite construction method invented long ago by Nešetřil and the second author (see Section 3 for a thorough introduction to this topic). Among all the partial results we mentioned in Section 1 the perhaps deepest one concerns graphs without four-cycles.

Theorem 2.1 (Rödl and Nešetřil)**.** *For every graph F with* $\mathrm{girth}(F) > 4$ *and every integer $r \geqslant 2$ there exists a graph H with* $\mathrm{girth}(H) > 4$ *and* $H \longrightarrow (F)_r$.

The proof utilises a strong form of the following fact: for every linear hypergraph M (of arbitrary uniformity k) and every number of colours r there is a linear hypergraph N such that $N \longrightarrow (M)_r$. In other words, one appeals to the case $g = 2$ (and arbitrary k) of

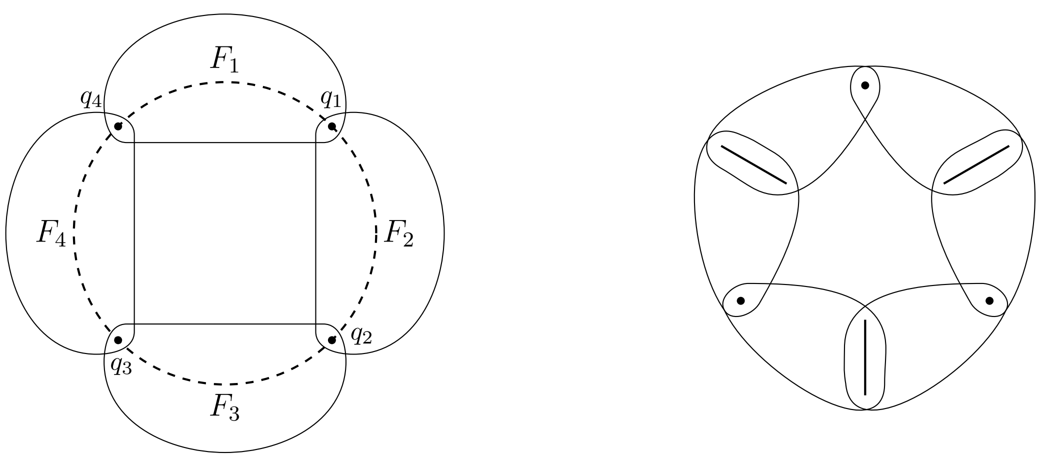
Theorem 1.4. This suggests that if one wants to continue along those lines, the following firm decisions ought to be made:

- The proof proceeds by induction on g.
- Even if ultimately one should only care about the graph case, one still needs to treat all values of k at the same time.

Thus the "smallest open case" before this work was the following.

Problem 2.2. *Extend Theorem 2.1 to* 3*-uniform hypergraphs.*

The solution to this problem is roughly as complicated as the proof of Theorem 1.4 itself and we would like to focus on another aspect of our proof strategy based on induction on g first. Suppose we have already understood everything about $g = 99$ and that we now want to handle a graph F with $\mathrm{girth}(F) > 100$. So we need to construct a graph H with $\mathrm{girth}(H) > 100$ possessing a set of copies $\mathscr{H} \subseteq \binom{H}{F}$ satisfying e.g. $\mathscr{H} \longrightarrow (F)_2$. Imagine that some four of the copies in $\mathscr{H}$ can be arranged cyclically such that any two consecutive copies share a vertex but nothing more (see Figure 2.1a). If for each $i \in \mathbb{Z}/4\mathbb{Z}$ the distance from q_{i-1} to q_i within the copy F_i was at most 20, then we could build a cycle in H through q_1, q_2, q_3, and q_4 whose length would be at most $4 \times 20 = 80$, contrary to $\mathrm{girth}(H) > 100$. We will therefore devote some effort into ensuring that the system $\mathscr{H}$ we are about to construct contains no four-cycles of copies as in Figure 2.1a. As a matter of fact, the existence of Ramsey systems $\mathscr{H}$ without these cycles can be proved as soon as $\mathrm{girth}(F) > 4$ and we shall obtain this together with our treatment of the case $g = 4$. In other words, the strength of the statement we shall actually prove by induction on g is somewhere between Theorem 1.4 and Theorem 1.7.



(a) A four-cycle of copies (b) A six-cycle with alternating connectors

FIGURE 2.1. Two cycles of copies.

For now we just want to say that due to the Ramsey property some copies in the systems $\mathscr{H} \subseteq \binom{H}{F}$ we plan to exhibit need to intersect in entire edges and not just in mere vertices. Thus we can also form cycles of copies of a type slightly more general than what we saw in Figure 2.1a. That is, we have to allow the so-called *connectors* between consecutive copies to be either vertices or edges (see Figure 2.1b). An important idea in our analysis of cycles of copies, which seems to be new, is that the difficulty of "excluding" such cycles does not only depend on their *length* (i.e., the number of copies they contain), but also on the number of times vertex connectors and edge connectors alternate: Having many such alternations will turn out to be helpful. Thus the cycle drawn in Figure 2.1b is the "easiest" possible cycle of length 6 and we shall deal with it before approaching the cycles of length 4 depicted in Figure 2.1a. In fact, for general linear hypergraphs cycles of six copies with alternating connectors are the most complex cycles that could have been handled with existing methods (even though apparently this has never been noticed before), while cycles with four vertex connectors require some genuinely new ideas. We shall discuss cycles of copies further in Section 4. In particular, we will introduce there our notion of the Girth of a system of copies (spelled with a capital G), which takes alternations of connectors into account. The ensuing Section 5 elaborates on the fact that partite constructions sometimes increase the Girth of our Ramsey systems of copies.

* * *

Let us return to Theorem 2.1. The partite construction method essentially reduces its proof to the bipartite case. Thus the main ingredient is a partite lemma preserving the girth assumption, i.e., a statement of the following form:

> *For every bipartite graph B with* $\mathrm{girth}(B) > 4$ *and every integer $r \geqslant 2$ there is a bipartite graph $B_\star$ with* $\mathrm{girth}(B_\star) > 4$ *and $B_\star \longrightarrow (B)_r$.* $(\star)$

It is the proof of this statement where the Ramsey theorem for linear hypergraphs is required. Denoting the vertex classes of B by X and Y we may assume that all vertices in X have the same degree $d \geqslant 2$. (If this is not the case already, we attach some pendant edges to the vertices in X). Now B can be viewed as a union of edge-disjoint stars $K_{1,d}$ whose centres are in X. This so-called *star decomposition* of B gives rise to a d-uniform hypergraph F on Y whose edges correspond to the non-central vertices of those stars, i.e.,

$$F = (Y, \{N(x)\colon x \in X\})\,. \tag{2.1}$$

The assumed absence of 4-cycles in B translates into the linearity of F.

It does not help much to apply the Ramsey theorem for linear hypergraphs directly to F itself. Rather, one first constructs an auxiliary, linear k-uniform hypergraph G, where $k = (d-1)r + 1$ has the property that for every r-colouring of a k-set there is a monochromatic d-set (Schubfachprinzip). Moreover, G is constructed together with some linear order of its vertex set and to render this notationally visible we shall write $G_<$ instead of G. We now apply an ordered version of the induction hypothesis[*] to $G_<$ with r^k colours, thus obtaining a linear, ordered, k-uniform hypergraph $H_<$ such that $(H_<) \longrightarrow (G_<)_{r^k}$. Going back to bipartite graphs we take for every edge e of H a new vertex v_e and join it to all members of e. This yields a bipartite graph $B_\star$ with vertex classes $\{v_e\colon e \in E(H_<)\}$ and $V(H_<)$. Since $H_<$ is linear, $B_\star$ is C_4-free and it turns out that $B_\star \longrightarrow (B)_r$ can be guaranteed by an appropriate choice of $G_<$. Roughly, this is because every colouring $\gamma\colon E(B_\star) \longrightarrow [r]$ associates with every edge $e \in E(H_<)$ one of r^k possible colour patterns, namely the k-tuple consisting of the colours of the k edges from v_e to e. Owing to the construction of $H_<$ there is some copy $G^\star_<$ of $G_<$ all of whose edges receive the same colour pattern. By our choice of k this colour pattern contains some colour $\varrho_\star \in [r]$ at least d times and an appropriate construction of $G_<$ ensures that the copy $G^\star_<$ we have just found contains a monochromatic copy of F (whose colour is $\varrho_\star$). This is all we want to say about the proof of the partite lemma at this juncture. An abstract version of the construction which leads us from B via F, $G_<$, and $H_<$ to $B^\star$, called the *extension process*, will be discussed in Section 6.

* * *

There is one further subtlety in the proof of Theorem 2.1 we would like to emphasise here. The question is why or under what circumstances a partite construction based on the above partite lemma does not create four-cycles. The worry one might have is whether forbidden cycles can arise when amalgamating graphs (previous "pictures") over a bipartite graph (as in the partite construction). For instance in Figure 2.2 we see two copies B_1, B_2 of the bipartite graph we subjected to the partite lemma, and two vertices x, y belonging to both of them. If at the previous stage of the construction x and y had common neighbours u, v, then altogether a four-cycle $x - u - y - v$ arises.

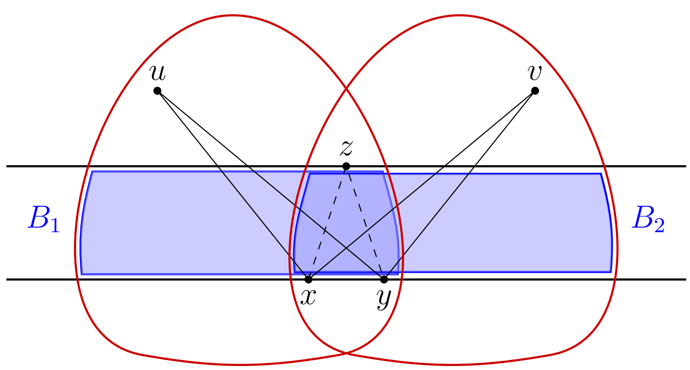


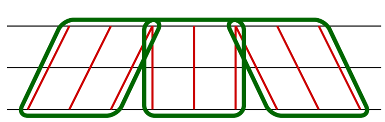
FIGURE 2.2. Four-cycles in amalgamations

[*]Keeping track of vertex orderings will often be important in the sequel; but these orderings never complicate the proofs, so we do not treat them carefully in this outline.

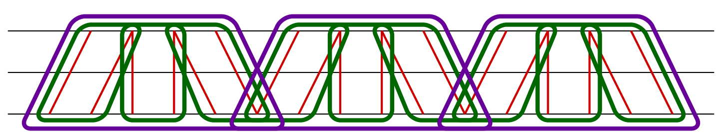
This problem was addressed in [24] by working with a very strong form of the Ramsey theorem for linear hypergraphs, which has the effect that we may assume B_1, B_2 to intersect in a star. More precisely, the partite lemma actually comes together with a system of copies $\mathscr{B} \subseteq \binom{B_\star}{B}$ such that $\mathscr{B} \longrightarrow (B)_r$ and any two distinct copies B_1, B_2 are either disjoint, or they intersect in a single vertex, or they intersect in a star. Thus the situation in Figure 2.2 requires the existence of a common neighbour z of x, y belonging to the intersection of B_1 and B_2. If $z \neq u$ then $x - u - y - z$ is a four-cycle in the left copy of the previous picture, which is absurd. Similarly, if $z = u$, then the cycle $x - u - y - v$ is entirely contained in the right copy of the previous picture. There are some other potential cases of four-cycles after the amalgamation that one needs to exclude before declaring Theorem 2.1 to be proved, but we shall not go into such details here.

* * *

When attempting to extend these ideas to 3-uniform hypergraphs one actually does not need a partite lemma applicable to *all* tripartite 3-uniform hypergraphs without 4-cycles. This is due to the fact that when executing the decisive partite construction one can ensure that all 3-partite hypergraphs one has to handle are highly structured. Indeed, the 3-partite hypergraphs the initial picture is composed of are just matchings. Moreover, using standard techniques one can ensure that the next picture is composed of 3-partite hypergraphs built from many copies of such matchings by gluing them together along single vertices belonging to the same vertex class (see Figure 2.3a). When one proceeds to the next picture, many copies of such hypergraphs get glued together along single vertices sitting on another common vertex class (see Figure 2.3b), and so it goes on. Later we shall call partite hypergraphs that can arise in this manner *trains* and we shall prove a partite lemma for trains by induction on their *height*, i.e., the number of times the gluing process needs to be iterated in their formation.



(a)



(b)

Figure 2.3. Two 3-uniform trains

For reasons of comparison it should perhaps be mentioned that in the 2-uniform case the distinction between bipartite graphs with train structure and general C_4-free bipartite graphs does not exist. This is because the star decomposition we used earlier (recall (2.1)) exemplifies that all C_4-free bipartite graphs are trains. In the 3-uniform case, however, the difference is quite pronounced and “most” tripartite hypergraphs without four-cycles fail to admit any train structure.

Dealing with trains gets quickly somewhat technical, but fortunately one can analyse a great portion of the general case by just looking at much simpler structures, which we call *pretrains*. These are hypergraphs equipped with an equivalence relation on their set of edges. The corresponding equivalence classes will be called the *wagons* of the pretrain. For instance, in the proof of the partite lemma for C_4-free bipartite graphs one can regard the given bipartite graph F as a pretrain, whose wagons are stars with centres in X.

It turns out that pretrains are the natural structures for describing the extension process the proof of the partite lemma ($\star$) was based on. The intersections of copies in stars we saw earlier will generalise to intersections in entire wagons. It will be a nontrivial, yet important problem to improve this to copies intersecting at most in edges. Of course one hopes to accomplish this by means of a further partite construction and, therefore, we shall study the behaviour of pretrains in such constructions in Section 7.

In an attempt to point to something essential we are still missing at this moment we would like to formulate a concrete test problem. We shall write pretrains in the form $(F, \equiv^F)$, where F is a hypergraph and $\equiv^F$ refers to an equivalence relation on $E(F)$. We say that $(F, \equiv^F)$ is an *induced subpretrain* of another pretrain $(H, \equiv^H)$ if F is an induced subhypergraph of H and, moreover, any two edges of F are $\equiv^F$-equivalent if and only if they are $\equiv^H$-equivalent. Given a pretrain $(F, \equiv^F)$ and a number of colours r we will be interested in constructing a pretrain $(H, \equiv^H)$ and a system $\mathscr{H}$ of copies of $(F, \equiv^F)$ in $(H, \equiv^H)$ such that, in an obvious sense, the partition relation $\mathscr{H} \longrightarrow (F, \equiv^F)$ holds. There are several known methods for obtaining such systems $\mathscr{H}$ (see e.g., Lemma 6.11 or Proposition 7.1).

But now assume, in addition, that the given pretrain $(F, \equiv^F)$ is *linear* in the sense that

- the hypergraph F is linear
- and any two wagons intersect in at most one vertex.

We would like to find $(H, \equiv^H)$ and $\mathscr{H}$ as above such that, moreover,

- the pretrain $(H, \equiv^H)$ is linear in the same sense
- and any two copies in $\mathscr{H}$ intersect at most in an edge.

Partite constructions are very good at meeting the second requirement, but they appear to have difficulties to maintain the linearity of the pretrains. In fact, the only way we know for achieving the linearity of $(H, \equiv^H)$ is to use the extension process instead, but this process is incapable of producing systems of copies intersecting in less than a wagon.

This conundrum will be resolved by the introduction of a new concept, called German $\mathfrak{Girth}$. It turns out that this generalises the Girth of systems of copies we studied earlier (before delving into pretrains) and interacts very well with partite constructions. $\mathfrak{Girth}$ itself will be introduced and studied in Section 8. Two of its most essential properties will then be demonstrated in Section 9. First, the extension process translates ordinary Girth properties of hypergraph constructions into the corresponding $\mathfrak{Girth}$ properties of pretrain constructions. Second, $\mathfrak{Girth}$ properties are indestructible by partite constructions.

We will then have all the ingredients necessary for proving a strong form of Theorem 1.4 in Sections 10 – 12, though it is still technically challenging to combine all these pieces together (cf. Theorem 10.19). Deducing Theorem 1.7 and Theorem 1.6 from this result will then be comparatively routine (see Section 13).

Annotated table of contents

Section 11. Trains in the extension process

§11.1 — Extensions of trains

[Preparing a generalisation of the extension process to quasitrains we define and study so-called 1-extensions of quasitrains.]

§11.2 — A generalised extension lemma

[Assuming $\diamondsuit_{\vec{g}}$ we describe a construction Υ applicable to trains of height $|\vec{g}|+1$.]

Section 12. Trains in partite constructions

§12.1 — Quasitrain constructions

[If Φ is a Ramsey construction for hypergraphs and Ξ denotes a partite lemma for quasitrains, then $\mathrm{PC}(\Phi, \Xi)$ is a Ramsey construction for quasitrains.]

§12.2 — Train amalgamations

[When we want to handle trains (as opposed to mere quasitrains), we will always start with a train construction Φ. We prove a general amalgamation lemma for train pictures.]

§12.3 — Amenable partite lemmata

[We decompose the problem of proving $\diamondsuit_{\vec{g}}$ into two simpler tasks.]

§12.4 — Girth resurrection

[Continuing §11.2 we show that $\mathrm{PC}(\Upsilon, \mathrm{PC}(\Upsilon, \Upsilon))$ exemplifies $\diamondsuit_{(2)\circ\vec{g}}$.]

§12.5 — Revisability

[We convert $\diamondsuit_{(g)^m\circ\vec{g}}$ into a partite lemma applicable to certain trains $\vec{F}$ with $\mathfrak{girth}(\vec{F}) > (g+1)\circ\vec{g}$ that we call m-revisable.]

§12.6 — Train constituents

[Performing a diagonal partite construction we complete the proof of Proposition 10.15.]

Section 13. Paradise

§13.1 — Forests of copies

[We relate Girth to forests of copies by showing that $\mathrm{Girth}(H, \mathscr{N}^+) > |\mathscr{N}|$ implies $\mathscr{N}$ to be a forest of copies (see Lemma 13.11).]

§13.2 — The final partite construction

[We prove a generalisation of Theorem 1.7, stating that if $\mathrm{girth}(F) > g$, then the upper bound on the number $|\mathscr{X}|$ of additional copies can be improved to $\frac{|\mathscr{N}|-2}{g-1}$. Finally, we deduce Theorem 1.6 (on the local structure of Ramsey hypergraphs).]

§3. The partite construction method

A common point of departure for many results in structural Ramsey theory is Ramsey's theorem [25] asserting that given natural numbers k, r, and m there exists a natural number M for which the partition relation

$$M \longrightarrow (m)^k_r$$

holds. Let us recall that this means: no matter how the edges of a complete k-uniform hypergraph $K^{(k)}_M$ on M vertices get coloured with r colours, there will always exist a monochromatic $K^{(k)}_m$. Thus if a k-uniform hypergraph F and a number of colours r are given, we may apply Ramsey's theorem to $m = v(F)$ and obtain a large clique $H = K^{(k)}_M$ arrowing F in the sense that the collection $\mathscr{H} = \binom{H}{F}_{\text{n.n.i.}}$ of subhypergraphs of H isomorphic to F has the partition property

$$\mathscr{H} \longrightarrow (F)_r\,,$$

where "n.n.i." abbreviates "not necessarily induced". The defect of non-induced copies can be remedied by an iterative amalgamation method called the *partite construction*. Introduced by Nešetřil and Rödl in [22], this method has since then been utilised in a variety of different contexts (see e.g. [3, 15, 16, 19, 24]). The present work heavily relies on the partite construction method as well and we shall therefore provide an overview in the remainder of this section.

3.1. **Partite lemmata.** The k-partite k-uniform hypergraphs F occurring in this article will always be accompanied by k-element *index sets* I and by fixed partitions

$$V(F) = \dot{\bigcup_{i\in I}} V_i(F)$$

of their vertex sets such that every edge $e \in E(F)$ intersects each *vertex class* $V_i(F)$ in exactly one vertex. The most natural choice for I is, of course, the set $[k]$. Later on, however, when we shall be executing partite constructions, we encounter k-partite hypergraphs whose vertex classes are already indexed differently and in order to avoid excessive relabelling it seems advisable to allow general index sets I from the very beginning.

Given two k-partite k-uniform hypergraphs F and H with the same index set I we say that F is a *partite subhypergraph* of H if F is a subhypergraph of H in the ordinary sense and, moreover, $V_i(F) \subseteq V_i(H)$ holds for every $i \in I$. The collection of all *partite copies* of F in H is denoted by $\binom{H}{F}^{\text{pt}}$.

Suppose now that a k-partite k-uniform hypergraph F with index set I as well as a number of colours r are given. A *partite lemma* is a construction delivering a k-partite k-uniform hypergraph H with the same index set I as well as a system of copies $\mathscr{H} \subseteq \binom{H}{F}^{\text{pt}}$ such that $\mathscr{H} \longrightarrow (F)_r$.

For the existence of such hypergraphs H we could simply refer to the literature. But on several later occasions we need to check that a particular construction based on the Hales-Jewett theorem [13] has certain additional properties (see [29] for a beautiful alternative proof of the Hales-Jewett theorem). For this reason, we briefly sketch the so-called *Hales-Jewett construction*.

In the degenerate case, where F has no edges, we simply set $H = F$ and $\mathscr{H} = \{F\}$. Otherwise we appeal to the Hales-Jewett theorem and obtain a natural number n such that every r-colouring of $E(F)^n$ contains a monochromatic combinatorial line. The vertex classes of H are defined by $V_i(H) = V_i(F)^n$ for every $i \in I$. The set $E(H)$ is constructed together with a *canonical bijection* $\lambda\colon E(F)^n \longrightarrow E(H)$. Given an n-tuple $(e_1, \dots, e_n)$ of edges of F, we write $x_{\nu i}$ for the common vertex of e_ν and $V_i(F)$, where $\nu \in [n]$ and $i \in I$, and we define the edge $e = \lambda(e_1, \dots, e_n)$ by demanding for every $i \in I$ that e intersects $V_i(H)$ in its vertex $(x_{1i}, \dots, x_{ni})$.

It is well known (and follows from Lemma 3.2 below) that every combinatorial line $L \subseteq E(F)^n$ gives rise to an induced subhypergraph F_L of H which is isomorphic to F and satisfies $E(F_L) = \{\lambda(\vec{e})\colon \vec{e} \in L\}$. Therefore

$$\mathscr{H} = \big\{F_L\colon L \text{ is a combinatorial line in } E(F)^n\big\}$$

is a system of copies of F in H and, owing to our choice of n involving the Hales-Jewett theorem, it has the desired property

$$\mathscr{H} \longrightarrow (F)_r\,.$$

To facilitate later references to this construction we set $\mathrm{HJ}_r(F) = (H, \mathscr{H})$, where HJ abbreviates “Hales-Jewett”.

3.2. **Pictures.** Suppose that F and G are two k-uniform hypergraphs and that $\mathscr{G} \subseteq \binom{G}{F}_{\text{n.n.i.}}$ is a system of subhypergraphs of G, which are isomorphic to F but not necessarily induced. A *picture over* $(G, \mathscr{G})$ is a triple $(\Pi, \mathscr{P}, \psi)$ consisting of a k-uniform hypergraph Π, a system $\mathscr{P} \subseteq \binom{\Pi}{F}$ of induced copies of F in Π, and a hypergraph homomorphism $\psi\colon \Pi \longrightarrow G$ mapping the copies in $\mathscr{P}$ onto copies in $\mathscr{G}$ (see Figure 3.1). Occasionally we shall encounter pictures carrying an additional structure compatible with additional structure that might be present on G.

Given a picture $(\Pi, \mathscr{P}, \psi)$ we call the preimages of the vertices of G under ψ *music lines* and put

$$V_x = V_x(\Pi) = \{v \in V(\Pi)\colon \psi(v) = x\}$$

for every $x \in V(G)$. Observe that $\{V_x\colon x \in V(G)\}$ partitions $V(\Pi)$, while the edges of Π cross each music line at most once. When visualising a picture we usually draw the music lines *horizontally* and we imagine $(G, \mathscr{G})$ to be drawn *vertically* next to the picture.

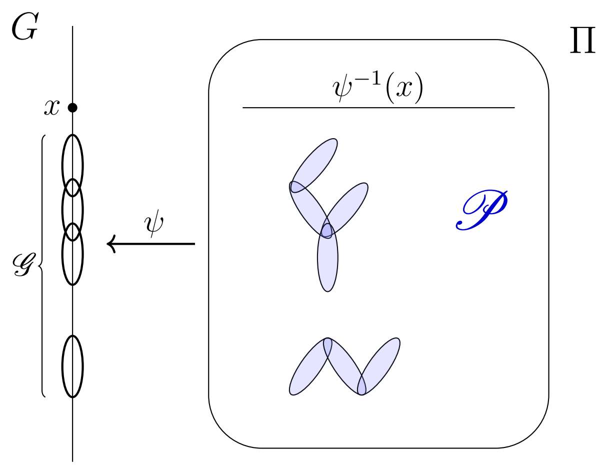


FIGURE 3.1. Picture $(\Pi, \mathscr{P}, \psi)$ over $(G, \mathscr{G})$.

Associated with every edge $e \in E(G)$ we have its *constituent* Π^e, which is the k-uniform k-partite hypergraph with index set e and vertex partition $\{V_x \colon x \in e\}$ whose set of edges is defined by

$$E(\Pi^e) = \{f \in E(\Pi) \colon \psi[f] = e\}\,.$$

A partite construction over $(G, \mathscr{G})$ always commences with the corresponding *picture zero* (see Figure 3.2). This is a picture $(\Pi_0, \mathscr{P}_0, \psi_0)$ having for each copy $F_\star \in \mathscr{G}$ a unique copy $\widehat{F}_\star$ projected to $F_\star$ by ψ_0. It is required that

- for distinct copies $F_\star, F_{\star\star} \in \mathscr{G}$ the corresponding copies $\widehat{F}_\star, \widehat{F}_{\star\star} \in \mathscr{P}$ be vertex-disjoint
- and that every vertex or edge of Π_0 belong to one of the copies $\widehat{F}_\star \in \mathscr{P}$.

Evidently such a picture zero can always be constructed and, in fact, it is uniquely determined up to isomorphism; it has $|\mathscr{G}| \cdot |V(F)|$ vertices and $|\mathscr{G}| \cdot |E(F)|$ edges.

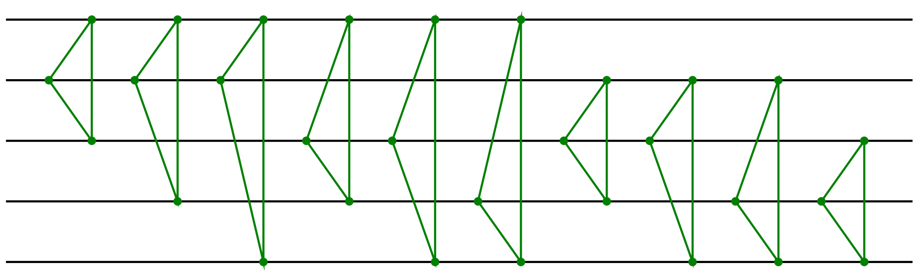


FIGURE 3.2. Picture zero over $(K_5, \mathscr{G})$, where $\mathscr{G} = \binom{K_5}{K_3}$.

Now suppose that Π^e is a constituent of some picture $(\Pi, \mathscr{P}, \psi_\Pi)$ and that, in addition, we have a k-partite k-uniform hypergraph H with index set e together with a system of induced partite copies $\mathscr{H} \subseteq \binom{H}{\Pi^e}^{\mathrm{pt}}$. We may then define a new picture $(\Sigma, \mathscr{Q}, \psi_\Sigma)$ by means of the following *partite amalgamation* (see Figure 3.3). Start with $\Sigma^e = H$ and extend each copy $\Pi^e_\star \in \mathscr{H}$ to its own copy $\Pi_\star$ of Π. These extensions are to be performed as disjointly as possible, so for any two distinct copies $\Pi^e_\star, \Pi^e_{\star\star} \in \mathscr{H}$ it is demanded that their extensions $\Pi_\star$, $\Pi_{\star\star}$ satisfy $V(\Pi_\star) \cap V(\Pi_{\star\star}) = V(\Pi^e_\star) \cap V(\Pi^e_{\star\star})$. From each of these extended copies $\Pi_\star$ there is a canonical isomorphism to the original hypergraph Π, which allows us to pull $\mathscr{P}$ and ψ_Π back onto $\Pi_\star$. Thereby we obtain a collection of pictures $\{(\Pi_\star, \mathscr{P}_\star, \psi_{\Pi_\star})\colon \Pi^e_\star \in \mathscr{H}\}$ each of which is isomorphic to $(\Pi, \mathscr{P}, \psi_\Pi)$. The new picture $(\Sigma, \mathscr{Q}, \psi_\Sigma)$ is defined to be their union with H. Explicitly

$$V(\Sigma) = \bigcup_{\Pi^e_\star \in \mathscr{H}} V(\Pi_\star) \cup V(H)\,, \qquad E(\Sigma) = \bigcup_{\Pi^e_\star \in \mathscr{H}} E(\Pi_\star) \cup E(H)\,,$$

$$\mathscr{Q} = \bigcup_{\Pi^e_\star \in \mathscr{H}} \mathscr{P}_\star\,, \qquad \text{and} \qquad \psi_\Sigma = \bigcup_{\Pi^e_\star \in \mathscr{H}} \psi_{\Pi_\star} \cup \psi_H\,,$$

where, as expected, $\psi_H\colon V(H) \longrightarrow e$ is the map sending for each $x \in e$ all vertices in $V_x(H)$ to x. One checks immediately that the copies in $\mathscr{Q}$ are induced, that ψ_Σ is a hypergraph homomorphism from Σ to G, and, finally, that the copies in $\mathscr{Q}$ project appropriately. Therefore $(\Sigma, \mathscr{Q}, \psi_\Sigma)$ is indeed a picture. The copies $\Pi_\star$ of Π obtained by extending the copies $\Pi^e_\star$ in $\mathscr{H}$ are called the *standard copies* of Π in Σ. In the sequel we shall indicate the partite amalgamation just explained by writing

$$(\Sigma, \mathscr{Q}, \psi_\Sigma) = (\Pi, \mathscr{P}, \psi_\Pi) * (H, \mathscr{H})\,.$$

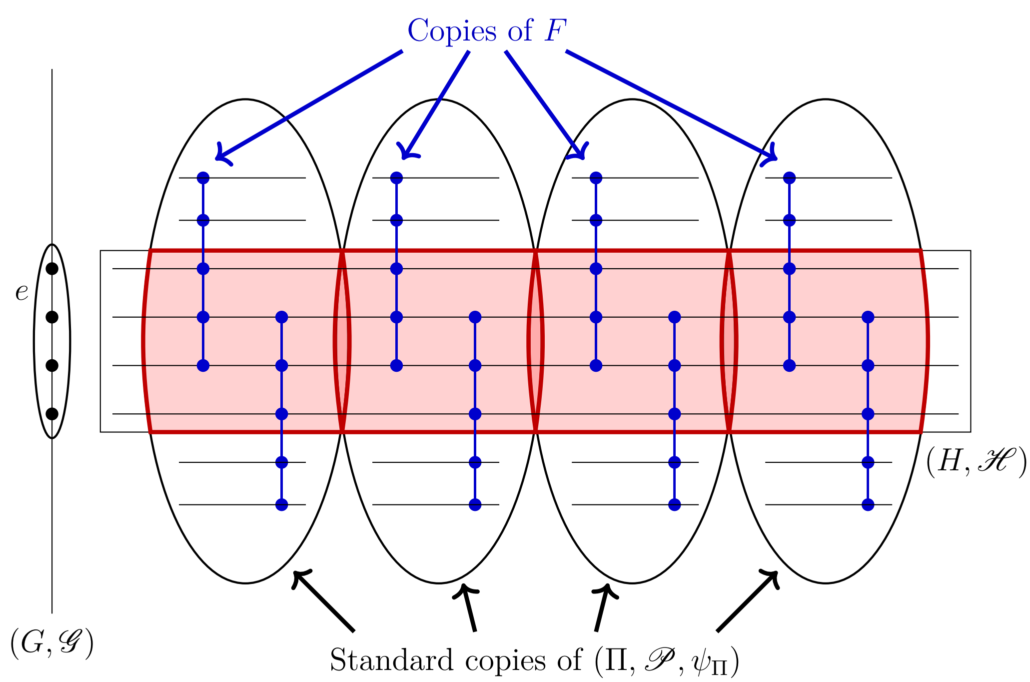


FIGURE 3.3. Partite amalgamation $(\Pi, \mathscr{P}, \psi_\Pi) * (H, \mathscr{H})$

3.3. **The induced Ramsey theorem.** To perform a partite construction, one needs a Ramsey theoretic result that gets applied vertically, as well as a partite lemma that one iteratively utilises horizontally. As an illustration of the method, we briefly describe one of the proofs of the *induced Ramsey theorem* from [23].

Given a k-uniform hypergraph F as well as a number of colours r we intend to construct a hypergraph H as well as a system of induced copies $\mathscr{H} \subseteq \binom{H}{F}$ such that $\mathscr{H} \longrightarrow (F)_r$. We begin by taking any pair $(G, \mathscr{G})$ consisting of a k-uniform hypergraph G and a system of not necessarily induced copies $\mathscr{G} \subseteq \binom{G}{F}_{\text{n.n.i.}}$ satisfying

$$\mathscr{G} \longrightarrow (F)_r\,. \tag{3.1}$$

For instance, by Ramsey's theorem we could take G to be a sufficiently large clique and set $\mathscr{G} = \binom{G}{F}_{\text{n.n.i.}}$. The precise choice of $(G, \mathscr{G})$ is, however, quite irrelevant and all that matters is the partition relation (3.1).

Let $\{e(1), \ldots, e(N)\}$ enumerate the edges of G in any order and perform the following recursive construction of pictures $(\Pi_\alpha, \mathscr{P}_\alpha, \psi_\alpha)_{0 \leqslant \alpha \leqslant N}$ over $(G, \mathscr{G})$. Begin with picture zero $(\Pi_0, \mathscr{P}_0, \psi_0)$. If for some positive $\alpha \leqslant N$ the picture $(\Pi_{\alpha-1}, \mathscr{P}_{\alpha-1}, \psi_{\alpha-1})$ has just been constructed, we apply any partite lemma to its constituent $\Pi^{e(\alpha)}_{\alpha-1}$, thus getting a pair $(H_\alpha, \mathscr{H}_\alpha)$ with

$$\mathscr{H}_\alpha \subseteq \binom{H_\alpha}{\Pi^{e(\alpha)}_{\alpha-1}}^{\text{pt}} \quad \text{and} \quad \mathscr{H}_\alpha \longrightarrow \big(\Pi^{e(\alpha)}_{\alpha-1}\big)_r\,. \tag{3.2}$$

For instance, at this moment we may just employ the Hales-Jewett construction from §3.1 and set

$$(H_\alpha, \mathscr{H}_\alpha) = \mathrm{HJ}_r\big(\Pi^{e(\alpha)}_{\alpha-1}\big)\,,$$

but any other choice validating (3.2) is equally fine. Having selected $(H_\alpha, \mathscr{H}_\alpha)$ we define the next picture by

$$(\Pi_\alpha, \mathscr{P}_\alpha, \psi_\alpha) = (\Pi_{\alpha-1}, \mathscr{P}_{\alpha-1}, \psi_{\alpha-1}) \ast (H_\alpha, \mathscr{H}_\alpha)\,.$$

It is known that the final picture $(\Pi_N, \mathscr{P}_N, \psi_N)$ has the property

$$\mathscr{P}_N \longrightarrow (F)_r\,, \tag{3.3}$$

i.e., that the system $(\Pi_N, \mathscr{P}_N)$ is as requested by the induced Ramsey theorem. Indeed, an easy proof by induction on α using (3.2) in the induction step shows that if for $0 \leqslant \alpha \leqslant N$ the edges of Π_α get coloured with r colours, then there is a copy $(\Pi^\star_0, \mathscr{P}^\star_0)$ of picture zero with the property that for every $\beta \in [\alpha]$ the constituent of $\Pi^\star_0$ over $e(\beta)$ is monochromatic. In particular, if one colours the edges of the final picture with r colours, then there exists a copy of picture zero all of whose constituents are monochromatic. The colours of these constituents project to an edge colouring of the vertical hypergraph G and (3.1) leads to a monochromatic copy of F in $\mathscr{P}_N$. Thereby (3.3) is proved, As the proof of the girth Ramsey theorem involves a great number of nested applications of the partite construction method, it will safe a considerable amount of space to establish an appropriate terminology.

Recall that in §3.1 we denoted the Hales-Jewett construction proving the partite lemma by HJ. Similarly, if we appeal to Ramsey's theorem to obtain a system $(G, \mathscr{G})$ with $\mathscr{G} \longrightarrow (F)_r$ whose copies not necessarily induced, then we shall write

$$\mathrm{Rms}_r(F) = (G, \mathscr{G}) \,.$$

Explicitly, this means that G is a sufficiently large clique whose size depends on $v(F)$ and r, and that $\mathscr{G} = \binom{G}{F}_{\text{n.n.i.}}$. So both Rms and HJ are examples of Ramsey theoretic *constructions*.

Pairs $(H, \mathscr{H})$ consisting of a hypergraph H and a set $\mathscr{H}$ of subhyergraphs of H will be called *systems of hypergraphs* and the members of $\mathscr{H}$ will be referred to as copies. It is neither required that the copies of a system be mutually isomorphic nor that they be induced.

In general, if a *construction* Φ is applied to a hypergraph F and a number of colours r, then it delivers a system of hypergraphs $\Phi_r(F) = (H, \mathscr{H})$ with $\mathscr{H} \subseteq \binom{H}{F}_{\text{n.n.i.}}$ and $\mathscr{H} \longrightarrow (F)_r$. Not every construction is applicable to every hypergraph. For instance, $\mathrm{HJ}_r(F)$ is only defined if F is a k-partite k-uniform hypergraph for some $k \geqslant 2$. It will be convenient to call any construction having this property and which, moreover, delivers systems of induced partite copies, a *partite lemma*.

The partite construction method is, strictly speaking, not a construction in the sense of the previous paragraph, but rather an *operation* capable of producing a new construction from two given ones. More precisely, for a construction Φ and a partite lemma Ξ the construction $\mathrm{PC}(\Phi, \Xi)$ is defined as follows. Given a hypergraph F as well a number of colours r, one sets $(G, \mathscr{G}) = \Phi_r(F)$ and generates a sequence of pictures $(\Pi_\alpha, \mathscr{P}_\alpha, \psi_\alpha)_{0 \leqslant \alpha \leqslant N}$ over $(G, \mathscr{G})$ in the same way as above but ensuring the partition relation (3.2) by means of the partite lemma Ξ, i.e., by setting

$$(H_\alpha, \mathscr{H}_\alpha) = \Xi_r\bigl(\Pi_{\alpha-1}^{e(\alpha)}\bigr) \,.$$

Finally one defines

$$\mathrm{PC}(\Phi, \Xi)_r(F) = (\Pi_N, \mathscr{P}_N) \,,$$

where $N = e(G)$ denotes the index of the final picture. For instance, the construction $\mathrm{PC}(\mathrm{Rms}, \mathrm{HJ})$ produces systems of hypergraphs that verify the induced Ramsey theorem.

Let us emphasise at this moment that it is not always easy to foresee the class of hypergraphs a construction of the form $\mathrm{PC}(\Phi, \Xi)$ is defined on. For instance, let Λ be a *linear partite lemma*, i.e., a partite lemma that is only applicable to linear k-partite k-uniform hypergraphs and delivers linear systems of k-partite k-uniform hypergraphs.

One might then come up with the idea of using the construction $\mathrm{PC}(\mathrm{Rms}, \Lambda)$ in order to obtain a Ramsey theorem for linear hypergraphs, i.e., the case $g = 2$ of Theorem 1.4. The main problem with this approach is that at this level of generality it is not completely clear whether the constituents of our pictures stay linear throughout the construction. There are two known ways of addressing this difficulty, one of which is explained in §3.7 later.

3.4. **Strong inducedness.** Let us explore some properties of constructions derivable from HJ and Rms by means of the partite construction method.

Definition 3.1. Given a hypergraph G and a subhypergraph F of G we say that F is *strongly induced* in G and write $F \blacktriangleleft G$ if for every edge $e \in E(G)$ there exists an edge $f \in E(F)$ with $e \cap V(F) \subseteq f$.

Evidently every strongly induced subhypergraph is, in particular, induced. One also checks easily that every hypergraph is a strongly induced subhypergraph of itself and that passing to a strongly induced subhypergraph is transitive, i.e., that $F \blacktriangleleft G \blacktriangleleft H$ implies $F \blacktriangleleft H$.

Lemma 3.2. *If F is a k-partite k-uniform hypergraph, $r \in \mathbb{N}$, and $\mathrm{HJ}_r(F) = (H, \mathscr{H})$, then every $F_\star \in \mathscr{H}$ is strongly induced in H.*

Proof. In the trivial case $E(F) = \varnothing$ we have $H = F = F_\star$ and the result is clear. Now suppose $E(F) \neq \varnothing$ and let n be the Hales-Jewett exponent entering the construction of $(H, \mathscr{H})$. As usual, F and H have index set I, and λ denotes the canonical bijection from $E(F)^n$ onto $E(H)$.

Let $\eta \colon E(F) \longrightarrow E(F)^n$ be the combinatorial embedding with $E(F_\star) = (\lambda \circ \eta)[E(F)]$. This means that there are a partition $[n] = C \cup M$ of the set of *coordinates* into a set C of *constant coordinates* and a nonempty set M of *moving coordinates* as well as a map $\widetilde{\eta} \colon C \longrightarrow E(F)$ such that for every $f \in E(F)$ and every $\nu \in [n]$ the ν^{th} coordinate of $\eta(f)$ is the edge

- $\widetilde{\eta}(\nu)$ if $\nu \in C$
- and f if $\nu \in M$.

Associated with η we have bijections $\eta_i \colon V_i(F) \longrightarrow V_i(F_\star)$ for all indices $i \in I$. Explicitly, if $x \in V_i(F)$, then $\eta_i(x) = (x_1, \ldots, x_n)$ satisfies $x_\nu = x$ for all $\nu \in M$, while for $\nu \in C$ the vertex x_ν is, independently of x, the vertex of $\widetilde{\eta}(\nu)$ belonging to $V_i(F)$. Observe that if $x \in V_i(F)$ and $f \in E(F)$ are incident, then so are $\eta_i(x)$ and $(\lambda \circ \eta)(f)$.

Now let $e = \lambda(f_1, \ldots, f_n)$ be an arbitrary edge of H. We are to exhibit an edge e' of $F_\star$ with $e \cap V(F_\star) \subseteq e'$. To this end we fix a moving coordinate $\nu(\star) \in M$ and define $e' = (\lambda \circ \eta)(f_{\nu(\star)})$. For any $i \in I$ let $\vec{x}$ be a vertex in $e \cap V_i(F_\star)$. We need to prove $\vec{x} \in e'$. Owing to $\vec{x} \in V_i(F_\star)$ there exists a vertex $x \in V_i(F)$ with $\vec{x} = \eta_i(x)$. Looking at the projection of $\vec{x} \in e$ to the $\nu(\star)^{\text{th}}$ coordinate we see $x \in f_{\nu(\star)}$ and thus we have indeed $\vec{x} = \eta_i(x) \in (\lambda \circ \eta)(f_{\nu(\star)}) = e'$. $\square$

The possible intersections of copies in $(H, \mathscr{H}) = \mathrm{HJ}_r(F)$ are somewhat complicated (and were explicitly described in [24]). We circumvent this issue by moving on to the construction $\mathrm{CPL} = \mathrm{PC}(\mathrm{HJ}, \mathrm{HJ})$ called the *clean partite lemma* (see Corollary 3.5).

Definition 3.3. A system of copies $(H, \mathscr{H})$ is said to have *clean intersections* if for any two distinct copies $F_\star, F_{\star\star} \in \mathscr{H}$ there exist edges $e_\star \in E(F_\star)$, $e_{\star\star} \in E(F_{\star\star})$ with

$$V(F_\star) \cap V(F_{\star\star}) = e_\star \cap e_{\star\star}\,.$$

We proceed with a general result saying that copies with clean intersections arise automatically when one performs a partite construction employing strongly induced copies vertically. (For historical reasons we refer to [3, Lemma 2.12]).

Lemma 3.4. *If Φ is a Ramsey construction delivering strongly induced copies, then for any partite lemma Ξ the construction $\mathrm{PC}(\Phi, \Xi)$ delivers systems of strongly induced copies whose intersections are clean.*

Proof. Let a hypergraph F as well as a number of colours r be given and set $\Phi_r(F) = (G, \mathscr{G})$. Due to the hypothesis on Φ the copies in $\mathscr{G}$ are strongly induced. When performing the partite construction $\mathrm{PC}(\Phi, \Xi)_r(F)$ we eventually reach a last picture $(\Pi_N, \mathscr{P}_N, \psi_N)$.

Let us show first that every copy $F_\star \in \mathscr{P}_N$ is strongly induced in Π_N. Given an arbitrary edge $e \in E(\Pi_N)$ we need to find an edge $f_\star \in E(F_\star)$ with

$$V(F_\star) \cap e \subseteq f_\star\,. \tag{3.4}$$

The projection ψ_N sends $F_\star$ and e to a copy $F' \in \mathscr{G}$ and an edge $e' \in E(G)$. As a consequence of $F' \blacktriangleleft G$ there exists an edge $f' \in E(F')$ with $V(F') \cap e' \subseteq f'$. Now the edge $f_\star \in E(F_\star)$ corresponding to f' satisfies (3.4), since ψ_N establishes a bijection between $V(F_\star)$ and $V(F')$.

It remains to show that the copies in $\mathscr{P}_N$ have clean intersections. To this end we argue by *induction along the partite construction*. That is we prove inductively that each of the successively constructed pictures has a system of copies with clean intersections. This is clear for picture zero, for disjoint copies have an empty intersection which is, in particular, clean.

For the induction step it suffices to show that if $(G, \mathscr{G})$ is a system with strongly induced copies, $(\Pi, \mathscr{P}, \psi_\Pi)$ is a picture over $(G, \mathscr{G})$ whose copies have clean intersections, $e \in E(G)$, $\Xi_r(\Pi^e) = (H, \mathscr{H})$, and $(\Sigma, \mathscr{Q}, \psi_\Sigma) = (\Pi, \mathscr{P}, \psi_\Pi) \ast (H, \mathscr{H})$, then the copies in $\mathscr{Q}$ have clean intersections as well.

Consider any two distinct copies $F_\star, F_{\star\star} \in \mathscr{Q}$. Recall that each of them belongs to a standard copy of Π in Σ. Let $\Pi_\star$ and $\Pi_{\star\star}$ be such standard copies. If they are equal, then the induction hypothesis shows that the intersection of $F_\star$ and $F_{\star\star}$ is clean.

So suppose $\Pi_\star \neq \Pi_{\star\star}$ from now on. The projection ψ_Σ sends $F_\star$ and $F_{\star\star}$ to certain copies F' and F'' in $\mathscr{G}$. Exploiting that these copies are strongly induced we obtain edges $e' \in E(F')$ and $e'' \in E(F'')$ satisfying $V(F') \cap e \subseteq e'$ and $V(F'') \cap e \subseteq e''$.

Let $e_\star \in E(F_\star)$ be the inverse image of e' under the projection from $F_\star$ to F' and define $e_{\star\star} \in E(F_{\star\star})$ similarly with respect to F''. Owing to

$$V(F_\star) \cap V(F_{\star\star}) \subseteq V(\Pi_\star) \cap V(\Pi_{\star\star}) \subseteq V(\Pi^e)$$

we have $V(F_\star) \cap V(F_{\star\star}) = e_\star \cap e_{\star\star}$, meaning that $e_\star$ and $e_{\star\star}$ witness that the intersection of $F_\star$ and $F_{\star\star}$ is clean. □

Corollary 3.5. *The clean partite lemma* $\mathrm{CPL} = \mathrm{PC}(\mathrm{HJ}, \mathrm{HJ})$ *delivers systems of strongly induced copies with clean intersections.*

Proof. Let a k-partite k-uniform hypergraph F with index set I and a number of colours r be given. Recall that in order to construct $\mathrm{CPL}_r(F)$ we first need to construct $(G, \mathscr{G}) = \mathrm{HJ}_r(F)$ and then we need to construct a sequence of pictures over $(G, \mathscr{G})$ using the partite lemma $\mathrm{HJ}_r(\cdot)$ in every step. Let us denote the final picture by $(\Pi, \mathscr{P}, \psi_\Pi)$.

By Lemma 3.2 the copies of F in $\mathscr{G}$ are strongly induced in G and thus, by Lemma 3.4, the copies in $\mathscr{P}$ are strongly induced and have clean intersections. It remains to explain why CPL can be regarded as a partite lemma. Recall from §3.1 that G is again a k-partite k-uniform hypergraph with index set I. This means that the map $\psi_G\colon V(G) \longrightarrow I$ with $\psi_G^{-1}(i) = V_i(G)$ for every $i \in I$ is a hypergraph homomorphism from G to $I^+ = (I, \{I\})$. Consequently $\psi_G \circ \psi_\Pi\colon V(\Pi) \longrightarrow I$ is a hypergraph homomorphism from Π to I^+, which shows that Π is indeed of the desired form. Since the copies in $\mathscr{P}$ project via ψ_Π to copies in $\mathscr{G} \subseteq \binom{G}{F}^{\mathrm{pt}}$, it is also clear that, in an obvious sense, $\mathscr{P} \subseteq \binom{\Pi}{F}^{\mathrm{pt}}$. □

Now it is natural to wonder whether we can gain anything by cleaning further. For instance, one may consider the partite lemma $\mathrm{CPL}^{(2)} = \mathrm{PC}(\mathrm{CPL}, \mathrm{CPL})$ and ask whether it has any desirable properties going beyond clean intersections. An affirmative answer to this question is obtained in Section 5 below. However, even the higher iterates such as $\mathrm{CPL}^{(3)} = \mathrm{PC}(\mathrm{CPL}^{(2)}, \mathrm{CPL}^{(2)})$ are insufficient for proving the girth Ramsey theorem due to the simple reason that when applied to the graph C_6 they always yield graphs containing a 4-cycle.

Our next result asserts that clean intersections delivered by a partite lemma are indestructible under further applications of the partite construction method. The Propositions 5.1 and 9.14 proved later vastly generalise this fact.

Lemma 3.6. *If* Φ *denotes an arbitrary Ramsey construction and* Ξ *is a partite lemma producing systems of strongly induced copies with clean intersections, then* $\mathrm{PC}(\Phi, \Xi)$ *has the same properties.*

Proof. We argue by induction along the partite construction. It is plain that the copies in picture zero are strongly induced and that their intersections are clean. For the induction step we assume that $(G, \mathscr{G})$ is a system of not necessarily induced hypergraphs, that $(\Pi, \mathscr{P}, \psi_\Pi)$ is a picture over $(G, \mathscr{G})$ whose copies are strongly induced and have clean intersections, that $(H, \mathscr{H})$ is a partite system of hypergraphs with strongly induced copies whose intersections are clean, and finally that $(\Sigma, \mathscr{Q}, \psi_\Sigma) = (\Pi, \mathscr{P}, \psi_\Pi) \ast (H, \mathscr{H})$. We need to show that the copies in $\mathscr{Q}$ are strongly induced and that their intersections are clean.

Beginning with the latter task we show first that the intersection of any two distinct copies $F_\star, F_{\star\star} \in \mathscr{Q}$ is clean. Let $\Pi_\star$ and $\Pi_{\star\star}$ be standard copies of Π containing $F_\star$ and $F_{\star\star}$, respectively. If those copies coincide we may appeal to the induction hypothesis, and thus it suffices to treat the case $\Pi_\star \neq \Pi_{\star\star}$. Since $V(\Pi_\star) \cap V(\Pi_{\star\star}) \subseteq V(H)$ and owing to the fact that in $\mathscr{H}$ the intersections are clean, there exist edges $e_\star \in E(\Pi_\star) \cap E(H)$ and $e_{\star\star} \in E(\Pi_{\star\star}) \cap E(H)$ with

$$V(F_\star) \cap V(F_{\star\star}) \subseteq V(\Pi_\star) \cap V(\Pi_{\star\star}) = e_\star \cap e_{\star\star}\,. \tag{3.5}$$

As the copies of $(\Pi_\star, \mathscr{P}_\star)$ and $(\Pi_{\star\star}, \mathscr{P}_{\star\star})$ are strongly induced, there exist edges $e' \in E(F_\star)$ and $e'' \in E(F_{\star\star})$ with

$$V(F_\star) \cap e_\star \subseteq e' \quad \text{and} \quad V(F_{\star\star}) \cap e_{\star\star} \subseteq e''\,. \tag{3.6}$$

Now we have

$$V(F_\star) \cap V(F_{\star\star}) \overset{(3.5)}{=} \big(V(F_\star) \cap e_\star\big) \cap \big(V(F_{\star\star}) \cap e_{\star\star}\big) \overset{(3.6)}{\subseteq} e' \cap e''$$

and, consequently, the edges e' and e'' exemplify that the intersection of $F_\star$ and $F_{\star\star}$ is clean.

It remains to show that every $F_\star \in \mathscr{Q}$ is strongly induced in Σ. That is, given an edge $e \in E(\Sigma)$ we need to prove that there exists an edge $e' \in E(F_\star)$ with $V(F_\star) \cap e \subseteq e'$. Again let $\Pi_\star$ be a standard copy containing $F_\star$. If there exists a standard copy $\Pi_{\star\star}$ containing e we may simply repeat the above argument ignoring e''. If no such standard copy exists, then necessarily $e \in E(H)$. Since the copy in $\mathscr{H}$ extended by $\Pi_\star$ is strongly induced in H, there exists an edge $e_0 \in E(\Pi_\star)$ with $V(\Pi_\star) \cap e \subseteq e_0$. Moreover, $F_\star \blacktriangleleft \Pi_\star$ leads to an edge $e' \in E(F_\star)$ with $V(F_\star) \cap e_0 \subseteq e'$. Now

$$V(F_\star) \cap e = V(F_\star) \cap V(\Pi_\star) \cap e \subseteq V(F_\star) \cap e_0 \subseteq e'$$

shows that e' has the desired property. $\square$

As an example, we may consider the construction $\Omega^{(2)} = \mathrm{PC}(\mathrm{Rms}, \mathrm{CPL})$, which given a hypergraph F and a number of colours r delivers a system $\Omega^{(2)}_r(F) = (H, \mathscr{H})$ of strongly induced copies with clean intersections such that $\mathscr{H} \longrightarrow (F)_r$. (The construction $\Omega^{(2)}$ occurs implicitly in [3]).

3.5. **Ordered constructions.** Suppose that $F_<$ is an *ordered hypergraph*, i.e., a hypergraph on whose set of vertices a linear ordering is imposed. Now for any number of colours r we want to construct an ordered hypergraph $H_<$ with

$$H_< \longrightarrow (F_<)_r \tag{3.7}$$

which means that the monochromatic induced copy of $F_<$ needs to occur with the correct ordering. If we were to omit the demand that the ordered monochromatic copy needs to be induced, we could just use Ramsey's theorem, thus getting a pair $\mathrm{Rms}_r(F_<) = (G_<, \mathscr{G})$ consisting of a sufficiently large ordered clique $G_<$ and the system $\mathscr{G} = \binom{G_<}{F_<}_{\text{n.n.i.}}$ of all not necessarily induced ordered subhypergraphs of $G_<$ isomorphic to $F_<$.

In order to obtain $H_<$ as in (3.7) we run the partite construction $\mathrm{PC}(\mathrm{Rms}, \mathrm{HJ})_r(F)$, thus getting a final picture $(\Pi, \mathscr{P}, \psi)$ over $(G_<, \mathscr{G})$, which is known to satisfy $\mathscr{P} \longrightarrow (F)_r$ in the sense of (unordered) hypergraphs. Recall that the copies in $\mathscr{P}$ are induced and project via ψ onto copies in $\mathscr{G}$, i.e., to copies that are ordered correctly in the vertical world. Thus if $\dashv$ denotes any linear ordering on $V(\Pi)$ with the property

$$\forall x, y \in V(\Pi) \ \big[\psi(x) < \psi(y) \implies x \dashv y\big], \tag{3.8}$$

then the copies in $\mathscr{P}$ become ordered copies of $F_<$ in $\Pi_{\dashv}$, meaning that $\Pi_{\dashv}$ is as desired.

Let us observe that this argument generalises as follows: If Φ denotes any *ordered* Ramsey construction, and Ξ is a partite lemma, then $\mathrm{PC}(\Phi, \Xi)$ is again an ordered construction. That is, if for some ordered hypergraph $F_<$ and $r \in \mathbb{N}$ the system $\mathrm{PC}(\Phi, \Xi)_r(F) = (H, \mathscr{H})$ is defined, then we may endow $V(H)$ with a linear ordering $\dashv$ as in (3.8), and $\mathscr{H} \subseteq \binom{H_{\dashv}}{F_<}$ will hold automatically. In other words, $\mathrm{PC}(\Phi, \Xi)$ becomes an ordered construction by setting $\mathrm{PC}(\Phi, \Xi)_r(F_<) = (H_{\dashv}, \mathscr{H})$.

Thus we can likewise consider $\Omega^{(2)} = \mathrm{PC}(\mathrm{Rms}, \mathrm{CPL})$ to be an ordered construction. It still has the benefits discussed in §3.4, i.e., it delivers strongly induced copies with clean intersections.

3.6. ***f*-partite hypergraphs.** The following concept interpolates between k-partite k-uniform hypergraphs and general k-uniform hypergraphs.

Definition 3.7. Let $f\colon I \longrightarrow \mathbb{N}$ be a function from a finite index set I to the positive integers such that $k = \sum_{i \in I} f(i)$ is at least 2. An *f-partite hypergraph* is a k-uniform hypergraph F together with a distinguished partition $V(F) = \dot\bigcup_{i\in I} V_i(F)$ of its vertex set satisfying

$$|e \cap V_i(F)| = f(i) \quad \text{for all } e \in E(F) \text{ and } i \in I\,. \tag{3.9}$$

For example, if $|I| = k$ and f is the constant function whose value is always 1, then an f-partite hypergraph is the same as a k-partite k-uniform hypergraph. On the other end of the spectrum there is the possibility that $|I| = 1$ and f attains the value k, in which case f-partite hypergraphs are the same as ordinary k-uniform hypergraphs.

Let us return to the case of a general function $f\colon I \longrightarrow \mathbb{N}$. Extending the terminology from §3.1 we say that an f-partite hypergraph F is an *f-partite subhypergraph* of an f-partite hypergraph H if F is a subhypergraph of H and $V_i(F) \subseteq V_i(H)$ holds for all $i \in I$. Moreover, for two f-partite hypergraphs F and H the symbol $\binom{H}{F}^{\mathrm{fpt}}$ refers to the set of all f-partite subhypergraphs of H that are isomorphic to F in the f-partite sense.

For $m \in \mathbb{N}$ we let K_m^f denote the *complete f-partite hypergraph* having for every index $i \in I$ a vertex class V_i of size m and having all edges compatible with (3.9). It is well known that given $m, r \in \mathbb{N}$ there exists a natural number M which is so large that no matter how the edges of K_M^f get coloured with r colours there will always exist a monochromatic copy of K_m^f. Indeed, for $|I| = 1$ this is just Ramsey's original theorem and the general version of this so-called *product Ramsey theorem* is easily established by induction on $|I|$ (see e.g., [12, Theorem 5.1.5]).

As usual, this result yields a Ramsey theorem for f-partite hypergraphs with non-induced copies. Extending our earlier notation we denote the corresponding construction by Rms as well. So explicitly for an f-partite hypergraph F and a number of colours r the formula $\mathrm{Rms}_r(F) = (G, \mathscr{G})$ indicates that G is a sufficiently large complete f-partite hypergraph, that $\mathscr{G} = \binom{G}{F}^{\mathrm{fpt}}_{\mathrm{n.n.i.}}$ is the collection of all f-partite subhypergraphs of G isomorphic to F and, finally, that $\mathscr{G} \longrightarrow (F)_r$.

We say that a *construction* Φ *is f-partite* if applied to an f-partite hypergraph F it delivers an *f-partite system of hypergraphs*, i.e., a system $(H, \mathscr{H})$ consisting of an f-partite hypergraph H and a set $\mathscr{H} \subseteq \binom{H}{F}^{\mathrm{fpt}}_{\mathrm{n.n.i.}}$ of f-partite subhypergraphs of H. For instance the version of Rms described in the previous paragraph is an f-partite construction.

Given an f-partite construction Φ as well as a partite lemma Ξ, we can utilise the partite construction method and form $\Theta = \mathrm{PC}(\Phi, \Xi)$. It is not hard to see that the construction Θ is again f-partite. (A similar argument occurred in the proof of Corollary 3.5.)

We may now regard $\Omega^{(2)} = \mathrm{PC}(\mathrm{Rms}, \mathrm{CPL})$ as an f-partite construction, which yields a rather strong form of the induced Ramsey theorem for f-partite hypergraphs. As explained in §3.5 we may actually apply this construction to ordered hypergraphs.

Proposition 3.8. *Given an ordered f-partite hypergraph $F_<$ and a number of colours r the ordered f-partite system $\Omega_r^{(2)}(F_<) = (H_<, \mathscr{H})$ satisfies $\mathscr{H} \longrightarrow (F_<)_r$, the copies in $\mathscr{H}$ are strongly induced, and the intersections of copies in $\mathscr{H}$ are clean.* □

3.7. **Linearity.** Clearly, if F is a non-trivial linear k-uniform hypergraph for some $k \geqslant 3$, and $r \geqslant 2$ is a number of colours, then $\mathrm{Rms}_r(F)$ fails to be linear. For the purposes of girth Ramsey theory it is important to know that the other constructions we have encountered so far behave better in this regard. We begin this discussion with a special case of a result in [24].

Lemma 3.9. *If F is a linear k-partite k-uniform hypergraph, $r \in \mathbb{N}$ is a number of colours, and $\mathrm{HJ}_r(F) = (H, \mathscr{H})$, then H is a linear hypergraph as well.*

Proof. The degenerate case $E(F) = \varnothing$ being clear we suppose $E(F) \neq \varnothing$ from now on. Let n be the Hales-Jewett exponent the construction of H is based on and let $\lambda\colon E(F)^n \longrightarrow E(H)$ denote the canonical bijection. For any two distinct edges

$$e = \lambda(e_1, \dots, e_n) \quad \text{and} \quad e' = \lambda(e'_1, \dots, e'_n)$$

of H we are to prove $|e \cap e'| \leqslant 1$. To this end we take an index $\nu \in [n]$ with $e_\nu \neq e'_\nu$. If e_ν and e'_ν are disjoint, then so are e and e'. Otherwise, the linearity of F discloses that e_ν and e'_ν have a unique vertex x in common. If i denotes the index with $x \in V_i(F)$, then all vertices that e and e' have in common belong to $V_i(H)$, whence $|e \cap e'| \leqslant |e \cap V_i(H)| = 1$. □

In the sequel, a *linear construction* will be a construction Φ, which, when applied to a linear hypergraph F and a number of colours r, yields a *linear system* $\Phi_r(F) = (H, \mathscr{H})$, i.e., a system of hypergraphs whose underlying hypergraph H is linear. So in other words the previous lemma asserts that HJ is a linear partite lemma. A picture $(\Pi, \mathscr{P}, \psi)$ is said to be *linear* if its underlying hypergraph Π is linear. In the linear case strong inducedness can be characterised as follows.

Fact 3.10. *A subhypergraph F of a linear hypergraph H is strongly induced if and only if it has the following three properties.*

- (*i*) *If an edge e of H intersects F in at least two vertices, then it belongs to F.*
- (*ii*) *If an edge e of H has only a single vertex x with F in common, then x is non-isolated in F.*
- (*iii*) *If F has no edges, then neither does H.* □

Clearly, of these conditions (*i*) is the most important one. The next question we would like to address is why the clean partite lemma CPL is linear.

Lemma 3.11. *If Φ and Ξ are linear, then so is $\mathrm{PC}(\Phi, \Xi)$.*

Proof. We prove inductively that all pictures encountered in the partite construction are linear. There is no problem with picture zero and, therefore, it suffices to establish the following statement.

> *If* $(\Pi, \mathscr{P}, \psi_\Pi)$ *is a linear picture over a linear system* $(G, \mathscr{G})$, $e \in E(G)$, *and the copies of the linear system* $(H, \mathscr{H})$ *are isomorphic to the constituent* Π^e, *then the picture*
>
> $$(\Sigma, \mathscr{Q}, \psi_\Sigma) = (\Pi, \mathscr{P}, \psi_\Pi) * (H, \mathscr{H})$$
>
> *is linear again.*

Given any two distinct edges $f', f'' \in E(\Sigma)$ we are to prove $|f' \cap f''| \leqslant 1$. If their projections $\psi_\Sigma[f']$ and $\psi_\Sigma[f'']$ are distinct, this follows from the fact that G is linear. So we may assume that $f = \psi_\Sigma[f'] = \psi_\Sigma[f'']$ holds for some $f \in E(G)$. In the special case $e = f$ we may appeal to the linearity of H, so suppose $f \neq e$ from now on. Now $f', f'' \notin E(H)$ implies that there are unique standard copies Π' and Π'' in Σ containing f' and f'', respectively.

If $\Pi' = \Pi''$, then the linearity of Π leads to the desired conclusion, so it remains to consider the case $\Pi' \neq \Pi''$. Given any two vertices $x, y \in f' \cap f''$ we need to prove $x = y$. Since the standard copies Π' and Π'' were constructed to be as disjoint as possible, we have

$$f' \cap f'' \subseteq V(\Pi') \cap V(\Pi'') \subseteq V(H)\,,$$

whence $\psi_\Sigma(x), \psi_\Sigma(y) \in e \cap f$. Owing to the linearity of H this yields $\psi_\Sigma(x) = \psi_\Sigma(y)$. In other words, x and y are on the same music line. But f' intersects this music line only once and, consequently, we have indeed $x = y$. $\square$

Corollary 3.12. *The clean partite lemma* CPL *is linear.*

Proof. By Lemma 3.9 the Hales-Jewett construction HJ is linear; so Lemma 3.11 tells us that CPL = PC(HJ, HJ) is linear as well. $\square$

Recall that by Corollary 3.5 the partite lemma CPL delivers systems whose copies have clean intersections. We observe that if $(H, \mathscr{H})$ is a *linear system*, then the copies in $\mathscr{H}$ have clean intersections if and only if for any two distinct copies $F_\star, F_{\star\star} \in \mathscr{H}$ the following three statements hold.

- (i) If $|V(F_\star) \cap V(F_{\star\star})| \geqslant 2$, then there exists an edge $e \in E(F_\star) \cap E(F_{\star\star})$ with $V(F_\star) \cap V(F_{\star\star}) = e$
- (ii) If $V(F_\star) \cap V(F_{\star\star})$ consists of a single vertex, then this vertex is non-isolated in $F_\star$ and $F_{\star\star}$.
- (iii) $E(F_\star), E(F_{\star\star}) \neq \varnothing$.

Again, the main property of relevance is (i).

Lemma 3.13. *If Ω is an arbitrary Ramsey construction and Ξ denotes a linear partite lemma delivering systems with strongly induced copies whose intersections are clean, then* $\mathrm{PC}(\Omega, \Xi)$ *is a linear construction.*

Proof. Arguing by induction along the partite construction it suffices to prove the following picturesque statement.

> *If $(\Pi, \mathscr{P}, \psi_\Pi)$ is a linear picture over a system of hypergraphs $(G, \mathscr{G})$ and $(H, \mathscr{H})$ is a linear k-partite k-uniform system whose copies are strongly induced and have clean intersections, then the picture*
>
> $$(\Sigma, \mathscr{Q}, \psi_\Sigma) = (\Pi, \mathscr{P}, \psi_\Pi) * (H, \mathscr{H})$$
>
> *is linear as well.*

Let us emphasise that while we are assuming here that the hypergraphs Π and H are linear, it is allowed that the vertical projection G fails to be linear. Assume for the sake of contradiction that Σ is nonlinear. Choose a pair of distinct edges $f_\star, f_{\star\star} \in E(\Sigma)$ whose intersection $t = f_\star \cap f_{\star\star}$ satisfies $|t| \geqslant 2$ and such that, subject to this, $\Lambda = |E(H) \cap \{f_\star, f_{\star\star}\}|$ is maximal.

Suppose first that $\Lambda = 0$, i.e., that $f_\star, f_{\star\star} \notin E(H)$. Denote the unique standard copies containing $f_\star$ and $f_{\star\star}$ by $\Pi_\star$ and $\Pi_{\star\star}$, respectively. Since Π is linear, we have $\Pi_\star \neq \Pi_{\star\star}$. Let $\Pi^e_\star, \Pi^e_{\star\star} \in \mathscr{H}$ be the copies of the constituent Π^e extended by $\Pi_\star$ and $\Pi_{\star\star}$. Recall that these copies are linear and that their intersection is clean. Together with $t \subseteq V(\Pi^e_\star) \cap V(\Pi^e_{\star\star})$ this proves that they have an edge f in common. But now $t \subseteq f_\star \cap f$ and $f \in E(H)$ show that the pair $\{f_\star, f\}$ contradicts the maximality of Λ.

Let us deal with the case $\Lambda = 1$ next. By symmetry we may suppose that $f_\star \notin E(H)$ and $f_{\star\star} \in E(H)$. Define the standard copy $\Pi_\star$ and $\Pi^e_\star \in \mathscr{H}$ as in the foregoing paragraph. Due to $t \subseteq V(\Pi^e_\star) \cap f_{\star\star}$ the fact that $\Pi^e_\star$ is strongly induced in H shows that the edge $f_{\star\star}$ belongs to $\Pi_\star$. Thus we get a contradiction to the linearity of Π.

Altogether we have thereby proved $\Lambda = 2$, i.e., that necessarily $f_\star, f_{\star\star} \in E(H)$. But this contradicts the linearity of H. □

Corollary 3.14. *The construction* $\Omega^{(2)} = \mathrm{PC}(\mathrm{Rms}, \mathrm{CPL})$ *is linear.*

Proof. The assumptions of Lemma 3.13 are satisfied by Corollary 3.5 and Corollary 3.12. □

3.8. ***A*-intersecting hypergraphs.** The definition of the train hypergraphs we shall study later (see Figure 2.3) will contain a demand that certain kinds of edges are allowed to intersect in certain vertex classes only. As a very modest step into this direction we show in this subsection that the construction $\Omega^{(2)}$ preserves such a property. Later this result will contribute to the base case of our main induction (see Lemma 10.13).

Definition 3.15. Given a finite index set I let $f\colon I \longrightarrow \mathbb{N}$ be a function such that $\sum_{i\in I} f(i)$ is at least 2. Further, let A be a subset of I and let F be an f-partite hypergraph.

(*a*) We set $V_A(F) = \bigcup_{i\in A} V_i(F)$.

(*b*) If $e \cap e' \subseteq V_A(F)$ holds for any two distinct edges e, e' of F we say that F is *A-intersecting.*

An f-partite Ramsey construction or a partite lemma Φ is said to be A-intersecting if whenever $\Phi_r(F) = (H, \mathscr{H})$ and F is A-intersecting for some subset A of the relevant index set, then so is H. The next three lemmata show that HJ, CPL, and $\Omega^{(2)}$ have this property.

Lemma 3.16. *The Hales-Jewett construction* HJ *is A-intersecting.*

Proof. Let $\mathrm{HJ}_r(F) = (H, \mathscr{H})$ for some A-intersecting k-partite k-uniform hypergraph F and for some number of colours r. As usual, we denote the implied Hales-Jewett exponent by n and we let $\lambda\colon E(F)^n \longrightarrow E(H)$ be the canonical bijection.

Given any two distinct edges e, e' of H we are to prove $e \cap e' \subseteq V_A(H)$. To this end we write $e = \lambda(e_1, \dots, e_n)$ and $e' = \lambda(e'_1, \dots, e'_n)$ with appropriate edges $e_1, \dots, e_n$ and $e'_1, \dots, e'_n$ of F. Due to $e \ne e'$ there exists a coordinate direction $\nu \in [n]$ such that $e_\nu \ne e'_\nu$. Since F is A-intersecting, we have $e_\nu \cap e'_\nu \subseteq V_A(F)$, and $e \cap e' \subseteq V_A(H)$ follows. □

Lemma 3.17. *The clean partite lemma* CPL *is A-intersecting.*

Proof. Let an A-intersecting k-partite k-uniform hypergraph F and a number of colours r be given. Due to $\mathrm{CPL} = \mathrm{PC}(\mathrm{HJ}, \mathrm{HJ})$ every picture encountered in the construction of $\mathrm{CPL}_r(F)$ possesses a k-partite structure and we shall prove inductively that all these pictures are A-intersecting. As this is clear for picture zero and the vertical system $\mathrm{HJ}_r(F)$ is A-intersecting, it thus suffices to prove the following statement.

Let G be an A-intersecting k-partite k-uniform hypergraph and suppose that

$$(\Sigma, \mathscr{Q}, \psi_\Sigma) = (\Pi, \mathscr{P}, \psi_\Pi) * (H, \mathscr{H})$$

holds for two pictures $(\Sigma, \mathscr{Q}, \psi_\Sigma)$, $(\Pi, \mathscr{P}, \psi_\Pi)$ over some system $(G, \mathscr{G})$, and for a k-partite k-uniform system $(H, \mathscr{H})$. If Π and H are A-intersecting, then so is Σ.

To verify this we consider any two distinct edges e', e'' of Σ. If their projections to G are distinct, then the assumption that G be A-intersecting yields $\psi_\Sigma(e') \cap \psi_\Sigma(e'') \subseteq V_A(G)$ and the desired inclusion $e' \cap e'' \subseteq V_A(\Sigma)$ follows.

So from now on we may assume $e_\star = \psi_\Sigma(e') = \psi_\Sigma(e'')$ for some edge $e_\star \in E(G)$. If $e_\star$ coincides with the edge $e \in E(G)$ over which the amalgamation happens, then $e' \cap e'' \subseteq V_A(H) \subseteq V_A(\Sigma)$ is a consequence of H being A-intersecting.

Thus we can assume $e \neq e_\star$ in the sequel, which implies $e \cap e_\star \subseteq V_A(G)$. If e' and e'' are in the same standard copy of Π, we just need to appeal to Π being A-intersecting and if those standard copies are distinct, then $e' \cap e''$ is contained in $V(H)$ and

$$\psi_\Sigma[e' \cap e''] \subseteq e \cap e_\star \subseteq V_A(G)$$

leads again to $e' \cap e'' \subseteq V_A(\Sigma)$. □

Lemma 3.18. *The construction $\Omega^{(2)}$ is A-intersecting.*

Proof. Again we argue by induction along the partite construction. There is no problem with picture zero. As $\Omega^{(2)} = \mathrm{PC}(\mathrm{Rms}, \mathrm{CPL})$ utilises the clean partite lemma, Corollary 3.5 and Lemma 3.17 show that it suffices to prove the following picturesque statement.

Suppose that

$$(\Sigma, \mathscr{Q}, \psi_\Sigma) = (\Pi, \mathscr{P}, \psi_\Pi) \ast (H, \mathscr{H})$$

holds for two pictures $(\Sigma, \mathscr{Q}, \psi_\Sigma)$, $(\Pi, \mathscr{P}, \psi_\Pi)$ over an f-partite system $(G, \mathscr{G})$ and for a k-partite k-uniform system $(H, \mathscr{H})$ with strongly induced copies whose intersections are clean. If Π and H are A-intersecting, then so is Σ.

Let e' and e'' be two distinct edges of Σ. Each of them is either in $E(H)$ or it is not, and by symmetry there are three possibilities to consider. If both edges belong to H, then $e' \cap e'' \subseteq V_A(H) \subseteq V_A(\Sigma)$ is clear.

Suppose next that $e' \notin E(H)$ and $e'' \in E(H)$. Let $\Pi_\star$ be the standard copy of Π containing e' and let $\Pi^e_\star \in \mathscr{H}$ be the copy extended by $\Pi_\star$. Due to $\Pi^e_\star \blacktriangleleft H$ there is an edge $e_\star \in E(\Pi^e_\star)$ such that $V(\Pi^e_\star) \cap e'' \subseteq e_\star$. In the special case $e'' = e_\star$ both edges e' and e'' belong to the same standard copy $\Pi_\star$ and $e' \cap e'' \subseteq V_A(\Pi_\star) \subseteq V_A(\Sigma)$ follows from the assumption that Π be A-intersecting. Moreover, if $e'' \neq e_\star$, then we have

$$e' \cap e'' \subseteq V(\Pi^e_\star) \cap e'' \subseteq e_\star \cap e'' \subseteq V_A(H) \subseteq V_A(\Sigma)\,.$$

It remains to deal with the case $e', e'' \notin E(H)$. Let $\Pi_\star$, $\Pi_{\star\star}$ be the standard copies of Π containing these two edges and let $\Pi^e_\star$, $\Pi^e_{\star\star}$ be the corresponding copies in $\mathscr{H}$. If they coincide we just need to appeal to the fact that Π is A-intersecting, so we can henceforth assume $\Pi^e_\star \neq \Pi^e_{\star\star}$. Now the intersection of these two copies is clean and thus there exists an edge $e_\star \in E(\Pi^e_\star)$ covering their intersection. The discussion of the previous paragraph shows $e_\star \cap e'' \subseteq V_A(\Sigma)$, whence $e' \cap e'' \subseteq V(\Pi^e_\star) \cap V(\Pi^e_{\star\star}) \cap e'' \subseteq e_\star \cap e'' \subseteq V_A(\Sigma)$. □

§4. Girth considerations

This section begins by enumerating, for the sake of completeness, some folkloric statements related to the classical girth concept introduced in Definition 1.3. From §4.2 onwards, however, we move on to new territory and study a concept of Girth applicable to linear systems of hypergraphs (see Definition 4.9 below). The notational difference between the two notions is that in the former case “girth” is written with a lower case “g”, whereas the capital “G” in “Girth” indicates that we consider the Girth of a linear system.

4.1. **Set systems and girth.** By a *set system* we mean a pair $S = (V, E)$ consisting of a set of *vertices* V and a collection $E \subseteq \wp(V)$ of subsets of V such that every *edge* $e \in E$ has at least two elements. Thus a hypergraph is a set system with the special property that its edges are of the same cardinality.

Definition 1.3 applies to set systems in place of hypergraphs as well and for reasons that will become apparent later we formulate the two facts that follow in this more general context. First, we study the effect of dropping condition $(C1)$.

Fact 4.1. *Let $g \geqslant 2$ be an integer and let $S = (V, E)$ be a set system with* $\mathrm{girth}(S) > g$. *If for some integer $n \in [2, g+1]$ we have a sequence*

$$e_1 v_1 \ldots e_n v_n$$

satisfying $(C2)$, $(C3)$, and $e_1, \ldots, e_n \in E$, then

(*a*) *either $e_1 = \cdots = e_n$*
(*b*) *or $n = g + 1$ and the edges $e_1, \ldots, e_n$ are distinct.*

Proof. Otherwise let $\mathscr{C} = e_1 v_1 \ldots e_n v_n$ be a counterexample with n minimum. We contend that there is an edge occurring at least twice in $\mathscr{C}$. If $n = g + 1$ this is immediate from the failure of (*b*). If $n \in [2, g]$, then $\mathrm{girth}(S) > g$ tells us that $\mathscr{C}$ cannot be an n-cycle, which in turn means that $(C1)$ fails, i.e., that $\mathscr{C}$ again contains two equal edges. Now by cyclic symmetry we may suppose that there exists an index $i \in [2, n]$ with $e_1 = e_i$.

Our plan is to prove

$$e_1 = \cdots = e_{i-1} \quad \text{and} \quad e_i = \cdots = e_n \,. \tag{4.1}$$

Together with our choice of i this will show that alternative (*a*) holds, thus concluding the proof.

For reasons of symmetry it suffices to establish the first part of (4.1). Our claim is obvious for $i = 2$ and in case $i \geqslant 3$ we can apply the minimality of n to the sequence $\mathscr{D} = e_1 v_1 \ldots e_{i-1} v_{i-1}$. As $\mathscr{D}$ contains $i - 1 \leqslant n - 1 \leqslant g$ edges, only option (*a*) can apply to $\mathscr{D}$, which has the desired consequence. □

Second, we immediately obtain the following well known “transitivity property” of girth that will assist us later when analysing the girth of trains (see Lemma 10.9 below).

Fact 4.2. *Let an integer $g \geqslant 2$ and a set system $S = (V, E)$ with* $\mathrm{girth}(S) > g$ *be given. If for every edge $e \in E$ we have a set system F_e with vertex set e and* $\mathrm{girth}(F_e) > g$*, then the set system $T = (V, E')$ defined by $E' = \bigcup_{e \in E} E(F_e)$ satisfies* $\mathrm{girth}(T) > g$ *as well.*

Moreover, if $\mathrm{girth}(F_e) > g + 1$ *holds for every $e \in E$ and $\mathscr{C} = f_1 v_1 \dots f_{g+1} v_{g+1}$ is a $(g+1)$-cycle in T, then $\mathscr{C}$ contains at most one edge from every system F_e.*

Proof. Assume first that contrary to $\mathrm{girth}(T) > g$ we have for some $n \in [2, g]$ an n-cycle

$$\mathscr{C} = f_1 v_1 \dots f_n v_n$$

in T. Owing to the linearity of S, there are uniquely determined edges $e(1), \dots, e(n) \in E$ such that $f_i \in E(F_{e(i)})$ holds for every $i \in \mathbb{Z}/n\mathbb{Z}$. Since $f_i \subseteq e(i)$, the cyclic sequence

$$\mathscr{D} = e(1) v_1 \dots e(n) v_n$$

has the properties $(C2)$, $(C3)$ of an n-cycle in S. Due to $\mathrm{girth}(S) > g \geqslant n$ Fact 4.1 applied to S and $\mathscr{D}$ informs us that for some $e \in E$ we have $e = e(1) = \dots = e(n)$. In other words, the sequence $\mathscr{C}$ is an n-cycle in F_e, contrary to $\mathrm{girth}(F_e) > g$. We have thereby proved that $\mathrm{girth}(T) > g$.

Now suppose moreover that $\mathrm{girth}(F_e) > g + 1$ holds for every $e \in E$ and that

$$\mathscr{C} = f_1 v_1 \dots f_{g+1} v_{g+1}$$

is a $(g+1)$-cycle in T. Choosing the edges $e(1), \dots, e(g+1) \in E$ as before we arrive again at a cyclic sequence $\mathscr{D} = e(1) v_1 \dots e(g+1) v_{g+1}$ and Fact 4.1 is still applicable. Its option *(a)* would lead to the same contradiction as before, so *(b)* holds and we are done. □

4.2. **Cycles of copies.** Suppose that $(H, \mathscr{H})$ is a linear system. Intuitively, a cycle in $\mathscr{H}$ is just a cyclic arrangement of copies any two consecutive ones of which are distinct but overlap. Here, "overlapping" means having at least a vertex and possibly even an edge in common. We are thus led to the following concept.

Definition 4.3. A *cycle of copies* in a linear system $(H, \mathscr{H})$ is a cyclic sequence

$$\mathscr{C} = F_1 q_1 F_2 q_2 \dots F_n q_n \tag{4.2}$$

such that $n \geqslant 2$ and the following conditions hold.

$(L1)$ The copies $F_1, \dots, F_n \in \mathscr{H}$ satisfy $F_i \neq F_{i+1}$ for all $i \in \mathbb{Z}/n\mathbb{Z}$.
$(L2)$ The vertices and edges $q_1, \dots, q_n \in V(H) \cup E(H)$ are distinct.
$(L3)$ If $i \in \mathbb{Z}/n\mathbb{Z}$ and q_i is a vertex, then $q_i \in V(F_i) \cap V(F_{i+1})$.
$(L4)$ If $i \in \mathbb{Z}/n\mathbb{Z}$ and q_i is an edge, then $q_i \in E(F_i) \cap E(F_{i+1})$.

We say that $q_1, \dots, q_n$ are the *connectors of* $\mathscr{C}$, while $F_1, \dots, F_n$ will be known as its *copies*.

Suppose now that $\mathscr{C} = F_1q_1F_2q_2 \dots F_nq_n$ is a cycle of copies. The number n will be called the *length* of $\mathscr{C}$ and denoted by $n = |\mathscr{C}|$. In most of our arguments, however, the length of a cycle of copies will only play a secondary rôle and a more central notion is that of its order, which we shall introduce next. To this end, we call an index $i \in \mathbb{Z}/n\mathbb{Z}$

- *pure* if either both of q_{i-1} and q_i are vertices or both are edges and
- *mixed* if one of q_{i-1} and q_i is a vertex while the other one is an edge.

Recall that by (*L2*) every $i \in \mathbb{Z}/n\mathbb{Z}$ is either pure or mixed. For parity reasons the number of mixed indices has to be even and, therefore, the quantity

$$\mathrm{ord}(\mathscr{C}) = \big|\{i \in \mathbb{Z}/n\mathbb{Z}\colon i \text{ is pure}\}\big| + \tfrac{1}{2}\big|\{i \in \mathbb{Z}/n\mathbb{Z}\colon i \text{ is mixed}\}\big|\,, \tag{4.3}$$

called the *order* of $\mathscr{C}$, has to be an integer. Evidently, the length of a cycle of copies can deviate from its order at most by a factor of 2, i.e., we have $|\mathscr{C}| \in [\mathrm{ord}(\mathscr{C}), 2\mathrm{ord}(\mathscr{C})]$. Occasionally we shall need to take both the order and the length into account and in such situations it is convenient to set

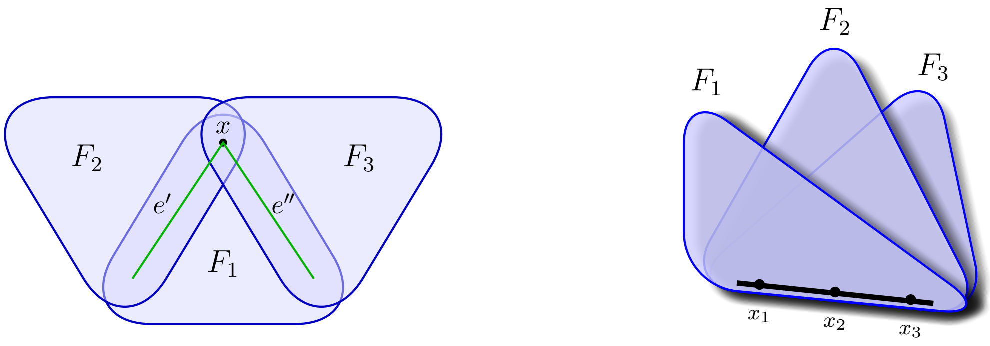
$$h(\mathscr{C}) = \big(\mathrm{ord}(\mathscr{C}), |\mathscr{C}|\big) \in \mathbb{N}^2\,. \tag{4.4}$$

When relating two ordered pairs of natural numbers by an inequality we always have the lexicographic ordering of $\mathbb{N}^2$ in mind. This convention puts greater emphasis on the order than on the length, for in (4.4) the order comes first. E.g., $h(\mathscr{C}) \leqslant (g, n)$ means that either $\mathrm{ord}(\mathscr{C}) < g$ or $\mathrm{ord}(\mathscr{C}) = g \,\&\, |\mathscr{C}| \leqslant n$.

One needs to be careful when defining the Girth of a linear system $(H, \mathscr{H})$ in terms of cycles of copies as introduced above. The reason for this is that one can take copies that are arranged like a tree and present them as a cycle of copies. It may be instructive to illustrate this point by means of two examples.

Example 4.4. Suppose that e' and e'' are two edges of a copy F_1 that have a vertex x in common (see Figure 4.1a). Let F_2 be a further copy having with F_1 only the edge e' in common and, similarly, let F_3 be a copy meeting F_1 only in e''. It is now forced that the copies F_2 and F_3 overlap in x and for transparency we assume that they are otherwise disjoint. This situation gives rise to the cycle of copies $\mathscr{A} = F_1e'F_2xF_3e''$ with $h(\mathscr{A}) = (2, 3)$. It should be clear, however, that such cycles are unavoidable in the Ramsey systems we seek to construct and, therefore, that they should have no bearing on the Girth of our systems.

Example 4.5. Let F_1, F_2, and F_3 be three copies which have an edge e in common but are otherwise disjoint (see Figure 4.1b). If x_1, x_2, and x_3 are any three distinct vertices of e, then $\mathscr{B} = F_1x_1F_2x_2F_3x_3$ is a valid example for a cycle of copies with $h(\mathscr{B}) = (3, 3)$ that is likewise unavoidable.



(a) $\mathscr{A} = F_1 e' F_2 x F_3 e''$ (b) $\mathscr{B} = F_1 x_1 F_2 x_2 F_3 x_3$

FIGURE 4.1. Two unavoidable cycles of copies

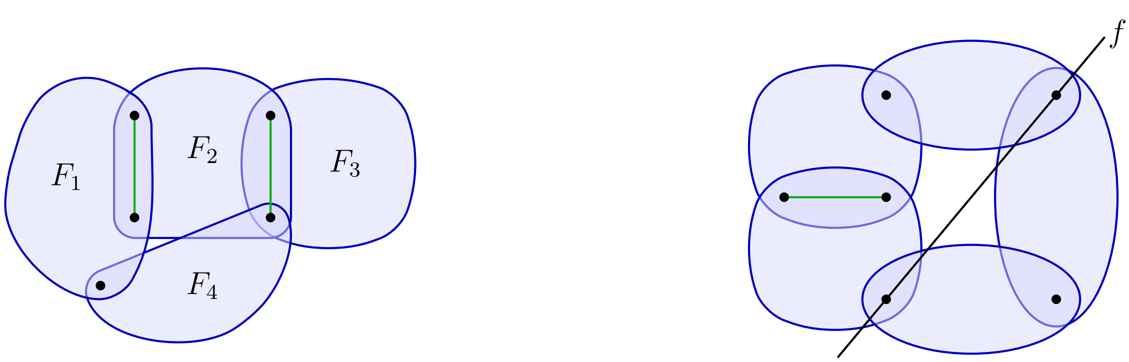
These circumstances suggest to declare cycles such as $\mathscr{A}$ and $\mathscr{B}$ to be "untidy" and to resolve that only tidy cycles of copies are allowed to affect the Girth of a system. Intuitively speaking, the reason why the above cycle $\mathscr{A}$ should not be tidy is that its connectors x and e' satisfy $x \in e'$. The untidiness of $\mathscr{B}$, on the other hand, derives from the existence of an edge e containing too many vertex connectors. The next definition renders these ideas in a more precise form (see Figure 4.2).

Definition 4.6. A cycle of copies $\mathscr{C} = F_1 q_1 \ldots F_n q_n$ in a linear system $(H, \mathscr{H})$ is said to be *tidy* if it has the following two properties.

($T1$) There do not exist connectors q_i and q_j with $q_i \in q_j$.
($T2$) For every edge $f \in E(H)$ the set

$$M(f) = \{i \in \mathbb{Z}/n\mathbb{Z} \colon q_i \text{ is a vertex belonging to } f\}$$

can be covered by a set of the from $\{i(\star), i(\star) + 1\}$, where $i(\star) \in \mathbb{Z}/n\mathbb{Z}$.



(a) An untidy cycle violating ($T1$) (b) Same for ($T2$)

FIGURE 4.2

Now it might be tempting to define a linear system to have large Girth if it contains no tidy cycles of copies of low order. However, there is one more phenomenon we did not take into account yet.

Example 4.7. Return to the cycle of copies $\mathscr{A}$ considered in Example 4.4. Take any vertices $y' \in e'$ and $y'' \in e''$ distinct from x. Now $\mathscr{A}_\star = F_1 y' F_2 x F_3 y''$ is a tidy cycle of copies. Indeed, $(T1)$ holds due to the absence of edge connectors while the failure of $(T2)$ would require the existence of an edge $f \in E(H)$ containing all three of x, y', and y'', which is absurd.

In such a situation we shall say that F_1 is a master copy of $\mathscr{A}_\star$. The intuitive reason for this is that everything of relevance happens within this copy. Treating the edges e' and e'' for the moment as if they were copies, we can form the cycle $F_1 y' e' x e'' y''$, which lives completely in its master copy F_1. Extending e' and e'' to the real copies F_2 and F_3 only conceals this situation.

Before making this precise in Definition 4.8 below we introduce a notational device for letting edges play the rôles of copies. With every edge $e \in E(H)$ we associate the subhypergraph $e^+ = (e, \{e\})$ of H. It will be convenient to write $E^+(H) = \{e^+ \colon e \in E(H)\}$. Now $\big(H, E^+(H)\big)$ is already a legitimate linear system and it will be sensible to investigate its Girth (see Lemma 4.12 below).

More interestingly, however, with every linear system $(H, \mathscr{H})$ we associate the system $(H, \mathscr{H}^+)$ defined by $\mathscr{H}^+ = \mathscr{H} \cup E^+(H)$. Whenever such systems occur, we call the members of $E^+(H)$ *edge copies*, while the other members of $\mathscr{H}^+$ will be referred to as *real copies*. When we want to direct attention to the fact that the linear systems we deal with are of the form $(H, \mathscr{H}^+)$, we call them *extended linear systems*. Let us now return to the question what master copies actually are.

Definition 4.8. Given a cycle of copies $\mathscr{C} = F_1 q_1 \dots F_n q_n$ in an extended linear system $(H, \mathscr{H}^+)$ as well as a copy $F_\star \in \{F_1, \dots, F_n\}$ we call $F_\star$ a *master copy* of $\mathscr{C}$ (see Figure 4.3), if there is a family

$$\big\{f_i \in E(F_\star) \colon i \in \mathbb{Z}/n\mathbb{Z} \text{ and } F_i \neq F_\star\big\}$$

of edges such that the cyclic sequence $\mathscr{D}$ obtained from $\mathscr{C}$ upon replacing every copy $F_i \neq F_\star$ by the edge copy f_i^+ is again a cycle of copies. When passing from $\mathscr{C}$ to $\mathscr{D}$ we say that the copies F_i distinct to $F_\star$ get *collapsed* to the edge copies f_i^+.

Notice that a master copy $F_\star$ is allowed to appear multiple times on the cycle $\mathscr{C}$. If there are two or more occurrences of $F_\star$ in $\mathscr{C}$, then the collapsing does not change this situation. Now we finally reach our Girth concept applying to extended linear systems.

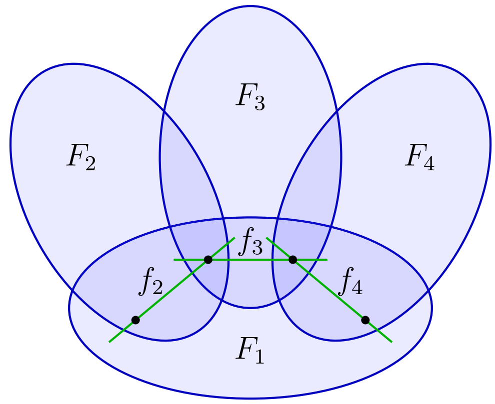


FIGURE 4.3. Master copy F_1

Definition 4.9. If $(H, \mathscr{H}^+)$ is an extended linear system and $g, n \in \mathbb{N}$, then

(a) $\mathrm{Girth}(H, \mathscr{H}^+) > (g, n)$ means that every tidy cycle of copies $\mathscr{C}$ in $\mathscr{H}^+$ satisfying $h(\mathscr{C}) \leqslant (g, n)$ has a master copy

(b) and $\mathrm{Girth}(H, \mathscr{H}^+) > g$ means that every tidy cycle of copies $\mathscr{C}$ in $\mathscr{H}^+$ with $\mathrm{ord}(\mathscr{C}) \leqslant g$ possesses a master copy.

Since the length of a cycle of copies is at most twice its order, $\mathrm{Girth}(H, \mathscr{H}^+) > g$ is equivalent to $\mathrm{Girth}(H, \mathscr{H}^+) > (g, 2g)$. We proceed with two simple facts that follow immediately from our definitions.

Fact 4.10. *Master copies are always real copies.*

Proof. If $F_\star$ is a master copy of the cycle $\mathscr{C} = F_1 q_1 \ldots F_n q_n$, then there needs to exist an index $i \in \mathbb{Z}/n\mathbb{Z}$ with $F_i = F_\star$. Due to (L1) we have $F_{i+1} \neq F_\star$ and, therefore, F_{i+1} is collapsible to an edge copy f_{i+1}^+ with $f_{i+1} \in E(F_\star)$. Now if $F_\star$ is an edge copy, then its only edge is f_{i+1} and the collapsed cycle violates (L1). □

Fact 4.11. *If $(H, \mathscr{H}^+)$ is an extended linear system with* $\mathrm{Girth}(H, \mathscr{H}^+) > (2, 2)$ *and $\mathscr{C} = F_1 q_1 \ldots F_n q_n$ denotes a cycle of copies in $(H, \mathscr{H}^+)$ having a master copy $F_\star$, then the edges f_i exemplifying this state this affair satisfy $f_i \in E(F_i) \cap E(F_\star)$.*

Proof. Since Definition 4.8 demands $f_i \in E(F_\star)$, we only have to check $f_i \in E(F_i)$, which is clear in the special case $F_i = f_i^+$. So we may assume $F_i \neq f_i^+$, whence $\mathscr{D} = F_i q_i f_i^+ q_{i-1}$ is a cycle of copies. If one of the connectors q_{i-1}, q_i is an edge, then it has to be equal to f_i and $f_i \in E(F_i)$ follows. If both of these connectors are vertices, then $\mathscr{D}$ is tidy, $h(\mathscr{D}) = (2, 2)$, and by the previous fact F_i is a master copy of $\mathscr{D}$. Moreover, the edge copy f_i^+ can only be collapsed to itself. □

Next we check that for the system of edge copies Girth is essentially the same as ordinary girth.

Lemma 4.12. *If H is a linear hypergraph and $g \geqslant 2$, then*

$$\operatorname{girth}(H) > g \iff \operatorname{Girth}\big(H, E^+(H)\big) > (g, g) \iff \operatorname{Girth}\big(H, E^+(H) \cup \{H\}\big) > g\,.$$

Proof. Clearly the last condition implies the middle one. Next we assume the middle condition and intend to derive the first. Suppose contrariwise that $e_1x_1 \dots e_nx_n$ is an n-cycle in H for some $n \in [2, g]$, chosen in such a way that n is as small as possible. Now $\mathscr{C} = e_1^+x_1 \dots e_n^+x_n$ is a cycle of copies in $E^+(H)$ and the minimality of n shows that $\mathscr{C}$ is tidy. Due to $h(\mathscr{C}) = (n, n) \leqslant (g, g)$ there has to exist a master copy of $\mathscr{C}$, contrary to Fact 4.10.

Thus it remains to prove that assuming $\operatorname{girth}(H) > g$ we can return to

$$\operatorname{Girth}\big(H, E^+(H) \cup \{H\}\big) > g\,.$$

Let $\mathscr{C}$ denote a tidy cycle of copies in $E^+(H) \cup \{H\}$ with $\operatorname{ord}(\mathscr{C}) \leqslant g$. If $\mathscr{C}$ involves the copy H, then H is a master copy of $\mathscr{C}$ and we are done. Now suppose that $\mathscr{C}$ is of the form $e_1^+q_1 \dots e_n^+q_n$. Notice that q_i cannot be an edge for any $i \in \mathbb{Z}/n\mathbb{Z}$, for then (*L*4) would imply $e_i = q_i = e_{i+1}$, contrary to (*L*1). Thus all connectors of $\mathscr{C}$ are vertices, all indices are pure and, consequently, we have $n = \operatorname{ord}(\mathscr{C}) \leqslant g$. But now Fact 4.1 applied to $e_1q_1 \dots e_nq_n$ yields $e_1 = \dots = e_n$, which is absurd. □

The lemma that follows relates Girth to concepts introduced earlier.

Lemma 4.13. *If a linear system $(H, \mathscr{H})$ has strongly induced copies with clean intersections, then* $\operatorname{Girth}(H, \mathscr{H}^+) > (2, 2)$.

Proof. Consider any cycle of two copies $\mathscr{C} = F_1q_1F_2q_2$ in $\mathscr{H}^+$. Due to (*L*2) and (*L*4) it cannot be the case that both connectors are edges.

Suppose next that exactly one of them, say q_1, is an edge, while q_2 is a vertex. If one of F_1 or F_2 is an edge copy, say $F_1 = f^+$, then $q_2 \in f = q_1$ shows that $\mathscr{C}$ fails to be tidy. If both F_1 and F_2 are real copies, then they have the edge q_1 in common and, as their intersection is clean, they have nothing else in common. Thus we have again $q_2 \in q_1$ and $\mathscr{C}$ is not tidy.

Finally, we consider the case that both q_1 and q_2 are vertices. By the linearity of H, at least one copy of $\mathscr{C}$, say F_1, is a real copy. We shall prove that F_1 is a master copy of $\mathscr{C}$. If $F_2 = f^+$ is an edge copy, then $F_1 \blacktriangleleft H$ yields $f \in E(F_1)$ and we just need to collapse F_2 to itself. If F_2 is a real copy, then the assumption that F_1 and F_2 intersect cleanly shows that these copies have an edge f in common. Now we can collapse F_2 to f^+. □

It may be observed that in the last case if both F_1 and F_2 are real copies, then both of them are master copies. Thus a cycle of copies of length 2 can have two master copies and part (*a*) of the following lemma elaborates on this fact. We also include a part (*b*) stating that longer cycles of copies can have at most one master copy. This observation is never going to be used in the sequel, but it may help to explain why cycles consisting of two copies will sometimes require a special treatment in later arguments.

Lemma 4.14. *Let $\mathscr{C}$ be a tidy cycle of copies in an extended linear system $(H, \mathscr{H}^+)$ with* $\mathrm{Girth}(H, \mathscr{H}^+) > (2,2)$.

(a) If $|\mathscr{C}| = 2$, then every real copy of $\mathscr{C}$ is a master copy.
(b) If $|\mathscr{C}| \geqslant 3$, then $\mathscr{C}$ has at most one master copy.

Proof. For the verification of (a) let $\mathscr{C} = F_1 q_1 F_2 q_2$. Due to $\mathrm{Girth}(H, \mathscr{H}^+) > (2,2) \geqslant h(\mathscr{C})$ we know that $\mathscr{C}$ has a master copy. Suppose that F_1 is such a master copy and that F_2 is collapsible to f^+. We need to prove that if F_2 is a real copy, then F_2 is a master copy of $\mathscr{C}$ as well. Since $F_2 \neq f^+$, this can be seen by looking at the cycle $f^+ q_1 F_2 q_2$.

Proceeding with (b) we write $\mathscr{C} = F_1 q_1 \dots F_n q_n$ and assume contrariwise that both F_1 and F_i are master copies of $\mathscr{C}$, where $i \in [2, n]$ satisfies $F_1 \neq F_i$. According to Fact 4.11 there are edges $f, f' \in E(F_1) \cap E(F_i)$ such that the master copy F_i allows to collapse F_1 to f^+ and, similarly, F_1 allows to collapse F_i to $(f')^+$. If $f' \neq f$ then $F_1 f F_i f'$ is a tidy cycle of copies without a master copy, contrary to $\mathrm{Girth}(H, \mathscr{H}^+) > (2,2)$.

Thus we have $f = f'$. Let us now look at the connectors q_1, q_n, q_{i-1}, and q_i. Due to $n = |\mathscr{C}| \geqslant 3$ their number is at least 3. Moreover, the collapsibility of F_1 and F_i to f^+ shows that all vertices among them are in f, while all edges among them are equal to f. So if $f \in \{q_1, q_n, q_{i-1}, q_i\}$ we get a contradiction to (T1), while $f \notin \{q_1, q_n, q_{i-1}, q_i\}$ causes the failure of (T2). This contradiction proves that $\mathscr{C}$ has indeed at most one master copy. □

We conclude this subsection with two lemmata that will help us later to analyse the Girth of systems arising by partite constructions.

Lemma 4.15. *Let $\mathscr{C}$ be a tidy cycle of copies in an extended linear system $(H, \mathscr{H}^+)$. If*

$$h(\mathscr{C}) < \mathrm{Girth}(H, \mathscr{H}^+)\,,$$

then all connectors of $\mathscr{C}$ are vertices.

Proof. Let $\mathscr{C} = F_1 q_1 \dots F_n q_n$ be such a tidy cycle of copies. Definition 4.9 tells us that it has a master copy $F_\star$. Let $i \in \mathbb{Z}/n\mathbb{Z}$ be an index for which we want to prove that q_i is a vertex. By (L1) at least one of its neighbouring copies F_i and F_{i+1} is distinct from $F_\star$ and by symmetry it suffices to treat the case $F_i \neq F_\star$. Let f_i be the edge F_i is collapsible to. Due to (L4) every edge among the connectors q_i, q_{i-1} is equal to f_i and by (L3) every vertex among them belongs to f_i. So by (L2) it cannot be the case that both connectors are edges and (T1) tells us that i cannot be mixed in $\mathscr{C}$. The only remaining case is that both of q_i and q_{i-1} are vertices. □

Lemma 4.16. *Let $\mathscr{C} = F_1q_1 \dots F_nq_n$ be a tidy cycle of copies of length $n \geqslant 3$ in an extended linear system $(H, \mathscr{H}^+)$. Suppose that for some set of indices $K \subseteq \mathbb{Z}/n\mathbb{Z}$ we are given a family of edge copies $\{f_k^+ : k \in K\}$.*

(*a*) *If the cyclic sequence $\mathscr{D}$ obtained from $\mathscr{C}$ upon replacing F_k by f_k^+ for every $k \in K$ has the properties (L2)–(L4), then $\mathscr{D}$ satisfies (L1) as well and, hence, it is a tidy cycle of copies.*

(*b*) *Moreover, in this case every master copy of $\mathscr{D}$, if there exists any, is a master copy of $\mathscr{C}$ as well.*

Proof. If part (*a*) fails, then $\mathscr{D}$ has an edge copy e^+ occurring twice in consecutive positions. Due to $n \geqslant 3$, there are three connectors next to these identical edge copies, say q_i, q_{i+1}, and q_{i+2}. As (*T*2) rules out the case $M(e) \supseteq \{i, i+1, i+2\}$, at least one of these connectors must be an edge and by (*L*4) this edge can only be e. By (*L*2) the edge e occurs exactly once among q_i, q_{i+1}, and q_{i+2} while the other two of these connectors are vertices. By (*L*3) these vertices are in e, contrary to $\mathscr{C}$ satisfying (*T*1). This concludes the proof of part (*a*).

Moving on to part (*b*) it is convenient to write $\mathscr{D} = \widetilde{F}_1q_1 \dots \widetilde{F}_nq_n$, where

$$\widetilde{F}_i = \begin{cases} F_i & \text{if } i \notin K \\ f_i^+ & \text{if } i \in K \end{cases}$$

for every $i \in \mathbb{Z}/n\mathbb{Z}$. Suppose now that $F_\star \in \{\widetilde{F}_1, \dots, \widetilde{F}_n\}$ is a master copy of $\mathscr{D}$ and that the family of edges $\mathscr{F} = \{e_i \in E(F_\star) : i \in \mathbb{Z}/n\mathbb{Z} \text{ and } \widetilde{F}_i \neq F_\star\}$ exemplifies this. Fact 4.10 shows that $F_\star$ is a real copy, whence $F_\star \in \{F_1, \dots, F_n\}$. Thus the underlying index set of $\mathscr{F}$ is a superset of K. By part (*a*) the subfamily $\{e_i \in E(F_\star) : i \in \mathbb{Z}/n\mathbb{Z} \text{ and } F_i \neq F_\star\}$ of $\mathscr{F}$ witnesses that $F_\star$ is indeed a master copy of $\mathscr{C}$. □

4.3. **Semitidiness.** Consider a cycle of copies $\mathscr{C} = F_1xF_2eF_3q_3 \dots F_nq_n$ in an extended linear system $(H, \mathscr{H}^+)$, where the connector x is a vertex, e is an edge, and $x \in e$. The last condition clearly violates (*T*1) and, therefore, it causes $\mathscr{C}$ to be untidy. So in case $\mathrm{ord}(\mathscr{C}) < \mathrm{Girth}(H, \mathscr{H}^+)$ Definition 4.9 does not tell us whether $\mathscr{C}$ has a master copy. Roughly speaking Lemma 4.18 below asserts that if $\mathscr{C}$ is "otherwise tidy", then the existence of such a master copy can nevertheless be proved. Any attempt to make this precise leads inevitably to the following concept.

Definition 4.17. Let $\mathscr{C} = F_1q_1 \dots F_nq_n$ be a cycle of copies in an extended linear system $(H, \mathscr{H}^+)$. Set

$$M(f) = \{i \in \mathbb{Z}/n\mathbb{Z} : q_i \text{ is a vertex belonging to } f\}$$

for every edge $f \in E(H)$. We say that $\mathscr{C}$ is *semitidy* if it has the following two properties.

(*S*1) If for some $i \in \mathbb{Z}/n\mathbb{Z}$ the connector q_i is an edge, then $M(q_i) \subseteq \{i-1, i+1\}$ and $|M(q_i)| \leqslant 1$.

($S2$) For every edge $f \in E(H)$ that fails to be a connector of $\mathscr{C}$ there exists an index $i(\star) \in \mathbb{Z}/n\mathbb{Z}$ with $M(f) \subseteq \{i(\star), i(\star)+1\}$.

Observe that every tidy cycle is, in particular, semitidy. Conversely, if $\mathscr{C}$ is semitidy and satisfies ($S1$) in the stronger form that $M(q_i) = \varnothing$ holds for every edge connector q_i, then $\mathscr{C}$ is tidy. Now the result we have been alluding to is the following.

Lemma 4.18. *Given an extended linear system $(H, \mathscr{H}^+)$ and $g \in \mathbb{N}$, we have*

$$\mathrm{Girth}(H, \mathscr{H}^+) > g$$

if and only if every semitidy cycle of copies $\mathscr{C}$ with $\mathrm{ord}(\mathscr{C}) \leqslant g$ possesses a master copy.

The proof of this lemma is essentially straightforward and, admittedly, more lengthy than illuminating. We would therefore like to say a few words on why we care to acquire this knowledge.

In Section 8 we shall study a generalisation of Girth applicable to systems of pretrains and it turns out that the characterisation of Girth provided by Lemma 4.18 fits better into this context. For instance, consider a system of pretrains with the special property that every wagon consists of a single edge. It is then desirable that its $\mathfrak{Girth}$ introduced later coincides with the Girth of the corresponding system of hypergraphs. It will turn out that owing to Lemma 4.18 this is indeed the case.

Now even if one has some trust that the characterisation of Girth provided here is really going to be relevant in the future, one may still ask why we bother with the proof of Lemma 4.18 rather than declaring it to be the official definition of Girth. This has the simple reason that the arguments in Section 5 dealing with Girth increments obtainable by means of the partite construction method become much more transparent if we only have to exhibit master copies of cycles that are actually tidy and not just semitidy.

The proof of Lemma 4.18 proceeds by induction on $h(\mathscr{C})$ and thus it requires a way to detect that another cycle $\mathscr{D}$ satisfies $h(\mathscr{D}) < h(\mathscr{C})$. For this purpose we shall always utilise the following observation.

Fact 4.19. *Let $n > i \geqslant 2$. If*

$$\mathscr{C} = F_1 q_1 \dots F_n q_n \quad \text{and} \quad \mathscr{D} = F_1 q_1 \dots F_i q_i$$

are cycles of copies, then $h(\mathscr{D}) < h(\mathscr{C})$.

Proof. Since $|\mathscr{D}| < |\mathscr{C}|$ is clear, it suffices to check that $\mathrm{ord}(\mathscr{D}) \leqslant \mathrm{ord}(\mathscr{C})$. Setting

$$\eta_j^{\mathscr{C}} = \begin{cases} 1 & \text{if } j \text{ is pure in } \mathscr{C}, \\ 1/2 & \text{if } j \text{ is mixed in } \mathscr{C} \end{cases}$$

we have

$$\operatorname{ord}(\mathscr{C}) = \eta_1^{\mathscr{C}} + \dots + \eta_n^{\mathscr{C}} \,. \tag{4.5}$$

Let the numbers $\eta_1^{\mathscr{D}}, \dots, \eta_i^{\mathscr{D}}$ be defined similarly with respect to $\mathscr{D}$. Since $\eta_j^{\mathscr{C}} = \eta_j^{\mathscr{D}}$ holds for every $j \in [2, i]$, we have

$$\begin{aligned}
\operatorname{ord}(\mathscr{D}) &= (\eta_2^{\mathscr{D}} + \dots + \eta_i^{\mathscr{D}}) + \eta_1^{\mathscr{D}} \\
&\leqslant (\eta_2^{\mathscr{C}} + \dots + \eta_i^{\mathscr{C}}) + 1 \\
&\leqslant (\eta_2^{\mathscr{C}} + \dots + \eta_i^{\mathscr{C}}) + (\eta_1^{\mathscr{C}} + \eta_n^{\mathscr{C}}) \\
&\leqslant \operatorname{ord}(\mathscr{C}) \,.
\end{aligned}$$

□

Proof of Lemma 4.18. As tidiness implies semitidiness, the backward implication is clear. For the forward implication we suppose that $\mathscr{C} = F_1 q_1 \dots F_n q_n$ is a semitidy cycle of copies in an extended linear system $(H, \mathscr{H}^+)$ with $\operatorname{ord}(\mathscr{C}) < \operatorname{Girth}(H, \mathscr{H}^+)$. We are to prove that $\mathscr{C}$ has a master copy. Arguing by induction on $h(\mathscr{C})$ we assume that every semitidy cycle of copies $\mathscr{C}'$ in $(H, \mathscr{H}^+)$ with $h(\mathscr{C}') < h(\mathscr{C})$ has a master copy. If $\mathscr{C}$ happens to be tidy we just need to appeal to Definition 4.9, so the interesting case is that, despite being semitidy, $\mathscr{C}$ fails to be tidy. Up to symmetry this is only possible if $q_1 = x$ is a vertex, $q_2 = e$ is an edge, and $x \in e$.

Suppose first that $n = 2$, i.e., that $\mathscr{C} = F_1 x F_2 e$. Now at least one of the copies F_1, F_2 needs to be distinct from e^+. If, say, $F_1 \neq e^+$, then the cycle $F_1 x e^+ e$ exemplifies that F_1 is a master copy of $\mathscr{C}$.

So henceforth we may suppose that $n \geqslant 3$, whence $g \geqslant 2$. Our goal is to prove that F_3 is a master copy of $\mathscr{C}$. To this end we first check that

$$F_3 \text{ is a real copy.} \tag{4.6}$$

Otherwise (*L*4) implies $F_3 = e^+$, so if q_3 is an edge we have $e = q_3$, contrary to (*L*2), and if q_3 is a vertex we have $|M(e)| \geqslant 2$, contrary to (*S*1). This proves (4.6).

First Case. $F_1 \neq F_3$

Now $\mathscr{D} = F_1 x F_3 q_3 \dots F_n q_n$ is a semitidy cycle of copies with $h(\mathscr{D}) < h(\mathscr{C})$, so by our induction hypothesis it has a master copy.

Assume first that F_3 fails to be such a master copy. Due to (4.6) and Lemma 4.14(*a*) we know that $\mathscr{D}$ consists of at least three copies, whence $n \geqslant 4$. Moreover, using the true master copy of $\mathscr{D}$ we can collapse F_3 to an appropriate edge copy f^+. Since we already have $x \in f$ and $\mathscr{C}$ is semitidy, it cannot be the case that q_3 is a vertex. But now only the case $q_3 = f$ remains and $x \in f$ contradicts (*S*1).

We have thereby proved that F_3 is a master copy of $\mathscr{D}$. This fact allows us to collapse, with the exception of F_2, all copies of $\mathscr{C}$ that are distinct from F_3. Provided we can finally collapse F_2 to e^+ this establishes that F_3 is indeed a master copy of $\mathscr{C}$. The only reason why this last collapse could be illegal is that it might cause the resulting "cycle" to violate $(L1)$. Owing to (4.6) this can only occur if we have just collapsed F_1 to e^+. In this case q_n cannot be a vertex, because this would imply $\{1, n\} \subseteq M(e)$, contrary to $(S1)$. But q_n cannot be an edge either, for then $(L4)$ implies $q_n = e$, contrary to $(L2)$. Altogether, F_3 is the desired master copy of $\mathscr{C}$.

Second Case. $F_1 = F_3$

Notice that this is only possible if $n \geqslant 4$. It will be convenient to set $F = F_1 = F_3$. Now $\mathscr{C}$ splits into the shorter cycles

$$\mathscr{A} = FxF_2e \quad \text{and} \quad \mathscr{B} = Fq_3 \dots F_nq_n\,,$$

and it suffices to prove that F is a common master copy of $\mathscr{A}$ and $\mathscr{B}$. By collapsing F_2 to e^+ we see that F is indeed a master copy of $\mathscr{A}$.

Assume towards contradiction that F fails to be a master copy of $\mathscr{B}$. Owing to (4.6) and Lemma 4.14(a) we have $n \geqslant 5$. Furthermore, Fact 4.19 yields $h(\mathscr{B}) < h(\mathscr{C})$ and, as one easily checks, $\mathscr{B}$ is semitidy. So our induction hypothesis shows that $\mathscr{B}$ has some master copy. In particular, F is collapsible to an appropriate edge copy f^+ in $\mathscr{B}$. Every edge connector among q_3 and q_n needs to be equal to f and every vertex connector among them needs to be incident with f. But in view of $n \geqslant 5$ this yields a contradiction to the semitidiness of $\mathscr{C}$. □

4.4. **Orientation.** The notion of Girth allows us to formulate a strengthening of Theorem 1.4 that will play a central rôle in our work.

> For every integer $g \geqslant 2$ there exists a Ramsey construction $\Omega^{(g)}$ that given an ordered f-partite hypergrsph F with $\mathrm{girth}(F) > g$ and a number of colours r produces an ordered f-partite system $\Omega^{(g)}_r(F) = (H, \mathscr{H})$ satisfying $\mathscr{H} \longrightarrow (F)_r$ and $\mathrm{Girth}(H, \mathscr{H}^+) > g$.

For clarity we would like to point out that due to Lemma 4.12 we have $\mathrm{girth}(H) > g$ in this situation, which shows that the existence of $\Omega^{(g)}$ is indeed a much stronger result than Theorem 1.4. In case one is willing to believe this statement without proof, one can now jump directly to Section 13 and read why it implies Theorem 1.7 and Theorem 1.6—so all prerequisites for this deduction have been covered already.

The construction $\Omega^{(2)} = \mathrm{PC}(\mathrm{Rms}, \mathrm{CPL})$ has been presented in the previous section, but we do not know yet that the systems $(H, \mathscr{H})$ it delivers satisfy $\mathrm{Girth}(H, \mathscr{H}^+) > 2$. We shall reach this knowledge in the next section when studying the Girth of pictures that arise in partite constructions (see Corollary 5.6).

In some sense, the existence of $\Omega^{(g)}$ is the statement we shall actually prove by induction on g, but the passage from $\Omega^{(g)}$ to $\Omega^{(g+1)}$ involves some further nested inductions whose intermediate stages deal with trains (see Figure 2.3). For a lack of an appropriate language we need to defer a more detailed description of our intended induction scheme to §10.3.

§5. Girth in partite constructions

As we saw in Section 3, the partite construction method allows us to pass from strongly induced copies to copies with clean intersections (Lemma 3.4) and clean intersections themselves are preserved under further applications of this method (Lemma 3.6). Both facts can, roughly speaking, be regarded as dealing with cycles consisting of two copies. The aim of the present section is twofold. First, we want to improve the latter result by addressing cycles of order two, rather than cycles of length two. Second, we develop generalistions to arbitrary Girth.

5.1. **From (g, g) to g.** Let us return to the question raised in §3.4 whether there is anything to gain if one attempts to clean the clean partite lemma CPL further. In the light of Lemma 4.13 we know that if F denotes a linear k-partite k-uniform hypergraph, r signifies a number of colours, and $\mathrm{CPL}_r(F) = (H, \mathscr{H})$, then $\mathrm{Girth}(H, \mathscr{H}^+) > (2, 2)$. It turns out that if we run any partite construction using CPL as our partite lemma, then the resulting final picture $(\Pi, \mathscr{P}, \psi)$ has the stronger property $\mathrm{Girth}(\Pi, \mathscr{P}^+) > 2$. In fact we shall show the following more general result, whose special case $g = 2$ corresponds to the aforementioned fact.

Proposition 5.1. *Suppose for any $g \geqslant 2$ that Ξ is a partite lemma*

- *applicable to k-partite k-uniform hypergraphs B with $\mathrm{girth}(B) > g$*
- *and delivering linear systems $\Xi_r(B) = (H, \mathscr{H})$ with $\mathrm{Girth}(H, \mathscr{H}^+) > (g, g)$.*

If Φ denotes any further Ramsey construction, then the construction $\Theta = \mathrm{PC}(\Phi, \Xi)$ applies to all hypergraphs F with $\mathrm{girth}(F) > g$ and yields linear systems $\Theta_r(F) = (\Pi, \mathscr{P})$ satisfying

$$\mathrm{Girth}(\Pi, \mathscr{P}^+) > g\,.$$

The heart of the matter is, of course, the following picturesque statement that will allow us to carry an inductive proof along the partite construction.

Lemma 5.2. *Suppose that $g \geqslant 2$, that $(G, \mathscr{G})$ is an arbitrary system of k-uniform hypergraphs, and that*

$$(\Sigma, \mathscr{Q}, \psi_\Sigma) = (\Pi, \mathscr{P}, \psi_\Pi) * (H, \mathscr{H})$$

holds for two pictures $(\Pi, \mathscr{P}, \psi_\Pi)$ and $(\Sigma, \mathscr{Q}, \psi_\Sigma)$ over $(G, \mathscr{G})$ as well as a linear k-partite system $(H, \mathscr{H})$. If

$$\mathrm{Girth}(\Pi, \mathscr{P}^+) > g \quad \text{and} \quad \mathrm{Girth}(H, \mathscr{H}^+) > (g, g)\,,$$

then $\mathrm{Girth}(\Sigma, \mathscr{Q}^+) > g$.

Proof. Consider a tidy cycle of copies $\mathscr{C} = F_1q_1 \dots F_nq_n$ in $(\Sigma, \mathscr{Q}^+)$ with $\mathrm{ord}(\mathscr{C}) \leqslant g$. We are to prove that $\mathscr{C}$ has a master copy.

By a *segment* we shall mean a subsequence $I = F_iq_i \dots F_j$ of $\mathscr{C}$ starting and ending with a copy and for which either

- (i) there exists a standard copy $(\Pi_\star, \mathscr{P}_\star^+)$ with $F_i, \dots, F_j \in \mathscr{P}_\star^+$,
- (ii) or $i = j$ and F_i is an edge copy corresponding to an edge of H.

In the former case we call the copy $\Pi_\star^e \in \mathscr{H}$ extended by the standard copy $\Pi_\star$ the *leader* of I and in the latter case F_i itself is considered to be its own *leader*. Observe that the leader of a segment is always a member of $\mathscr{H}^+$. A *segmentation* of $\mathscr{C}$ is a sequence of the form

$$\mathscr{C} = I_1r_1 \dots I_tr_t\,, \tag{5.1}$$

where $I_1, \dots, I_t$ are segments and $\{r_1, \dots, r_t\} \subseteq \{q_1, \dots, q_n\}$. Such segmentations exist, for each of the copies $F_1, \dots, F_n$ forms a segment on its own. From now on we let the segmentation (5.1) be chosen in such a way that t is minimal.

Denote the leaders of $I_1, \dots, I_t$ by $\Pi_1^e, \dots, \Pi_t^e$, respectively. If $\Pi_i^e = \Pi_{i+1}^e$ holds for some $i \in [t-1]$, then $I_ir_iI_{i+1}$ is again a segment, contrary to the minimality of t. Similarly, we may assume by cyclic symmetry that in case $t \geqslant 2$ we have $\Pi_1^e \ne \Pi_t^e$.

If $t = 1$, then the segment I_1 contains at least two copies from $\mathscr{C}$, wherefore $\Pi_1^e \in \mathscr{H}$. The standard copy Π_1 extending Π_1^e hosts the entire cycle $\mathscr{C}$ and $\mathrm{Girth}\big(\Pi_1, \mathscr{P}_1^+\big) > g$ yields the desired master copy.

Now suppose $t \geqslant 2$ and, consequently, $\Pi_i^e \ne \Pi_{i+1}^e$ for all $i \in \mathbb{Z}/t\mathbb{Z}$. It follows that the vertices and edges $r_1, \dots, r_t$ belong to H, whence

$$\mathscr{D} = \Pi_1^er_1 \dots \Pi_t^er_t$$

is a cycle of copies in $(H, \mathscr{H}^+)$.

Claim 5.3. *The length of $\mathscr{D}$ is at most g.*

Proof. Define the numbers $\eta_1, \dots, \eta_n$ by

$$\eta_i = \begin{cases} 1 & \text{if } i \text{ is pure in } \mathscr{C}, \\ 1/2 & \text{if } i \text{ is mixed in } \mathscr{C}. \end{cases}$$

For each $\tau \in \mathbb{Z}/t\mathbb{Z}$ let ϑ_τ be the sum of all η_k for which F_k belongs to the segment I_τ. Owing to

$$\vartheta_1 + \dots + \vartheta_t = \eta_1 + \dots + \eta_n = \mathrm{ord}(\mathscr{C}) \leqslant g$$

it suffices to show $\vartheta_1, \dots, \vartheta_t \geqslant 1$. Fix an arbitrary index $\tau \in \mathbb{Z}/t\mathbb{Z}$. If the segment I_τ contains at least two copies from $\mathscr{C}$, then $\vartheta_\tau \geqslant 2/2 = 1$ is clear. It remains to deal with the case that the segment I_τ consists of a single copy, say F_k. Observe that $q_{k-1} = r_{\tau-1}$, $q_k = r_\tau$, and $\vartheta_\tau = \eta_k$. There is no problem if k is pure in $\mathscr{C}$, so assume towards a contradiction that k is mixed. In other words, one of q_{k-1} and q_k is a vertex, the other one is an edge, and both belong to F_k. Since the copies in $\mathscr{Q}^+$ cross the music lines of Σ at most once, all this can only happen if the edge among q_{k-1}, q_k contains the vertex. But this option is ruled out by $(T1)$. $\square$

We infer $\mathrm{ord}(\mathscr{D}) \leqslant g$, and $h(\mathscr{D}) \leqslant (g, g) < \mathrm{Girth}(H, \mathscr{H}^+)$. Moreover, the tidiness of $\mathscr{C}$ implies that $\mathscr{D}$ is tidy as well. Therefore

$$\mathscr{D} \text{ has a master copy,} \tag{5.2}$$

Lemma 4.15 discloses that the connectors $r_1, \ldots, r_t$ are vertices, and by Fact 4.10 the copies in $\mathscr{H}$ need to be real.

Claim 5.4. *We may assume that $n \geqslant t \geqslant 3$.*

Proof. Let us first look at the case that $n = 2$ and, consequently, $t = 2$. By symmetry and (5.2) we may suppose that Π_1^e is a master copy of $\mathscr{D}$, i.e., that there is an edge $f \in E(\Pi_1^e)$ such that $\mathscr{E} = \Pi_1^e r_1 f^+ r_2$ is a cycle of copies in $(H, \mathscr{H}^+)$. Now we have $\mathscr{C} = F_1 q_1 F_2 q_2$ and, if F_2 belongs to the segment with leader Π_2^e, then $F_1 q_1 f^+ q_2$ is a cycle of copies witnessing that F_1 is a master copy of $\mathscr{C}$.

So henceforth we may assume that $n \geqslant 3$. If $t = 2$, then $\mathscr{D}$ has a copy leading a segment containing at least two copies of $\mathscr{C}$. Suppose that Π_1^e is such a copy and that Π_1 denotes the standard copy extending it. Now Π_1^e needs to be a real copy of $\mathscr{D}$ and Lemma 4.14(a) reveals that, actually, Π_1^e is a master copy of $\mathscr{D}$. So there is some edge $f \in E(\Pi_1)$ such that $\Pi_e^1 r_1 f^+ r_2$ is a cycle of copies in $(H, \mathscr{H}^+)$. Since $\mathscr{C}$ satisfies $(T2)$ with respect to f, the connectors r_1 and r_2 occur in consecutive positions on $\mathscr{C}$ and by our choice of Π_1^e it follows that the segment I_2 contains a unique copy of $\mathscr{C}$. Now by Lemma 4.16(a) we can collapse this copy of $\mathscr{C}$ to f^+, thus obtaining a new cycle which lives completely in the standard copy Π_1. Since $\mathrm{Girth}(\Pi_1, \mathscr{P}_1^+) > g$, the new cycle has a master copy and by Lemma 4.16(b) it follows that $\mathscr{C}$ has a master copy as well. $\square$

Utilising (5.2) we now pick a copy $\Pi_\star^e$ of $\mathscr{D}$ and a family of edges

$$\{f_i \in E(\Pi_\star^e)\colon i \in \mathbb{Z}/t\mathbb{Z} \text{ and } \Pi_i^e \neq \Pi_\star^e\}$$

exemplifying that $\Pi_\star^e$ is a master copy of $\mathscr{D}$. Denote the standard copy extending $\Pi_\star^e$ by $\Pi_\star$.

Claim 5.5. *If $i \in \mathbb{Z}/t\mathbb{Z}$ is an index with $\Pi_i^e \neq \Pi_\star^e$, then the segment I_i consists of a single copy $\widehat{F}_i$.*

Proof. Recall that $\mathscr{C}$ satisfies $(T2)$. Applying this fact to the edge f_i we learn that the vertices r_{i-1} and r_i occur in consecutive positions on $\mathscr{C}$. Due to $t \geqslant 3$ this proves the claim. $\square$

Consider the cyclic sequence $\mathscr{E}$ obtained from $\mathscr{C}$ upon collapsing every copy $\widehat{F}_i$ that Claim 5.5 delivers to the corresponding edge copy f_i^+. Since $\mathscr{C}$ is tidy and $n \geqslant 3$, Lemma 4.16(*a*) shows that $\mathscr{E}$ is again a tidy cycle of copies. As this cycle belongs entirely to $(\Pi_\star, \mathscr{P}_\star^+)$, it needs to have a master copy and owing to Lemma 4.16(*b*) we infer that $\mathscr{C}$ has a master copy as well. □

Proof of Proposition 5.1. On first sight it may not be obvious whether $\Theta_r(F)$ is defined for every hypergraph F with $\mathrm{girth}(F) > g$. Given any such hypergraph F as well as a number of colours r we set $\Phi_r(F) = (G, \mathscr{G})$ and construct the corresponding picture zero $(\Pi_0, \mathscr{P}_0, \psi_0)$. Every copy $F_0 \in \mathscr{P}_0$ is isomorphic to F, which by Lemma 4.12 implies

$$\mathrm{Girth}\big(F_0, E^+(F_0) \cup \{F_0\}\big) > g\,.$$

Since the copies in $\mathscr{P}_0$ are disjoint, it follows that $\mathrm{Girth}(\Pi_0, \mathscr{P}_0^+) > g$. As usual we take an enumeration $E(G) = \{e(1), \dots, e(N)\}$ and start running the partite construction.

Consider any positive $\alpha \leqslant N$ for which

- the picture $(\Pi_{\alpha-1}, \mathscr{P}_{\alpha-1}, \psi_{\alpha-1})$ exists
- and satisfies $\mathrm{Girth}(\Pi_{\alpha-1}, \mathscr{P}_{\alpha-1}^+) > g$.

A further application of Lemma 4.12 shows, in particular, that $\mathrm{girth}\big(\Pi_{\alpha-1}^{e(\alpha)}\big) > g$. Thus the constituent $\Pi_{\alpha-1}^{e(\alpha)}$ can be plugged into the partite lemma Ξ and we obtain a linear system $\Xi_r\big(\Pi_{\alpha-1}^{e(\alpha)}\big) = (H, \mathscr{H})$ with $\mathrm{Girth}(H, \mathscr{H}^+) > (g, g)$. By Lemma 5.2 the new picture

$$(\Pi_\alpha, \mathscr{P}_\alpha, \psi_\alpha) = (\Pi_{\alpha-1}, \mathscr{P}_{\alpha-1}, \psi_{\alpha-1}) \ast (H, \mathscr{H})$$

satisfies $\mathrm{Girth}(\Pi_\alpha, \mathscr{P}_\alpha^+) > g$ again.

This completes an induction on α. In the last step we learn that the final picture $(\Pi_N, \mathscr{P}_N, \psi_N)$ is well defined and, as desired, that $\mathrm{Girth}\big(\Pi_N, \mathscr{P}_N^+\big) > g$. □

Corollary 5.6. *The ordered f-partite Ramsey construction $\Omega^{(2)} = \mathrm{PC}(\mathrm{Rms}, \mathrm{CPL})$ is applicable to linear hypergraphs and delivers linear systems $(H, \mathscr{H})$ with $\mathrm{Girth}(H, \mathscr{H}^+) > 2$. In particular, there is a partite lemma producing such systems.*

Proof. By Corollary 3.5 and Lemma 4.13 the clean partite lemma CPL satisfies the assumptions of Proposition 5.1 for $g = 2$. □

5.2. **From $g-1$ to (g, g).** Imagine that we have a partite lemma $\Xi^{(4)-}$ applicable to k-partite, k-uniform hypergraphs F with $\mathrm{girth}(F) > 4$ and delivering systems of hypergraphs $(H, \mathscr{H}^+)$ with $\mathrm{girth}(H) > 4$ and $\mathrm{Girth}(H, \mathscr{H}^+) > 3$. Can we then derive the case $g = 4$ of Theorem 1.4 by means of the partite construction method? If so, can we even build Ramsey systems without the four-cycles of copies shown in Figure 2.1a? As the next result shows, the answers so these questions become affirmative when we resolve to utilise strongly induced copies vertically.

Proposition 5.7. *Let Ω denote a linear Ramsey construction delivering systems of strongly induced copies and let $g \geqslant 3$ be an integer. Suppose that Ξ refers to a partite lemma*

- *applicable to k-partite k-uniform hypergraphs B with $\mathrm{girth}(B) > g$*
- *yielding systems of copies $\Xi_r(B) = (H, \mathscr{H})$ with*

$$\mathrm{girth}(H) > g \quad \textit{and} \quad \mathrm{Girth}(H, \mathscr{H}^+) > g - 1\,.$$

If $\Theta = \mathrm{PC}(\Omega, \Xi)$, then for every hypergraph F with $\mathrm{girth}(F) > g$ and every $r \in \mathbb{N}$ the system $\Theta_r(F) = (H, \mathscr{H})$ exists and satisfies $\mathrm{Girth}(H, \mathscr{H}^+) > (g, g)$.

The corresponding picturesque statement reads as follows.

Lemma 5.8. *Suppose $g \geqslant 3$ and that*

$$(\Sigma, \mathscr{Q}, \psi_\Sigma) = (\Pi, \mathscr{P}, \psi_\Pi) * (H, \mathscr{H})$$

holds for two pictures $(\Pi, \mathscr{P}, \psi_\Pi)$ and $(\Sigma, \mathscr{Q}, \psi_\Sigma)$ over a linear system $(G, \mathscr{G})$ with strongly induced copies and a k-partite k-uniform system $(H, \mathscr{H})$. If

$$\mathrm{Girth}(\Pi, \mathscr{P}^+) > (g, g)\,, \quad \mathrm{girth}(H) > g\,, \quad \textit{and} \quad \mathrm{Girth}(H, \mathscr{H}^+) > g - 1\,,$$

then $\mathrm{Girth}(\Sigma, \mathscr{Q}^+) > (g, g)$.

The deduction of Proposition 5.7 from Lemma 5.8 is very similar to the proof of Proposition 5.1 based on Lemma 5.2 and we omit the details.* During the proof of Lemma 5.8 it is helpful to keep in mind that a cycle of copies $\mathscr{C}$ satisfies $h(\mathscr{C}) \leqslant (g, g)$ if and only if either $h(\mathscr{C}) = (g, g)$ or $\mathrm{ord}(\mathscr{C}) \leqslant g - 1$. In the former case, $\mathscr{C}$ has length g and all connectors of $\mathscr{C}$ are of the same type. On the whole, the proofs of Lemma 5.2 and Lemma 5.8 are very similar and, therefore, it suffices to indicate the appropriate changes.

Proof of Lemma 5.8. Consider a tidy cycle of copies $\mathscr{C} = F_1 q_1 \dots F_n q_n$ in $(\Sigma, \mathscr{Q}^+)$ with $h(\mathscr{C}) \leqslant (g, g)$. We need to prove that $\mathscr{C}$ possesses a master copy. Define segments of $\mathscr{C}$, their leaders, and segmentations of $\mathscr{C}$ in the same way as in the proof of Lemma 5.2. Again let

$$\mathscr{C} = I_1 r_1 \dots I_t r_t$$

be a segmentation of $\mathscr{C}$ such that t is minimal. As we are done otherwise, we may suppose $t \geqslant 2$ and that the leaders of any two consecutive segments are distinct. Let

$$\mathscr{D} = \Pi_1^e r_1 \dots \Pi_t^e r_t$$

denote the associated cycle of leaders, which lives in $(H, \mathscr{H}^+)$. The point of Claim 5.3 was to verify $\mathrm{ord}(\mathscr{D}) < \mathrm{Girth}(H, \mathscr{H}^+)$, which requires a different reasoning now.

*Strictly speaking, Lemma 5.2 and Lemma 5.8 are the only results of this section that will be quoted later.

Claim 5.9. *We have* $\mathrm{ord}(\mathscr{D}) \leqslant g-1$.

Proof. Repeated applications of Fact 4.19 show $h(\mathscr{D}) \leqslant h(\mathscr{C})$, so the claim could only fail if $h(\mathscr{D}) = h(\mathscr{C}) = (g, g)$, which we assume from now on. In particular, this entails that every segment I_τ with $\tau \in \mathbb{Z}/t\mathbb{Z}$ consists of a single copy and that the connectors of $\mathscr{C}$ and $\mathscr{D}$ are the same. In other words, we have $n = t = g$ and $q_i = r_i$ for all $i \in \mathbb{Z}/n\mathbb{Z}$.

If all these connectors are edges, then F_1 contains two distinct edges q_1 and q_n from H, contrary to the fact that F_1 crosses every music line of Σ at most once.

It remains to consider the case that all connectors are vertices. Now for every $i \in \mathbb{Z}/n\mathbb{Z}$ the copy F_i projects via ψ_Σ to a copy $\widetilde{F}_i$ in $\mathscr{G}^+$. This copy contains the distinct vertices $\psi_\Sigma(q_{i-1})$ and $\psi_\Sigma(q_i)$. Owing to $\widetilde{F}_i \blacktriangleleft G$ this leads to an edge $\widetilde{e}_i$ with $\psi_\Sigma(q_{i-1}), \psi_\Sigma(q_i) \in \widetilde{e}_i \in E(\widetilde{F}_i)$. But G is linear, so $\widetilde{e}_i$ is actually the edge H is projected to by ψ_Σ. By looking at the inverse of the projection map one obtains an edge $e_i \in E(F_i) \cap E(H)$ with $q_{i-1}, q_i \in e_i$. In view of Fact 4.1 the only possibility how the cyclic sequence $e_1 q_1 \dots e_n q_n$ can fail to violate the assumption $\mathrm{girth}(H) > g$ is that we have $e = e_1 = \dots = e_n$ for some edge $e \in E(H)$. But now $q_1, \dots, q_n \in M(e)$ and $n = g \geqslant 3$ contradict the tidiness of $\mathscr{C}$. □

Mutatis mutandis the rest of the proof carries over. □

§6. The extension process

This section deals with a Ramsey theoretic construction pioneered by Nešetřil and Rödl, who utilised it for proving the 2-uniform case of Theorem 1.4 for $g = 4$ in [24]. The process is applied there in a rather concrete manner, exploiting that every C_4-free bipartite graph F is expressible as an edge-disjoint union of stars, any two of which intersect in at most one vertex. Such a decomposition of the set of edges contains, of course, the same information as an equivalence relation on $E(F)$ that regards any two edges of F to be equivalent if they pertain to the same star. This perspective leads us to the concept of a pretrain and thereby to a rather general abstract framework for discussing the extension process.

6.1. **Pretrains.** A *pretrain* is a pair $(H, \equiv)$ consisting of a hypergraph H and an equivalence relation $\equiv$ on $E(H)$. If H is an ordered hypergraph, we call $(H, \equiv)$ an *ordered pretrain*. Similarly one defines the notion of an *f-partite pretrain*. A *wagon* of a pretrain $(H, \equiv)$ is a subhypergraph of H without isolated vertices whose set of edges forms an equivalence class of $\equiv$.

Definition 6.1. Given two pretrains $(F, \equiv^F)$ and $(H, \equiv^H)$, the former is said to be a *subpretrain* of the latter if

- (i) F is a subhypergraph of H
- (ii) and any two edges of F are equivalent with respect to $\equiv^F$ if and only if they are equivalent with respect to $\equiv^H$.

Condition *(i)* is the only one that will occasionally receive further specifications. For instance, it may happen that F is an induced subhypergraph of H and then $(F, \equiv^F)$ is said to be an *induced subpretrain* of $(H, \equiv^H)$. If $(F, \equiv^F)$ and $(H, \equiv^H)$ are ordered pretrains, then for $(F, \equiv^F)$ to be an *ordered subpretrain* of $(H, \equiv^H)$ we require *(i)* to hold in the stronger sense that F is an ordered subhypergraph of H. A similar modification of *(i)* is required for *f-partite subpretrains*.

The demand *(ii)* tells us that $\equiv^F$ is entirely determined by F and $\equiv^H$, and thus we shall frequently suppress the equivalence relation when talking about subpretrains. Notice that when passing from $(H, \equiv^H)$ to $(F, \equiv^F)$ every wagon W_H of H either *vanishes* in the sense that $E(W_H) \cap E(F) = \varnothing$, or it *contracts* to a wagon W_F of F with $E(W_F) = E(W_H) \cap E(F)$.

It should be clear that every pretrain is a subpretrain of itself and that the subpretrain relation is transitive.

A *system of pretrains* is a triple $(H, \equiv, \mathscr{H})$ consisting of a pretrain $(H, \equiv)$ and of a collection $\mathscr{H}$ of subpretrains of $(H, \equiv)$. Moreover $\binom{(H, \equiv^H)}{(F, \equiv^F)}$ refers to the collection of all induced subpretrains of $(H, \equiv^H)$ *isomorphic* to $(F, \equiv^F)$, where the notion of *pretrain isomorphism* is assumed to be self-explanatory. The specifiers n.n.i., pt, and fpt may be attached to $\binom{(H, \equiv^H)}{(F, \equiv^F)}$ in the usual way. For $\mathscr{H} \subseteq \binom{(H, \equiv^H)}{(F, \equiv^F)}$ and $r \in \mathbb{N}$ the partition relation

$$\mathscr{H} \longrightarrow (F, \equiv^F)_r$$

means that no matter how the edges of H are coloured with r colours, there will always exist a copy $(F_\star, \equiv^{F_\star}) \in \mathscr{H}$ whose edges are the same colour.

To aid the readers orientation we remark that the induced Ramsey theorem does also hold for pretrains in place of hypergraphs and that, in fact, this result can be shown by means of the same partite construction (see Proposition 7.1 below).

6.2. **Extensions.** For a concise description of the extension process we require a little bit of preparation. First of all, suppose that we start with a pretrain $(F, \equiv^F)$ and enlarge its wagons as disjointly as possible, thus obtaining a new pretrain $(H, \equiv^H)$. We then say that $(H, \equiv^H)$ is an extension of $(F, \equiv^F)$. Let us say the same thing again in a more precise way.

Definition 6.2. Given a subpretrain $(F, \equiv^F)$ of a pretrain $(H, \equiv^H)$ we say that $(H, \equiv^H)$ is an *extension* of $(F, \equiv^F)$ provided the following conditions hold.

(i) The hypergraphs F and H have the same isolated vertices.*

*Here and at several other places that follow one might get the impression that we are overly pedantic in our treatment of isolated vertices. After all, for results such as Theorem 1.4 it does not matter whether F is allowed to have isolated vertices or not. (If necessary, one could first remove the isolated vertices from F, apply the theorem, and put the isolated vertices back at the end.) Nevertheless, hypergraphs with isolated (*ii*) Every wagon W of H contracts to a wagon of F, i.e., satisfies $E(W) \cap E(F) \neq \varnothing$.

(*iii*) If two distinct wagons $W^\star_H$ and $W^{\star\star}_H$ of H contract to the wagons $W^\star_F$ and $W^{\star\star}_F$ of F, then $V(W^\star_H) \cap V(W^{\star\star}_H) = V(W^\star_F) \cap V(W^{\star\star}_F)$.

If $(H, \equiv^H)$ is an extension of $(F, \equiv^F)$, then $\binom{(H,\equiv^H)}{(F,\equiv^F)}$ can have more than one element. As it turns out to be useful later, we call $(F, \equiv^F)$ itself the *standard copy* of $(F, \equiv^F)$ in $(H, \equiv^H)$. If $(F, \equiv^F)$ and $(H, \equiv^H)$ are ordered and a pretrain $(H_\star, \equiv^{H_\star})$ is order-isomorphic to $(H, \equiv^H)$, then it should be clear what we mean by the *standard copy* of $(F, \equiv^F)$ in $(H_\star, \equiv^{H_\star})$.

We proceed with an easy statement that follows directly from the definition of extensions.

Lemma 6.3. *Suppose that the pretrain $(H, \equiv^H)$ is an extension of the pretrain $(F, \equiv^F)$ and that the wagon X of $(H, \equiv^H)$ contracts to the wagon W of $(F, \equiv^F)$. If $e \in E(X)$, then*

$$e \cap V(W) = e \cap V(F)\,.$$

Proof. Since $V(W) \subseteq V(F)$, the left side is a subset of the right side. For the converse direction we consider an arbitrary vertex $x \in e \cap V(F)$. Because of $x \in e$ and clause (*i*) of Definition 6.2 we know that x cannot be isolated in F and, hence, there is a wagon W' of $(F, \equiv^F)$ with $x \in V(W')$. If $W = W'$ we are done immediately, so suppose $W \neq W'$ from now on. Denoting the wagon of $(H, \equiv^H)$ contracting to W' by X' we have

$$x \in V(X) \cap V(X') = V(W) \cap V(W')$$

by Definition 6.2(*iii*) and thus, in particular, $x \in V(W)$. □

Due to the next fact we can often restrict our attention to pretains all of whose wagons are isomorphic to each other.

Fact 6.4. *Every ordered pretrain $(F, \equiv^F)$ has an extension $(\widehat{F}, \equiv^{\widehat{F}})$ all of whose wagons are order-isomorphic to the disjoint union of all wagons of $(F, \equiv^F)$.* □

In this situation we say that $(\widehat{F}, \equiv^{\widehat{F}})$ arises from $(F, \equiv^F)$ by *wagon assimilation* (see Figure 6.1).

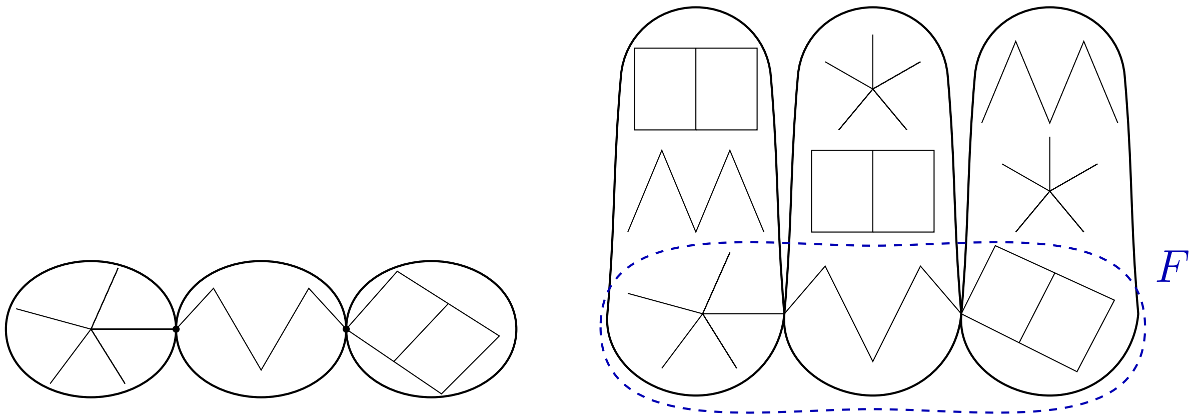
When extending wagons we sometimes want to say in an exact manner how the old wagons are supposed to "sit" in the new wagons and the definition that follows will help us to verbalise our intentions.

Definition 6.5. We say that (X, W) is an *ordered hypergraph pair* if X is an ordered hypergraph and W is an ordered subhypergraph of X. Two ordered hypergraph pairs (X, W)

---

vertices frequently arise in an auxiliary rôle throughout our constructions. For instance the picture zero shown in Figure 3.2, even though it has no isolated vertices in itself, consists of ten constituents each of which possesses six isolated vertices. Therefore, when starting with this picture we occasionally have to apply a partite lemma to a bipartite graph with isolated vertices.

and (X', W') are called *isomorphic* if X is isomorphic to X' and the unique isomorphism from X to X' maps W onto W'.

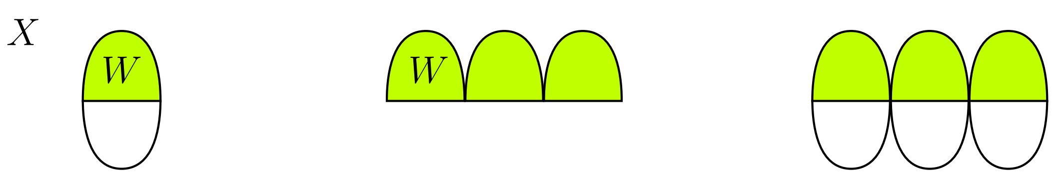


(a) $(F, \equiv^F)$ (b) $(\widehat{F}, \equiv^{\widehat{F}})$

FIGURE 6.1. The pretrain $(\widehat{F}, \equiv^{\widehat{F}})$ arises from $(F, \equiv^F)$ by wagon assimilation. Notice that F is disconnected from the rest of $\widehat{F}$.

We may now say what it means to extend all wagons of an ordered pretrain "in the same way".

Definition 6.6. Let (X, W) be an ordered hypergraph pair such that neither X nor W has isolated vertices, and let $(F, \equiv^F)$ be an ordered pretrain all of whose wagons are order-isomorphic to W. We write $(F, \equiv^F) \ltimes (X, W)$ for the extension of $(F, \equiv^F)$ having the following property: If $X_\star$ is a wagon of $(F, \equiv^F) \ltimes (X, W)$ contracting to the wagon $W_\star$ of $(F, \equiv^F)$, then the ordered hypergraph pair $(X_\star, W_\star)$ is isomorphic to (X, W) (see Figure 6.2).



(a) The pair (X, W) (b) The pretrain $(F, \equiv^F)$ (c) $(F, \equiv^F) \ltimes (X, W)$

FIGURE 6.2. Illustration of Definition 6.6

Strictly speaking the ordered pretrain $(F, \equiv^F) \ltimes (X, W)$ has thereby not been defined in a unique manner, for there are no rules as to how two new vertices from distinct wagons compare under the order relation. The ambiguity that remains, however, has no bearing on later developments and, therefore, we shall ignore it in the sequel. That is we talk about $(F, \equiv^F) \ltimes (X, W)$ as if it were uniquely determined. A later collapsing argument will hinge on a certain tameness property of this construction that we shall introduce next.

Definition 6.7. Suppose that the pretrain $(H, \equiv^H)$ is an extension of $(F, \equiv^F)$. We say that this extension is *tame* if F is strongly induced in H and, moreover, the following holds: For every edge $e \in E(H)$ and every vertex $x \in V(F) \cap e$ there exists an edge e' such that $x \in e' \in E(F)$ and $e \equiv^H e'$.

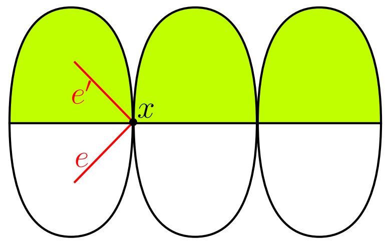


FIGURE 6.3. A tame extension

Lemma 6.8. *Let $(\widehat{F}, \equiv^{\widehat{F}})$ be the ordered pretrain that arises from $(F, \equiv^F)$ by wagon assimilation. Suppose further that all wagons of $(\widehat{F}, \equiv^{\widehat{F}})$ are isomorphic to W and that (X, W) is an ordered hypergraph pair without isolated vertices. If W is strongly induced in X, then $(\widehat{F}, \equiv^{\widehat{F}}) \ltimes (X, W)$ is a tame extension of $(F, \equiv^F)$.*

Proof. We begin with the strong inducedness. Given an edge e of $(\widehat{F}, \equiv^{\widehat{F}}) \ltimes (X, W)$ we need to find an edge e' of F such that $V(F) \cap e \subseteq e'$.

Let us first deal with the special case that $V(F) \cap e = \varnothing$. Now we can take an arbitrary edge e' of F and, hence, our claim could only fail if F has no edges. But then none of the hypergraphs W, X, and $(\widehat{F}, \equiv^{\widehat{F}}) \ltimes (X, W)$ can have any edges (recall that $W \blacktriangleleft X$), meaning that there is no edge e to consider.

So we may assume $V(F) \cap e \ne \varnothing$ from now on. Let $X_\star$ be the wagon of $(\widehat{F}, \equiv^{\widehat{F}}) \ltimes (X, W)$ containing e and denote its contraction to $(\widehat{F}, \equiv^{\widehat{F}})$ by $W_\star$. Since the ordered hypergraph pairs $(X_\star, W_\star)$ and (X, W) are isomorphic, the assumption $W \blacktriangleleft X$ implies $W_\star \blacktriangleleft X_\star$. Thus there exists an edge $e' \in E(W_\star)$ such that $V(W_\star) \cap e \subseteq e'$. Owing to Lemma 6.3 we infer $V(\widehat{F}) \cap e \subseteq e'$. In particular, $e' \in E(\widehat{F})$ covers the nonempty intersection $V(F) \cap e$. Since F is disconnected from the rest of $\widehat{F}$ (cf. Figure 6.1) and $V(F) \cap e \ne \varnothing$, this entails $e' \in E(F)$. So altogether e' is as required, i.e., F is indeed strongly induced in $(\widehat{F}, \equiv^{\widehat{F}}) \ltimes (X, W)$.

Notice that in the case $V(F) \cap e \ne \varnothing$ the edge e' we have just exhibited lies in the same wagon of $(\widehat{F}, \equiv^{\widehat{F}}) \ltimes (X, W)$ as the given edge e. This proves that the moreover-part of Definition 6.7 is satisfied as well. $\square$

6.3. **Linear pretrains.** There are two possible linearity conditions one can impose on pretrains depending on whether one wants the edges or the wagons to avoid intersections in two or more vertices. The linear pretrains considered in this article avoid both possibilities at the same time.

Definition 6.9. A pretrain $(H, \equiv^H)$ is said to be *linear* if

(*i*) the hypergraph H is linear
(*ii*) and $|V(W') \cap V(W'')| \leqslant 1$ holds for any two distinct wagons W', W''.

A system of pretrains $(H, \equiv, \mathscr{H})$ is *linear* if its underlying pretrain $(H, \equiv)$ is linear.

For later use we record an easy consequence of linearity.

Fact 6.10. *Every wagon W of a linear pretrain $(H, \equiv^H)$ is strongly induced in H.*

Proof. Due to the linearity of H it suffices to check the three statements (*i*)–(*iii*) in Fact 3.10. The last two of them are clear, since W has at least one edge and no isolated vertices. Now suppose that some edge e of H intersects $V(W)$ in at least two vertices. If W' denotes the wagon to which e belongs, then $|V(W) \cap V(W')| \geqslant |V(W) \cap e| \geqslant 2$ and Definition 6.9(*ii*) imply $W = W'$, whence $e \in E(W)$. □

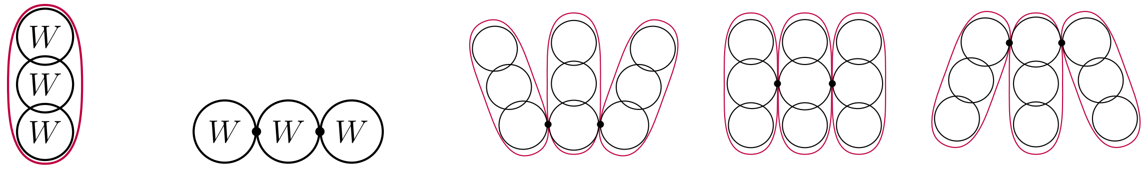
The *extension process*, which we shall now describe, is an operation transforming two given ordered linear Ramsey construction Φ and Ψ applicable to hypergraphs into a Ramsey construction $\mathrm{Ext}(\Phi, \Psi)$ applicable to ordered linear pretrains.

Given Φ and Ψ, an ordered linear pretrain $(F, \equiv^F)$, and a number of colours r, we explain how to construct the system of pretrains $\mathrm{Ext}(\Phi, \Psi)_r(F, \equiv^F) = (H, \equiv^H, \mathscr{H})$ in eight steps.

(*1*) Let $(\widehat{F}, \equiv^{\widehat{F}})$ be obtained from $(F, \equiv^F)$ by wagon assimilation (see Fact 6.4). So all wagons of $(\widehat{F}, \equiv^{\widehat{F}})$ are isomorphic to the same ordered hypergraph W. Moreover, $(\widehat{F}, \equiv^{\widehat{F}})$ is again a linear pretrain.
(*2*) Construct $\Phi_r(W) = (X, \mathscr{X})$ and assume, without loss of generality, that X has no isolated vertices. Notice that, by hypothesis on Φ, the hypergraph X is ordered and linear.
(*3*) Define the ordered pretrain $(G, \equiv^G)$ to be the disjoint union of

$$\left\{(\widehat{F}, \equiv^{\widehat{F}}) \ltimes (X, W_\star) \colon W_\star \in \mathscr{X}\right\}.$$

This is a linear pretrain containing $|\mathscr{X}|$ standard copies of $(F, \equiv^F)$. All of its wagons are order-isomorphic to X (see Figure 6.4).
(*4*) Define M to be the ordered $|V(X)|$-uniform linear hypergraph with $V(M) = V(G)$ whose edges correspond to the wagons of $(G, \equiv^G)$.
(*5*) Construct the linear system $(N, \mathscr{N}) = \Psi_{r^{e(X)}}(M)$. Notice that, in particular, N is a $|V(X)|$-uniform hypergraph arrowing M with $r^{e(X)}$ colours.



(a) $(X, \mathscr{X})$ (b) $(\widehat{F}, \equiv^{\widehat{F}})$ (c) $(G, \equiv^G)$ and M

FIGURE 6.4. Steps (1)–(4) of the extension process

(6) Let $(H, \equiv^H)$ be the ordered linear pretrain obtained by inserting ordered copies of X into the edges of N and declaring them to be wagons.

(7) Now every copy $M_\star \in \mathscr{N}$ gives rise to a copy of $(G, \equiv^G)$ in $(H, \equiv^H)$, which has the same vertex set. Let $\mathscr{H}_\bullet \subseteq \binom{(H,\equiv^H)}{(G,\equiv^G)}$ be the system of all these copies. Every copy in $\mathscr{H}_\bullet$ contains a system $|\mathscr{X}|$ standard copies of $(F, \equiv^F)$. We write $\mathscr{H}$ for the system of all (at most) $|\mathscr{N}| \cdot |\mathscr{X}|$ copies of $(F, \equiv^F)$ in $(H, \equiv^H)$ that arise in this manner.

(8) Finally, we set $\mathrm{Ext}(\Phi, \Psi)_r(F, \equiv^F) = (H, \equiv^H, \mathscr{H})$.

The next result explains our interest in this extension process.

Lemma 6.11. *If Φ and Ψ denote linear ordered Ramsey constructions, $(F, \equiv^F)$ is an ordered linear pretrain, $r \in \mathbb{N}$, and $\mathrm{Ext}(\Phi, \Psi)_r(F, \equiv^F) = (H, \equiv^H, \mathscr{H})$, then*

$$\mathscr{H} \longrightarrow (F, \equiv^F)_r\,.$$

Proof. Let $\gamma\colon E(H) \longrightarrow [r]$ be an arbitrary colouring. Each wagon of $(H, \equiv^H)$ is isomorphic to X and thus it receives one of $r^{e(X)}$ possible colour patterns. These colour patterns induce an auxiliary colouring of $E(N)$ and by our construction of $(N, \mathscr{N})$ in Step (5) there exists a copy $M_\star \in \mathscr{N}$ which is monochromatic with respect to this auxiliary colouring. The common colour pattern of its wagons can be regarded as a colouring $\delta\colon E(X) \longrightarrow [r]$. By Step (2) there exists a copy $W_\star \in \mathscr{X}$ that is monochromatic with respect to δ. The copy of $(G, \equiv^G)$ corresponding to $M_\star$ contains a copy of $(\widehat{F}, \equiv^{\widehat{F}}) \ltimes (X, W_\star)$. The standard copy of $(\widehat{F}, \equiv^{\widehat{F}})$ therein is monochromatic with respect to γ. In particular, there exists a monochromatic copy of $(F, \equiv^F)$ belonging to $\mathscr{H}$. □

It should be clear that

- if Φ and Ψ deliver systems of induced subhypergraphs, then $\mathrm{Ext}(\Phi, \Psi)$ delivers systems of induced pretrains
- and that Φ and Ψ are f-partite, then so is $\mathrm{Ext}(\Phi, \Psi)$.

§7. PRETRAINS IN PARTITE CONSTRUCTIONS

We organise the material in this section in such a manner that it constitutes a proof of the induced Ramsey theorem for pretrains. This result, however, only serves as a point of reference and the arguments occurring in the proof will be more relevant in the sequel than the theorem itself.

Proposition 7.1. *Given an ordered f-partite pretrain $(F, \equiv^F)$ as well as a number of colours r, there exists an ordered f-partite system of pretrains $(H, \equiv^H, \mathscr{H})$ such that*

$$\mathscr{H} \longrightarrow (F, \equiv^F)_r \,.$$

Observe that this result is incomparable in strength to Lemma 6.11. For instance, we defined constructions of the form $\mathrm{Ext}(\Omega, \Phi)$ as applying to linear pretrains only.* On the other hand, the proof of Proposition 7.1 presented in §7.4 relies on a construction that often yields nonlinear pretrains $(H, \equiv^H)$ even when the given pretrain $(F, \equiv^F)$ is linear.

7.1. **A partite lemma for pretrains.** Let $(F, \equiv^F)$ be a k-partite k-uniform pretrain and let $r \in \mathbb{N}$ be a number of colours. The Hales-Jewett construction introduced in §3.1 leads to a system $\mathrm{HJ}_r(F) = (H, \mathscr{H})$.

On $E(H)$ we can define an equivalence relation $\equiv^H$ by taking, essentially, the product of the equivalence relations $\equiv^F$ that we have on the "factors" of H. More precisely, if H is the n^{th} Hales-Jewett power of F and $\lambda\colon E(F)^n \longrightarrow E(H)$ denotes the canonical bijection, then we define

$$\lambda(e_1, \dots, e_n) \equiv^H \lambda(e'_1, \dots, e'_n) \iff \forall \nu \in [n]\ e_\nu \equiv^F e'_\nu$$

for all $e_1, \dots, e_n, e'_1, \dots, e'_n \in E(F)$.

Claim 7.2. *If $F_\star \in \mathscr{H}$ and $\equiv^{F_\star}$ denotes the equivalence relation on $E(F_\star)$ rendering $(F, \equiv^F)$ and $(F_\star, \equiv^{F_\star})$ naturally isomorphic, then $(F_\star, \equiv^{F_\star})$ is an induced subpretrain of $(H, \equiv^H)$.*

Proof. We already saw in Lemma 3.2 that $F_\star$ is an induced subhypergraph of H. It remains to show that any two edges of $F_\star$ are equivalent with respect to $\equiv^{F_\star}$ if and only if they are equivalent with respect to $\equiv^H$.

Suppose that $F_\star$ is given by the combinatorial embedding $\eta\colon E(F) \longrightarrow E(F)^n$, which in turn depends, as in the proof of Lemma 3.2, on the partition $[n] = C \cup M$ of $[n]$ into constant and moving coordinates and on the map $\tilde{\eta}\colon C \longrightarrow E(F)$. Let $e, e' \in E(F)$ be arbitrary and write $\eta(e) = (e_1, \dots, e_n)$ as well as $\eta(e') = (e'_1, \dots, e'_n)$. Now it remains to observe

$$\begin{aligned}(\lambda \circ \eta)(e) \equiv^H (\lambda \circ \eta)(e') &\iff \forall \nu \in [n]\ e_\nu \equiv e'_\nu \\ &\iff \forall \nu \in M\ e_\nu \equiv e'_\nu \\ &\iff e \equiv^F e' \,,\end{aligned}$$

where the last equivalence exploits $M \neq \varnothing$. □

*One can relax this requirement somewhat, but not indefinitely.

Concerning notation, it seems best to denote the system of pretrains $\{(F_\star, \equiv^{F_\star})\colon F_\star \in \mathscr{H}\}$ by $\mathscr{H}$ again, so that we may refer to the system of pretrains $(H, \equiv^H, \mathscr{H})$. Besides, we write

$$\mathrm{HJ}_r(F, \equiv^F) = (H, \equiv^H, \mathscr{H})$$

for the above construction; so from now on $\mathrm{HJ}_r(\cdot)$ applies to pretrains as well.

7.2. **Pretrain pictures.** Consider a pretrain $(F, \equiv^F)$ as well as a system of hypergraphs $(G, \mathscr{G})$ with $\mathscr{G} \subseteq \binom{G}{F}_{\text{n.n.i.}}$. A *pretrain picture over* $(G, \mathscr{G})$ is defined to be a quadruple $(\Pi, \equiv, \mathscr{P}, \psi)$ such that

- $(\Pi, \mathscr{P}, \psi)$ is a picture over $(G, \mathscr{G})$ (in the sense of §3.2),
- $(\Pi, \equiv)$ is a pretrain,
- and every copy $(F_\star, \equiv^{F_\star}) \in \mathscr{P}$ is a subpretrain of $(\Pi, \equiv)$.

In this context, the *pretrain picture zero* $(\Pi_0, \equiv_0, \mathscr{P}_0, \psi_0)$ is defined in the expected way: One starts with picture zero $(\Pi_0, \mathscr{P}_0, \psi_0)$ as defined in §3.2 and determines the equivalence relation $\equiv_0$ on $E(\Pi)$ in such a manner that

- all copies in $\mathscr{P}_0$ become isomorphic to $(F, \equiv^F)$ as pretrains,
- and edges belonging to different copies are nonequivalent with respect to $\equiv_0$.

The second bullet may seem arbitrary at this moment, but proves to be useful later. Essentially, there will be notions of cycles and $\mathfrak{Girth}$ for systems of pretrains, and in those cycles wagons can serve as connectors. Now if the wagons were allowed to spread over several copies in $\mathscr{P}_0$, then we could jump from one copy to the next using wagon connectors and already the $\mathfrak{Girth}$ of picture zero could be out of control.

7.3. **Partite amalgamations.** Now suppose that $(\Pi, \equiv^\Pi, \mathscr{P}, \psi_\Pi)$ is a pretrain picture over a system of hypergraphs $(G, \mathscr{G})$, that $e \in E(G)$, and that $(H, \equiv^H, \mathscr{H})$ is a system of pretrains all of whose copies are isomorphic to $\big(\Pi^e, \equiv^{\Pi^e}\big)$. As demanded by the partite construction method, we aim at defining a new picture

$$(\Sigma, \equiv^\Sigma \mathscr{Q}, \psi_\Sigma) = (\Pi, \equiv^\Pi, \mathscr{P}, \psi_\Pi) * (H, \equiv^H, \mathscr{H})$$

over $(G, \mathscr{G})$. As in §3.2 we construct

$$(\Sigma, \mathscr{Q}, \psi_\Sigma) = (\Pi, \mathscr{P}, \psi_\Pi) * (H, \mathscr{H})$$

and it remains to define an equivalence relation $\equiv^\Sigma$ on $E(\Sigma)$. For every standard copy $\Pi_\star$ in Σ we copy $\equiv^\Pi$ onto $\Pi_\star$, thus getting a pretrain $(\Pi_\star, \equiv^{\Pi_\star})$. Essentially $\equiv^\Sigma$ is going to be the transitivisation of the free amalgamation of $\{\equiv^{\Pi_\star} : \Pi_\star \text{ is a standard copy}\}$ over $\equiv^H$.

Our official definition of this equivalence relation is somewhat lengthy, but its main properties can be summarised as follows.

Lemma 7.3. *If $(\Pi, \equiv^{\Pi}, \mathscr{P}, \psi_\Pi)$ is a pretrain picture over $(G, \mathscr{G})$, $(H, \equiv^H, \mathscr{H})$ is a system of pretrains, and*

$$(\Sigma, \mathscr{Q}, \psi_\Sigma) = (\Pi, \mathscr{P}, \psi_\Pi) * (H, \mathscr{H}),$$

then there is an equivalence relation $\equiv^\Sigma$ on $E(\Sigma)$ with the following properties.

(*a*) *If $\Pi_\star$ is a standard copy in Σ, then $(\Pi_\star, \equiv^{\Pi_\star})$ is a subpretrain of $(\Sigma, \equiv^\Sigma)$.*
(*b*) *$(H, \equiv^H)$ is a subpretrain of $(\Sigma, \equiv^\Sigma)$.*
(*c*) (*i*) *If $\Pi_\star$ is a standard copy and $e_\star \in E(\Pi_\star)$, $e_0 \in E(H)$ satisfy $e_\star \equiv^\Sigma e_0$, then there exists an edge $e' \in E(H) \cap E(\Pi_\star)$ with $e_\star \equiv^\Sigma e' \equiv^\Sigma e_0$.*
(*ii*) *Similarly, if $\Pi_\star$, $\Pi_{\star\star}$ are distinct standard copies and two edges $e_\star \in E(\Pi_\star)$, $e_{\star\star} \in E(\Pi_{\star\star})$ satisfy $e_\star \equiv^\Sigma e_{\star\star}$, then there exist edges $e' \in E(H) \cap E(\Pi_\star)$ and $e'' \in E(H) \cap E(\Pi_{\star\star})$ with $e_\star \equiv^\Sigma e' \equiv^\Sigma e'' \equiv^\Sigma e_{\star\star}$.*
(*d*) *$(\Sigma, \equiv^\Sigma, \mathscr{Q}, \psi_\Sigma)$ is a pretrain picture over $(G, \mathscr{G})$.*

Proof. Since the members of $\mathscr{H}$ are subpretrains of $(H, \equiv^H)$, every standard copy $\Pi_\star$ satisfies

$$\forall e', e'' \in E(\Pi_\star) \cap E(H) \ \big[e' \equiv^{\Pi_\star} e'' \iff e' \equiv^H e''\big]. \tag{7.1}$$

Let us call the edges in $E(\Sigma) \smallsetminus E(H)$ *new* and the edges of H *old*. Observe that every new edge belongs to a unique standard copy. Our first step is to define a relation $\equiv^\Sigma$ on $E(\Sigma)$. To this end we consider any two edges $e_\star$ and $e_{\star\star}$ of Σ. If $e_\star$ happens to be new, we denote the standard copy it belongs to by $\Pi_\star$. Similarly, if $e_{\star\star}$ is new, its standard copy is denoted by $\Pi_{\star\star}$. We define $e_\star \equiv^\Sigma e_{\star\star}$ to hold if one of the following five cases occurs.

(α) Both $e_\star$ and $e_{\star\star}$ are new, $\Pi_\star \neq \Pi_{\star\star}$, and there exist edges $e' \in E(H) \cap E(\Pi_\star)$ and $e'' \in E(H) \cap E(\Pi_{\star\star})$ with $e_\star \equiv^{\Pi_\star} e' \equiv^H e'' \equiv^{\Pi_{\star\star}} e_{\star\star}$.
(β) Both $e_\star$ and $e_{\star\star}$ are new, $\Pi_\star = \Pi_{\star\star}$, and $e_\star \equiv^{\Pi_\star} e_{\star\star}$.
(γ) The edge $e_\star$ is new, $e_{\star\star}$ is old, and there is an edge $e' \in E(H) \cap E(\Pi_\star)$ with $e_\star \equiv^{\Pi_\star} e' \equiv^H e_{\star\star}$.
(δ) The edge $e_\star$ is old, $e_{\star\star}$ is new, and there is an edge $e'' \in E(H) \cap E(\Pi_{\star\star})$ with $e_\star \equiv^H e'' \equiv^{\Pi_{\star\star}} e_{\star\star}$.
(ε) Both $e_\star$ and $e_{\star\star}$ are old and $e_\star \equiv^H e_{\star\star}$.

Observe that the hypotheses of these five cases are mutually exclusive and cover all possibilities.

Claim 7.4. *The relation $\equiv^\Sigma$ is indeed an equivalence relation.*

Proof. Reflexivity and symmetry are clear. The proof of transitivity is not difficult but requires to look at a large number of cases depending on whether the three edges under consideration are old or new and on which of the standard copies the new ones belong to coincide. Leaving the other cases as exercises we will only display the case of three new edges $e_1 \equiv^{\Sigma} e_2 \equiv^{\Sigma} e_3$ living in three standard copies Π_1, Π_2, and Π_3 with $\Pi_1 \ne \Pi_2 \ne \Pi_3$. Now both equivalences are in case (α) and we obtain four auxiliary edges $e_1' \in E(H) \cap E(\Pi_1)$, $e_2', e_2'' \in E(H) \cap E(\Pi_2)$, and $e_3'' \in E(H) \cap E(\Pi_3)$ with

$$e_1 \equiv^{\Pi_1} e_1' \equiv^{H} e_2' \equiv^{\Pi_2} e_2 \quad \text{ and } \quad e_2 \equiv^{\Pi_2} e_2'' \equiv^{H} e_3'' \equiv^{\Pi_3} e_3\,.$$

As $\equiv^{\Pi_2}$ is an equivalence relation, it follows that $e_2' \equiv^{\Pi_2} e_2''$, which in view of (7.1) yields $e_2' \equiv^{H} e_2''$. The transitivity of $\equiv^{H}$ leads to $e_1 \equiv^{\Pi_1} e_1' \equiv^{H} e_3'' \equiv^{\Pi_3} e_3$ and in case $\Pi_1 \ne \Pi_3$ this, together with (α), proves the desired relation $e_1 \equiv^{\Sigma} e_3$. In the special case $\Pi_1 = \Pi_3$ we appeal to (7.1) again and obtain $e_1 \equiv^{\Pi_1} e_3$, which due to (β) implies $e_1 \equiv^{\Sigma} e_3$. □

Let us prove part (*a*) next. Given a standard copy $\Pi_\star$ and two edges $e_\star, e_{\star\star} \in E(\Pi_\star)$, we are to prove that

$$e_\star \equiv^{\Sigma} e_{\star\star} \quad \Longleftrightarrow \quad e_\star \equiv^{\Pi_\star} e_{\star\star}\,. \tag{7.2}$$

Depending on whether $e_\star$ and $e_{\star\star}$ are old or new the clauses (β)–(ε) provide a statement equivalent to $e_\star \equiv^{\Sigma} e_{\star\star}$ and in all four cases (7.1) shows that the forward implication holds.

The backward implication in (7.2) is clear if $e_\star$ and $e_{\star\star}$ are either both old or both new. If only $e_\star$ is new, (γ) asks for a witness e' and we can simply take $e' = e_{\star\star}$. Similarly, if only $e_{\star\star}$ is new, then $e'' = e_\star$ exemplifies (δ). This concludes the proof of (7.2) and, hence, of part (*a*) of the lemma.

Part (*b*) is much easier and follows from the fact that $\equiv^{\Sigma}$-equivalence of old edges is decided by (ε) alone.

Condition (*c*)(*i*) follows from (γ) if $e_\star$ is new and if $e_\star$ is old we just need to set $e' = e_\star$. Similarly, for the verification of (*c*)(*ii*) one needs to consider four possibilities depending on whether $e_\star$ and $e_{\star\star}$ are old or new. The main case is that both are new and then (α) yields the desired edges. If only $e_\star$ is new but $e_{\star\star}$ is old we use (γ) and set $e'' = e_{\star\star}$. The cases where $e_\star$ is old are similar using (δ), (ε), and $e' = e_\star$. This completes the proof of (*c*).

The only thing we need to prove for (*d*) is that the copies in $\mathscr{Q}$ are subpretrains of $(\Sigma, \equiv^{\Sigma})$. Owing to the transitivity of the subpretrain relation this is a direct consequence of (*a*). □

7.4. **Proof of Proposition 7.1.** Given an ordered f-partite pretrain $(F, \equiv^{F})$ together with a number of colours r we can construct the system $\mathrm{Rms}_r(F) = (G, \mathscr{G})$ as in Section 3 and enumerate $E(G) = \{e(1), \dots, e(N)\}$ as usual. Starting with the pretrain picture zero $(\Pi_0, \equiv_0, \mathscr{P}_0, \psi_0)$ introduced in §7.2 we construct recursively a sequence

$$(\Pi_\alpha, \equiv_\alpha, \mathscr{P}_\alpha, \psi_\alpha)_{\alpha \leqslant N}$$

of pretrain pictures in the expected way. That is, if the pretrain picture

$$(\Pi_{\alpha-1}, \equiv_{\alpha-1}, \mathscr{P}_{\alpha-1}, \psi_{\alpha-1})$$

has just been constructed for some positive $\alpha \leqslant N$, we apply the pretrain construction $\mathrm{HJ}_r(\cdot)$ to its constituent over $e(\alpha)$, thus getting a system of pretrains $(H_\alpha, \equiv^{H_\alpha}, \mathscr{H}_\alpha)$, and as explained in §7.3 we construct the next pretrain picture

$$(\Pi_\alpha, \equiv_\alpha, \mathscr{P}_\alpha, \psi_\alpha) = (\Pi_{\alpha-1}, \equiv_{\alpha-1}, \mathscr{P}_{\alpha-1}, \psi_{\alpha-1}) \ast (H_\alpha, \equiv^{H_\alpha}, \mathscr{H}_\alpha) .$$

For the usual reason, the final picture satisfies the partition relation

$$(\Pi_N, \equiv_N, \mathscr{P}_N) \longrightarrow (F, \equiv^F)_r ,$$

and thereby Proposition 7.1 is proved. □

Similarly, whenever we have a Ramsey construction Φ for hypergraphs and a partite lemma Ξ for pretrains, we obtain the Ramsey construction $\mathrm{PC}(\Phi, \Xi)$ applicable to pretrains. For instance, we can now regard $\mathrm{CPL} = \mathrm{PC}(\mathrm{HJ}, \mathrm{HJ})$ also as a partite lemma for pretrains and then we can move on to the pretrain construction $\Omega^{(2)} = \mathrm{PC}(\mathrm{Rms}, \mathrm{CPL})$. This construction provides an alternative proof of Proposition 7.1, which has the obvious advantage to yield systems of pretrains whose underlying hypergraphs have clean intersections.

7.5. **Orientation.** To motivate the material in the next two sections we would briefly like to discuss the following problem, which seems rather important to us: Given a linear pretrain $(F, \equiv^F)$ and a number of colours r, can we find a linear system of pretrains $(H, \equiv^H, \mathscr{H})$ such that

- $\mathscr{H} \longrightarrow (F, \equiv^F)_r$
- and any two copies in $\mathscr{H}$ have a clean intersection? (Recall that due to the linearity of F this means, roughly speaking, that any two copies intersect "at most in an edge"—cf. the discussion before Lemma 3.13).

As the aforementioned construction $\Omega^{(2)}$ yields such clean intersections, it may be tempting to just set $(H, \equiv^H, \mathscr{H}) = \Omega^{(2)}_r(F, \equiv^F)$. However, it can happen for linear pretrains $(F, \equiv^F)$ that the pretrain $(H, \equiv^H)$ obtained by means of $\Omega^{(2)}$ fails to be linear. Indeed, Corollary 3.14 only tells us that the hypergraph H needs to be linear, but there is no reason why any two wagons of $\equiv^H$ should intersect in at most one vertex. As a matter of fact, it could be shown that already our initial construction HJ does not preserve this kind of linearity and the same holds for the constructions CPL and $\Omega^{(2)}$ derived from it.

Fortunately we already know a pretrain construction yielding linear pretrains, namely $\mathrm{Ext}(\Omega^{(2)}, \Omega^{(2)})$ (see Lemma 6.11 and the eight-step definition preceding it). But the copies provided by this construction can intersect in entire wagons and, hence, their intersections are in general quite far from being clean. The method of the present subsection suggests a way to remedy this situation by looking instead at the construction $\mathrm{PC}\big(\Omega^{(2)}, \mathrm{Ext}(\Omega^{(2)}, \Omega^{(2)})\big)$. At this moment, however, it is not yet clear whether this is a sensible construction, for it is difficult to foresee whether at some moment we encounter a picture whose constituents are nonlinear pretrains, in which case it would be impossible to apply the partite lemma $\mathrm{Ext}(\Omega^{(2)}, \Omega^{(2)})$.

An important insight we shall only gain in Section 9 is that actually this obstruction does not arise. So, in other words, $\mathrm{PC}\big(\Omega^{(2)}, \mathrm{Ext}(\Omega^{(2)}, \Omega^{(2)})\big)$ applied to a linear pretrain yields a linear Ramsey pretrain and it is immediately clear that this construction solves our problem.

§8. Basic properties of $\mathfrak{Girth}$

In systems of pretrains there are certain undesirable kinds of "short cyclic configurations of copies", such as two copies intersecting the same two wagons in edges, but these cannot be detected by Girth as defined in §4.2. The goal of the present section is to define and study a concept of $\mathfrak{Girth}$ applicable to systems of pretrains which, roughly speaking, takes care of cyclic configurations of copies that become visible at the level of wagons. In the next section, we shall then see that $\mathfrak{Girth}$ is highly compatible both with the extension process and with the partite construction method.

8.1. **Basic concepts.** Recall that a linear system of hypergraphs has small Girth if all its cycles of copies of low order have one of two properties rendering them negligible: either they fail to be semitidy or they have a master copy. In the same way we shall define a system of pretrains to have small $\mathfrak{Girth}$ if all its so-called big cycles of low order behave similarly. The notion of a big cycle is defined as follows.

Definition 8.1. A *big cycle* in a system of pretrains $(H, \equiv, \mathscr{H})$ is a cyclic sequence

$$\mathscr{C} = F_1 q_1 F_2 q_2 \dots F_n q_n$$

with $n \geqslant 2$ satisfying the following conditions.

$(B1)$ The *copies* $F_1, \dots, F_n$ are in $\mathscr{H}$ and we have $F_i \neq F_{i+1}$ for every $i \in \mathbb{Z}/n\mathbb{Z}$.

$(B2)$ The *connectors* $q_1, \dots, q_n$ are distinct and each of them is either a vertex or a wagon of $(H, \equiv)$.

$(B3)$ If $i \in \mathbb{Z}/n\mathbb{Z}$ and q_i is a vertex, then $q_i \in V(F_i) \cap V(F_{i+1})$.

$(B4)$ If $i \in \mathbb{Z}/n\mathbb{Z}$ and q_i is a wagon, then $E(q_i) \cap E(F_i) \neq \varnothing$ and $E(q_i) \cap E(F_{i+1}) \neq \varnothing$.

The main difference between big cycles and the cycles of copies introduced in Definition 4.3 is that now wagons rather than edges act as connectors, which causes a corresponding change in the fourth condition from $(L4)$ to $(B4)$. A similar modification allows us to define the order of a big cycle. Notably, if $\mathscr{C} = F_1 q_1 \dots F_n q_n$ is a big cycle, we call an index $i \in \mathbb{Z}/n\mathbb{Z}$

- *pure* if either both of q_{i-1} and q_i are wagons or both are vertices and
- *mixed* if one of q_{i-1} and q_i is a wagon while the other one is a vertex,

and then we call the integer

$$\operatorname{ord}(\mathscr{C}) = \big|\{i \in \mathbb{Z}/n\mathbb{Z} \colon i \text{ is pure}\}\big| + \tfrac{1}{2}\big|\{i \in \mathbb{Z}/n\mathbb{Z} \colon i \text{ is mixed}\}\big|$$

the *order* of $\mathscr{C}$.

Extended systems can be introduced in the same way as in Section 4. Explicitly, if $(H, \equiv)$ is a pretrain, then every edge $e \in E(H)$ gives rise to a subpretrain with vertex set e, having e as its only edge, and endowed with the only possible equivalence relation. There cannot arise confusion if we denote this subpretrain again by e^+. Moreover, we keep using the notation $E^+(H)$ for $\{e^+ \colon e \in E(H)\}$, but this time meaning the collection of all subpretrains of $(H, \equiv)$ of the form e^+. Besides, if $(H, \equiv, \mathscr{H})$ is a system of pretrains, we shall again write $\mathscr{H}^+ = \mathscr{H} \cup E^+(H)$ and call $(H, \equiv, \mathscr{H}^+)$ an *extended system of pretrains*. Finally, we keep referring to copies of the form e^+ as *edge copies*, while other copies of extended systems of pretrains will be called *real copies*. The $M(f)$-notation we have already seen in Section 4 can now be applied to wagons.

Definition 8.2. Given a big cycle $\mathscr{C} = F_1 q_1 \dots F_n q_n$ in an extended system of pretrains $(H, \equiv, \mathscr{H}^+)$ we set

$$M^{\mathscr{C}}(W) = \big\{i \in \mathbb{Z}/n\mathbb{Z} \colon q_i \text{ is a vertex and } q_i \in V(W)\big\}$$

for every wagon W of the pretrain $(H, \equiv)$. If $\mathscr{C}$ is clear from the context, we may omit it and just write $M(W)$ instead.

The semitidy cycles of copies considered earlier lead us to acceptable big cycles.

Definition 8.3. A big cycle

$$\mathscr{C} = F_1 q_1 \dots F_n q_n$$

in an extended system of pretrains $(H, \equiv, \mathscr{H}^+)$ is said to be *acceptable* if it has the following three properties.

($A1$) If $\operatorname{ord}(\mathscr{C}) = 1$, then at least one copy of $\mathscr{C}$ is real.

($A2$) If $i \in \mathbb{Z}/n\mathbb{Z}$ and q_i is a wagon, then $M(q_i) \subseteq \{i-1, i+1\}$. Moreover, if $|M(q_i)| = 2$, then there exists no edge $f \in E(H)$ with $q_{i-1}, q_{i+1} \in f$ (see Figure 8.1).

($A3$) If a wagon W of $(H, \equiv)$ does not appear among $q_1, \dots, q_n$, then there exists some $i(\star) \in \mathbb{Z}/n\mathbb{Z}$ with $M(W) \subseteq \{i(\star), i(\star)+1\}$.

Let us offer two reasons why we believe that acceptability is a natural concept. First, it turns out in Claim 8.19 below that if all wagons consist of single edges, then in an obvious sense acceptable cycles are the same as semitidy cycles. Second, if a big cycle $\mathscr{C} = F_1 q_1 \dots F_n q_n$ fails to have the properties ($A2$) and ($A3$), then it "decomposes" into two or more shorter cycles of at most the same order.

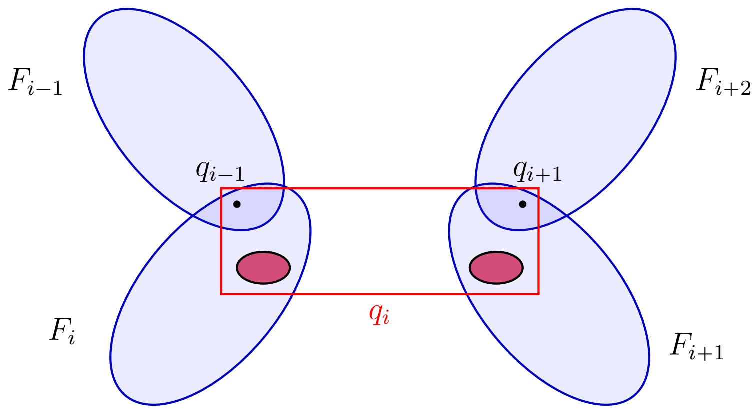


FIGURE 8.1. $|M(q_i)| = 2$ and there is no $f \in E(H)$ such that $\{q_{i-1}, q_{i+1}\} \subseteq f$. The purple ellipses represent edges that q_i has in common with F_i and F_{i+1}.

For instance, if contrary to *(A2)* the connector $W = q_n$ is a wagon with $M(W) = \{i\}$ for some $i \neq 1, n-1$, then there exists an edge $e \in E(W)$ with $q_i \in e$ and $\mathscr{C}$ decomposes into the cycles $F_1 q_1 \dots F_i q_i e^+ W$ and $e^+ q_i F_{i+1} q_{i+1} \dots F_n W$. Similar decompositions of $\mathscr{C}$ can be found whenever $M(W) \nsubseteq \{1, n-1\}$. Moreover, if $M(W) = \{1, n-1\}$ holds and an edge $f \in E(H)$ with $q_1, q_{n-1} \in f$ exists, then one can decompose $\mathscr{C}$ into the cycles $\mathscr{A} = f^+ q_1 F_2 q_2 \dots F_{n-1} q_{n-1}$ and $\mathscr{B} = F_1 q_1 f^+ q_{n-1} F_n W$. Without giving details we remark that for linear systems of pretrains $(H, \equiv, \mathscr{H})$ condition *(A3)* can be motivated in a similar way. Finally, *(A1)* just deals with the simplest possible case and the reason for declaring big cycles of order 1 consisting of two edge copies to be inacceptable will become clearer in Example 8.10. The next concept has no analogue in Section 4.

Definition 8.4. Given an extended system of pretrains $(H, \equiv, \mathscr{H}^+)$ we say that a sequence P is a *piece* if

(i) either $P = f^+$ consists of a single edge copy
(ii) or $P = f_1^+ W f_2^+$, where W is a wagon and f_1^+, f_2^+ are distinct edge copies with $f_1, f_2 \in E(W)$.

Pieces of the form *(i)* are called *short*, while pieces of type *(ii)* are said to be *long*. If $F_\star \in \mathscr{H}^+$ is a copy and P is a piece, we shall refer to P as an $F_\star$*-piece* if either $P = f^+$ is short and $f \in E(F_\star)$, or if $P = f_1^+ W f_2^+$ is long and $f_1, f_2 \in E(F_\star)$.

The natural generalisation of master copies to the present context allows to collapse copies to such pieces, but the rules concerning long pieces are rather restrictive.

Definition 8.5. Let $\mathscr{C} = F_1 q_1 \dots F_n q_n$ be a big cycle in an extended system of pretrains $(H, \equiv, \mathscr{H}^+)$. A copy $F_\star$ occurring in $\mathscr{C}$ is said to be a *supreme copy* of $\mathscr{C}$ if there exists a family of $F_\star$-pieces $\{P_i : i \in \mathbb{Z}/n\mathbb{Z} \text{ and } F_i \neq F_\star\}$ with the following properties.

(i) The cyclic sequence $\mathscr{D}$ obtained from $\mathscr{C}$ upon replacing every copy F_i distinct from $F_\star$ by the corresponding $F_\star$-piece P_i is again a big cycle.

(ii) If $i \in \mathbb{Z}/n\mathbb{Z}$, $F_i \neq F_\star$, and $P_i = (f_i')^+ W_i (f_i'')^+$ is long, then

(α) the connectors q_{i-1}, q_i are vertices

(β) and there is no edge f with $q_{i-1}, q_i \in f \in E(H)$.

As this definition is central to everything that follows, we would like to illustrate it with two figures.

Example 8.6. Suppose first that $\mathscr{C} = F_1 q_1 F_2 q_2 F_3 q_3$ is a big cycle with vertex connectors q_1, q_2 and a wagon connector $W = q_3$ (see Figure 8.2). Under what circumstances is F_2 a supreme copy of $\mathscr{C}$? Certainly this requires F_1 and F_3 to be collapsible to some F_2-pieces P_1, P_3. Due to condition (ii)(α) these pieces need to be short. Therefore $P_1 = f_1^+$ and $P_3 = f_3^+$ have to be edge copies with $f_1, f_3 \in E(F_2)$. Moreover, $f_1^+ q_1 F_2 q_2 f_3^+ q_3$ has to be a big cycle, which requires $q_1 \in f_1$, $q_2 \in f_3$, and $f_1, f_3 \in E(q_3)$.

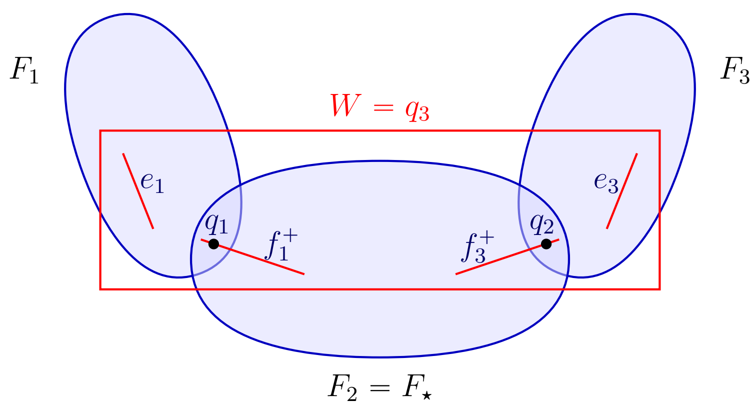


FIGURE 8.2. A cycle with supreme copy F_2. Since the connector q_3 is a wagon, the copies F_1, F_3 need to get collapsed to short F_2-pieces.

Example 8.7. Suppose next that $\mathscr{C} = F_1 q_1 F_2 q_2 F_3 q_3 F_4 q_4$ is a big cycle all of whose connectors are vertices (see Figure 8.3). In order to determine whether F_2 is a supreme copy of $\mathscr{C}$ we first ask ourselves which of the pairs $\{q_4, q_1\}$, $\{q_2, q_3\}$, and $\{q_3, q_4\}$ are covered by edges. Assume that there are edges f_1, f_3 such that $q_4, q_1 \in f_1$ and $q_2, q_3 \in f_3$, but that no edge contains q_3 and q_4. Due to condition (ii)(β) the copies F_1, F_3 need to get collapsed to short pieces and by ($B3$) these pieces need to be f_1^+, f_3^+. Moreover, there exists no short piece F_4 could be collapsed to and thus there needs to be some long F_2-piece $(f_4')^+ W (f_4'')^+$ we can use for this purpose.

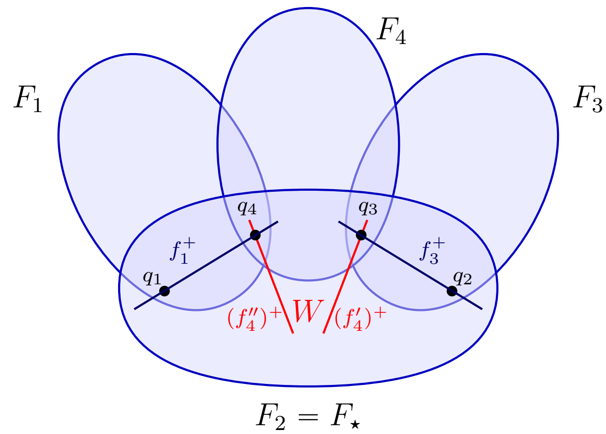


FIGURE 8.3. A big cycle with supreme copy F_2. The copies F_1, F_3 are collapsed to the short pieces f_1^+, f_3^+, respectively. Moreover, F_4 gets collapsed to the long piece $(f_4')^+W(f_4'')^+$. This requires that no edge through q_3, q_4 exists.

Remark 8.8. *(1)* In the situation of Definition 8.5 one can prove with the help of *(ii)*(α) that the new big cycle $\mathscr{D}$ obtained in *(i)* has the same order as $\mathscr{C}$. The point of condition *(ii)*(β) is that in case such an edge f exists it should be better to choose $P_i = f^+$. In other words, long pieces are only allowed if they preserve the order and are unavoidable.
(2) The same argument as in Fact 4.10 shows that supreme copies are always real copies.

We are now sufficiently prepared for the central concept of this section. Recall that in Definition 6.9 we defined a system of pretrains to be linear if its underlying pretrain is linear, which in turn means that neither edges nor wagons can intersect in more than one vertex.

Definition 8.9. Given an extended system of pretrains $(H, \equiv, \mathscr{H}^+)$ as well as a positive integer g we write $\mathfrak{Girth}(H, \equiv, \mathscr{H}^+) > g$ if

- the pretrain $(H, \equiv)$ is linear
- and every acceptable big cycle $\mathscr{C}$ in $(H, \equiv, \mathscr{H}^+)$ whose order is at most g has a supreme copy.

Let us clarify that the notions "acceptability" and "supreme copy" are defined in all systems of pretrains no matter whether they are linear or not. Accordingly, one can also ask for nonlinear systems whether they satisfy the statement "every big cycle of order at most g has a supreme copy". The answer can very well be affirmative for a system $(H, \equiv, \mathscr{H}^+)$ that fails to be linear, but in such cases we do not regard $\mathfrak{Girth}(H, \equiv, \mathscr{H}^+) > g$ as being true.

Example 8.10. Let $(H, \equiv, \mathscr{H}^+)$ be an extended linear system of pretrains. Up to symmetry big cycles of order 1 are of the form $\mathscr{C} = F_1 x F_2 W$, where x is a vertex and W is a wagon. Such cycles always satisfy the acceptability conditions *(A2)* and *(A3)*, so the only possibility for them to be inacceptable is that *(A1)* fails and both F_1, F_2 are edge copies. This configuration is tantamount to a vertex $x \in V(W)$ having at least the degree 2 in W. As a matter of fact, we only impose condition *(A1)* in the definition of acceptability for the reason that we want this to be permissible without impairing the $\mathfrak{Girth}$ of $(H, \equiv, \mathscr{H}^+)$.

Next, we figure out what it means to say that F_1 is a supreme copy of $\mathscr{C}$. By Remark 8.8(*2*) this requires F_1 to be real. Moreover, due to Definition 8.5(*ii*)(α) the copy F_2 can only be collapsed to a short piece f_1^+ and an edge $f_1 \in E(F_1)$ is appropriate for this purpose if $x \in f_1 \in E(W)$. Suppose now that this happens and that F_2 is a real copy as well. Then $f_1^+ x F_2 W$ is an acceptable big cycle too and F_2 is a supreme copy of this cycle if there exists a further edge $f_2 \in E(F_2)$ with $x \in f_2 \in E(W)$.

For later reference we summarise this discussion as follows.

Lemma 8.11. *Given an extended linear system of pretrains $(H, \equiv^H, \mathscr{H}^+)$ the following two statements are equivalent.*

(*a*) $\mathfrak{Girth}(H, \equiv^H, \mathscr{H}^+) > 1$*;*

(*b*) *For every big cycle $\mathscr{C} = F_1 x F_2 W$ there are edges*

$$f_1 \in E(F_1) \cap E(W) \quad \text{and} \quad f_2 \in E(F_2) \cap E(W)$$

such that $x \in f_1 \cap f_2$.

Proof. Suppose first that (*a*) holds and let $\mathscr{C} = F_1 x F_2 W$ be a big cycle. If F_1 is a real copy the existence of f_1 was proved in Example 8.10 and if F_1 is an edge copy it suffices to take f_1 to be the only edge of F_1. Similarly, the desired edge f_2 exists as well and thereby (*b*) is proved.

For the converse direction we let $\mathscr{C} = F_1 x F_2 W$ be an acceptable big cycle of order 1. Due to *(A1)* and symmetry we can suppose that the copy F_1 is real. If the edge f_1 is obtained from assumption (*b*), then F_2 is collapsible to the short F_1-piece f_1^+ and thus F_1 is a supreme copy of $\mathscr{C}$. $\square$

We proceed with some further observations on cycles of length 2.

Lemma 8.12. *Let $(H, \equiv^H, \mathscr{H}^+)$ be an extended linear system of pretrains. Provided that $\mathfrak{Girth}(H, \equiv^H, \mathscr{H}^+) > 2$, there are no big cycles of length 2 both of whose connectors are wagons.*

Proof. Assume that $\mathscr{C} = F_1 W_1 F_2 W_2$ is a big cycle in $(H, \equiv^H, \mathscr{H}^+)$ with wagons W_1, W_2. Since $\mathscr{C}$ has no vertex connectors, it satisfies all three acceptability conditions, and thus it needs to have a supreme copy, say F_1. Now we can collapse F_2 to a short piece f^+ (see Definition 8.5*(ii)*(α)) and the fact that $F_1W_1f^+W_2$ is again a big cycle yields the contradiction $f \in E(W_1) \cap E(W_2)$. $\square$

In analogy with Lemma 4.14*(a)* we have the following result.

Lemma 8.13. *If $\mathscr{C}$ is a big cycle of length 2 in an extended linear system of pretrains $(H, \equiv, \mathscr{H}^+)$ with $\mathfrak{Girth}(H, \equiv, \mathscr{H}^+) > 2$, then every real copy of $\mathscr{C}$ is a supreme copy of $\mathscr{C}$.*

Proof. The case $\mathrm{ord}(\mathscr{C}) = 1$ has already been discussed in Example 8.10 and the case that both connectors of $\mathscr{C}$ are wagons was excluded in the previous lemma.

Thus it remains to treat the case that $\mathscr{C}$ is of the form F_1xF_2y for some vertices x and y, and that, without loss of generality, F_1 is a real copy. Evidently, $\mathscr{C}$ is acceptable and, therefore, it needs to possess a supreme copy. The interesting case occurs if F_2 is a supreme copy of $\mathscr{C}$. Let the F_2-piece P exemplify this fact. This means that PxF_2y is a big cycle, and it follows that $\mathscr{D} = F_1xPy$ is a big cycle, too. Clearly, $\mathscr{D}$ has order 2.

First Case. The piece $P = f^+$ is short.

Now $\mathscr{D}$ is acceptable and due to Remark 8.8*(2)* its supreme copy is F_1. If the F_1-piece Q witnesses this state of affairs, then Q exemplifies the supremacy of F_1 in $\mathscr{C}$ as well.

Second Case. The piece $P = (f')^+W(f'')^+$ is long.

Again $\mathscr{D}$ is acceptable, the moreover-part of *(A2)* being ensured by Definition 8.5*(ii)*(β). As before, F_1 is a supreme copy of $\mathscr{D}$ and by Definition 8.5*(ii)*(α) there exist short F_1-pieces $f_\star^+$, $f_{\star\star}^+$ such that $F_1xf_\star^+Wf_{\star\star}^+y$ is a big cycle. Now $f_\star^+Wf_{\star\star}^+$ is a long F_1-piece verifying that F_1 is a supreme copy of $\mathscr{C}$. $\square$

$\mathfrak{Girth}$ relates to strong inducedness in the following way.

Lemma 8.14. *If an extended system of pretrains $(H, \equiv, \mathscr{H}^+)$ satisfies*

$$\mathfrak{Girth}(H, \equiv, \mathscr{H}^+) > 2\,,$$

then every edge f intersecting a copy $F_\star \in \mathscr{H}^+$ in at least two vertices belongs to that copy.

Notice that due to the linearity of H the three items *(i)*–*(iii)* listed in Fact 3.10 tell us what it means to say that a copy in $\mathscr{H}^+$ is strongly induced in H. The first of these conditions coincides with the conclusion of the lemma above. The remaining conditions, *(ii)* and *(iii)*, are usually satisfied in practice but we cannot deduce them from the assumption $\mathfrak{Girth}(H, \equiv, \mathscr{H}^+) > 2$.

Proof of Lemma 8.14. Suppose that $x, y \in V(F_\star) \cap f$ are distinct. If $F_\star = f^+$, then $f \in E(F_\star)$ is clear, so suppose $F_\star \neq f^+$ from now on. The big cycle $\mathscr{C} = F_\star xf^+y$ is acceptable, its order is 2 and, therefore, it possesses a supreme copy. By Remark 8.8*(2)* this supreme copy can only be $F_\star$ and the existence of an edge containing x and y ensures that f^+ collapses to a short piece $(f')^+$ with $f' \in E(F_\star)$. Since $x, y \in f \cap f'$, the linearity of H implies $f = f'$ and thus we have indeed $f \in E(F_\star)$. □

We conclude our list of preliminary observations on $\mathfrak{Girth}$ as follows.

Fact 8.15. *If a big cycle $\mathscr{C}$ in an extended linear system of pretrains has a supreme copy $F_\star$, then all vertex connectors of $\mathscr{C}$ belong to $F_\star$.*

Proof. Let q_i be a vertex connector of $\mathscr{C} = F_1 q_1 \dots F_n q_n$. Due to $(B1)$ at least one of the copies F_i, F_{i+1} is distinct from $F_\star$, so by symmetry we may assume $F_i \ne F_\star$. Consider the $F_\star$-piece P_i to which F_i collapses. If $P_i = f^+$ is short, then $q_i \in f \subseteq V(F_\star)$ follows. Similarly, if $P_i = (f')^+ W (f'')^+$ is long, then we have $q_i \in f'' \subseteq V(F_\star)$. □

8.2. **Special cases.** The next item on our agenda are simplified characterisations of $\mathfrak{Girth}(H, \equiv, \mathscr{H}^+)$ applicable to systems of pretrains with special properties. First, the simplest case for $\mathscr{H}^+$ is that $\mathscr{H}^+ = E^+(H)$ and we address this situation in Lemma 8.17 below. Second, the case where $\mathscr{H}^+$ is arbitrary and every wagon consists of a single edge is dealt with in Lemma 8.18; it turns out that $\mathfrak{Girth}$ is then essentially the same as Girth.

Definition 8.16. For a pretrain $(H, \equiv)$ and an integer $g \geqslant 2$ we write $\mathfrak{girth}(H, \equiv) > g$ if H is linear and the girth of the set system one obtains from $(H, \equiv)$ upon replacing the wagons by new edges (and deleting the original edges) exceeds g.

Due to Fact 4.1 this means, explicitly, that if for some $n \in [2, g]$ we have a cyclic sequence $W_1 q_1 \dots W_n q_n$ such that

- $W_1, \dots, W_n$ are wagons of $(H, \equiv)$,
- the vertices $q_1, \dots, q_n$ of H are distinct,
- and $q_i \in V(W_i) \cap V(W_{i+1})$ holds for every $i \in \mathbb{Z}/n\mathbb{Z}$,

then $W_1 = \dots = W_n$. Notice that a pretrain $(H, \equiv)$ is linear in the sense of Definition 6.9 if and only if $\mathfrak{girth}(H, \equiv) > 2$.

Let us emphasise that the lowercase "$\mathfrak{g}$" in Definition 8.16 as opposed to the capital "$\mathfrak{G}$" in Definition 8.9 indicates that here we deal with individual objects as opposed to the systems treated there. So the rule on capitalisation is the same as in Section 4. It might further be helpful to bear in mind that German letters highlight the importance of equivalence relations on the edge sets, while Roman letters are used in cases where such pretrain structures are absent or ignored.

Lemma 8.17. *If $(H, \equiv)$ is a linear pretrain and $g \geqslant 2$, then*

$$\mathfrak{girth}(H, \equiv) > g \iff \mathfrak{Girth}\big(H, \equiv, E^+(H)\big) > g \iff \mathfrak{Girth}\big(H, \equiv, E^+(H) \cup \{(H, \equiv)\}\big) > g\,.$$

Proof. Evidently, the last condition implies the middle one. Conversely, the middle condition also implies the last one, for $(H, \equiv)$ is a supreme copy of every big cycle in the system of pretrains $\big(H, \equiv, E^+(H) \cup \{(H, \equiv)\}\big)$ that contains $(H, \equiv)$. It therefore remains to show that the first and middle statement are equivalent.

Suppose first that $\mathfrak{Girth}\big(H, \equiv, E^+(H)\big) > g$, but that for some $n \in [2, g]$ there exists a cyclic sequence $W_1 q_1 \dots W_n q_n$ violating $\mathfrak{girth}(H, \equiv) > g$. If we choose this counterexample with n as small as possible, then $W_1, \dots, W_n$ are distinct. Moreover, the linearity of $(H, \equiv)$ yields $n \geqslant 3$.

Let us select for every index $i \in \mathbb{Z}/n\mathbb{Z}$ a piece P_i as follows. If there exists an edge $f_i \in W_i$ with $q_{i-1}, q_i \in f_i$, then let $P_i = f_i^+$. If there is no such edge, then we pick edges $f_i', f_i'' \in E(W_i)$ with $q_{i-1} \in f_i'$ and $q_i \in f_i''$, which is possible due to the fact that wagons have no isolated vertices. Moreover, the absence of f_i entails $f_i' \neq f_i''$ and thus $P_i = (f_i')^+ W_i (f_i'')^+$ is a long piece. Now $\mathscr{C} = P_1 q_1 \dots P_n q_n$ is a big cycle in $\big(H, \equiv, E^+(H)\big)$ with $\mathrm{ord}(\mathscr{C}) = n \in [3, g]$. By $n \geqslant 3$ it has the property *(A1)* of acceptability and one checks easily that any violation of *(A2)* or *(A3)* would yield a contradiction to the minimality of n. Thus $\mathscr{C}$ is acceptable and Definition 8.5 entails that $\mathscr{C}$ has a supreme copy. However, $\mathscr{C}$ does not even contain a real copy, so we arrive at a contradiction to Remark 8.8(*2*).

It remains to prove that, conversely, assuming $\mathfrak{girth}(H, \equiv) > g$ we can derive

$$\mathfrak{Girth}\big(H, \equiv, E^+(H)\big) > g\,.$$

Consider any acceptable big cycle $\mathscr{C} = f_1^+ q_1 \dots f_n^+ q_n$ in $\big(H, \equiv, E^+(H)\big)$ with $\mathrm{ord}(\mathscr{C}) \leqslant g$. Due to *(A1)* the order of $\mathscr{C}$ is at least 2. Notice that owing to *(B2)* and *(B4)* is cannot be the case that two consecutive connectors of $\mathscr{C}$ are wagons. Therefore, the order of $\mathscr{C}$ is the number of its vertex connectors. By symmetry we may suppose that

$$1 \leqslant i(1) < i(2) < \dots < i(m) = n$$

are the indices of the vertex connectors of $\mathscr{C}$, where $m = \mathrm{ord}(\mathscr{C}) \in [2, g]$. As q_1 and q_2 cannot be consecutive wagons, $i(1)$ is either 1 or 2 and in both cases there exists a wagon W_1 containing all edges f_i with $i \in [1, i(1)]$. Similarly, for every $\mu \in [2, m]$ there exists a wagon W_μ containing all edges f_i with $i \in [i(\mu - 1) + 1, i(\mu)]$. Now $W_1 q_{i(1)} \dots W_m q_{i(m)}$ is a "wagon cycle" and $\mathfrak{girth}(H, \equiv) > g \geqslant m$ tells us that there exists a wagon W with

$$W = W_1 = \dots = W_m\,.$$

In particular,

$$f_1, \dots, f_n \in E(W)\,, \qquad q_{i(1)}, \dots, q_{i(m)} \in V(W)\,,$$

and the only wagon possibly appearing on $\mathscr{C}$ is W.

If W does not appear on $\mathscr{C}$, then (A3) yields $n = m = |M^{\mathscr{C}}(W)| \leqslant 2$, contrary to the linearity of H. This shows that W appears on $\mathscr{C}$ and (A2) discloses $m = 1$, contrary to (A1). □

The next result essentially says that if every edge is its own wagon, then $\mathfrak{Girth}$ is the same as Girth.

Lemma 8.18. *Let $(H, \mathscr{H})$ be a linear system and let $g \geqslant 2$ be a natural number. If $\equiv$ denotes the equivalence relation on $E(H)$ whose equivalence classes are single edges, then*

$$\mathrm{Girth}(H, \mathscr{H}^+) > g \iff \mathfrak{Girth}(H, \equiv, \mathscr{H}^+) > g\,.$$

Proof. For simplicity we treat cycles of copies in $(H, \mathscr{H}^+)$ as if they were big cycles in $(H, \equiv, \mathscr{H}^+)$ and vice versa. Pedantically speaking, this is not quite precise, for in the former case the non-vertex connectors are just edges of H, while in the latter case they are wagons and thus of the form $(e, \{e\})$ with $e \in E(H)$. Moreover, in the former case the copies are just subhypergraphs of H belonging to $\mathscr{H}^+$, whereas in the latter case they are, officially, subpretrains of $(H, \equiv)$ and thus accompanied by equivalence relations on their sets of edges. Ignoring these extremely minor differences it will be convenient to identify the two concepts for the current purposes; thus when speaking of cycles in the remainder of this proof we shall mean either cycles of copies in $(H, \mathscr{H}^+)$ or the corresponding big cycles in $(H, \equiv, \mathscr{H}^+)$. It turns out that there is a quite literal translation from Definition 4.17 to Definition 8.9.

Claim 8.19. *A cycle is semitidy if and only if it is acceptable.*

Proof. Consider a semitidy cycle $\mathscr{C}$. Clearly, $\mathscr{C}$ satisfies (A1), and (S1), (S2) imply (A2) and (A3), respectively. Thus semitidiness does indeed imply acceptability.

Conversely, let $\mathscr{C}$ be an acceptable cycle. If (S1) fails for some edge connector e, then (A2) shows that the connectors next to e are two distinct vertices in e. However, this causes e itself to violate the moreover-part of (A2). This proves (S1). Condition (S2) follows from (A3). □

Claim 8.20. *A cycle has a master copy if and only if it has a supreme copy.*

Proof. If $F_\star$ is a master copy of a cycle $\mathscr{C}$, then the family of edges of $F_\star$ exemplifying this fact leads to a corresponding family of short $F_\star$-pieces exemplifying that $F_\star$ is a supreme copy of $\mathscr{C}$. Taking into account that our special choice of $\equiv$ precludes the existence of long pieces the converse direction is proved similarly. □

Owing to Lemma 4.18 our claims show that we are done. □

8.3. **Two further facts.** We conclude this section by returning to general systems of pretrains and proving two statements concerning big cycles and $\mathfrak{Girth}$. The first of them asserts that not too many edge copies of an acceptable big cycle can belong to the same wagon.

Lemma 8.21. *Given an acceptable big cycle $\mathscr{C} = F_1q_1 \dots F_nq_n$ of length $n \geqslant 3$ in an extended system of pretrains $(H, \equiv, \mathscr{H}^+)$ and a wagon W of $(H, \equiv)$ there exists an index $i(\star)$ such that the set*

$$Q(W) = \left\{i \in \mathbb{Z}/n\mathbb{Z} \colon F_i = e_i^+ \text{ is an edge copy with } e_i \in E(W)\right\}$$

satisfies $Q(W) \subseteq \{i(\star), i(\star)+1\}$. Moreover, if $|Q(W)| = 2$, then $q_{i(\star)} = W$ and the connectors $q_{i(\star)-1}$, $q_{i(\star)+1}$ are vertices (see Figure 8.4). In particular, if a vertex connector is between two edge copies, then their underlying edges belong to distinct wagons.

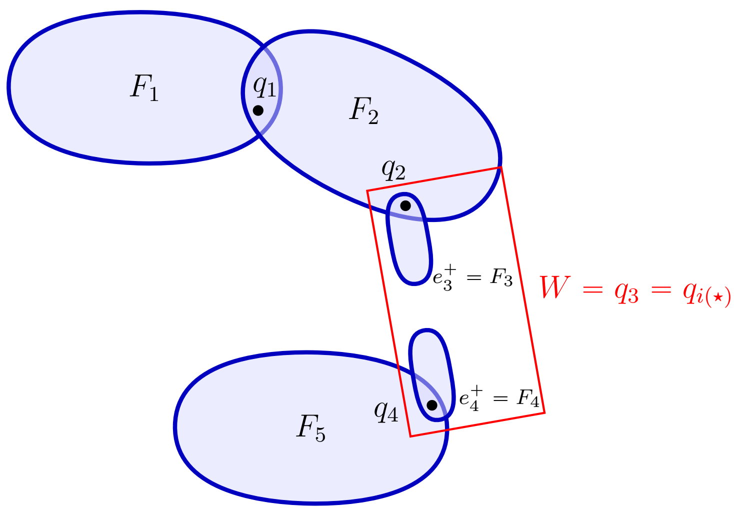


FIGURE 8.4. The case $Q(W) = \{i(\star), i(\star)+1\}$ (and $i(\star) = 3$) of Lemma 8.21

Proof. Notice that if $i \in Q(W)$, then by $(B3)$ and $(B4)$ each of the connectors q_{i-1}, q_i is either a vertex belonging to $V(W)$ or W itself. Thus if W does not appear on $\mathscr{C}$, then the acceptability condition $(A3)$ implies $|Q(W)| \leqslant 1$. Furthermore, if $W = q_{i(\star)}$ appears on $\mathscr{C}$, then $(A2)$ discloses $Q(W) \subseteq \{i(\star), i(\star)+1\}$. Finally, if this holds with equality, then $q_{i(\star)-1}$ and $q_{i(\star)+1}$ are indeed vertices. $\square$

We proceed with a natural analogue of Lemma 4.16 that reads as follows.

Lemma 8.22. *Let $\mathscr{C} = F_1q_1 \dots F_nq_n$ be an acceptable big cycle of length $n \geqslant 3$ in a linear extended system of pretrains $(H, \equiv, \mathscr{H}^+)$. Suppose that for some set of indices $K \subseteq \mathbb{Z}/n\mathbb{Z}$ we have a family of pieces $\{P_k \colon k \in K\}$ such that if for some $k \in K$ the piece $P_k = (f_k')^+ W_k (f_k'')^+$ is long, then*

(α) *the connectors q_{k-1}, q_k are vertices,*
(β) *there is no edge $f \in E(H)$ with $q_{k-1}, q_k \in f$,*
(γ) *and $W_k \notin \{q_1, \dots, q_n\}$.*

Let $\mathscr{D}$ be the cyclic sequence obtained from $\mathscr{C}$ upon replacing every copy F_k with $k \in K$ by the corresponding piece P_k.

(*a*) *If $\mathscr{D}$ satisfies (B3) and (B4), then it is an acceptable big cycle.*
(*b*) *Moreover, if $\mathscr{D}$ has a supreme copy, then so does $\mathscr{C}$.*

Notice that the conditions (α) and (β) are the same as in Definition 8.5(*ii*).

Proof of Lemma 8.22. A wagon connector of $\mathscr{D}$ will be called *new* if it occurs in the middle of a long piece P_k and otherwise, i.e., if has been inherited from $\mathscr{C}$, it will be called *old*. This distinction is not necessary for vertex connectors, since all of them are "old". Notice that condition (γ) imposed on admissible long pieces P_k says that no wagon is at the same time old and new.

Proof of part (*a*). The two claims that follow will establish that $\mathscr{D}$ is indeed a big cycle.

Claim 8.23. *The wagons of $\mathscr{D}$ are distinct and, hence, $\mathscr{D}$ satisfies (B2).*

Proof. This could only fail, if some new wagon occurs twice, i.e., if there are distinct indices $k, \ell \in K$ such that the pieces P_k and P_ℓ are long and contain the same wagon $W_k = W_\ell$ in the middle. However, this would necessitate $\{k-1, k, \ell-1, \ell\} \subseteq M^{\mathscr{C}}(W_k)$ and, hence, $|M^{\mathscr{C}}(W_k)| \geqslant 3$, contrary to (A3). $\square$

Claim 8.24. *The cyclic sequence $\mathscr{D}$ satisfies (B1) and, hence, it is a big cycle.*

Proof. Otherwise, some edge copy e^+ occurs twice in consecutive positions, i.e., $\mathscr{D}$ has a subsequence of the form $r_{i-1}e^+r_ie^+r_{i+1}$. Let $W_\star$ denote the unique wagon with $e \in E(W_\star)$. Notice that all wagons among r_{i-1}, r_i, and r_{i+1} are equal to $W_\star$, while all vertices among these connectors are in e. Thus, if all three of these connectors are vertices, then $|M^{\mathscr{C}}(W_\star)| \geqslant 3$, contrary to (A2) or (A3). Owing to (B2) it remains to consider the case that two of these connectors are vertices, while the third one is equal to $W_\star$.

Suppose first that $W_\star = r_i$. Now $W_\star$ cannot be new, for $e^+W_\star e^+$ does not qualify as a piece. However, if $W_\star$ is old, then the edge e contradicts the moreover-part of $W_\star$ satisfying (A2).

By symmetry, it only remains to discuss the case that $W_\star = r_{i-1}$. If $W_\star$ is old, then r_i, r_{i+1} correspond to indices in $M^{\mathscr{C}}(W_\star)$ and we obtain a contradiction to (A2). Finally, let $W_\star$ be new. The existence of a new wagon implies $|\mathscr{D}| > |\mathscr{C}| \geqslant 3$, i.e., $|\mathscr{D}| \geqslant 4$. So r_{i-2}, r_i, and r_{i+1} are three distinct vertices of $W_\star$ that witness $\big|M^{\mathscr{C}}(W_\star)\big| \geqslant 3$. This contradiction to (A3) concludes the proof that $\mathscr{D}$ is indeed a big cycle. $\square$

It remains to show that $\mathscr{D}$ is acceptable. Notice that condition (A1) is clear.

Claim 8.25. *The big cycle $\mathscr{D}$ satisfies (A2).*

Proof. The first part of (A2) holds for old wagons because it holds in $\mathscr{C}$ and it holds for new wagons because of (A3). If for some wagon W of $\mathscr{D}$ there is an edge f as in the moreover-part of (A2), then (β) shows that W is old and we reach a contradiction to the fact that $\mathscr{C}$ satisfies (A2). $\square$

Claim 8.26. *Moreover, $\mathscr{D}$ satisfies (A3) and is, hence, acceptable.*

Proof. Let W be a wagon not appearing in $\mathscr{D}$. Since W cannot belong to $\mathscr{C}$ either and $\mathscr{C}$ satisfies (A3), the only problem that could arise is that the set $M^{\mathscr{C}}(W)$ consists of two consecutive indices and that we want to insert a long piece $P = (f')^+W_\star(f'')^+$ between the corresponding vertex connectors. If this happens, then these connectors belong to both wagons W and $W_\star$. Due to the linearity of $(H, \equiv)$ it follows that $W = W_\star$. As $\mathscr{D}$ contains the piece P, this contradicts the assumption that W lies outside $\mathscr{D}$. $\square$

Proof of part (*b*). Rewrite $\mathscr{D} = G_1r_1 \dots G_mr_m$ and suppose that

$$\{Q_i \colon i \in \mathbb{Z}/m\mathbb{Z} \text{ and } G_i \neq F_\star\}$$

is a family of $F_\star$-pieces exemplifying that $F_\star$ is a supreme copy of $\mathscr{D}$. Recall that $F_\star$ is a real copy and, consequently, it appears in $\mathscr{C}$. In the light of part (*a*) it suffices to prove that for every index $k \in \mathbb{Z}/n\mathbb{Z}$ with $F_k \neq F_\star$ there exists an $F_\star$-piece R_k such that

(*1*) the cyclic sequence arising from $\mathscr{C}$ if one exchanges F_k by R_k satisfies (B3) and (B4)
(*2*) and if R_k is long, then it has the properties (α), (β), and (γ).

If $k \notin K$, then in the passage from $\mathscr{C}$ to $\mathscr{D}$ the copy F_k is preserved and receives a new index $i(k) \in \mathbb{Z}/m\mathbb{Z}$. Moreover, we can take $R_k = Q_{i(k)}$. Suppose next that $k \in K$ and that the piece P_k is short. Now the copy F_k of $\mathscr{C}$ got replaced by an edge copy $P_k = G_{i(k)}$ in $\mathscr{D}$. Since $F_\star$ is a real copy, we have $F_\star \neq G_{i(k)}$ and, therefore, we can again take $R_k = Q_{i(k)}$.

Suppose finally that $k \in K$ and that the piece $P_k = (f'_k)^+W_k(f''_k)^+$ is long. In $\mathscr{D}$ we have new indices $(f'_k)^+ = G_{i(k)}$, $W_k = r_{i(k)}$, and $(f''_k)^+ = G_{i(k)+1}$. Moreover, the pieces $Q_{i(k)}$ and $Q_{i(k)+1}$ are short and $R_k = Q_{i(k)}W_kQ_{i(k)+1}$ is a long $F_\star$-piece with the desired properties. $\square$

§9. $\mathfrak{Girth}$ in constructions

As we shall see in this section, the extension process and partite constructions are in perfect harmony with $\mathfrak{Girth}$.

9.1. **The extension lemma.** Roughly speaking, we shall prove now that the extension process described in Section 6 converts Girth into $\mathfrak{Girth}$—in the sense that if a hypergraph construction Ψ yields systems with large Girth, then the pretrain construction $\mathrm{Ext}(\Phi, \Psi)$ yields systems with large $\mathfrak{Girth}$, provided that Φ generates strongly induced copies. In other words, German $\mathfrak{Girth}$ seems to be the correct notion for analysing cycles in systems of pretrains obtained by means of the extension process.

Lemma 9.1. *Suppose that $g \geqslant 2$,*

- *that Φ is a linear ordered Ramsey construction for hypergraphs delivering systems with strongly induced copies*
- *and that Ψ denotes a Ramsey construction applicable to ordered hypergraphs M with $\mathrm{girth}(M) > g$ and producing ordered systems of hypergraphs $\Psi_r(M) = (N, \mathscr{N})$ with $\mathrm{Girth}(N, \mathscr{N}^+) > g$.*

If $(F, \equiv^F)$ denotes a pretrain with $\mathfrak{girth}(F, \equiv^F) > g$, then for every number of colours r the system

$$\mathrm{Ext}(\Phi, \Psi)_r(F, \equiv^F) = (H, \equiv^H, \mathscr{H})$$

is defined and satisfies $\mathfrak{Girth}(H, \equiv^H, \mathscr{H}^+) > g$.

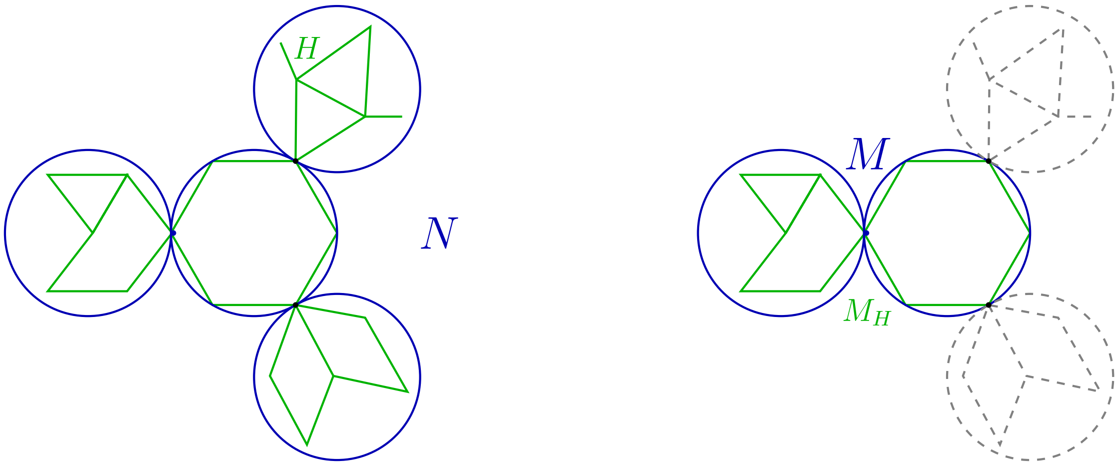
The proof of any statement addressing pretrain systems of the form $\mathrm{Ext}(\Phi, \Psi)_r(F, \equiv^F)$ ultimately needs to refer back to the eight-step description of the extension process given immediately before Lemma 6.11. It turns out, however, that our argument becomes both more transparent and more reusable on later occasions if we look at the construction from the perspective of the system called $(N, \equiv^N, \mathscr{N})$ there. For this reason some additional concepts seem to be useful. The first of them describes the relationship between the hypergraphs H and N.

Definition 9.2. Let H and N be two hypergraphs on the same vertex set. We say that H is *living* in N (see Figure 9.1a) if

(*i*) for every edge e of H there exists an edge f of N covering it,
(*ii*) and every $f \in E(N)$ induces a subhypergraph of H without isolated vertices.

Clearly if N is linear, then the edge f guaranteed by (*i*) is uniquely determined by e. Next we look at the transition from $\mathscr{N}$ to $\mathscr{H}_\bullet$ in Step (7) of the extension process.

Definition 9.3. Let $(N, \mathscr{N})$ be a linear system of hypergraphs. Suppose further that the hypergraph H is living in N. For every copy $M \in \mathscr{N}$ the subhypergraph M_H of H defined by $V(M_H) = V(M)$ and $E(M_H) = \bigcup_{f \in E(M)} E(H[f])$ is said to be *derived* from M (see Figure 9.1b). Setting $\mathscr{H} = \{M_H \colon M \in \mathscr{N}\}$ we call $(H, \mathscr{H})$ a system of copies *derived* from $(N, \mathscr{N})$.



(a) A graph H living in a 6-uniform hypergraph N

(b) The copy M_H is derived from M (which has two edges)

FIGURE 9.1

Let us briefly pause and check that being strongly induced is a property of copies preserved under such derivations.

Fact 9.4. *Let $(N, \mathscr{N})$ be a linear system of hypergraphs with strongly induced copies. If a hypergraph H is living in N, then the copies of the derived system $(H, \mathscr{H})$ are strongly induced as well.*

Proof. Consider an arbitrary copy $M \in \mathscr{N}$, its derived copy $F \in \mathscr{H}$, as well as an edge $e \in E(H)$. We are to prove that the set $x = V(F) \cap e$ can be covered by an appropriate edge e' of F. To this end we let $f \in E(N)$ be the edge covering e. Due to $M \blacktriangleleft N$ there is an edge $f' \in E(M)$ such that $x \subseteq V(M) \cap f \subseteq f'$.

If $|x| \geqslant 2$, then the linearity of N and $x \subseteq f \cap f'$ imply $f = f'$, whence $e \in E(F)$, which allows us to take $e' = e$. If, on the other hand, $|x| \leqslant 1$, then the fact that $H[f']$ has no isolated vertices yields an edge $e' \in E(H[f']) \subseteq E(F)$ such that $x \subseteq e'$. $\square$

Definition 9.5. Let $(N, \equiv^N)$ be a pretrain whose underlying hypergraph N is linear. Suppose further that the hypergraph H is living in N. The equivalence relation $\equiv^H$ on $E(H)$ *derived* from $\equiv^N$ is defined by declaring $e \equiv^H e'$ for two edges $e, e' \in E(H)$ if there are edges $f, f' \in E(N)$ such that $e \subseteq f$, $e' \subseteq f'$, and $f \equiv^N f'$. In this situation the pretrain $(H, \equiv^H)$ is said to be *derived* from $(N, \equiv^N)$.

Another way to think about this construction is that to every wagon W_N of $(N, \equiv^N)$ there corresponds a unique wagon W_H of $(H, \equiv^H)$ such that $E(W_H) = \bigcup_{f \in E(W_N)} E(H[f])$, called the wagon *derived* from W_N. Owing to Definition 9.2(*ii*) we have $V(W_N) = V(W_H)$. Combined with the fact that, conversely, every wagon of $(H, \equiv^H)$ is derived from some wagon of $(N, \equiv^N)$ this shows that if $(N, \equiv^N)$ and H are linear, then so is $(H, \equiv^H)$. Roughly speaking, our next result asserts that derivations cannot decrease $\mathfrak{Girth}$.

Lemma 9.6. *Let the linear hypergraph H be living in the linear hypergraph N. If a system of pretrains $(N, \equiv^N, \mathscr{N})$ satisfies $\mathfrak{Girth}(N, \equiv^N, \mathscr{N}^+) > g$ for some integer $g \geqslant 1$, then the system $(H, \equiv^H, \mathscr{H})$ derived from it satisfies $\mathfrak{Girth}(H, \equiv^H, \mathscr{H}^+) > g$ as well.*

Proof. Let δ be the bijective map assigning to each wagon of $(H, \equiv^H)$ the wagon of $(N, \equiv^N)$ it is derived from. We already know that the pretrain $(H, \equiv^H)$ is linear. Thus it remains to show that every acceptable big cycle

$$\mathscr{C} = F_1 q_1 \ldots F_n q_n$$

in $(H, \equiv^H, \mathscr{H}^+)$ with $\mathrm{ord}(\mathscr{C}) \leqslant g$ has a supreme copy. Assume for the sake of contradiction that $\mathscr{C}$ is a counterexample to this statement and, moreover, that among all possibilities $\mathscr{C}$ has been chosen in such a way that

$$\nu = \big|\big\{i \in \mathbb{Z}/n\mathbb{Z} \colon F_i \text{ is a real copy}\big\}\big|$$

is minimal.

Claim 9.7. *If $n \geqslant 3$, the index $i \in \mathbb{Z}/n\mathbb{Z}$ is mixed, and the vertex among q_{i-1}, q_i belongs to the wagon, then F_i is an edge copy.*

Proof. By symmetry it suffices to deal with the case that q_{i-1} is a vertex, q_i is a wagon, and $q_{i-1} \in V(q_i)$. Assume towards a contradiction that F_i is a real copy. Pick an edge $e_i \in E(H)$ with $q_{i-1} \in e_i \in E(q_i)$ and denote the cyclic sequence obtained from $\mathscr{C}$ upon collapsing F_i to e_i^+ by $\mathscr{D}$. Owing to Lemma 8.22(*a*) applied to $\{i\}$ and e_i^+ here in place of K and P_i there $\mathscr{D}$ is an acceptable big cycle. (Notice that the clauses (α)–(γ) of Lemma 8.22 are irrelevant here, because the piece e_i^+ is short).

Now $\mathrm{ord}(\mathscr{D}) = \mathrm{ord}(\mathscr{C}) \leqslant g$ and $\mathscr{D}$ contains fewer real copies than $\mathscr{C}$. So by the minimality of ν we know that $\mathscr{D}$ possesses a supreme copy. Due to Lemma 8.22(*b*) the cycle $\mathscr{C}$ has a supreme copy as well, contrary to the choice of $\mathscr{C}$. □

After these preliminaries we briefly describe our strategy for finding a supreme copy of $\mathscr{C}$, which consists of three major steps.

(*1*) We translate $\mathscr{C}$ into a cyclic sequence $\mathscr{D}$ with respect to the system $(N, \equiv^N, \mathscr{N}^+)$.

(*2*) Second we check that $\mathscr{D}$ is an acceptable big cycle whose order is at most g. Now the hypothesis $\mathfrak{Girth}(N, \equiv^N, \mathscr{N}^+) > g$ yields a supreme copy $M_\star$ of $\mathscr{D}$ and a family $\mathscr{Q}$ of pieces witnessing the supremacy of $M_\star$.

(*3*) Finally, we need to translate $M_\star$ and $\mathscr{Q}$ back to a supreme copy $F_\star$ of $\mathscr{C}$ and a family $\mathscr{P}$ of $F_\star$-pieces witnessing the supremacy of $F_\star$.

The first step of this plan depends on the notion of a twin in $\mathscr{C}$, that we shall now explain. Suppose that $Z = (e')^+W(e'')^+$ is a subsequence of $\mathscr{C}$, where $e', e'' \in E(H)$ are edges and W is a wagon connector. Owing to (B2) and (B4) the other connectors of $\mathscr{C}$ next to $(e')^+$ and $(e'')^+$ are vertices.

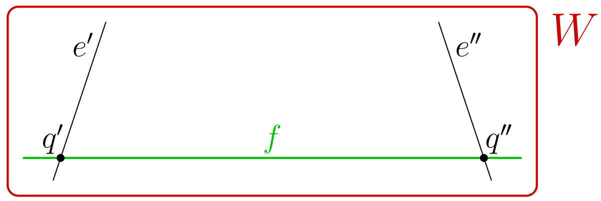


FIGURE 9.2. A twin $(e')^+W(e'')^+$ with conductor f.

In other words, $\mathscr{C}$ actually has a subsequence of the form $q'Zq''$, where q' and q'' are vertices. If there exists an edge $f \in E(N)$ containing q' and q'' we call Z a *twin* and an edge f verifying this fact is said to be a *conductor* of Z (see Figure 9.2).

Due to (A1) there cannot exist any twins if $n = 2$. Moreover, if for $n \geqslant 3$ we have a twin $Z = (e')^+X(e'')^+$, then the vertices q', q'' mentioned in the previous paragraph are distinct and by the linearity of N the conductor of Z is uniquely determined.

Clearly the twins are mutually non-overlapping in the sense that each edge copy F_i belongs to at most one twin. So by symmetry we may suppose that F_1 and F_n do not appear together in a twin.

We say that

$$\mathscr{C} = Z_1 r_1 \dots Z_m r_m \tag{9.1}$$

is the *twin decomposition* of $\mathscr{C}$ if $\{r_1, \dots, r_m\} \subseteq \{q_1, \dots, q_n\}$, every Z_j is either a single edge copy, a single real copy, or a twin, and conversely every twin is in the set $\{Z_1, \dots, Z_m\}$. We have $m \geqslant 2$, as there are no twins in the case $n = 2$. Set

$$\overline{r}_j = \begin{cases} \delta(r_j) & \text{if } r_j \text{ is a wagon} \\ r_j & \text{if } r_j \text{ is a vertex} \end{cases}$$

for every $j \in \mathbb{Z}/m\mathbb{Z}$. According to the rules that follow we shall determine certain copies $M_1, \dots, M_m \in \mathscr{N}^+$, the intention being that

$$\mathscr{D} = M_1 \overline{r}_1 \dots M_m \overline{r}_m$$

will turn out to be the desired big cycle in $(N, \equiv^N, \mathscr{N}^+)$. Let $j \in \mathbb{Z}/m\mathbb{Z}$ be given.

(*a*) If Z_j is a real copy, let $M_j \in \mathscr{N}$ be the copy it is derived from.
(*b*) If $Z_j = e_j^+$ is an edge copy, let $f_j \in E(N)$ be the edge covering e_j and set $M_j = (f_j)^+$.
(*c*) Finally, if $Z_j = (e_j')^+W_j(e_j'')^+$ is a twin with conductor $f_j \in E(N)$, then we set $M_j = f_j^+$.

Having thus defined $\mathscr{D}$ we proceed with the second part of our plan.

Claim 9.8. *The cyclic sequence $\mathscr{D}$ is a big cycle in $(N, \equiv^N, \mathscr{N}^+)$.*

Proof. The demands $(B2)$, $(B3)$ are clear and $(B4)$ follows easily from the fact that twins are surrounded by vertex connectors. So it remains to prove $(B1)$.

Assume contrariwise that some copy $M_\star$ occurs twice in consecutive positions in $\mathscr{D}$, say $M_\star = M_j = M_{j+1}$ for some $j \in \mathbb{Z}/m\mathbb{Z}$. Due to (a) and the fact that $\mathscr{C}$ satisfies $(B1)$ we know that $M_\star = f_\star^+$ is an edge copy. Let $X_\star$ be the wagon of $(N, \equiv^N)$ containing $f_\star$ and write $W_\star = \delta^{-1}(X_\star)$ for its derived wagon. Each of the three rules (a)–(c) could in principle give rise to the edge copy $f_\star^+$. Depending on which of them was used in the definitions of M_j, M_{j+1} there arise three possibilities for each of Z_j, Z_{j+1}. Both of them are either the copy $H[f_\star]$ derived from $f_\star^+$ (as in (a)), a single edge copy covered by $f_\star$ (as in (b)), or a twin conducted by $f_\star$ (as in (c)). Furthermore, each of the connectors r_{j-1}, r_j, and r_{j+1} is either $W_\star$ or a vertex belonging to $f_\star$.

First case: We have $m = 2$

Suppose first that none of Z_j, Z_{j+1} is a twin, so that $n = 2$. Due to the linearity of H and $(A1)$ it cannot be the case that $\mathscr{C}$ consists of two edge copies. Thus $H[f]$ appears in $\mathscr{C}$ and by collapsing the other copy, which has to be an edge copy, to itself we see that $H[f]$ is a supreme copy of $\mathscr{C}$, contrary to the choice of $\mathscr{C}$.

Thus we can assume by symmetry that Z_j is a twin, which causes the connectors of $\mathscr{D}$ to be vertices. As the wagon in the middle of Z_j contains those vertices, the linearity of $(N, \equiv^N)$ shows that this wagon has to be $W_\star$. Now Z_{j+1} cannot be a twin as well (because it also would need to contain the wagon $W_\star$), and by $(A2)$ Z_{j+1} cannot be a single edge copy; so only the case $Z_{j+1} = H[f]$ remains. Utilising the fact that this copy has no isolated vertices we find two edges $e', e'' \in E(H[f])$ containing the connectors r_{j-1}, r_j. By collapsing the two edge copies in Z_j to $(e')^+$ and $(e'')^+$ we see that $H[f]$ is a supreme copy, which again contradicts the choice of $\mathscr{C}$.

Second case: We have $n \geqslant m \geqslant 3$

Now r_{j-1}, r_j, and r_{j+1} are three distinct connectors and due to $|M^{\mathscr{C}}(W_\star)| \leqslant 2$ one of them is equal to $W_\star$, while the two other ones are vertices in $f_\star$. If $r_j = W_\star$, then Claim 9.7 tells us that Z_j, Z_{j+1} are single edge copies. So $Z_j r_j Z_{j+1}$ is a twin conducted by $f_\star$, which contradicts the fact that (9.1) is the twin decompositions of $\mathscr{C}$.

By symmetry it remains to consider the case that $r_{j-1} = W_\star$, whilst r_j and r_{j+1} are vertices. Since the wagon $W_\star$ satisfies $(A2)$ with respect to $\mathscr{C}$, this is only possible if r_{j-1} and r_{j+1} are consecutive in $\mathscr{C}$. So $m = 3$ and Claim 9.7 implies that Z_{j-1}, Z_j are single edge copies, which in turn causes $Z_{j-1} r_{j-1} Z_j$ to be a twin with conductor $f_\star$. Again this contradicts (9.1) being the twin decomposition of $\mathscr{C}$. □

Claim 9.9. *The big cycle $\mathscr{D}$ is acceptable.*

Proof. Assume first that contrary to (A1) we have $\mathscr{D} = f_1^+ X f_2^+ x$, where $f_1, f_2 \in E(N)$, the connector X is a wagon of $(N, \equiv^N)$, and x is a vertex. Since twins are surrounded by vertex connectors, there cannot be any twins in $\mathscr{C}$. Thus $\mathscr{C}$ is of the form $\mathscr{C} = F_1 W F_2 x$, where W is derived from X. As $\mathscr{C}$ is required to satisfy (*A*1), we may assume that F_1 is real, and by (*a*) $F_1 = H[f_1]$ is the copy derived from $f_1^+ \in \mathscr{N}^+$. So F_1 has no isolated vertices and due to $x \in f_1$ there exists an edge $e_2 \in E(F_1)$ passing through x. By collapsing F_2 to e_2^+ we see that F_1 is a supreme copy of $\mathscr{C}$, which contradicts the assumption that $\mathscr{C}$ be a counterexample.

Working towards (A2) we consider any wagon connector $\overline{r}_j$ of $\mathscr{D}$. If $r_j = q_i$, then the acceptability of $\mathscr{C}$ yields $M^{\mathscr{C}}(q_i) \subseteq \{i-1, i+1\}$, whence $M^{\mathscr{D}}(\overline{r}_j) \subseteq \{j-1, j+1\}$. Now suppose that $|M^{\mathscr{D}}(\overline{r}_j)| = 2$ and that some edge $f \in E(N)$ satisfies $r_{i-1}, r_{i+1} \in f$. Evidently this is only possible if $n \geqslant m \geqslant 3$ and $M^{\mathscr{D}}(\overline{r}_j) = \{j-1, j+1\}$, which in turn implies $M^{\mathscr{C}}(q_i) = \{i-1, i+1\}$. Claim 9.7 tells us that F_i, F_{i+1} are edge copies and, therefore, $F_i q_i F_{i+1}$ is a twin conducted by f. This contradiction to the fact that (9.1) is the twin decomposition of $\mathscr{C}$ establishes that $\mathscr{D}$ satisfies the moreover-part of (*A*2) as well.

It remains to verify (*A*3). To this end we consider an arbitrary wagon $W_\star$ of $(H, \equiv^H)$ such that $X_\star = \delta(W_\star)$ fails to be a wagon connector of $\mathscr{D}$. If $W_\star$ is a connector of $\mathscr{C}$, then it appears in the middle of a twin and by (*A*2) the set $M^{\mathscr{D}}(X_\star)$ can be covered by a pair of consecutive indices. If, on the other hand, $W_\star$ is absent from $\mathscr{C}$, we can appeal to (*A*3). Thereby Claim 9.9 is proved. □

Owing to $\mathrm{ord}(\mathscr{D}) = \mathrm{ord}(\mathscr{C}) \leqslant g$ and $\mathfrak{Girth}(N, \equiv^N, \mathscr{N}^+) > g$ Claim 9.9 shows that $\mathscr{D}$ has a supreme copy $M_\star$. Let the family of $M_\star$-pieces

$$\mathscr{Q} = \{Q_j \colon j \in \mathbb{Z}/m\mathbb{Z} \text{ and } M_j \neq M_\star\}$$

exemplify the supremacy of $M_\star$. Recall that $M_\star$ is a real copy by Remark 8.8(*2*), whence there is a real copy $F_\star \in \mathscr{H}$ derived from it and appearing in $\mathscr{C}$. Coming to Step (*3*) of the plan outlined above we shall now show that $F_\star$ is a supreme copy of $\mathscr{C}$.

Claim 9.10. *For the index set $K = \{i \in \mathbb{Z}/n\mathbb{Z} \colon F_i \neq F_\star\}$ there is a family*

$$\mathscr{P} = \{P_i \colon i \in K\}$$

of $F_\star$-pieces such that for every $i \in K$ the following holds.

(*1*) *The cyclic sequence obtained from $\mathscr{C}$ upon replacing F_i by P_i has the properties (B3) and (B4).*

(*2*) *If the piece P_i is long, then it satisfies the clauses (α)–(γ) from Lemma 8.22.*

Proof. Given any $i \in K$ we need to explain how to find the required piece P_i. Let us start with the case that for some twin $Z_j = (e'_j)^+ W_j (e''_j)^+$, say with conductor f_j, we have $F_i \in \{(e'_j)^+, (e''_j)^+\}$. By Definition 8.5(*ii*)(β) applied to $\mathscr{D}$ and $(N, \equiv^N, \mathscr{N}^+)$ here rather than $\mathscr{C}$ and $(H, \equiv, \mathscr{H}^+)$ there the piece Q_j to which $M_j = f_j^+$ collapses is short; due to the linearity of N the only possibility is $Q_j = f_j^+$, which implies $f_j \in E(M_\star)$. Since $H[f_j]$ has no isolated vertices, we can collapse $(e'_j)^+$, $(e''_j)^+$ to two edge copies $(e^\star_j)^+$, $(e^{\star\star}_j)^+$ with $r_{j-1} \in e^\star_j \in E(H[f_j])$ and $r_j \in e^{\star\star}_j \in E(H[f_j])$.

So from now on we can assume that $q_{i-1} F_i q_i = r_{j-1} Z_j r_j$ holds for some $j \in \mathbb{Z}/m\mathbb{Z}$. Because of $i \in K$ we have $M_j \neq M_\star$ and, therefore, $\mathscr{Q}$ provides an $M_\star$-piece Q_j.

First Case. The $M_\star$-piece $Q_j = f_j^+$ is short.

Suppose first that one of the connectors q_{i-1} and q_i, say q_i, is a wagon. Now q_{i-1} needs to be a vertex, there exists an edge $e_i \in H[f_j]$ with $q_{i-1} \in e_i$, and the short $F_\star$-piece $P_i = e_i^+$ is as desired.

It remains to consider the case that q_{i-1} and q_i are vertices. If there exists an edge $e_i \in E(H)$ containing both of them, then the linearity of N yields $e_i \subseteq f_j$ and it is permissible to set $P_i = e_i^+$. From now on we assume that such an edge e_i does not exist. If e'_i, e''_i are two edges of $H[f_j]$ with $q_{i-1} \in e'_i$ as well as $q_i \in e''_i$, and W_i denotes the unique wagon of $(H, \equiv^H)$ with $f_j \in E(\delta(W_i))$, then $P_i = (e'_i)^+ W_i (e''_i)^+$ is a long $F_\star$-piece satisfying (*1*) and the demands (α), (β) mentioned in (*2*). If (γ) fails, then the first part of (*A2*) discloses $n = 3$ and $q_{i+1} = W_i$. But then $q_{i-1}, q_i \in V(q_{i+1})$, so by Claim 9.7 both F_{i+1} and F_{i+2} are edge copies. Thus the supreme copy $M_\star$ fails to appear on $\mathscr{D}$, which is absurd.

Second Case. The $M_\star$-piece $Q_j = (f'_j)^+ X_j (f''_j)^+$ is long.

Here X_j is a wagon of $(N, \equiv^N)$ that fails to appear on $\mathscr{D}$, whence $W_j = \delta^{-1}(X_j)$ is a wagon of $(H, \equiv^H)$ that is distinct from $r_1, \ldots, r_m$. As $H[f'_j]$ and $H[f''_j]$ have no isolated vertices, there are edges $e'_j, e''_j \in E(F_\star)$ such that $r_{j-1} \in e'_j$ and $r_j \in e''_j$. Now $P_i = (e'_j)^+ W_j (e''_j)^+$ is an $F_\star$-piece satisfying (α) and (β), because Q_j has these properties as well. Proceeding with (γ) we assume contrariwise that $W_j \in \{q_1, \ldots, q_n\}$. As we already know $W_j \notin \{r_1, \ldots, r_m\}$ this is only possible if W_j is in the middle of some twin Z_k, where $k \in (\mathbb{Z}/m\mathbb{Z}) \smallsetminus \{j\}$. Now $\{j-1, j, k-1, k\} \subseteq M^{\mathscr{D}}(X_j)$ and the acceptability of $\mathscr{D}$ yield $m = 2$. Combined with $M_\star \neq M_k, M_j$ this contradicts the fact that $M_\star$ appears in $\mathscr{D}$. □

If $n = 2$ this shows immediately that $F_\star$ is a supreme copy of $\mathscr{C}$ and for $n \geqslant 3$ we need to point out additionally that owing to Lemma 8.22(*a*) all collapses suggested by the claim can be carried out simultaneously. □

The next concept is motivated by the relationship between the systems $\mathscr{H}_\bullet$ and $\mathscr{H}$ occurring in Step (*7*) of the extension process.

Definition 9.11. *(a)* If a pretrain $(G, \equiv^G)$ is the disjoint union of the family of pretrains $\{(G_i, \equiv^i) \colon i \in I\}$ and for every $i \in I$ the pretrain $(G_i, \equiv^i)$ is a tame extension of its subpretrain F_i, then the system of pretrains $\mathscr{S}_G = \{F_i \colon i \in I\}$ is said to be *scattered* in G.
(b) Let $(H, \equiv^H, \mathscr{H}_\bullet)$ be a system of pretrains. If for every copy $G \in \mathscr{H}_\bullet$ we have a system $\mathscr{S}_G$ of subpretrains scattered in G, then the system $\mathscr{H} = \bigcup_{G \in \mathscr{H}_\bullet} \mathscr{S}_G$ is said to be *scattered* in $\mathscr{H}_\bullet$.

Lemma 9.12. *Let $(H, \equiv^H, \mathscr{H}_\bullet)$ be a system of pretrains satisfying $\mathfrak{Girth}(H, \equiv^H, \mathscr{H}_\bullet^+) > g$ for some integer $g \geqslant 1$. If $\mathscr{H}$ is scattered in $\mathscr{H}_\bullet$, then $\mathfrak{Girth}(H, \equiv^H, \mathscr{H}^+) > g$ follows.*

Proof. As in Definition 9.11*(b)* we write $\mathscr{H} = \bigcup_{G \in \mathscr{H}_\bullet} \mathscr{S}_G$, where for every copy $G \in \mathscr{H}_\bullet$ the system $\mathscr{S}_G$ is scattered in G.

Claim 9.13. *Suppose $(G_\star, \equiv^{G_\star}) \in \mathscr{H}_\bullet$ and that $\mathscr{C} = F_1 q_1 \dots F_n q_n$ is an acceptable big cycle in $(H, \equiv^H, \mathscr{H}^+)$ with $\mathrm{ord}(\mathscr{C}) \leqslant g$. If all copies of $\mathscr{C}$ belong to $\mathscr{S}_{G_\star} \cup E^+(G_\star)$, then $\mathscr{C}$ has a supreme copy.*

Proof. Since $\mathscr{S}_{G_\star}$ is scattered in $G_\star$, we can express $G_\star$ as a disjoint union of a family $\{G_i \colon i \in I\}$ of subpretrains such that for every member of $\mathscr{S}_{G_\star}$ there is a unique G_i tamely extending it. The entire cycle $\mathscr{C}$ cannot jump from one G_i to another one and thus there is a pretrain $G_\circ$ in this family such that all edge copies of $\mathscr{C}$ correspond to edges of $G_\circ$ and all real copies of $\mathscr{C}$ coincide with the unique copy $F_\circ \in \mathscr{S}_{G_\star}$ contained in $G_\circ$. Due to $\mathfrak{Girth}(H, \equiv^H, \mathscr{H}_\bullet^+) > g$ only the case that $F_\circ$ is a real copy appearing on $\mathscr{C}$ is interesting. We shall show that then $F_\circ$ itself is a supreme copy of $\mathscr{C}$.

As a first step towards this goal we prove that all vertex connectors of $\mathscr{C}$ belong to $F_\circ$. Assume (reductio ad absurdum) that this fails for some vertex connector q_i. Now F_i, F_{i+1} are edge copies and $n \geqslant 3$, so Lemma 8.21 shows that the underlying edges of F_i, F_{i+1} belong to distinct wagons, say W_i, W_{i+1}. Since $G_\circ$ is an extension of $F_\circ$, we have indeed $q_i \in V(W_i) \cap V(W_{i+1}) \subseteq V(F_\circ)$.

Now let $K = \{i \in \mathbb{Z}/n\mathbb{Z} \colon F_i \ne F_\circ\}$. We shall show that for every $i \in K$ there is a short $F_\circ$-piece P_i such that in $\mathscr{C}$ we can collapse F_i to P_i. If $n = 2$ this will show immediately that $F_\circ$ is a supreme copy of $\mathscr{C}$ (because then we have $|K| = 1$), and if $n \geqslant 3$ Lemma 8.22*(a)* tells us that all these collapses can be carried out simultaneously.

Consider an arbitrary index $i \in K$. We already know that $F_i = f_i^+$ is an edge copy. If the connectors q_{i-1} and q_i are vertices, then $F_\circ \blacktriangleleft G_\circ$ and $q_{i-1}, q_i \in V(F_\circ) \cap f_i$ imply $f_i \in E(F_\circ)$, for which reason it is permissible to set $P_i = F_i$.

If, on the other hand, one of the connectors q_{i-1}, q_i, say q_i, is a wagon, then q_{i-1} is a vertex and the moreover-part of Definition 6.7 shows that there is an edge e_i such that $q_{i-1} \in e_i \in E(q_i) \cap E(F_\circ)$. Clearly $P_i = e_i^+$ is as desired. $\square$

Now let $\mathscr{C} = F_1q_1 \dots F_nq_n$ be an arbitrary acceptable big cycle in $(H, \equiv^H, \mathscr{H}^+)$ whose order is at most g. We would like to construct an auxiliary acceptable big cycle

$$\mathscr{D} = G_1q_1 \dots G_nq_n$$

in $(H, \equiv^H, \mathscr{H}_\bullet^+)$, which inherits its connectors from $\mathscr{C}$. To this end we associate with every copy $F_i \in \mathscr{H}^+$ a copy $G_i \in \mathscr{H}_\bullet^+$ such that

- either $F_i = G_i$ is an edge copy
- or F_i is a real copy and $F_i \in \mathscr{S}_{G_i}$.

Evidently $\mathscr{D}$ has the properties (*B*2)–(*B*4) of an acceptable big cycle in $(H, \equiv^H, \mathscr{H}_\bullet^+)$. Moreover, if some copy $G_\star \in \mathscr{H}_\bullet^+$ appears twice in consecutive positions of $\mathscr{D}$, then $G_\star$ is a real copy and we get a contradiction to the fact that $\mathscr{H}_{G_\star}$ is scattered in $G_\star$. So altogether, $\mathscr{D}$ is a big cycle.

The acceptability of $\mathscr{D}$ and $\mathrm{ord}(\mathscr{D}) = \mathrm{ord}(\mathscr{C}) \leqslant g$ are clear. Thus our assumption $\mathfrak{Girth}(H, \equiv^H, \mathscr{H}_\bullet^+) > g$ leads to a supreme copy $G_\star$ of $\mathscr{D}$. Recall that $G_\star$ is a real copy by Remark 8.8(*2*). Set $K = \{i \in \mathbb{Z}/n\mathbb{Z} \text{ and } G_i \neq G_\star\}$ and let the family of $G_\star$-pieces $\{P_i : i \in K\}$ exemplify the supremacy of $G_\star$ in $\mathscr{D}$. Starting with $\mathscr{C}$ we can still replace every copy F_i with $i \in K$ by the piece P_i. If $n = 2$ this shows immediately that $\mathscr{C}$ has a supreme copy and for $n \geqslant 3$ Lemma 8.22(*a*) tells us that we obtain an acceptable big cycle $\mathscr{E}$ in this manner. Due to Claim 9.13 $\mathscr{E}$ has a supreme copy and by Lemma 8.22(*b*) so does $\mathscr{C}$. $\square$

Proof of Lemma 9.1. We assume that the reader has the eight-step description (*1*)–(*8*) of $\mathrm{Ext}(\Phi, \Psi)$ given immediately before Lemma 6.11 on the desk and without further explanation we use the same notation as there. Let us remark that the pretrain $(G, \equiv^G)$ generated in Step (*3*) satisfies $\mathfrak{girth}\big(G, \equiv^G\big) > g$, whence $\mathrm{girth}(M) > g$. So the application of $\Psi_{r^{e(X)}}$ in Step (*5*) is justified, meaning that the system $(H, \equiv^H, \mathscr{H}^+)$ does indeed exist. It remains to establish that its $\mathfrak{Girth}$ exceeds g.

Let $\equiv^N$ be the equivalence relation on $E(N)$ whose equivalence classes are single edges. By Step (*6*) the hypergraph H is living in N and the pretrain $(H, \equiv^H)$ is derived from $(N, \equiv^N)$. Similarly, the first part of Step (*7*) says that the system $\mathscr{H}_\bullet$ is derived from $\mathscr{N}$. Due to $\mathrm{Girth}(N, \mathscr{N}^+) > g$ and Lemma 8.18 we know $\mathfrak{Girth}(N, \equiv^N, \mathscr{N}^+) > g$ and, therefore, Lemma 9.6 discloses $\mathfrak{Girth}(H, \equiv^H, \mathscr{H}_\bullet^+) > g$.

For every copy $(G_\star, \equiv^{G_\star}) \in \mathscr{H}_\bullet$ the $|\mathscr{X}|$ standard copies of $(F, \equiv^F)$ form a scattered system (cf. Lemma 6.8) and thus $\mathscr{H}$ is scattered in $\mathscr{H}_\bullet$. Now $\mathfrak{Girth}(H, \equiv^H, \mathscr{H}^+) > g$ follows from Lemma 9.12. $\square$

9.2. **$\mathfrak{Girth}$ preservation.** The next result and its proof share strong similarities with the material in Section 5. However, the focus there was on making incremental progress, whereas here we only care about maintaining what we already have.

Proposition 9.14. *Suppose that $g \geqslant 2$ and that Ξ is a partite lemma*

- *applicable to k-partite k-uniform pretrains $(B, \equiv^B)$ with $\mathfrak{girth}(B, \equiv^B) > g$*
- *and producing k-partite k-uniform systems of pretrains $(H, \equiv^H, \mathscr{H})$ with*

$$\mathfrak{Girth}(H, \equiv^H, \mathscr{H}^+) > g\,.$$

If Φ denotes an arbitrary linear Ramsey construction for hypergraphs generating systems of strongly induced copies, then the pretrain construction $\mathrm{PC}(\Phi, \Xi)$

- *applies to all pretrains $(F, \equiv^F)$ with $\mathfrak{girth}(F, \equiv^F) > g$*
- *and delivers systems of pretrains $(H, \equiv^H, \mathscr{H})$ with $\mathfrak{Girth}(H, \equiv^H, \mathscr{H}^+) > g$.*

One may briefly wonder why this is a useful result. After all, the main property $\mathrm{PC}(\Phi, \Xi)$ is shown to have is already assumed for Ξ. Indicating only one example of a successful application of Proposition 9.14 we would like to point out that by Lemma 3.4 the copies in the systems produced by $\mathrm{PC}(\Phi, \Xi)$ will always have clean intersections, while this need not be the case for the systems generated by Ξ (whose copies can intersect in entire wagons). For this reason, $\mathrm{PC}(\Phi, \Xi)$ can have more desirable properties than Ξ itself. As usual, the main work going into the proof of Proposition 9.14 concerns a picturesque lemma.

Lemma 9.15. *Suppose that $g \geqslant 2$ and that*

$$(\Sigma, \equiv^\Sigma, \mathscr{Q}, \psi_\Sigma) = (\Pi, \equiv^\Pi, \mathscr{P}, \psi_\Pi) * (H, \equiv^H, \mathscr{H}) \tag{9.2}$$

holds for two pretrain pictures $(\Pi, \equiv^\Pi, \mathscr{P}, \psi_\Pi)$ and $(\Sigma, \equiv^\Sigma, \mathscr{Q}, \psi_\Sigma)$ over a linear system $(G, \mathscr{G})$ with strongly induced copies and for a k-partite k-uniform system of pretrains $(H, \equiv^H, \mathscr{H})$. If

$$\mathfrak{Girth}(\Pi, \equiv^\Pi, \mathscr{P}^+) > g \quad \text{and} \quad \mathfrak{Girth}(H, \equiv^H, \mathscr{H}^+) > g\,,$$

then $\mathfrak{Girth}(\Sigma, \equiv^\Sigma, \mathscr{Q}^+) > g$.

Proof. The picturesque statement in the proof of Lemma 3.11 reveals that Σ is a linear hypergraph. So it remains to establish

(*a*) that any two wagons of $(\Sigma, \equiv^\Sigma)$ intersect in at most one vertex
(*b*) and that every acceptable big cycle $\mathscr{C}$ in $(\Sigma, \equiv^\Sigma, \mathscr{Q}^+)$ with $\mathrm{ord}(\mathscr{C}) \leqslant g$ has a supreme copy.

Stage A: First observations. The assumption that the copies in $\mathscr{G}$ be strongly induced will be used in the following way.

Claim 9.16. *If $F_\star \in \mathscr{Q}^+$ and $x, y \in V(F_\star) \cap V(H)$ are two distinct vertices, then there exists an edge f with $x, y \in f \in E(F_\star) \cap E(H)$.*

Proof. Let the amalgamation (9.2) be constructed over the edge $e \in E(G)$. Since $F_\star$ intersects every music line of the pretrain picture $(\Sigma, \equiv^\Sigma, \mathscr{Q}, \psi_\Sigma)$ at most once, the vertices $\psi_\Sigma(x), \psi_\Sigma(y) \in e \cap \psi_\Sigma[V(F_\star)]$ are distinct. So if $F_\star = f_\star^+$ is an edge copy, then the linearity of G yields $\psi_\Sigma[f_\star] = e$, for which reason $f = f_\star$ is as desired. If the copy $F_\star$ is real, then ψ_Σ projects it into $\mathscr{G}$ and thus onto a strongly induced subhypergraph of G. In particular, this projection needs to contain the edge e. The edge $f \in E(F_\star)$ projected to e has the desired property. □

We proceed by transferring the hypothesis $\mathfrak{Girth}(\Pi, \equiv^\Pi, \mathscr{P}^+) > g$ to standard copies. More precisely, we deal with the special case of cycles in $(\Sigma, \equiv^\Sigma, \mathscr{Q}^+)$ all of whose copies belong to the same standard copy of the previous picture.

Claim 9.17. *Let $(\Pi_\star, \equiv^{\Pi_\star}, \mathscr{P}_\star)$ be a standard copy. If all copies of an acceptable big cycle $\mathscr{C}$ in $(\Sigma, \equiv^\Sigma, \mathscr{Q}^+)$ with $\mathrm{ord}(\mathscr{C}) \leqslant g$ belong to $\mathscr{P}_\star^+$, then $\mathscr{C}$ possesses a supreme copy.*

Proof. Let $\mathscr{C} = F_1 q_1 \ldots F_n q_n$. Strictly speaking, $\mathscr{C}$ does not need to be a big cycle in the pretrain system $(\Pi_\star, \equiv^{\Pi_\star}, \mathscr{P}_\star^+)$, for the wagon connectors of $\mathscr{C}$ are wagons of $\equiv^\Sigma$ rather than wagons of $\equiv^{\Pi_\star}$. However, (*B*4) shows that every wagon among $q_1, \ldots, q_n$ contracts to $(\Pi_\star, \equiv^{\Pi_\star})$. Thus we can define

$$\overline{q}_i = \begin{cases} \text{the contraction of } q_i \text{ to } (\Pi_\star, \equiv^{\Pi_\star}), & \text{if } q_i \text{ is a wagon} \\ q_i, & \text{if } q_i \text{ is a vertex} \end{cases}$$

for every $i \in \mathbb{Z}/n\mathbb{Z}$ and then

$$\overline{\mathscr{C}} = F_1 \overline{q}_1 \ldots F_n \overline{q}_n$$

will be a big cycle in the system of pretrains $(\Pi_\star, \equiv^{\Pi_\star}, \mathscr{P}_\star^+)$ with $\mathrm{ord}(\overline{\mathscr{C}}) = \mathrm{ord}(\mathscr{C}) \leqslant g$. It is easily confirmed that $\overline{\mathscr{C}}$ is acceptable and, therefore, $\overline{\mathscr{C}}$ possesses a supreme copy $F_\star \in \{F_1, \ldots, F_n\}$. Let

$$\overline{\mathscr{M}} = \{\overline{P}_i \colon i \in \mathbb{Z}/n\mathbb{Z} \text{ and } F_i \neq F_\star\}$$

be a family of $F_\star$-pieces with respect to the ambient system $(\Pi_\star, \equiv^{\Pi_\star}, \mathscr{P}_\star^+)$ that exemplifies the supremacy of $F_\star$ in $\overline{\mathscr{C}}$.

We intend to show that $F_\star$ is a supreme copy of $\mathscr{C}$ as well and to this end we need to convert $\overline{\mathscr{M}}$ into an appropriate family of $F_\star$-pieces with respect to $(\Sigma, \equiv^\Sigma, \mathscr{Q}^+)$. For every index i such that $\overline{P}_i$ is short we just set $P_i = \overline{P}_i$. If $\overline{P}_i = (f_i')^+ \overline{W}_i (f_i'')^+$ is long, however, then we put $P_i = (f_i')^+ W_i (f_i'')^+$, where W_i denotes the wagon of $(\Sigma, \equiv^\Sigma)$ contracting to $\overline{W}_i$.

It will turn out that

$$\mathscr{M} = \{P_i \colon i \in \mathbb{Z}/n\mathbb{Z} \text{ and } F_i \neq F_\star\}$$

is the desired family of $F_\star$-pieces. The main point to be checked here is that the long pieces in this family still satisfy Definition 8.5(*ii*)(*β*) with respect to the larger pretrain $(\Sigma, \equiv^\Sigma)$. Assume for the sake of contradiction that P_i is a long piece and that some edge f

in $E(\Sigma) \smallsetminus E(\Pi_\star)$ contains the connectors q_{i-1}, q_i. Due to the linearity of G this requires $f \in E(H)$. But now Lemma 8.14 applied to the system $(H, \equiv^H, \mathscr{H}^+)$ shows that f belongs to the copy $\Pi^e_\star \in \mathscr{H}$ extended by $\Pi_\star$, contrary to $f \notin E(\Pi_\star)$. This concludes the proof that $F_\star$ is a supreme copy of $\mathscr{C}$. □

Next we study a peculiar kind of big cycles in our amalgamation that will occasionally require a special treatment later.

Claim 9.18. *Let $(\Pi_\star, \mathscr{P}_\star)$ and $(\Pi_{\star\star}, \mathscr{P}_{\star\star})$ be two distinct standard copies of $(\Pi, \mathscr{P})$. If*

$$\mathscr{C} = F_1 W_\star F_2 x F_3 W_{\star\star} F_4 y$$

is a big cycle in $(\Sigma, \equiv^\Sigma, \mathscr{Q}^+)$ such that the connectors x, y are vertices, $W_\star$, $W_{\star\star}$ are wagons, and $F_1, F_2 \in \mathscr{P}^+_\star$ as well as $F_3, F_4 \in \mathscr{P}^+_{\star\star}$, then there exists an edge $f \in E(H)$ with $x, y \in f$.

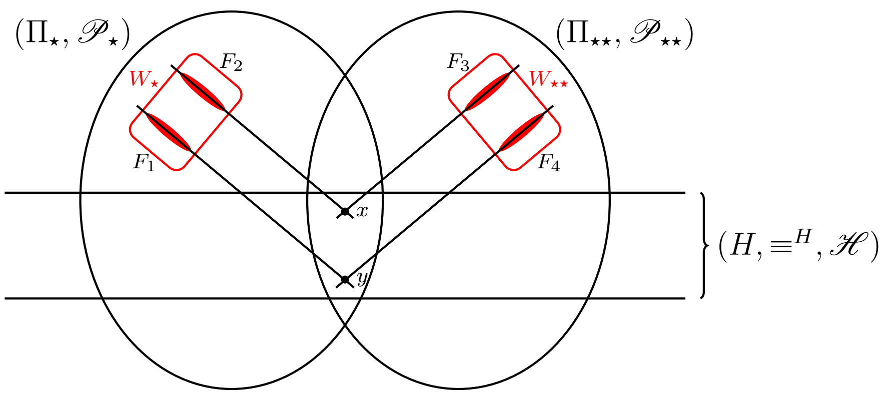


FIGURE 9.3. The configuration analysed in Claim 9.18

Proof. Notice that $x, y \in V(H)$. Due to the linearity of the pretrain $(H, \equiv^H)$ it cannot be the case that $W_\star$, $W_{\star\star}$ contract to wagons $W^H_\star$, $W^H_{\star\star}$ of $(H, \equiv^H)$ with $x, y \in V(W^H_\star) \cap V(W^H_{\star\star})$. By symmetry we may therefore suppose that

$$\text{if the contraction } W^H_\star \text{ exists, then } \{x, y\} \nsubseteq V(W^H_\star). \tag{9.3}$$

Let $\Pi^e_\star, \Pi^e_{\star\star} \in \mathscr{H}$ be the copies extended by $\Pi_\star$, $\Pi_{\star\star}$. Now $\mathscr{D} = \Pi^e_\star x \Pi^e_{\star\star} y$ is a big cycle in $(H, \mathscr{H}^+)$. If either of its copies is an edge copy, then the existence of the desired edge f is clear, so we can assume that $\Pi^e_\star$, $\Pi^e_{\star\star}$ are both real.

Owing to $\mathfrak{Girth}(H, \equiv^H, \mathscr{H}^+) > g \geqslant 2$ Lemma 8.13 tells us that $\Pi^e_\star$ is a supreme copy of $\mathscr{D}$. Let the $\Pi^e_\star$-piece P exemplify this fact. If $P = f^+$ is short we have found the desired edge f. It remains to consider the case that $P = (f')^+ W^H (f'')^+$ is long, where, let us emphasise, W^H is a wagon of $(H, \equiv^H)$.

By $x, y \in V(W^H)$ and (9.3) it cannot be the case that $W^H = W^H_\star$. Thus the wagon W of $(\Sigma, \equiv^\Sigma)$ contracting to W^H is distinct from $W_\star$. A fortiori the contractions $\overline{W}$, $\overline{W}_\star$ of W, $W_\star$ to $(\Pi_\star, \equiv^{\Pi_\star})$ are distinct as well and, therefore,

$$\mathscr{E} = F_1 \overline{W}_\star F_2 x (f')^+ \overline{W} (f'')^+ y$$

(see Figure 9.4) is a big cycle in $(\Pi_\star, \equiv^{\Pi_\star}, \mathscr{P}^+_\star)$.

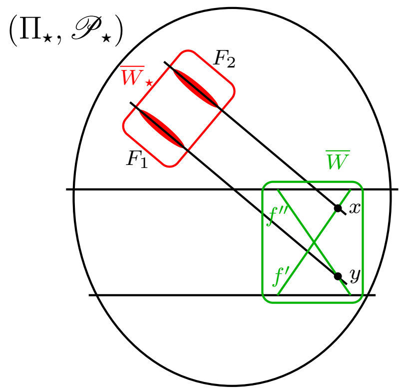


FIGURE 9.4. The big cycle $\mathscr{E}$.

Clearly, it satisfies the acceptability condition $(A1)$, and $(A2)$ could only fail if the desired edge f exists. Moreover, $(A3)$ holds due to the fact that the only wagon of the linear pretrain $(\Pi_\star, \equiv^{\Pi_\star})$ that contains x and y is $\overline{W}$ and, hence, a connector of $\mathscr{E}$.

Altogether we may assume that $\mathscr{E}$ is acceptable. Due to

$$\mathfrak{Girth}(\Pi_\star, \equiv^{\Pi_\star}, \mathscr{P}^+) > g \geqslant 2 = \mathrm{ord}(\mathscr{E})$$

this implies that $\mathscr{E}$ has a supreme copy. By Fact 8.15 such a supreme copy needs to contain both x and y and thus Claim 9.16 leads again to the desired edge f. □

Stage B: Segmentations. When considering any (not necessarily acceptable) big cycle $\mathscr{C} = F_1 q_1 \dots F_n q_n$ in $(\Sigma, \equiv^\Sigma)$ we can define *segments* of $\mathscr{C}$, their *leaders*, and *segmentations* of $\mathscr{C}$ as in the proof of Lemma 5.2. Unless the copies of $\mathscr{C}$ are contained in a single standard copy of $(\Pi, \mathscr{P}^+)$ we obtain a segmentation

$$\mathscr{C} = I_1 r_1 \dots I_t r_t$$

with $t \geqslant 2$ such that any two consecutive ones among the leaders $\Pi^e_1, \dots, \Pi^e_t$ are distinct. As usual we want to pass to a big cycle in $(H, \equiv^H, \mathscr{H}^+)$ by replacing every segment I_τ by its leader Π^e_τ.

The resulting cyclic sequence clearly has the properties $(B1)$–$(B3)$. However, the wagons among $r_1, \dots, r_t$ usually fail to be wagons of $(H, \equiv^H)$ and thus they are, pedantically speaking, not allowed to serve as wagons of such cycles. Due to the clauses (b) and (c) of Lemma 7.3, however, we know that such wagon connectors r_τ contract to $(H, \equiv^H)$. More precisely, if the leaders Π^e_τ and $\Pi^e_{\tau+1}$ are in $\mathscr{H}$, then (c)(ii) shows that they contain edges of r_τ. Similarly, if only one of them is in $\mathscr{H}$ while the other one is an edge copy, we appeal to (c)(i) and the case that both are edge copies is unproblematic due to (b).

Therefore we can define

$$\overline{r}_\tau = \begin{cases} \text{the contraction of } r_\tau \text{ to } (H, \equiv^H) & \text{if } r_\tau \text{ is a wagon} \\ r_\tau & \text{if } r_\tau \text{ is a vertex} \end{cases}$$

for every $\tau \in \mathbb{Z}/t\mathbb{Z}$ and the cyclic sequence

$$\mathscr{D} = \Pi^e_1 \overline{r}_1 \ldots \Pi^e_t \overline{r}_t$$

will be a big cycle in $(H, \equiv^H, \mathscr{H}^+)$. Let us observe that Fact 4.19 also holds for big cycles instead of cycles of copies, because its proof is based on a computation that can be carried out in both scenarios. Utilising the thus modified form of Fact 4.19 repeatedly we obtain $\mathrm{ord}(\mathscr{D}) \leqslant \mathrm{ord}(\mathscr{C})$. It will be convenient to call $\mathscr{D}$ a *reduct* of $\mathscr{C}$.

Stage C: The proof of (a). Recall that by Fact 6.10 the wagons of linear pretrains are strongly induced. Thus the following claim can be regarded as a first step towards proving that $(\Sigma, \equiv^\Sigma)$ is linear.

Claim 9.19. *Every wagon of $(\Sigma, \equiv^\Sigma)$ is strongly induced.*

Proof. Let W denote an arbitrary wagon of $(\Sigma, \equiv^\Sigma)$. Due to the definition of wagons we have $E(W) \neq \varnothing$ and no vertex of W is isolated. Hence it remains to check that if an edge $f \in E(\Sigma)$ intersects $V(W)$ in two distinct vertices x and y, then $f \in V(W)$. To this end we consider pairs of edges $e_x, e_y \in E(W)$ with $x \in e_x$ and $y \in e_y$. If e_x and e_y can be chosen in such a way that any two of the three edges e_x, e_y, and f coincide, we conclude $f \in E(W)$ either directly or by appealing to the linearity of the hypergraph Σ. Assuming that this is not the case we obtain, for each choice of (e_x, e_y), a big cycle

$$\mathscr{C} = f^+ x e^+_x W e^+_y y$$

of order 2. It should perhaps be pointed out that $\mathscr{C}$ is always inacceptable, for the edge f exemplifies that the wagon W violates the moreover-part of (A2). Nevertheless, $\mathscr{C}$ can have acceptable reducts and we can learn something by looking at supreme copies of such reducts.

If the edges e_x, e_y can be selected in such a way that together with f they belong to a common standard copy $(\Pi_\star, \equiv^{\Pi_\star})$, then $f \in E(W)$ follows from the fact that the contraction of W to $(\Pi_\star, \equiv^{\Pi_\star})$ is strongly induced in $\Pi_\star$.

So we can suppose that such a choice of e_x, e_y is impossible. Now let these edges and a segmentation

$$\mathscr{C} = I_1 r_1 \ldots I_t r_t \tag{9.4}$$

be determined in such a manner that $t \in \{2, 3\}$ is minimal. Write $\mathscr{D} = \Pi^e_1 \overline{r}_1 \dots \Pi^e_t \overline{r}_t$ for the corresponding reduct. Up to symmetry one of the following three cases occurs.

First Case. $\mathscr{D} = f^+ x \Pi^e_2 y$

Let Π_2 be the standard copy of Π extending Π^e_2 and containing the edges e_x, e_y. Lemma 8.14 applied to the system $(H, \equiv^H, \mathscr{H}^+)$, the copy Π^e_2, and the edge f yields $f \in E(\Pi^e_2)$. Thus all copies of $\mathscr{C}$ can be found in the same standard copy Π_2 and we are in the case discussed immediately before (9.4).

Second Case. $\mathscr{D} = \Pi^e_1 \overline{W} \Pi^e_2 y$

Here Π^e_1 extends to a standard copy Π_1 of Π that contains the edges f, e_x. Moreover, $\overline{W}$ denotes the contraction of W to $(H, \equiv^H)$. Due to Lemma 8.11 there exists an edge e'_y in $E(\Pi^e_1) \cap E(\overline{W})$ passing through y. By choosing e'_y instead of e_y we again arrive at a situation where everything relevant happens within the same standard copy, namely Π_1.

Third Case. $\mathscr{D} = f^+ x \Pi^e_2 \overline{W} \Pi^e_3 y$ *and, hence,* $t = 3$.

If $x, y \in V(\overline{W})$, then $\overline{W} \blacktriangleleft H$ yields $f \in E(\overline{W}) \subseteq E(W)$ and we are done immediately. Now assume for the sake of contradiction that $\{x, y\} \nsubseteq V(\overline{W})$, which causes $\mathscr{D}$ to be acceptable. Without loss of generality we may suppose that Π^e_2 is a supreme copy of $\mathscr{D}$. The Π^e_2-piece Π^e_3 collapses to needs to be short. If $(e'_y)^+$ denotes that piece, then by choosing e'_y instead of e_y we are led to a case with $t = 2$, contrary to the minimality of t. $\square$

Now we are ready to proceed to the linearity of the pretrain $(\Sigma, \equiv^\Sigma)$. Given any two vertices x and y, we are to prove that the set $\mathscr{W}$ of all wagons W with $x, y \in V(W)$ has at most one element. If there exists an edge $f \in E(\Sigma)$ with $x, y \in f$, then Claim 9.19 discloses $f \in E(W)$ for every $W \in \mathscr{W}$ and $|\mathscr{W}| \leqslant 1$ follows. Thus we may assume that

$$\text{no } f \in E(\Sigma) \text{ contains both } x \text{ and } y. \tag{9.5}$$

If $x, y \in V(H)$, then the linearity of $(H, \equiv^H)$ implies that at most one wagon $W^H_\star$ of this pretrain satisfies $x, y \in V(W^H_\star)$. If such a wagon exists, then we denote the unique wagon of $(\Sigma, \equiv^\Sigma)$ contracting to it by $W_\star$.

Now assume for the sake of contradiction that there exist distinct $W', W'' \in \mathscr{W}$. In case $W_\star$ exists we suppose, additionally, that $W_\star \in \{W', W''\}$. For every quadruple $\vec{e} = (e'_x, e'_y, e''_x, e''_y)$ of edges with

$$x \in e'_x \in E(W')\,, \quad y \in e'_y \in E(W')\,, \quad x \in e''_x \in E(W'')\,, \quad \text{and} \quad y \in e''_y \in E(W'')\,,$$

the statement (9.5) shows that

$$\mathscr{C} = (e'_x)^+ W' (e'_y)^+ y (e''_y)^+ W'' (e''_x)^+ x$$

is a big cycle. As the standard copies of $(\Pi, \equiv^{\Pi})$ are linear pretrains, there is no such quadruple $\vec{e}$ whose four edges are in the same standard copy. Thus every choice of $\vec{e}$ gives rise to at least one segmentation

$$\mathscr{C} = I_1 r_1 \dots I_t r_t \tag{9.6}$$

of the associated cycle $\mathscr{C}$. For the rest of the argument we fix $\vec{e}$ and the segmentation (9.6) in such a manner that $t \in \{2, 3, 4\}$ is minimal. Let $\mathscr{D} = \Pi_1^e \overline{r}_1 \dots \Pi_t^e \overline{r}_t$ denote the corresponding reduct of $\mathscr{C}$.

First Case. $t = 2$

Let us consider the connectors of $\mathscr{D}$. Due to Claim 9.18 and (9.5) it cannot be the case that both of them are vertices. Moreover, Lemma 8.12 excludes the case that r_1, r_2 are the two wagons of $\mathscr{C}$. Up to symmetry this means that only the possibility $\mathscr{D} = \Pi_1^e \overline{W'} \Pi_2^e x$ remains, where Π_1^e denotes the leader of $(e'_x)^+$.

Now Lemma 8.11 provides an edge f such that $x \in f \in E(\overline{W'}) \cap E(\Pi_2^e)$ and the quadruple (f, e'_y, e''_x, e''_y) can play the rôle of $\vec{e}$. However, all four edges of this quadruple pertain to the same standard copy Π_2, which is absurd.

Second Case. $t \in \{3, 4\}$

We contend that $\mathscr{D}$ is acceptable. The property *(A1)* is clear, *(A2)* follows from (9.5), and *(A3)* only requires attention if $t = 4$ and the wagon W we want to test contains both x and y. We already know, however, that there exists at most one such wagon, namely $W_\star^H$. By our choice of W', W'' and by $t = 4$ this wagon is, if it exists, a connector of $\mathscr{D}$ and thus irrelevant for *(A3)*.

SO $\mathscr{D}$ is indeed acceptable and due to $\mathrm{ord}(\mathscr{D}) = 2$ it follows that $\mathscr{D}$ has a supreme copy, say Π_2^e. If both segments I_1, I_3 contained more than one copy, then the contradiction $2 \leqslant |\mathscr{C}| - |\mathscr{D}| = 4 - t \leqslant 1$ would arise. Consequently we may assume that I_1 consists of a single copy, which causes the corresponding index to be mixed in $\mathscr{D}$. If $P_1 = f^+$ denotes the short Π_2^e-piece Π_1^e collapses to, then by including f into the quadruple $\vec{e}$ (instead of the edge corresponding to I_1) we reach a contradiction to the minimality of t.

This concludes the proof that $(\Sigma, \equiv^{\Sigma})$ is linear.

Stage D: The proof of *(b)*. Consider any acceptable big cycle $\mathscr{C} = F_1 q_1 \dots F_n q_n$ in $(\Sigma, \equiv^{\Sigma}, \mathscr{Q}^+)$ whose order is at most g. We are to prove that $\mathscr{C}$ has a supreme copy.

Assuming that Claim 9.17 does not imply this immediately, we pick a segmentation

$$\mathscr{C} = I_1 r_1 \dots I_t r_t$$

of $\mathscr{C}$ whose length $t \geqslant 2$ is minimal and pass to the associated reduct

$$\mathscr{D} = \Pi_1^e \overline{r}_1 \dots \Pi_t^e \overline{r}_t \,.$$

We commence with the simplest case.

Claim 9.20. *If* $\mathrm{ord}(\mathscr{D}) = 1$*, then* $\mathscr{C}$ *has a supreme copy.*

Proof. Suppose $\mathscr{D} = \Pi_1^e x \Pi_2^e \overline{W}$, where x is a vertex and $\overline{W}$ is a wagon of $(H, \equiv^H)$.

Let us first deal with the special case $n = |\mathscr{C}| = 2$, where $\mathscr{C} = F_1 x F_2 W$. Due to (*A*1) and symmetry we may assume that the copy F_1 is real. Now Lemma 8.11 applied to $\mathscr{D}$ and Π_1^e yields an edge f such that $x \in f \in E(\overline{W}) \cap E(\Pi_1^e)$. Since all copies of the acceptable big cycle $\mathscr{E} = F_1 x f^+ W$ are in the same standard copy of Π, Claim 9.17 shows that $\mathscr{E}$ has a supreme copy. Remark 8.8(*2*) tells us that this supreme copy must be F_1. The short F_1-piece Q to which we can collapse f^+ witnesses that F_1 is a supreme copy of $\mathscr{C}$ as well.

It remains to treat the case $n \geqslant 3$. If both segments I_1, I_2 contained at least two copies of $\mathscr{C}$, then the wagon W would exemplify that $\mathscr{C}$ violates (*A*2). So we may assume that I_1 consists of a single copy, say F_1. This time we utilise Lemma 8.11 for obtaining an edge f with $x \in f \in E(\overline{W}) \cap E(\Pi_2^e)$. Lemma 8.22(*a*) tells us that in $\mathscr{C}$ we can replace F_1 by f^+, thus arriving at another acceptable big cycle $\mathscr{E} = f^+ x I_2 W$. The advantage of $\mathscr{E}$ is that all its copies belong to the same standard copy of Π. Thus Claim 9.17 tells us that $\mathscr{E}$ has some supreme copy and owing to Lemma 8.22(*b*) so does $\mathscr{C}$. $\square$

So in the sequel we can assume $\mathrm{ord}(\mathscr{D}) \in [2, \mathrm{ord}(\mathscr{C})]$. In particular, $\mathscr{D}$ satisfies (*A*1) and one confirms easily that it inherits the properties (*A*2), (*A*3) from $\mathscr{C}$. Altogether $\mathscr{D}$ is an acceptable big cycle whose order is at most g; thus

$$\mathscr{D} \text{ has a supreme copy.} \tag{9.7}$$

Next, we isolate the following special case.

Claim 9.21. *If* $t = 2$*, then* $\mathscr{C}$ *has a supreme copy.*

Proof. As a consequence of Lemma 8.12 it cannot be the case that both connectors of $\mathscr{D}$ are wagons. So Claim 9.20 allows us to assume that r_1, r_2 are vertices. We distinguish several cases depending on whether the segments I_1, I_2 consist of single copies. Up to symmetry there are three possibilities.

First Case. $n = 2$

Claim 9.16 leads to an edge $f \in E(F_1) \cap E(F_2) \cap E(H)$. By (*B*1) it cannot be the case that both copies of $\mathscr{C}$ are equal to f^+ and, hence, we may suppose that $F_1 \neq f^+$. So we can collapse F_2 to the F_1-piece f^+, thus inferring that F_1 is a supreme copy of $\mathscr{C}$.

Second Case. $n \geqslant 3$ *and* I_2 *consists of a singly copy.*

To simplify the notation we assume $I_2 = F_n$. Due to Claim 9.16 there exists an edge $f \in E(F_n) \cap E(H)$. Now Lemma 8.22(*a*) informs us that $\mathscr{E} = I_1 r_1 f^+ r_2$ is an acceptable big cycle in $(\Sigma, \equiv^\Sigma, \mathscr{Q}^+)$. Moreover, $r_1, r_2 \in f \cap V(\Pi_1^e)$ and Lemma 8.14 imply $f \in E(\Pi_1^e)$, for which reason all copies of $\mathscr{E}$ belong to a common standard copy of Π. By Claim 9.17 it follows that $\mathscr{E}$ has a supreme copy and, finally, by Lemma 8.22(*b*) so does $\mathscr{C}$.

Third Case. Both I_1, I_2 contain at least two copies.

Suppose for the sake of contradiction that some edge f of Σ covers $\{r_1, r_2\}$. Let W be the wagon of $(\Sigma, \equiv^{\Sigma})$ containing f. Due to (*A*2) we know that W cannot serve as a connector of $\mathscr{C}$ and thus the description of the present case entails that W violates (*A*3). This proves that

$$\text{there is no } f \in E(\Sigma) \text{ such that } r_1, r_2 \in f. \tag{9.8}$$

In particular, the copies Π^e_1 and Π^e_2 need to be real and due to Lemma 8.13 both of them are supreme copies of $\mathscr{D}$. By Claim 9.18 and symmetry we may suppose that I_1 does not consist of two copies and an interposed wagon. Let P be a Π^e_1-piece with the property that collapsing Π^e_2 to P establishes the supremacy of Π^1_e in $\mathscr{D}$. Because of (9.8) we know that P is long. Write $P = (f')^+\overline{W}(f'')^+$ and denote the wagon of $(\Sigma, \equiv^{\Sigma})$ contracting to $\overline{W}$ by W. Owing to (*A*3) this wagon needs to appear in $\mathscr{C}$. Now (*A*2) tells us that the connectors of $\mathscr{C}$ next to W are r_1 and r_2.

So our special choice of I_1 guarantees that I_2 consists of two copies and the wagon W between them. Without loss of generality we may suppose that $I_2 = F_{n-1}WF_n$. Now Lemma 8.22 applies to $K = \{n-1, n\}$, $P_{n-1} = (f')^+$, and $P_n = (f'')^+$. So by part (*a*) of the lemma $I_1 r_1 (f')^+ W (f'')^+ r_2$ is an acceptable big cycle, by Claim 9.17 this cycle has a supreme copy, and by Lemma 8.22(*b*) $\mathscr{C}$ has a supreme copy as well. □

Throughout the remainder of this proof we suppose that $n \geqslant t \geqslant 3$. Let $\Pi^e_\star$ be a supreme copy of $\mathscr{D}$ (see (9.7)) and let

$$\big\{Q_\tau \colon \tau \in \mathbb{Z}/t\mathbb{Z} \text{ and } \Pi^e_\tau \neq \Pi^e_\star\big\}$$

be a family of $\Pi^e_\star$-pieces acting as supremacy witnesses. Our goal is to associate with every copy F_k of $\mathscr{C}$ that belongs to a segment I_τ whose leader Π^e_τ is distinct from $\Pi^e_\star$ a $\Pi^e_\star$-piece P_k such that

(*1*) if one eliminates F_k from $\mathscr{C}$ and inserts P_k instead of it, then the resulting cyclic sequence satisfies (*B*3) and (*B*4),

(*2*) and if P_k is long, then it has the properties (α), (β), and (γ) from Lemma 8.22.

Once we have such pieces P_k at our disposal, we can perform all replacements indicated in (*1*) simultaneously and Lemma 8.22(*a*) informs us that the cyclic sequence that results is an acceptable big cycle. All copies of this cycle will be contained in the same standard copy $(\Pi_\star, \equiv^{\Pi_\star}, \mathscr{P}^+_\star)$, where $\Pi_\star$ extends $\Pi^e_\star$. Owing to Claim 9.17 the modified cycle needs to have a supreme copy, so by Lemma 8.22(*b*) the original big cycle $\mathscr{C}$ has a supreme copy as well.

Thus it suffices to exhibit the $\Pi^e_\star$-pieces described above. Notice that the connectors $\overline{r}_{\tau-1}$ and $\overline{r}_\tau$ cannot be wagons at the same time, for then the piece Q_τ could not exist. Suppose next that one of those connectors, say $\overline{r}_{\tau-1}$, is a vertex while the other one, i.e. $\overline{r}_\tau$, is a wagon. In this case the piece Q_τ needs to be short. Besides, since the wagon r_τ satisfies *(A2)* in $\mathscr{C}$, the segment I_τ has length 1 and we can set $P_k = Q_\tau$.

The last case we need to consider is that both $r_{\tau-1}$ and r_τ are vertices. Suppose first that the piece $Q_\tau = f^+$ is short and let W be the wagon of $(\Sigma, \equiv^\Sigma)$ with $f \in E(W)$. An easy argument using *(A2)* shows that W cannot be a connector of $\mathscr{C}$. So *(A3)* applies to W. We deduce that the segment I_τ has length 1 and again we can set $P_k = Q_\tau$.

So henceforth we may assume that the piece $Q_\tau = (f')^+\overline{W}(f'')^+$ is long. The easiest possibility would be that the interval I_τ is of the form $F_j W F_{j+1}$, where W denotes the wagon of $(\Sigma, \equiv^\Sigma)$ contracting to $\overline{W}$, for then we just need to set $P_j = (f')^+$ and $P_{j+1} = (f'')^+$.

Let us now assume for the sake of contradiction that I_τ is not of this special form. Observe first that the admissibility of Q_τ as a supremacy witness entails that

$$\text{no edge of } H \text{ contains both } r_{\tau-1} \text{ and } r_\tau. \tag{9.9}$$

If the connectors $r_{\tau-1}$ and r_τ are consecutive in $\mathscr{C}$, then Claim 9.16 applied to the copy between them yields an edge that contradicts (9.9). This shows that

$$\text{the connectors } r_{\tau-1} \text{ and } r_\tau \text{ fail to be consecutive in } \mathscr{C}. \tag{9.10}$$

Now condition *(A3)* tells us that the wagon W has to appear somewhere in $\mathscr{C}$. On the other hand, $\overline{W}$ cannot appear in $\mathscr{D}$, for then the piece Q_τ would again not be admissible. In other words, W is hidden in one of the segments. But by *(A2)* and $t \geqslant 3$ this is only possible if I_τ is indeed of the desired form. □

The following should now be straightforward, but let us elaborate.

Proof of Proposition 9.14. Consider a pretrain $(F, \equiv^F)$ with $\mathfrak{girth}(F, \equiv^F) > g$ and a number of colours r. Construct the linear system $\Phi_r(F) = (G, \mathscr{G})$ with strongly induced copies and enumerate the edges of G arbitrarily as $E(G) = \{e(1), \dots, e(N)\}$. By Lemma 8.17 the picture zero $(\Pi_0, \equiv_0, \mathscr{P}_0, \psi_0)$ corresponding to this situation satisfies $\mathfrak{Girth}(\Pi_0, \equiv_0, \mathscr{P}_0^+) > g$.

Starting with picture zero we intend to execute a partite construction, i.e., to construct recursively a sequence of pictures $(\Pi_\alpha, \equiv_\alpha, \mathscr{P}_\alpha, \psi_\alpha)_{\alpha \leqslant N}$ over $(G, \mathscr{G})$. Suppose that for some $\alpha \in [N]$ we have just constructed a picture $(\Pi_{\alpha-1}, \equiv_{\alpha-1}, \mathscr{P}_{\alpha-1}, \psi_{\alpha-1})$ with

$$\mathfrak{Girth}(\Pi_{\alpha-1}, \equiv_{\alpha-1}, \mathscr{P}^+_{\alpha-1}) > g\,.$$

In particular, we have $\mathfrak{Girth}\big(\Pi_{\alpha-1}, \equiv_{\alpha-1}, E^+(\Pi_{\alpha-1})\big) > g$ and Lemma 8.17 reveals

$$\mathfrak{girth}(\Pi_{\alpha-1}, \equiv_{\alpha-1}) > g\,.$$

This in turn implies that $\Xi_r(\cdot)$ applies to the constituent $\big(\Pi^{e(\alpha)}_{\alpha-1}, \equiv^{e(\alpha)}_{\alpha-1}\big)$, where $\equiv^{e(\alpha)}_{\alpha-1}$ denotes the restriction of $\equiv_{\alpha-1}$ to $E\big(\Pi^{e(\alpha)}_{\alpha-1}\big)$. In other words,

$$\Xi_r\big(\Pi^{e(\alpha)}_{\alpha-1}, \equiv^{e(\alpha)}_{\alpha-1}\big) = (H_\alpha, \equiv^{H_\alpha}, \mathscr{H}_\alpha)$$

is defined and satisfies $\mathfrak{Girth}(H_\alpha, \equiv^{H_\alpha}, \mathscr{H}^+_\alpha) > g$. By Lemma 9.15 the new picture

$$(\Pi_\alpha, \equiv_\alpha, \mathscr{P}_\alpha, \psi_\alpha) = (\Pi_{\alpha-1}, \equiv_{\alpha-1}, \mathscr{P}_{\alpha-1}\psi_{\alpha-1}) \ast (H_\alpha, \equiv^{H_\alpha}, \mathscr{H}_\alpha)$$

has the property $\mathfrak{Girth}(\Pi_\alpha, \equiv_\alpha, \mathscr{P}^+_\alpha) > g$ and thus the partite construction goes on.

Eventually we obtain the final picture $\mathrm{PC}(\Phi, \Xi)_r(F, \equiv^F) = (\Pi_N, \equiv_N, \mathscr{P}_N)$ with

$$\mathfrak{Girth}(\Pi_N, \equiv_N, \mathscr{P}^+_N) > g\,. \qquad \square$$

9.3. **Orientation.** The main results of this section lead to a quick solution of the problem discussed in §7.5. The construction $\Upsilon = \mathrm{Ext}(\Omega^{(2)}, \Omega^{(2)})$ applies to all linear, ordered, f-partite pretrains. Due to the case $g = 2$ of Lemma 9.1 the systems of pretrains $(H, \equiv^H, \mathscr{H})$ generated by Υ satisfy $\mathfrak{Girth}(H, \equiv^H, \mathscr{H}^+) > 2$ (the hypotheses of this lemma were verified in Proposition 3.8 and Corollary 5.6).

Now we can use Υ as a partite lemma and look at the construction $\Lambda = \mathrm{PC}(\Omega^{(2)}, \Upsilon)$. The case $g = 2$ of Proposition 9.14 informs us that for every linear, ordered, f-partite pretrain $(F, \equiv^F)$ and every number of colours r the system of pretrains $\Lambda_r(F, \equiv^F) = (H, \equiv^H, \mathscr{H})$ is defined and satisfies $\mathfrak{Girth}(H, \equiv^H, \mathscr{H}^+) > 2$. So, in particular, $(H, \equiv^H)$ is a linear pretrain arrowing $(F, \equiv^F)$ with r colours. Moreover, and this is the main advantage of Λ over Υ, the copies in $\mathscr{H}$ have clean intersections due to Lemma 3.4.

§10. Trains

Let us start this section with an observation of central importance to our girth Ramsey theoretic endeavour: a "typical" k-partite k-uniform hypergraph can never occur as a constituent of a picture that arises when the partite construction method is carried out over a linear system $(G, \mathscr{G})$.

For instance, all constituents of picture zero consist of a matching together with some isolated vertices (cf. Figure 3.2). The first picture is obtained by applying the intended partite lemma to the constituent over some edge $e(1) \in E(G)$ and performing the usual amalgamation. Evidently, every constituent of the new picture over an edge $e \in E(G)$ with $e(1) \cap e = \varnothing$ is still a matching augmented by some isolated vertices. More interestingly, if an edge $e \neq e(1)$ intersects $e(1)$, then by the linearity of G this intersection occurs in a single vertex x and, accordingly, the new constituent over e is obtained by taking several disjoint copies of the old constituent and identifying a couple of vertices on the music line V_x. However, there occur no such identifications on the other music lines V_z with $z \in e \smallsetminus \{x\}$. Notice that we can endow the new constituent with a pretrain structure by declaring two edges to be equivalent if they belong to the same copy of the old constituent. It should also be clear that this pretrain is going to be linear if our partite lemma delivers copies with clean intersections.

Similarly, the most complicated possibility for a constituent of the second picture is that one takes plenty of disjoint copies of such a pretrain and identifies some of their vertices, all identified vertices belonging to a common music line. As the partite construction progresses, there iteratively arise more and more complex objects all of which fit into the framework we shall now develop (see also Figure 2.3).

10.1. **Quasitrains and parameters.** We begin by encoding the recursive formation of our objects in terms of a sequence of nested equivalence relations.

Definition 10.1. Given a positive integer m we say that $\vec{F} = (F, \equiv_0, \dots, \equiv_m)$ is a *quasitrain of height m* if F is a hypergraph and $\equiv_0, \dots, \equiv_m$ are equivalence relations on $E(F)$ such that the following holds.

- (i) Every wagon of the pretrain $(F, \equiv_0)$ consists of a single edge.
- (ii) If $e', e'' \in E(F)$, $\mu \in [0, m)$, and $e' \equiv_\mu e''$, then $e' \equiv_{\mu+1} e''$.
- (iii) Any two edges of F are equivalent with respect to $\equiv_m$.

For every $\mu \in [0, m]$ the wagons of the pretrain $(F, \equiv_\mu)$ are called the *μ-wagons* of $\vec{F}$.

Example 10.2. Quasitrains of height 1 are triples $(F, \equiv_0, \equiv_1)$, where F is a hypergraph, the 0-wagons are single edges (by (i)), and all edges belong to the same 1-wagon (by (iii)). Consequently, there is a natural bijective correspondence between quasitrains of height 1 and hypergraphs. At a later moment it will be convenient to call $(F, \equiv_0, \equiv_1)$ the quasitrain of height 1 *associated with* the hypergraph F.

Example 10.3. Similarly, quasitrains of height 2 are quadruples $(F, \equiv_0, \equiv_1, \equiv_2)$ with the properties that $(F, \equiv_0, \equiv_2)$ is a quasitrain of height 1 and $(F, \equiv_1)$ is a pretrain. In particular, quasitrains of height 2 are in natural bijective correspondence with pretrains.

The train concept to which we proceed next is taking into account what we said earlier about amalgamations occurring only in prespecified music lines. Recall that in Definition 3.15(a) we introduced the following convenient piece of notation for such situations: given an f-partite hypergraph F with index set I and a subset $A \subseteq I$ the union $\bigcup_{i \in A} V_i(F)$ of the vertex classes whose indices belong to A is denoted by $V_A(F)$.

Definition 10.4. Suppose that $m \in \mathbb{N}$, that $f\colon I \longrightarrow \mathbb{N}$ is a function from a finite index set I to $\mathbb{N}$ with $\sum_{i \in I} f(i) \geqslant 2$, and that $\vec{A} = (A_1, \dots, A_m) \in \wp(I)^m$ is a sequence of subsets of I. A structure $\vec{F} = (F, \equiv_0, \dots, \equiv_m)$ is said to be an *f-partite train of height m with parameter $\vec{A}$* (see Figure 10.1) if it has the following two properties.

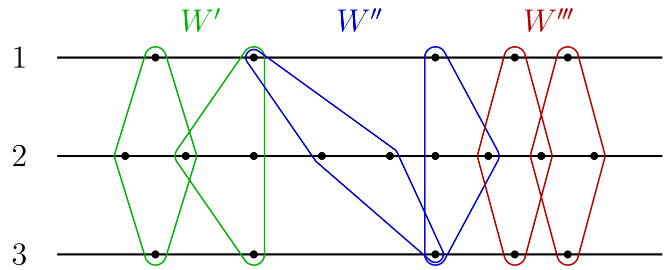


FIGURE 10.1. An f-partite train of height 1 with parameter (A_1), where $f\colon [3] \longrightarrow \mathbb{N}$ is defined by $f(1) = f(3) = 1$, $f(2) = 2$, and $A_1 = \{1, 2\}$.

(*i*) $\vec{F}$ is an f-partite quasitrain of height m.

(*ii*) If $\mu \in [m]$ and $W'_{\mu-1}$, $W''_{\mu-1}$ are two distinct $(\mu - 1)$-wagons of $\vec{F}$ included in a common μ-wagon, then $V(W'_{\mu-1}) \cap V(W''_{\mu-1}) \subseteq V_{A_\mu}(F)$.

Example 10.5. Trains of height 1 have one-term sequences serving as their parameters. If $\vec{F} = (F, \equiv_0, \equiv_1)$ is an f-partite quasitrain of height 1 (cf. Example 10.2), I denotes the domain of f, and $A_1 \subseteq I$, then $\vec{F}$ is a train with parameter (A_1) if and only if $e' \cap e'' \subseteq V_{A_1}(F)$ holds for any two distinct edges $e', e'' \in E(F)$, i.e., if F is A_1-intersecting in the sense of Definition 3.15(*b*).

When compared to the discussion at the beginning of this section, there are two directions in which our definition of trains might appear to be too general. First, we only motivated the need for k-partite k-uniform trains, whereas Definition 10.4 allows the underlying hypergraphs of trains to be f-partite for general functions f. The reason for this is that we intend to subject trains to the extension process and, as we saw in Section 6, when we apply a construction of the form $\mathrm{Ext}(\Phi, \Psi)$ to a k-partite k-uniform pretrain, the construction Ψ needs to be applicable to general f-partite hypergraphs.

Another surprise in Definition 10.4 might be that we allow the sets $A_1, \dots, A_m$ constituting the parameter of the train to be arbitrary subsets of I. After all, we have just tried to convey the idea that for trains created by partite constructions we can demand $|A_\mu| \leqslant 1$ for every $\mu \in [m]$, and that this very fact will be responsible for the girth increment the achievement of which is the reason for starting to consider trains at all. On the other hand, when we have some construction applicable to f-partite trains and attempt to clean it by means of the partite construction method, then from the point of view of the constituents of the involved pictures the sizes of the sets in the parameter can appear to have been enlarged. This tension between conflicting demands on the parameters will completely resolve itself in Section 12.

It is probably clear what we mean by subtrains, but let us elaborate for the sake of completeness.

Definition 10.6. Given two quasitrains

$$\vec{F} = (F, \equiv_0^F, \dots, \equiv_m^F) \quad \text{and} \quad \vec{G} = (G, \equiv_0^G, \dots, \equiv_m^G)$$

of the same height m we say that $\vec{F}$ is a *subquasitrain* of $\vec{G}$ if

(*i*) F is a subhypergraph of G

(*ii*) and $\forall e', e'' \in E(F)\ \forall \mu \in [0, m]\ \big[e' \equiv_\mu^F e'' \iff e' \equiv_\mu^G e''\big]$.

If, moreover, both $\vec{F}$ and $\vec{G}$ are f-partite trains having the same parameter $\vec{A}$ and (*i*) holds in the stronger form that F is an f-partite subhypergraph of G, then $\vec{F}$ is called a *subtrain* of $\vec{G}$.

The subtrain relation is, of course, reflexive and transitive. A *quasitrain system of height* m is a structure of the form

$$(H, \equiv_0^H, \dots, \equiv_m^H, \mathscr{H}),$$

where $\vec{H} = (H, \equiv_0^H, \dots, \equiv_m^H)$ is a quasitrain of height m and $\mathscr{H}$ is a collection of subquasitrains of $\vec{H}$. It will be convenient to denote this system by $(\vec{H}, \mathscr{H})$ as well. We regard the notion of an *extended quasitrain system* $(\vec{H}, \mathscr{H}^+)$ to be self-explanatory.

10.2. **More German girth.** Next we need to generalise $\mathfrak{girth}$ to quasitrains, and $\mathfrak{Girth}$ to systems of quasitrains. In both contexts we work with the free monoid $\mathfrak{M}$ generated by $\mathbb{N}_{\geqslant 2}$. Thus the elements of $\mathfrak{M}$ can be thought of as sequences of elements from $\mathbb{N}_{\geqslant 2}$ and the composition of $\mathfrak{M}$, denoted by $\circ$, is the concatenation of sequences. The *empty sequence* $\varnothing$ is the neutral element of $\mathfrak{M}$. For a sequence $\vec{g} = (g_1, \dots, g_m) \in \mathfrak{M}$ we call $m = |\vec{g}|$ the *length* of $\vec{g}$ and we use $\inf(\vec{g})$ as an abbreviation for $\inf\{g_1, \dots, g_m\}$. So we have $\inf(\varnothing) = \infty$ and in all other cases the infimum is just a minimum. For integers $g \geqslant 2$ and $m \geqslant 1$ the sequence $(g, \dots, g)$ consisting of m terms equal to g is the m^{th} power of the one-term sequence (g) and thus we shall denote it by $(g)^m$. Finally, we write $\mathfrak{M}_{\leqslant}$ for the subset of $\mathfrak{M}$ consisting of all nondecreasing sequences (including $\varnothing$), and we set $\mathfrak{M}_{\leqslant}^{\times} = \mathfrak{M}_{\leqslant} \smallsetminus \{\varnothing\}$.

A straightforward adaptation of Definition 8.16 leads to the following $\mathfrak{girth}$ notion for quasitrains.

Definition 10.7. Given a quasitrain $\vec{F} = (F, \equiv_0, \dots, \equiv_m)$ of height m and a sequence $\vec{g} = (g_1, \dots, g_m) \in \mathfrak{M}$ of length m we write $\mathfrak{girth}(\vec{F}) > \vec{g}$ if for every $\mu \in [m]$ and every μ-wagon W_μ we have

$$\mathfrak{girth}(W_\mu, \equiv_{\mu-1}^{W_\mu}) > g_\mu,$$

where $\equiv_{\mu-1}^{W_\mu}$ denotes the restriction of $\equiv_{\mu-1}$ to $E(W_\mu)$. A quasitrain $\vec{F}$ of height m with $\mathfrak{girth}(\vec{F}) > (2)^m$ is said to be *linear*.

The transitivity of ordinary girth leads to the following statement.

Lemma 10.8. *Let $\vec{g} = (g_1, \ldots, g_m) \in \mathfrak{M}^{\times}_{\leqslant}$ be given. If $\vec{F} = (F, \equiv^F_0, \ldots, \equiv^F_m)$ denotes a quasitrain of height m with $\mathfrak{girth}(\vec{F}) > \vec{g}$, then $\mathfrak{girth}(F, \equiv_{\mu-1}) > g_\mu$ holds for every $\mu \in [m]$. In particular, we have* $\mathrm{girth}(F) > g_1$.

Proof. The case $\mu = m$ is clear, for by Definition 10.1(*iii*) there is only one m-wagon of $\vec{F}$ and by Definition 10.7 applied to that wagon we obtain indeed $\mathfrak{girth}(F, \equiv_{m-1}) > g_m$.

Arguing by decreasing induction on μ we now assume that some $\mu \in [m-1]$ has the property $\mathfrak{girth}(F, \equiv_\mu) > g_{\mu+1} \geqslant g_\mu$, meaning that the μ-wagons of $\vec{F}$ form a set system whose girth exceeds g_μ. According to Definition 10.7 each of these μ-wagons is composed of $(\mu - 1)$-wagons the vertex sets of which form a set system whose girth is larger than g_μ as well. Due to Fact 4.2 this implies $\mathfrak{girth}(F, \equiv_{\mu-1}) > g_\mu$ and the induction is complete.

As the 0-wagons consist of single edges, the special case $\mu = 1$ yields $\mathrm{girth}(F) > g_1$. □

The reason why we are keeping track of suitable parameters when discussing trains is that they are needed for the following important yet innocent looking variant of this argument.

Lemma 10.9. *Let $\vec{F} = (F, \equiv_0, \ldots, \equiv_m)$ be a k-partite k-uniform train whose parameter $\vec{B} = (B_1, \ldots, B_m)$ satisfies $|B_\mu| \leqslant 1$ for every $\mu \in [m]$. If $\mathfrak{girth}(\vec{F}) > (g)^m$ holds for some integer $g \geqslant 2$, then* $\mathrm{girth}(F) > g + 1$.

Proof. We are to prove that the girth of the unique m-wagon of F exceeds $g + 1$. Arguing indirectly, we let $\mu \in [0, m]$ be minimal with the property that for some μ-wagon W_μ of $\vec{F}$ the statement $\mathrm{girth}(W_\mu) > g + 1$ fails.

As the 0-wagons of $\vec{F}$ are single edges, we have $\mu > 0$. Therefore W_μ is comprised of $(\mu - 1)$-wagons and by the minimality of μ the girth of each of them is larger than $g + 1$. According to Definition 10.7 the vertex sets of these $(\mu - 1)$-wagons form a set system whose girth is larger than g. Due to Fact 4.2 it follows that W_μ contains a $(g + 1)$-cycle $e_1 v_1 \ldots e_{g+1} v_{g+1}$ and distinct $(\mu - 1)$-wagons $W^1_{\mu-1}, \ldots, W^{g+1}_{\mu-1}$ with $e_i \subseteq V(W^i_{\mu-1})$ for every $i \in \mathbb{Z}/(g + 1)\mathbb{Z}$. Now clause (*ii*) of Definition 10.4 implies $v_1, \ldots, v_{g+1} \in V_{B_\mu}$. Together with $|B_\mu| \leqslant 1$ this shows, in particular, that v_1, v_{g+1} are in the same vertex class of F. On the other hand, these two vertices are distinct and belong to the edge e_1. We have thereby reached a contradiction to the assumption that F be k-partite and k-uniform for some $k \geqslant 2$, and the proof is complete. □

Definition 10.10. Given an extended quasitrain system $(H, \equiv_0, \ldots, \equiv_m, \mathscr{H}^+) = (\vec{H}, \mathscr{H}^+)$ of height m, and a sequence $\vec{g} = (g_\ell, \ldots, g_m) \in \mathfrak{M}$ whose length is at most m we write $\mathfrak{Girth}(\vec{H}, \mathscr{H}^+) > \vec{g}$ if

- for every $\mu \in [\ell, m]$ we have $\mathfrak{Girth}(H, \equiv_{\mu-1}, \mathscr{H}^+) > g_\mu$,
- and $\mathfrak{Girth}(H, \equiv_m, \mathscr{H}^+) > 1$.

In practice we will always have $\ell \in \{1, 2\}$ when working with this definition. The case $\ell = 1$ is certainly more natural, as it corresponds to having a demand on $\mathfrak{Girth}(H, \equiv_{\mu-1}, \mathscr{H}^+)$ for every possible value of μ. If a train system $(\vec{H}, \mathscr{H})$ is created by means of the extension process, however, there will be copies intersecting in entire 1-wagons (see §11.2) and thus there is nothing interesting that could be said about $\mathfrak{Girth}(H, \equiv_0, \mathscr{H}^+)$. This lack of information is the reason why we sometimes have to resort to the case $\ell = 2$. Fortunately, the partite construction method allows us to regain control over $\mathfrak{Girth}(H, \equiv_0, \mathscr{H}^+)$ and thus we can prevent this deficit from spreading (see §12.4). So we never need to deal with the possibility $\ell > 2$. Let us end this discussion by recording a direct consequence of Lemma 8.17.

Corollary 10.11. *If a sequence $\vec{g} \in \mathfrak{M}$ and a quasitrain $\vec{H} = (H, \equiv_0, \dots, \equiv_m)$ of height $m = |\vec{g}|$ satisfy $\mathfrak{Girth}\big(\vec{H}, E^+(H)\big) > \vec{g}$, then $\mathfrak{girth}(\vec{H}) > \vec{g}$ holds as well.* □

10.3. **Diamonds.** We shall now give a precise description of the induction scheme the proof of the girth Ramsey theorem is based on. The central statement concerns trains of large $\mathfrak{girth}$ and reads as follows.

Definition 10.12. For every sequence $\vec{g} \in \mathfrak{M}^{\times}_{\leqslant}$ of length m the Ramsey theoretic principle $\diamondsuit_{\vec{g}}$ asserts that there exists a construction $\Psi^{\vec{g}}$ associating with every ordered f-partite train

$$\vec{F} = (F, \equiv^F_0, \dots, \equiv^F_m)$$

of height m satisfying $\mathfrak{girth}(\vec{F}) > \vec{g}$ and with every number of colours r an ordered f-partite train system

$$\Psi^{\vec{g}}_r(\vec{F}) = (H, \equiv^H_0, \dots, \equiv^H_m, \mathscr{H}) = (\vec{H}, \mathscr{H})$$

with the same parameter as $\vec{F}$ such that the copies in $\mathscr{H}$ are strongly induced,

$$\mathscr{H} \longrightarrow (\vec{F})_r\,, \qquad \text{and} \qquad \mathfrak{Girth}(\vec{H}, \mathscr{H}^+) > \vec{g}\,.$$

Essentially we shall prove all these principles "by induction on $\vec{g}$". Up to some changes in the language, the base case has already been analysed.

Lemma 10.13. *The principle $\diamondsuit_{(2)}$ holds.*

Proof. Let an ordered f-partite train $(F, \equiv^F_0, \equiv^F_1)$ of height 1 with parameter (A) and a number of colours r be given. For reasons of simplicity we assume first that F has no isolated vertices.

Now we form the hypergraph system $\Omega^{(2)}_r(F) = (H, \mathscr{H})$, and consider the quasitrain $\vec{H} = (H, \equiv^H_0, \equiv^H_1)$ associated with H (as in Example 10.2). Corollary 5.6 informs us that H is an ordered f-partite hypergraph, $\mathscr{H} \longrightarrow (F)_r$, and $\mathrm{Girth}(H, \mathscr{H}^+) > 2$. Since F is

A-intersecting (cf. Example 10.5), Lemma 3.18 shows that H is A-intersecting as well or, in other words, that $\vec{H}$ is a train with parameter (A).

By Proposition 3.8 the copies in $\mathscr{H}$ are strongly induced in H. Moreover, Lemma 8.18 yields $\mathfrak{Girth}(H, \equiv_0^H, \mathscr{H}^+) > 2$. Since F has no isolated vertices, Lemma 8.11 implies $\mathfrak{Girth}(H, \equiv_1^H, \mathscr{H}^+) > 1$. According to Definition 10.10 this shows $\mathfrak{Girth}(\vec{H}, \mathscr{H}^+) > (2)$ and altogether the system $(\vec{H}, \mathscr{H})$ is as demanded by $\diamondsuit_{(2)}$. This concludes our discussion of the case that F has no isolated vertices.

In the general case we remove the isolated vertices from F, perform the construction we have just seen, and then we put the isolated vertices back. This needs to be done in such a way that each copy in $\mathscr{H}$ ends up getting its own isolated vertices, so that $\mathfrak{Girth}(H, \equiv_1^H, \mathscr{H}^+) > 1$ remains valid. □

There will be two quite distinct kinds of induction steps for traversing $\mathfrak{M}_{\leqslant}^{\times}$. First, the extension process together with some subsequent cleaning steps will allow us to prove the following implication.

Proposition 10.14. *For every nonempty nondecreasing sequence $\vec{g} \in \mathfrak{M}_{\leqslant}^{\times}$ the principle $\diamondsuit_{\vec{g}}$ implies $\diamondsuit_{(2)\circ\vec{g}}$.*

Second, we shall later describe a diagonal variant of the partite construction method and establish the following result.

Proposition 10.15. *Suppose that $\vec{g} \in \mathfrak{M}_{\leqslant}$ is a (possibly empty) nondecreasing sequence and that g is an integer satisfying $\inf(\vec{g}) > g \geqslant 2$. If for every positive integer m the principle $\diamondsuit_{(g)^m\circ\vec{g}}$ holds, then $\diamondsuit_{(g+1)\circ\vec{g}}$ is likewise valid.*

Throughout the remainder of this subsection, we assume that these two propositions are true and explore some of their consequences. In particular, we show that they really yield all karo principles.

Lemma 10.16. *For every sequence $\vec{g} \in \mathfrak{M}_{\leqslant}^{\times}$ and every integer g such that $\inf(\vec{g}) \geqslant g \geqslant 2$ the principle $\diamondsuit_{\vec{g}}$ implies $\diamondsuit_{(g)\circ\vec{g}}$.*

Proof. We argue by induction on g. Proposition 10.14 provides the base case $g = 2$. Now suppose that the lemma holds for some integer $g \geqslant 2$ and that some nonempty nondecreasing sequence $\vec{g} \in \mathfrak{M}_{\leqslant}^{\times}$ satisfying $\diamondsuit_{\vec{g}}$ and $\inf(\vec{g}) \geqslant g+1$ is given. Iterative applications of the induction hypothesis establish $\diamondsuit_{(g)^m\circ\vec{g}}$ for every positive integer m and thus Proposition 10.15 yields the desired principle $\diamondsuit_{(g+1)\circ\vec{g}}$. □

Lemma 10.17. *For every integer $g \geqslant 2$ the principle $\diamondsuit_{(g)}$ holds.*

Proof. Referring to Lemma 10.13 as a base case we argue by induction on g. In the induction step we suppose that $g \geqslant 2$ denotes an integer such that $\diamondsuit_{(g)}$ is true. Repeated applications of the previous lemma disclose $\diamondsuit_{(g)^m}$ for every positive integer m and appealing once more to Proposition 10.15, this time with the empty sequence, we infer $\diamondsuit_{(g+1)}$. $\square$

It should be clear that the two foregoing lemmata suffice for proving the following statement by induction on $|\vec{g}|$.

Corollary 10.18. *For every sequence $\vec{g} \in \mathfrak{M}^{\times}_{\leqslant}$ the principle $\diamondsuit_{\vec{g}}$ is valid.* $\square$

Moreover, when unravelling the meaning of Lemma 10.17 one arrives at a fairly strong form of the girth Ramsey theorem first announced in §4.4.

Theorem 10.19. *For every integer $g \geqslant 2$ there exists a Ramsey construction $\Omega^{(g)}$ that given an ordered f-partite hypergraph F with $\mathrm{girth}(F) > g$ and a number of colours r produces an ordered f-partite system $\Omega^{(g)}_r(F) = (H, \mathscr{H})$ satisfying $\mathscr{H} \longrightarrow (F)_r$ and $\mathrm{Girth}(H, \mathscr{H}^+) > g$.*

Proof. Let $\vec{F} = (F, \equiv^F_0, \equiv^F_1)$ be the ordered f-partite quasitrain associated to F (as in Example 10.2). We can view $\vec{F}$ as a train with parameter (I), where I denotes the domain of f. Since $\mathfrak{girth}(\vec{F}) > (g)$, the principle $\diamondsuit_{(g)}$ yields a train system $(\vec{H}, \mathscr{H})$ such that $\mathscr{H} \longrightarrow (\vec{F})_r$ and $\mathfrak{Girth}(\vec{H}, \mathscr{H}^+) > (g)$. Writing $\vec{H} = (H, \equiv^H_0, \equiv^H_1)$ it remains to observe that $\mathfrak{Girth}(H, \equiv^H_0, \mathscr{H}^+) > g$ implies $\mathrm{Girth}(H, \mathscr{H}^+) > g$ (see Lemma 8.18). $\square$

In analogy to Lemma 3.18 we could add that the construction $\Omega^{(g)}$ also preserves being A-intersecting, but due to a lack of known applications we do not state this more carefully here. Let us finally recall that every k-uniform hypergraph F can be viewed as an f-partite hypergraph, where f denotes any function from a one-element set to $\{k\}$. Therefore, Theorem 10.19 immediately yields a Ramsey theorem for arbitrary hypergraphs without short cycles. As the conclusion $\mathrm{Girth}(H, \mathscr{H}^+) > g$ implies $\mathrm{girth}(H) > g$ (see Lemma 4.12) we thereby see that Theorem 10.19 yields Theorem 1.4. Let us collect all these implications into a single statement.

Summary 10.20. *The conjunction of Proposition 10.14 and Proposition 10.15 leads to Corollary 10.18, Theorem 10.19, and to the girth Ramsey theorem as formulated in Theorem 1.4.*

Remark 10.21. With the help of ordinal numbers our induction scheme can be reformulated in a, perhaps, more transparent way. The map $\varphi\colon \mathfrak{M}_{\leqslant} \longrightarrow \omega^\omega$ defined by

$$\varphi(g_1, \dots, g_m) = \omega^{g_m - 2} + \dots + \omega^{g_1 - 2}$$

is bijective. Thus Corollary 10.18 asserts that $\diamondsuit_{\varphi^{-1}(\alpha)}$ holds for every positive ordinal α beneath ω^ω. The proof is by induction on α, its base case $\alpha = 1$ agrees with Lemma 10.13, and Proposition 10.14 takes care of the successor step $\diamondsuit_{\varphi^{-1}(\alpha)} \Longrightarrow \diamondsuit_{\varphi^{-1}(\alpha+1)}$. Finally, Proposition 10.15 provides the limit step, for in the notation employed there the sequence $\langle\varphi((g)^m \circ \vec{g})\colon m < \omega\rangle$ converges to the limit number $\varphi((g+1)\circ\vec{g})$.

§11. Trains in the extension process

In this section we take the first major step towards proving Proposition 10.14. Notice that we are given there a construction applicable to certain trains of height $m = |\vec{g}|$, and we are to exhibit another construction capable of handling certain trains of height $m+1$. The plan for accomplishing this height increment is to use the extension process; the main result of this section describes how far we can go with this idea (see Lemma 11.10). The main deficit of the construction we obtain is that the copies it produces can still intersect in entire 1-wagons. Following the arguments we saw in §9.3 this can be remedied by means of the partite construction method, but we defer the details of this cleaning step to §12.4.

11.1. **Extensions of trains.** The extension process discussed in Section 6 generalises from pretrains to quasitrains of arbitrary height and our next immediate goal is to develop an appropriate language for the description of such constructions. In other words, we repeat §6.2 in a more intricate setting.

Definition 11.1. Given a subquasitrain $\vec{F} = (F, \equiv^F_0, \dots, \equiv^F_m)$ of a quasitrain

$$\vec{H} = (H, \equiv^H_0, \dots, \equiv^H_m)$$

we call $\vec{H}$ a 1-*extension* of $\vec{F}$ provided that $(H, \equiv^H_1)$ is an extension of $(F, \equiv^F_1)$.

Let us recall that the latter notion was introduced in Definition 6.2 and in the present situation it means that the following three conditions hold.

(i) The hypergraphs F and H have the same isolated vertices.
(ii) Every 1-wagon W of $\vec{H}$ contracts to a 1-wagon of $\vec{F}$, i.e., satisfies $E(W)\cap E(F) \ne \varnothing$.
(iii) If two distinct 1-wagons $W^\star_1$ and $W^{\star\star}_1$ of $\vec{H}$ contract to the 1-wagons $\overline{W}^\star_1$ and $\overline{W}^{\star\star}_1$ of $\vec{F}$, then $V(W^\star_1) \cap V(W^{\star\star}_1) = V(\overline{W}^\star_1) \cap V(\overline{W}^{\star\star}_1)$.

The next lemma expresses the intuitively obvious fact that there is a natural bijective correspondence between the 1-extensions of a quasitrain $\vec{F} = (F, \equiv^F_0, \dots, \equiv^F_m)$ and the extensions of the pretrain $(F, \equiv^F_1)$.

Lemma 11.2. *Let $\vec{F} = (F, \equiv^F_0, \dots, \equiv^F_m)$ be a quasitrain. If the pretrain $(H, \equiv^H_1)$ is an extension of $(F, \equiv^F_1)$, then there exists a unique quasitrain structure*

$$\vec{H} = (H, \equiv^H_0, \dots, \equiv^H_m)$$

such that $\vec{F}$ is a subquasitrain of $\vec{H}$.

Notice that in this situation $\vec{H}$ is a 1-extension of $\vec{F}$.

Proof of Lemma 11.2. Dealing with the uniqueness first, we consider any such quasitrain $\overrightarrow{H} = (H, \equiv^H_0, \dots, \equiv^H_m)$. Definition 10.1(*i*) determines the relation $\equiv^H_0$. For every $\mu \in [2, m]$ we contend that

$$\forall e', e'' \in E(H)\ \big[e' \equiv^H_\mu e'' \iff \exists e_\star, e_{\star\star} \in E(F)\ \ e' \equiv^H_1 e_\star \equiv^F_\mu e_{\star\star} \equiv^H_1 e''\big]. \tag{11.1}$$

Indeed, let $\mu \in [2, m]$ and $e', e'' \in E(H)$. For the backwards implication we just need to observe that because of Definition 10.1(*ii*) and Definition 10.6(*ii*) the formula $e' \equiv^H_1 e_\star \equiv^F_\mu e_{\star\star} \equiv^H_1 e''$ yields $e' \equiv^H_\mu e_\star \equiv^H_\mu e_{\star\star} \equiv^H_\mu e''$, whence $e' \equiv^H_\mu e''$.

For the forwards direction, we suppose $e' \equiv^H_\mu e''$ and take two edges $e_\star, e_{\star\star} \in E(F)$ which are in the same 1-wagons of H as e', e'', respectively. Since $(H, \equiv^H_1)$ is an extension of $(F, \equiv^F_1)$, Definition 6.2(*ii*) informs us that such edges do indeed exist. Now $e_\star \equiv^H_1 e' \equiv^H_\mu e'' \equiv^H_1 e_{\star\star}$ implies $e_\star \equiv^H_\mu e_{\star\star}$. Due to the fact that $\vec{F}$ is a subquasitrain of $\overrightarrow{H}$ we can conclude $e_\star \equiv^F_\mu e_{\star\star}$, and altogether the edges $e_\star$, $e_{\star\star}$ are as desired. This proves (11.1) and the uniqueness of $\overrightarrow{H}$ follows.

Addressing the claim on the existence of $\overrightarrow{H}$ we define $\equiv^H_0$ as demanded by Definition 10.1(*i*) and for $\mu \in [2, m]$ we define $\equiv^H_\mu$ by (11.1). We omit the easy proof that these relations are indeed equivalence relations satisfying the clauses (*ii*) and (*iii*) of Definition 10.1. □

The following special cases are going to be important soon.

Example 11.3. Suppose that $\vec{F} = (F, \equiv^F_0, \dots, \equiv^F_m)$ is an arbitrary ordered quasitrain of height m and that $(H, \equiv^H_1)$ denotes the ordered pretrain obtained from $(F, \equiv^F_1)$ by wagon assimilation as discussed in Example 6.4. Lemma 11.2 leads us to a (unique) ordered quasitrain $\overrightarrow{H} = (H, \equiv^H_0, \dots, \equiv^H_m)$ of height m which is a 1-extension of $\vec{F}$. It will be convenient to say that $\overrightarrow{H}$ arises from $\vec{F}$ by *assimilation of its* 1*-wagons*.

Example 11.4. Consider an ordered hypergraph pair (X, W) such that neither X nor W has isolated vertices. Given an ordered quasitrain $\vec{F} = (F, \equiv^F_0, \dots, \equiv^F_m)$ all of whose 1-wagons are order-isomorphic to W we can form the ordered pretrain

$$(G, \equiv^G_1) = (F, \equiv^F_1) \ltimes (X, W)$$

as in Definition 6.6. Since $(G, \equiv^G_1)$ is an extension of $(F, \equiv^F_1)$, Lemma 11.2 gives rise to a (unique) 1-extension $\vec{G} = (G, \equiv^G_0, \dots, \equiv^G_m)$ of $\vec{F}$. We shall write $\vec{G} = \vec{F} \ltimes (X, W)$ for this construction (see Figure 11.1).

Lemma 11.5. *If the quasitrain $\overrightarrow{H} = (H, \equiv^H_0, \dots, \equiv^H_m)$ of height m is a 1-extension of the quasitrain $\vec{F} = (F, \equiv^F_0, \dots, \equiv^F_m)$, then for every $\mu \in [m]$ the pretrain $(H, \equiv^H_\mu)$ is an extension of $(F, \equiv^F_\mu)$.*

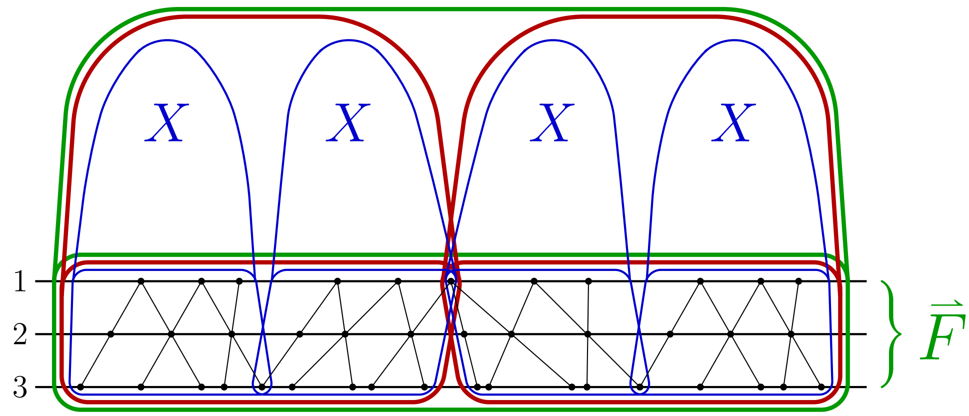


FIGURE 11.1. The train $\vec{F} \ltimes (X, W)$, where $\vec{F}$ is 3-partite and has the parameter $(\{1,2\},\{3\},\{1\})$. For space reasons, the 3-partite structure of X is not shown in this figure.

Proof. Given $\mu \in [m]$ we check the three conditions in Definition 6.2. There is no problem with *(i)*. For *(ii)* we consider an arbitrary μ-wagon W_μ of $\vec{H}$ and take an edge $e \in E(W_\mu)$. Since the 1-wagon of e contracts to $\vec{F}$, there is an edge $e' \in E(F)$ with $e' \equiv^H_1 e$. In particular, we have $e' \in E(W_\mu) \cap E(F)$, meaning that W_μ indeed contracts.

Proceeding with *(iii)* we let $\overline{W}'_\mu$, $\overline{W}''_\mu$ be the contractions of two distinct μ-wagons W'_μ, W''_μ of $\vec{H}$. Now $V(\overline{W}'_\mu) \cap V(\overline{W}''_\mu) \subseteq V(W'_\mu) \cap V(W''_\mu)$ is clear and for the reverse inclusion we look at an arbitrary vertex $x \in V(W'_\mu) \cap V(W''_\mu)$. Pick two edges $e' \in E(W'_\mu)$, $e'' \in E(W''_\mu)$ with $x \in e' \cap e''$ and denote the 1-wagons of $\vec{H}$ these edges belong to by W'_1, W''_1, respectively. Since $\vec{H}$ is a 1-extension of $\vec{F}$, the contractions $\overline{W}'_1$, $\overline{W}''_1$ of these wagons satisfy

$$x \in V(W'_1) \cap V(W''_1) = V(\overline{W}'_1) \cap V(\overline{W}''_1)$$

and, consequently, there exist edges $e_\star \in E(\overline{W}'_1)$, $e_{\star\star} \in E(\overline{W}''_1)$ with $x \in e_\star \cap e_{\star\star}$. As e' and $e_\star$ are in the same 1-wagon of $\vec{H}$, they are also in the same μ-wagon, for which reason $e_\star \in E(W'_\mu) \cap E(F) = E(\overline{W}'_\mu)$. Similarly we have $e_{\star\star} \in E(\overline{W}''_\mu)$ and altogether $x \in V(\overline{W}'_\mu) \cap V(\overline{W}''_\mu)$ follows. □

In general, 1-extensions of trains are only known to be quasitrains, but they may fail to be trains. This is for the reason that two edges of an enlarged 1-wagon might intersect each other in a 'wrong' vertex class. The lemma that follows shows that, actually, this is the only obstacle.

Lemma 11.6. *Suppose that $\vec{F} = (F, \equiv^F_0, \dots, \equiv^F_m)$ is an f-partite train of height m with parameter $\vec{A} = (A_1, \dots, A_m)$ and that the f-partite quasitrain $\vec{H} = (H, \equiv^H_0, \dots, \equiv^H_m)$ is a 1-extension of $\vec{F}$. If every 1-wagon X of $\vec{H}$ has the property that $e' \cap e'' \subseteq V_{A_1}(H)$ holds for any two distinct edges $e', e'' \in E(X)$, then $\vec{H}$ is a train with parameter $\vec{A}$.*

Proof. We need to check that Definition 10.4(*ii*) holds for $\overrightarrow{H}$. For $\mu = 1$ this was stated as a hypothesis, so suppose $\mu \in [2, m]$ from now on. Let $W'_{\mu-1}$ and $W''_{\mu-1}$ be two distinct $(\mu-1)$-wagons of $\overrightarrow{H}$ contained in the same μ-wagon and denote their contractions to $\vec{F}$ by $\overline{W}'_{\mu-1}$, $\overline{W}''_{\mu-1}$. Lemma 11.5 informs us that $(H, \equiv^H_{\mu-1})$ is an extension of $(F, \equiv^F_{\mu-1})$ and thus we have indeed

$$V(W'_{\mu-1}) \cap V(W''_{\mu-1}) = V(\overline{W}'_{\mu-1}) \cap V(\overline{W}''_{\mu-1}) \subseteq V_{A_\mu}(F) \subseteq V_{A_\mu}(H)\,. \qquad \square$$

Taking a 1-extension can affect the $\mathfrak{girth}$ of a quasitrain, for the new 1-wagons might contain shorter cycles than the original 1-wagons. The next result shows, that if there are no problems with 1-wagons, then 1-extensions inherit the $\mathfrak{girth}$ of the original quasitrain.

Lemma 11.7. *Suppose that $\vec{g} = (g_1, \dots, g_m) \in \mathfrak{M}$ is a sequence of some positive length m, that $\vec{F} = (F, \equiv^F_0, \dots, \equiv^F_m)$ is a quasitrain with $\mathfrak{girth}(\vec{F}) > \vec{g}$ and that $\overrightarrow{H} = (H, \equiv^H_0, \dots, \equiv^H_m)$ is a 1-extension of $\vec{F}$. If every 1-wagon X of $\overrightarrow{H}$ satisfies* $\mathrm{girth}(X) > g_1$, *then* $\mathfrak{girth}(\overrightarrow{H}) > \vec{g}$.

Proof. Given $\mu \in [m]$ and a μ-wagon W_μ of $\overrightarrow{H}$ we are to prove that $\mathfrak{girth}(W_\mu, \equiv^{W_\mu}_{\mu-1}) > g_\mu$. In the special case $\mu = 1$ this was stated as an assumption, so it remains to consider the case $\mu \in [2, m]$. Assume for the sake of contradiction that for some $n \in [2, g_\mu]$ there exists an n-cycle of distinct $(\mu-1)$-wagons

$$\mathscr{C} = W^1_{\mu-1} q_1 \dots W^n_{\mu-1} q_n$$

all of which are contained in W_μ. Owing to Lemma 11.5 these $(\mu-1)$-wagons contract to $\vec{F}$ and the connecting vertices $q_1, \dots, q_n$ are in $V(F)$. Therefore, the contraction $\overline{W}_\mu$ of W_μ fails to have the property $\mathfrak{girth}(\overline{W}_\mu, \equiv^{\overline{W}_\mu}_{\mu-1}) > g_\mu$, contrary to $\mathfrak{girth}(\vec{F}) > \vec{g}$. $\square$

Let us state an immediate consequence of Lemma 11.6 and Lemma 11.7.

Corollary 11.8. *If $\vec{g} \in \mathfrak{M}$ is a nonempty sequence, $\vec{F}$ denotes a train of height $|\vec{g}|$ with parameter $\vec{A}$ and $\mathfrak{girth}(\vec{F}) > \vec{g}$, and the quasitrain $\overrightarrow{H}$ arises from $\vec{F}$ by assimilation of its 1-wagons, then $\overrightarrow{H}$ is again a train of height m with parameter $\vec{A}$ and $\mathfrak{girth}(\overrightarrow{H}) > \vec{g}$.* $\square$

We conclude this subsection with a brief discussion of disjoint unions of quasitrains. Suppose first that $\{\vec{G}_j : j \in J\}$ is a family of mutually vertex-disjoint f-partite quasitrains of the same height $m \in \mathbb{N}$, say $\vec{G}_j = (G_j, \equiv^j_0, \dots, \equiv^j_m)$ for every $j \in J$. By the *union* of this family we mean the f-partite quasitrain $\vec{G} = (G, \equiv^G_0, \dots, \equiv^G_m)$ with

$$V(G) = \dot{\bigcup_{j \in J}} V(G_j) \quad \text{and} \quad E(G) = \dot{\bigcup_{j \in J}} E(G_j)$$

whose equivalence relations are defined as follows.

- If $e', e'' \in E(G)$ and $\mu \in [0, m)$, then $e' \equiv^G_\mu e''$ means that there is an index $j \in J$ with $e', e'' \in E(G_j)$ and $e' \equiv^j_\mu e''$.

- Moreover, $e' \equiv^G_m e''$ holds for all edges $e', e'' \in E(G)$.

One checks immediately that $\vec{G}$ is indeed a quasitrain of height m and that $\vec{G}_j$ is a subquasitrain of $\vec{G}$ for every $j \in J$. If the quasitrains $\vec{G}_j$ are ordered, we order the vertex classes of $\vec{G}$ in such a way that $\vec{G}_j$ is an ordered subquasitrain of $\vec{G}$ for every $j \in J$.

If a family $\mathfrak{G} = \{\vec{G}_j : j \in J\}$ of not necessarily vertex-disjoint ordered quasitrains of the same height m is given, we can take a family of mutually vertex-disjoint ordered quasitrains $\{\vec{G}^\star_j : j \in J\}$ such that $\vec{G}^\star_j$ is order-isomorphic to $\vec{G}_j$ for every $j \in J$, and then we can form its union $\vec{G}$ as explained above. In this situation $\vec{G}$ is called the *disjoint union* of the family $\mathfrak{G}$. One readily confirms that if all members of $\mathfrak{G}$ are trains with the same parameter $\vec{A}$, then $\vec{G}$ is a train with parameter $\vec{A}$ as well. Moreover, if $\vec{g} \in \mathfrak{M}$ is a sequence of length m and $\mathfrak{girth}(\vec{G}_j) > \vec{g}$ holds for every $j \in J$, then $\mathfrak{girth}(\vec{G}) > \vec{g}$ follows.

We conclude this discussion with an easy fact on disjoint unions that will help us in Section 12 to analyse the $\mathfrak{Girth}$ of train picture zero.

Fact 11.9. *Suppose that a sequence $\vec{g} \in \mathfrak{M}^\times_\leqslant$ and a quasitrain $\vec{F}$ of height $|\vec{g}|$ satisfy $\mathfrak{girth}(\vec{F}) > \vec{g}$. If $\mathscr{P}$ denotes a set of mutually vertex-disjoint copies of $\vec{F}$ and the quasitrain $\vec{P}$ is their union, then $\mathfrak{Girth}(\vec{P}, \mathscr{P}^+) > \vec{g}$.*

Proof. Writing $\vec{g} = (g_1, \dots, g_m)$ we shall prove first

$$\mathfrak{Girth}(P, \equiv^P_{\mu-1}, \mathscr{P}^+) > g_\mu \qquad \text{for every } \mu \in [m]. \tag{11.2}$$

To this end we recall that Lemma 10.8 tells us $\mathfrak{girth}(F, \equiv^F_{\mu-1}) > g_\mu$, which due to Lemma 8.17 implies $\mathfrak{Girth}\big(F, \equiv^F_{\mu-1}, E^+(F) \cup \{(F, \equiv^F_{\mu-1})\}\big) > g_\mu$. Since big cycles of the pretrain system $(P, \equiv^P_{\mu-1}, \mathscr{P}^+)$ cannot jump from one copy in $\mathscr{P}$ to another copy, this confirms (11.2).

Moreover, Lemma 8.11 immediately implies $\mathfrak{Girth}(P, \equiv^P_m, \mathscr{P}^+) > 1$. $\square$

11.2. **A generalised extension lemma.** The proof of Proposition 10.14 starts with the assumption that for some sequence $\vec{g} = (g_1, \dots, g_m) \in \mathfrak{M}^\times_\leqslant$ we have a construction $\Psi^{\vec{g}}$ exemplifying $\diamondsuit_{\vec{g}}$. The extension process then gives rise to a construction $\Upsilon = \mathrm{Ext}(\Omega^{(2)}, \Psi^{\vec{g}})$, which is applicable to trains of height $m+1$ whose $\mathfrak{girth}$ exceeds $(2) \circ \vec{g}$. Roughly speaking, this construction starts by applying $\Omega^{(2)}$ to the (assimilated) 1-wagons and forming a disjoint union over all possible extensions. We then apply $\Psi^{\vec{g}}$ to an auxiliary train of height m with "larger edges", and insert the earlier 1-wagons back into the current edges.

We proceed with a full description as to how this construction is carried out. Suppose that $\vec{F} = (F, \equiv^F_0, \dots, \equiv^F_{m+1})$ is an ordered f-partite train of height $m+1$ with parameter $\vec{A} = (A_1, \dots, A_{m+1})$ satisfying $\mathfrak{girth}(\vec{F}) > (2) \circ \vec{g}$, and that r signifies a number of colours. Now the train system $\Upsilon_r(\vec{F}) = (\vec{H}, \mathscr{H})$ is constructed by means of the following eight steps paralleling the discussion in §6.3.

(1) Let $(\widehat{F}, \equiv_0^{\widehat{F}}, \dots, \equiv_{m+1}^{\widehat{F}})$ be obtained from $\vec{F}$ by assimilating its 1-wagons. Due to Corollary 11.8 this is a train of height $m+1$ with parameter $\vec{A}$ containing a standard copy of $\vec{F}$ and satisfying $\mathfrak{girth}(\widehat{F}, \equiv_0^{\widehat{F}}, \dots, \equiv_{m+1}^{\widehat{F}}) > (2) \circ \vec{g}$. Let W denote an ordered f-partite hypergraph all 1-wagons of this train are isomorphic to. Notice that W is a linear A_1-intersecting hypergraph without isolated vertices.

(2) Construct $\Omega_r^{(2)}(W) = (X, \mathscr{X})$ and assume, without loss of generality, that X has no isolated vertices. Now X is an ordered f-partite hypergraph which is linear and A_1-intersecting (by Proposition 3.8, Corollary 3.14, and Lemma 3.18). Furthermore, Corollary 5.6 entails $\mathrm{Girth}(X, \mathscr{X}^+) > 2$.

(3) Construct the family

$$\mathfrak{G} = \left\{(\widehat{F}, \equiv_0^{\widehat{F}}, \dots, \equiv_{m+1}^{\widehat{F}}) \ltimes (X, W_\star) \colon W_\star \in \mathscr{X}\right\}$$

of ordered f-partite quasitrains as in Example 11.4. By Lemma 11.6 and Lemma 11.7 every $\vec{G}_\star \in \mathfrak{G}$ is a train with parameter $\vec{A}$ that satisfies $\mathfrak{girth}(\vec{G}_\star) > (2) \circ \vec{g}$.

Let $\vec{G} = (G, \equiv_0^G, \dots, \equiv_{m+1}^G)$ be the disjoint union of the trains in $\mathfrak{G}$ as defined at the end of §11.1. Thus $\vec{G}$ is a train with parameter $\vec{A}$ satisfying $\mathfrak{girth}(\vec{G}) > (2) \circ \vec{g}$ and all 1-wagons of $\vec{G}$ are order-isomorphic to X. Moreover, $\vec{G}$ contains $|\mathscr{X}|$ standard copies of $(F, \equiv^F)$.

(4) Let I be the index set of the given train $\vec{F}$ and define the function $x\colon I \longrightarrow \mathbb{N}$ by $x(i) = |V_i(X)|$ for every $i \in I$. Let M be the ordered x-partite hypergraph with $V(M) = V(G)$ whose edges correspond to the wagons of $(G, \equiv_1^G)$ (so that G is living in M). Moreover, let $\overrightarrow{M} = (M, \equiv_0^M, \dots, \equiv_m^M)$ be the train of height m whose μ-wagons correspond to the $(\mu+1)$-wagons of $\vec{G}$. In more precise terms, this means that for every $\mu \in [0, m]$ the pretrain $(G, \equiv_{\mu+1}^G)$ is derived from $(M, \equiv_\mu^M)$. We remark that $\overrightarrow{M}$ has the parameter $\vec{A}_\bullet = (A_2, \dots, A_{m+1})$ and satisfies $\mathfrak{girth}(\overrightarrow{M}) > \vec{g}$.

(5) Construct the ordered x-partite train system

$$\Psi^{\vec{g}}_{r^{e(X)}}(\overrightarrow{M}) = (\overrightarrow{N}, \mathscr{N}) = (N, \equiv_0^N, \dots, \equiv_m^N, \mathscr{N})$$

of height m. Due to $\diamondsuit_{\vec{g}}$ the copies of this system are strongly induced and we have $\mathfrak{Girth}(\overrightarrow{N}, \mathscr{N}^+) > \vec{g}$. Moreover, the parameter of $\overrightarrow{N}$ is again $\vec{A}_\bullet$.

(6) Let H be the ordered f-partite hypergraph obtained from N by inserting ordered copies of X into its edges. We endow H with the following train structure

$$\overrightarrow{H} = (H, \equiv_0^H, \dots, \equiv_{m+1}^H)$$

of height $m+1$. For $\mu \in [m+1]$ we declare two edges $e', e'' \in E(H)$ to be in the same μ-wagon of $\overrightarrow{H}$ if the edges $f', f'' \in E(N)$ with $e' \subseteq f'$ and $e'' \subseteq f''$ satisfy $f' \equiv_{\mu-1}^N f''$. In other words, we demand that the pretrain $(H, \equiv_\mu^H)$ be derived from

$(N, \equiv^N_{\mu-1})$. Recall that the 0-wagons of $\overrightarrow{H}$ need to be determined according to Definition 10.1(i).

(7) Every copy $\overrightarrow{M}_\star \in \mathscr{N}$ gives rise to a derived copy $\vec{G}_\star \in \binom{\overrightarrow{H}}{\vec{G}}$ and we write $\mathscr{H}_\bullet$ for the system of all $|\mathscr{N}|$ copies arising in this manner. Each member of $\mathscr{H}_\bullet$ contains $|\mathscr{X}|$ standard copies of $\vec{F}$. Let $\mathscr{H} \subseteq \binom{\overrightarrow{H}}{\vec{F}}$ denote the system of all these copies.

(8) Finally, we set $\Upsilon_r(\vec{F}) = (\overrightarrow{H}, \mathscr{H})$.

The next result summarises all properties of this construction we shall need in the sequel.

Lemma 11.10. *Assume $\diamondsuit_{\vec{g}}$ for some sequence $\vec{g} \in \mathfrak{M}^\times_\leqslant$ of length m. There exists a construction Υ which generates for every ordered f-partite train $\vec{F}$ of height $m+1$ with $\mathfrak{girth}(\vec{F}) > (2) \circ \vec{g}$ and every number of colours r a train system $\Upsilon_r(\vec{F}) = (\overrightarrow{H}, \mathscr{H})$ with the same parameter as $\vec{F}$ and strongly induced copies such that $\mathscr{H} \longrightarrow (\vec{F})_r$ and $\mathfrak{Girth}(\overrightarrow{H}, \mathscr{H}^+) > \vec{g}$.*

It might be helpful to point out that the conclusion of this lemma implies

$$\mathfrak{girth}(\overrightarrow{H}) > (2) \circ \vec{g}\,.$$

This is because $\mathfrak{Girth}(\overrightarrow{H}, \mathscr{H}^+) > \vec{g}$ presupposes that H is linear, whence all 1-wagons of H are linear. Moreover, for $\mu \in [2, m+1]$ the required statement on cycles of $(\mu-1)$-wagons within the same μ-wagon follows from $\mathfrak{Girth}(\overrightarrow{H}, E^+(H)) > \vec{g}$ via Lemma 8.17.

Proof of Lemma 11.10. Given a train $\vec{F} = (F, \equiv^F_0, \dots, \equiv^F_{m+1})$ of height $m+1$ with parameter $\vec{A}$ such that $\mathfrak{girth}(\vec{F}) > (2) \circ \vec{g}$ and a number of colours r we follow the above Steps (1)–(8) and construct the train system $\Upsilon_r(\vec{F}) = (\overrightarrow{H}, \mathscr{H})$.

Lemma 6.11 (applied to $\Omega^{(2)}$ and $\Psi^{\vec{g}}$ here in place of Φ and Ψ there) yields the partition relation $\mathscr{H} \longrightarrow (\vec{F})_r$. Thus it remains to prove

(a) that the copies in $\mathscr{H}$ are strongly induced,

(b) and $\mathfrak{Girth}(\overrightarrow{H}, \mathscr{H}^+) > \vec{g}$.

We begin by taking a closer look at the train $\vec{G}$ constructed in Step (3). Let $\mathscr{S}_G$ be the system consisting of the $|\mathscr{X}|$ standard copies of $\vec{F}$ in $\vec{G}$. Each of them is contained in a unique train of the form $\vec{G}_\star = (\widehat{F}, \equiv^{\widehat{F}}_0, \dots, \equiv^{\widehat{F}}_{m+1}) \ltimes (X, W_\star)$, where $W_\star \in \mathscr{X}$.* Since the copies in $\mathscr{X}$ are strongly induced, Lemma 6.8 informs us that $(G_\star, \equiv^{G_\star}_1)$ is a tame extension of $(F, \equiv^F_1)$. In particular,

$$\text{the system } (G, \mathscr{S}_G) \text{ has strongly induced copies.} \tag{11.3}$$

*For the sake of transparency we are ignoring here the isomorphism implied in the formation of the disjoint union.

Moreover, for every $\mu \in [m+1]$ the equivalence relation $\equiv_1^{G_\star}$ refines $\equiv_\mu^{G_\star}$ and thus the pretrain $(G_\star, \equiv_\mu^{G_\star})$ is a tame extension of $(F, \equiv_\mu^F)$ as well. Consequently,

$$\text{for every } \mu \in [m] \text{ the system } \mathscr{S}_G \text{ is scattered in } (G, \equiv_\mu^G). \tag{11.4}$$

For clarity we point out that we cannot claim this for $\mu = m+1$; the difference is that the unique $(m+1)$-wagon of $\vec{G}$ is the union of the $(m+1)$-wagons of the various trains $\vec{G}_\star$, while for $\mu \in [m]$ the μ-wagons of these trains "remain separate from each other".

Later we will need to know that

$$\text{if } F_\star \in \mathscr{S}_G \text{ and } x \text{ is an isolated vertex of } F_\star, \text{ then } x \text{ is isolated in } G. \tag{11.5}$$

Indeed, by the definition of wagon assimilation, x is also isolated in $\widehat{F}$ and thus in every member of $\mathfrak{G}$.

We proceed by discussing the hypergraphs H and N. Clearly, H is living in N (cf. Step *(6)*) and the system derived from $\mathscr{N}$ is $\mathscr{H}_\bullet$ (cf. Step *(7)*). By Step *(5)* the copies in $\mathscr{N}$ are strongly induced and, therefore, Fact 9.4 tells us that the system $(H, \mathscr{H}_\bullet)$ has strongly induced copies, too. Together with (11.3) and the fact that strong inducedness is a transitive relation this proves *(a)*.

Let us now write $\vec{g} = (g_1, \dots, g_m)$ and fix some $\mu \in [m]$. Due to

$$\mathfrak{Girth}(\overrightarrow{N}, \mathscr{N}^+) > \vec{g} \tag{11.6}$$

we know $\mathfrak{Girth}(N, \equiv_{\mu-1}^N, \mathscr{N}^+) > g_\mu$. As the pretrain $(H, \equiv_\mu^H)$ is derived from $(N, \equiv_{\mu-1}^N)$ (cf. Step *(6)*), Lemma 9.6 translates this to $\mathfrak{Girth}(H, \equiv_\mu^H, \mathscr{H}_\bullet^+) > g_\mu$, which together with (11.4) and Lemma 9.12 reveals $\mathfrak{Girth}(H, \equiv_\mu^H, \mathscr{H}^+) > g_\mu$.

So in order to establish *(b)* it only remains to prove $\mathfrak{Girth}(H, \equiv_{m+1}^H, \mathscr{H}^+) > 1$. The related statement

$$\mathfrak{Girth}(H, \equiv_{m+1}^H, \mathscr{H}_\bullet^+) > 1 \tag{11.7}$$

can be shown as in the previous paragraph by observing that (11.6) contains the information $\mathfrak{Girth}(N, \equiv_m^N, \mathscr{N}^+) > 1$ and invoking Lemma 9.6.

Now let $F_1, F_2 \in \mathscr{H}^+$ be two distinct copies having a vertex x in common. According to Lemma 8.11 and symmetry it suffices to exhibit an edge $e \in E(F_1)$ such that $x \in e$. Assume contrariwise that x is an isolated vertex of F_1.

In particular, F_1 is a real copy. Let $G_1 \in \mathscr{H}_\bullet$ be the copy one of whose standard copies is $\vec{F}_1$. If F_2 is a real copy as well we determine $G_2 \in \mathscr{H}_\bullet$ similarly and if F_2 is an edge copy we set $G_2 = F_2$.

Due to (11.5) we know that x is isolated in G_1. Moreover, the copies $G_1, G_2 \in \mathscr{H}_\bullet^+$ have the vertex x in common. If they were distinct, then (11.7) and Lemma 8.11 would lead to the contradiction that x is non-isolated in G_1.

So altogether $G_1 = G_2 \in \mathscr{H}_\bullet$ is a real copy and, therefore, F_1 and F_2 are two distinct standard copies in G_1. But now $x \in V(F_1) \cap V(F_2)$ contradicts $F_1 \neq F_2$ because of the construction of $\vec{G}$. □

§12. Trains in partite constructions

In this section we complete the proof of Proposition 10.14 and establish Proposition 10.15. Both tasks are accomplished by means of the partite construction method and thus we begin with some general remarks on train pictures.

12.1. **Quasitrain constructions.** Since it allows us to ignore parameters, it will be easier to study quasitrains in partite constructions first. Suppose that Φ denotes a Ramsey construction for hypergraphs and that Ξ is a partite lemma applicable to k-partite k-uniform quasitrains of a fixed height $m \in \mathbb{N}$. As we shall see below, we can then define a construction $\mathrm{PC}(\Phi, \Xi)$ applicable to quasitrains of height m.

Let us first introduce some terminology for such situations. Suppose that $(G, \mathscr{G})$ is a system of hypergraphs, where $\mathscr{G} \subseteq \binom{G}{F}$ holds for some hypergraph F endowed with a fixed quasitrain structure $\vec{F} = (F, \equiv_0^F, \dots, \equiv_m^F)$ of height m. A *quasitrain picture over* $(G, \mathscr{G})$ is a structure of the form $(\vec{\Pi}, \mathscr{P}, \psi) = (\Pi, \equiv_0^\Pi, \dots, \equiv_m^\Pi, \mathscr{P}, \psi)$, where

- $\vec{\Pi}$ is a quasitrain of height m,
- $(\Pi, \mathscr{P}, \psi)$ is a picture over $(G, \mathscr{G})$,
- and $\mathscr{P} \subseteq \binom{\vec{\Pi}}{\vec{F}}$, i.e., every copy $(F_\star, \equiv_0^{F_\star}, \dots, \equiv_m^{F_\star}) \in \mathscr{P}$ is a subquasitrain of $\vec{\Pi}$ isomorphic to $\vec{F}$.

For instance, the picture zero $(\Pi_0, \mathscr{P}_0, \psi_0)$ introduced in §3.2 expands uniquely to the corresponding *quasitrain picture zero* $(\vec{\Pi}_0, \mathscr{P}_0, \psi_0)$ with the property that for $\mu \in [0, m)$ every μ-wagon of $\vec{\Pi}_0$ is contained in exactly one copy from $\mathscr{P}_0$. Recall that owing to Definition 10.1*(iii)* all edges of Π_0 need to be in the same m-wagon of $\vec{\Pi}_0$. In other words, the quasitrain $\vec{\Pi}_0$ is constructed to be the disjoint union of the quasitrains in $\mathscr{P}_0$.

Now suppose that $(\Pi, \equiv_0^\Pi, \dots, \equiv_m^\Pi, \mathscr{P}, \psi_\Pi)$ is such a quasitrain picture and that $e \in E(G)$. The constituent Π^e induces a subquasitrain $\vec{\Pi}^e$ of $\vec{\Pi}$. For reasons that will become apparent later we only define amalgamations over e when

$$E(\Pi^e) \neq \varnothing\,, \tag{12.1}$$

which will never cause problems in practice. Given a k-partite k-uniform quasitrain system

$$(\vec{H}, \mathscr{H}) = (H, \equiv_0^H, \dots, \equiv_m^H, \mathscr{H})$$

(where $k = |e|$) with $\mathscr{H} \subseteq \binom{\vec{H}}{\vec{\Pi}^e}$ we can construct a structure

$$(\Sigma, \equiv_0^\Sigma, \dots, \equiv_m^\Sigma, \mathscr{Q}, \psi_\Sigma) = (\Pi, \equiv_0^\Pi, \dots, \equiv_m^\Pi, \mathscr{P}, \psi_\Pi) \ast (H, \equiv_0^H, \dots, \equiv_m^H, \mathscr{H}) \tag{12.2}$$

by forming, in a first step, the ordinary picture $(\Sigma, \mathscr{Q}, \psi_\Sigma) = (\Pi, \mathscr{P}, \psi_\Pi) \ast (H, \mathscr{H})$ and then defining the equivalence relations $\equiv_0^\Sigma, \dots, \equiv_m^\Sigma$ on $E(\Sigma)$ in such a way that

$$(\Sigma, \equiv_\mu^\Sigma, \mathscr{Q}, \psi_\Sigma) = (\Pi, \equiv_\mu^\Pi, \mathscr{P}, \psi_\Pi) \ast (H, \equiv_\mu^H, \mathscr{H})$$

holds for every $\mu \in [0, m]$ (see §7.3 and Lemma 7.3). One verifies easily that $(\vec{\Sigma}, \mathscr{Q}, \psi_\Sigma)$ is again a quasitrain picture over $(G, \mathscr{G})$, where $\vec{\Sigma} = (\Sigma, \equiv_0^\Sigma, \dots, \equiv_m^\Sigma)$. In particular, one has to check here that the construction in §7.3 causes the equivalence classes of $\equiv_0^\Sigma$ to consist of single edges. Moreover, one needs to convince oneself that all edges of Σ are in the same wagon with respect to $\equiv_m^\Sigma$, which requires (12.1).

Let us now return to the discussion of $\mathrm{PC}(\Phi, \Xi)$, where, let us recall, Φ is a Ramsey construction for hypergraphs and Ξ denotes a partite lemma for quasitrains of height m. Given a quasitrain $\vec{F} = (F, \equiv_0^F, \dots, \equiv_m^F)$ of height m and a number of colours r the quasitrain system $\mathrm{PC}(\Phi, \Xi)_r(\vec{F})$ is constructed as follows. Set $\Phi_r(F) = (G, \mathscr{G})$ and assume, without loss of generality, that every edge of G appears in at least one copy of $\mathscr{G}$ (other edges of G are not needed for ensuring the partition relation $\mathscr{G} \longrightarrow (F)_r$). Now picture zero $(\vec{\Pi}_0, \mathscr{P}_0, \psi_0)$ over $(G, \mathscr{G})$ satisfies $E(\Pi_0^e) \neq \varnothing$ for every $e \in E(G)$ and thus we never need to worry about (12.1) throughout the ensuing partite construction. As usual we fix an enumeration $E(G) = \{e(1), \dots, e(N)\}$ and recursively we construct a sequence $(\vec{\Pi}_\alpha, \mathscr{P}_\alpha, \psi_\alpha)_{\alpha \leqslant N}$ of quasitrain pictures over $(G, \mathscr{G})$. Whenever we have just obtained $(\vec{\Pi}_{\alpha-1}, \mathscr{P}_{\alpha-1}, \psi_{\alpha-1})$ for some $\alpha \in [N]$ we generate the k-partite k-uniform quasitrain system $\Xi_r(\vec{\Pi}_{\alpha-1}^{e(\alpha)}) = (\vec{H}_\alpha, \mathscr{H}_\alpha)$ and amalgamate

$$(\vec{\Pi}_\alpha, \mathscr{P}_\alpha, \psi_\alpha) = (\vec{\Pi}_{\alpha-1}, \mathscr{P}_{\alpha-1}, \psi_{\alpha-1}) \ast (\vec{H}_\alpha, \mathscr{H}_\alpha)\,.$$

Finally, when the final quasitrain picture $(\vec{\Pi}_N, \mathscr{P}_N, \psi_N)$ has been reached, we stipulate

$$\mathrm{PC}(\Phi, \Xi)_r(\vec{F}) = (\vec{\Pi}_N, \mathscr{P}_N)\,.$$

12.2. **Train amalgamations.** The next problem is whether the partite construction method can handle trains instead of quasitrains as well. That is, we shall need to know suitable conditions on constructions Φ, Ξ, and trains $\vec{F}$ guaranteeing that for every number of colours r the quasitrain $\mathrm{PC}(\Phi, \Xi)_r(\vec{F})$ turns out to be a train. As usual, our strategy is to enforce that all quasitrain pictures $(\vec{\Pi}, \mathscr{P}, \psi)$ generated along the way have the property that their underlying quasitrains $\vec{\Pi}$ are trains. This is already somewhat problematic for picture zero and we resolve to deal with this situation by demanding that Φ be a train construction, so that vertically we have a train system $(\vec{G}, \mathscr{G})$ and not just a hypergraph system. We are thus led to the following notion of train pictures.

Definition 12.1. Suppose that $\vec{F}$ is a train and that $(\vec{G}, \mathscr{G})$ is a train system all of whose copies are isomorphic to $\vec{F}$. We say that $(\vec{\Pi}, \mathscr{P}, \psi_\Pi)$ is a *train picture over* $(\vec{G}, \mathscr{G})$ if (*i*) the trains $\vec{F}$, $\vec{G} = (G, \equiv_0^G, \dots, \equiv_m^G)$, and $\vec{\Pi} = (\Pi, \equiv_0^\Pi, \dots, \equiv_m^\Pi)$ are f-partite for the same function f, have the same height m, and the same parameter $\vec{A}$;

(*ii*) $(\vec{\Pi}, \mathscr{P}, \psi_\Pi)$ is a quasitrain picture over $(G, \mathscr{G})$;

(*iii*) and $\forall \mu \in [0, m]\ \forall e, e' \in E(\Pi)\ \ [e \equiv_\mu^\Pi e' \implies \psi_\Pi(e) \equiv_\mu^G \psi_\Pi(e')]$.

In this situation, we call m and $\vec{A}$ the *height* and the *parameter* of the picture $(\vec{\Pi}, \mathscr{P}, \psi_\Pi)$.

A more intuitive way of thinking about condition (*iii*) is that the projection ψ_Π is required to be a "train homomorphism" from $\vec{\Pi}$ to $\vec{G}$ (rather than just a mere hypergraph homomorphism). Notice that given $\vec{F}$ and $(\vec{G}, \mathscr{G})$ with $\mathscr{G} \subseteq \binom{\vec{G}}{\vec{F}}$ as in (*i*) we can always form the *train picture zero* $(\vec{\Pi}_0, \mathscr{P}_0, \psi_0)$ in the usual way. The next statement takes a brief look at the parameters of constituents of train pictures.

Fact 12.2. *Let $(\vec{\Pi}, \mathscr{P}, \psi)$ be an f-partite train picture over the train system $(\vec{G}, \mathscr{G})$. If m and $\vec{A} = (A_1, \dots, A_m)$ denote the height and the parameter of this picture, then for every edge $e \in E(G)$ the constituent $\vec{\Pi}^e$ is a train of height m whose parameter $\vec{D} = (D_1, \dots, D_m)$ is given by $D_\mu = e \cap V_{A_\mu}(G)$ for every $\mu \in [m]$.*

As usual, the constituent $\vec{\Pi}^e$ is regarded here as a k-partite k-uniform train with index set e, where $k = |e|$.

Proof. Recall that the f-partite structure of Π is defined by $V_i(\Pi) = \psi^{-1}\big(V_i(G)\big)$ for every index i in the domain of f. Thus the sets $D_\mu = e \cap V_{A_\mu}(G)$ satisfy

$$V_{D_\mu}(\Pi^e) = \psi^{-1}(D_\mu) = \psi^{-1}(e) \cap \psi^{-1}\big(V_{A_\mu}(G)\big) = V(\Pi^e) \cap V_{A_\mu}(\Pi)$$

for every $\mu \in [m]$.

Now if two edges $e_\star$, $e_{\star\star} \in E(\Pi^e)$ belong to the same μ-wagon but not to the same $(\mu - 1)$-wagon of $\vec{\Pi}^e$, then $e_\star \cap e_{\star\star} \subseteq V(\Pi^e) \cap V_{A_\mu}(\Pi) = V_{D_\mu}(\Pi^e)$. $\square$

When executing partite constructions with train pictures, we can maintain parameters by appealing to the following result.

Lemma 12.3. *Let $(\vec{G}, \mathscr{G})$ be a train system of height m with parameter $\vec{A} = (A_1, \dots, A_m)$. Suppose further that $(\vec{\Pi}, \mathscr{P}, \psi_\Pi)$ is a train picture over $(\vec{G}, \mathscr{G})$ and that*

$$(\vec{\Sigma}, \mathscr{Q}, \psi_\Sigma) = (\vec{\Pi}, \mathscr{P}, \psi_\Pi) \ast (\vec{H}, \mathscr{H})$$

holds for a train system $(\vec{H}, \mathscr{H}) = (H, \equiv_0^H, \dots, \equiv_m^H, \mathscr{H})$ and a quasitrain picture $(\vec{\Sigma}, \mathscr{Q}, \psi_\Sigma)$. If the amalgamation occurs over the edge $e \in E(G)$,

- *the parameter $\vec{D} = (D_1, \dots, D_m)$ of $\vec{H}$ is given by $D_\mu = e \cap V_{A_\mu}(G)$ for every $\mu \in [m]$,*
- *and $\mathfrak{Girth}(H, \equiv_\mu, \mathscr{H}^+) > 1$ for every $\mu \in [m]$,*

then $(\vec{\Sigma}, \mathscr{Q}, \psi_\Sigma)$ is a train picture over $(\vec{G}, \mathscr{G})$.

Let us emphasise that our demand on the parameter $\vec{D}$ agrees with Fact 12.2. As we saw in the above proof, it leads to $V_{D_\mu}(H) = V_{A_\mu}(\Sigma) \cap V(H)$ for every $\mu \in [m]$.

Proof of Lemma 12.3. We have to show that $\vec{A}$ is a legitimate parameter for $\vec{\Sigma}$ and that clause *(iii)* of Definition 12.1 holds for Σ instead of Π. The latter condition is clear for $\mu = 0$, because $\equiv^\Sigma_0$ is the same as equality. So it remains to consider an index $\mu \in [m]$, a μ-wagon W_μ of $\vec{\Sigma}$, and two edges $e', e'' \in E(W_\mu)$. We are to prove that

(*a*) if $e' \not\equiv^\Sigma_{\mu-1} e''$ and $x \in e' \cap e''$, then $x \in V_{A_\mu}(\Sigma)$;
(*b*) and $\psi_\Sigma(e') \equiv^G_\mu \psi_\Sigma(e'')$.

For both edges e' and e'' we distinguish the cases whether they belong to H or not. By symmetry there are three possibilities.

First Case: Neither e' nor e'' is in $E(H)$.

Let $\Pi_\star$ and $\Pi_{\star\star}$ denote the standard copies of Π with $e' \in E(\Pi_\star)$ and $e'' \in E(\Pi_{\star\star})$, respectively. In the special case $\Pi_\star = \Pi_{\star\star}$ both claims follow from $(\vec{\Pi}, \mathscr{P}, \psi_\Pi)$ being a train picture and $\vec{\Pi}_\star$ being a subtrain of $\vec{\Sigma}$, so we may assume $\Pi_\star \ne \Pi_{\star\star}$ from now on.

Starting with (*a*) we observe that the vertex x needs to belong to $V(H)$. Lemma 7.3(*c*)(*ii*) tells us that the copies $\Pi^e_\star, \Pi^e_{\star\star} \in \mathscr{H}$ extended by the standard copies $\Pi_\star$, $\Pi_{\star\star}$ intersect the wagon W_μ and, for this reason, W_μ contracts to a wagon $\overline{W}_\mu$ of $(H, \equiv^H_\mu)$. Clearly $\Pi^e_\star x \Pi^e_{\star\star} \overline{W}_\mu$ is a big cycle of order 1 in $(H, \equiv^H_\mu, \mathscr{H}^+)$. Due to $\mathfrak{Girth}(H, \equiv^H_\mu, \mathscr{H}^+) > 1$ and Lemma 8.11 there are two edges $e_\star \in E(\Pi^e_\star) \cap E(\overline{W}_\mu)$ and $e_{\star\star} \in E(\Pi^e_{\star\star}) \cap E(\overline{W}_\mu)$ with $x \in e_\star \cap e_{\star\star}$. By now we know four edges containing x and belonging to W_μ, namely e', $e_\star$, $e_{\star\star}$, and e''. Owing to $e' \not\equiv^\Sigma_{\mu-1} e''$ at least one of the three cases

$$e' \not\equiv^\Sigma_{\mu-1} e_\star\,, \quad e_\star \not\equiv^\Sigma_{\mu-1} e_{\star\star}\,, \quad \text{or} \quad e_{\star\star} \not\equiv^\Sigma_{\mu-1} e''$$

occurs. As $\vec{A}$ parametrises $\Pi_\star$, the first alternative implies indeed $x \in V_{A_\mu}(\Pi_\star) \subseteq V_{A_\mu}(\Sigma)$, and the third case is similar. Moreover, the second case rewrites as $e_\star \not\equiv^H_{\mu-1} e_{\star\star}$ and $x \in V_{D_\mu}(H) \subseteq V_{A_\mu}(\Sigma)$ follows (see Figure 12.1). This completes the proof of (*a*).

Proceeding with (*b*) we again invoke Lemma 7.3(*c*)(*ii*), thus obtaining two edges $e_\star \in E(H) \cap E(\Pi_\star)$ and $e_{\star\star} \in E(H) \cap E(\Pi_{\star\star})$ with $e' \equiv^\Sigma_\mu e_\star \equiv^\Sigma_\mu e_{\star\star} \equiv^\Sigma_\mu e''$. Since the train picture $(\vec{\Pi}, \mathscr{P}, \psi_\Pi)$ satisfies Definition 12.1(*iii*), we have $\psi_\Sigma(e') \equiv^G_\mu \psi_\Sigma(e_\star)$ and $\psi_\Sigma(e'') \equiv^G_\mu \psi_\Sigma(e_{\star\star})$. Together with $\psi_\Sigma(e_\star) = e = \psi_\Sigma(e_{\star\star})$ this yields the desired equivalence $\psi_\Sigma(e') \equiv^G_\mu \psi_\Sigma(e'')$.

Second Case: We have $e' \notin E(H)$ and $e'' \in E(H)$.

For the proof of (*a*) we again observe $x \in V(H)$, denote the standard copy of Π containing e' by $\Pi_\star$, and let $\Pi^e_\star \in \mathscr{H}$ be the copy extended by $\Pi_\star$. Invoking Lemma 7.3(*c*)(*i*) we infer that $\Pi^e_\star x (e'')^+ \overline{W}_\mu$ is a big cycle of order 1 in $(H, \equiv^H_\mu, \mathscr{H}^+)$ and as before we find

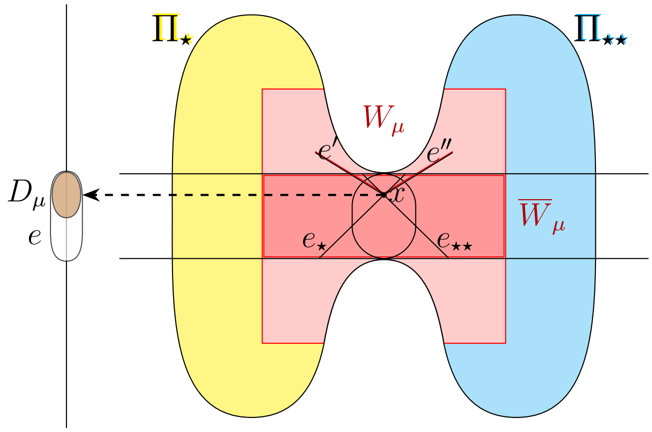


Figure 12.1. Part *(a)* in the first case.

an edge $e_\star \in E(\Pi_\star^e) \cap E(\overline{W}_\mu)$ with $x \in e_\star$. Since at least one of the statements

$$e' \not\equiv^{\Sigma}_{\mu-1} e_\star \quad \text{or} \quad e_\star \not\equiv^{\Sigma}_{\mu-1} e''$$

holds, we can conclude $x \in V_{A_\mu}(\Sigma)$ as in the first case.

For dealing with *(b)* we appeal to Lemma 7.3*(c)(i)* again, this time getting an edge $e_\star \in E(H) \cap E(\Pi_\star)$ with $e' \equiv^\Sigma_\mu e_\star \equiv^\Sigma_\mu e''$. Due to $\psi_\Sigma(e') \equiv^G_\mu \psi_\Sigma(e_\star)$ and $\psi_\Sigma(e_\star) = e = \psi_\Sigma(e'')$ we have indeed $\psi_\Sigma(e') \equiv^G_\mu \psi_\Sigma(e'')$.

Third Case: Both e' and e'' are in $E(H)$.

Now *(a)* follows from $x \in V_{D_\mu}(H) \subseteq V_{A_\mu}(\Sigma)$, and $\psi_\Sigma(e') = e = \psi_\Sigma(e'')$ yields *(b)*. □

Next we adapt the $\mathfrak{Girth}$ preservation lemma from §9.2 to train pictures.

Lemma 12.4. *Let $(\vec{G}, \mathscr{G})$ be a train system of height m with strongly induced copies. Suppose further that*

$$(\vec{\Sigma}, \mathscr{Q}, \psi_\Sigma) = (\vec{\Pi}, \mathscr{P}, \psi_\Pi) * (\vec{H}, \mathscr{H})$$

holds for two train pictures $(\vec{\Sigma}, \mathscr{Q}, \psi_\Sigma)$ and $(\vec{\Pi}, \mathscr{P}, \psi_\Pi)$ over $(\vec{G}, \mathscr{G})$, and a k-partite k-uniform train system $(\vec{H}, \mathscr{H})$. If for some sequence $\vec{g} = (g_\ell, \dots, g_m) \in \mathfrak{M}$ whose length is at most m we have $\mathfrak{Girth}(\vec{\Pi}, \mathscr{P}^+) > \vec{g}$ and $\mathfrak{Girth}(\vec{H}, \mathscr{H}^+) > \vec{g}$, then $\mathfrak{Girth}(\vec{\Sigma}, \mathscr{Q}^+) > \vec{g}$.

Proof. According to Definition 10.10 the $\mathfrak{Girth}$ assumptions mean

(1) $\mathfrak{Girth}(\Pi, \equiv^\Pi_{\mu-1}, \mathscr{P}^+) > g_\mu$ for every $\mu \in [\ell, m]$;
(2) $\mathfrak{Girth}(\Pi, \equiv^\Pi_m, \mathscr{P}^+) > 1$;
(3) $\mathfrak{Girth}(H, \equiv^H_{\mu-1}, \mathscr{H}^+) > g_\mu$ for every $\mu \in [\ell, m]$;
(4) and $\mathfrak{Girth}(H, \equiv^H_m, \mathscr{H}^+) > 1$.

Due to Lemma 9.15 the statements *(1)* and *(3)* entail $\mathfrak{Girth}(\Sigma, \equiv^\Sigma_{\mu-1}, \mathscr{Q}^+) > g_\mu$ for every $\mu \in [\ell, m]$ and thus it only remains to prove $\mathfrak{Girth}(\Sigma, \equiv^\Sigma_m, \mathscr{Q}^+) > 1$.

To this end we consider a big cycle

$$\mathscr{C} = F_1 x F_2 W_\Sigma$$

in $(\Sigma, \equiv_m^\Sigma)$, where $F_1, F_2 \in \mathscr{Q}^+$ are distinct copies, x is a vertex, and W_Σ denotes the unique m-wagon of $\vec{\Sigma}$. Due to Lemma 8.11 and symmetry it suffices to show that there exists an edge $f_1 \in E(F_1)$ passing through x.

If $F_1 \notin E(H)^+$ we denote the standard copy of $(\Pi, \mathscr{P}^+)$ to which F_1 belongs by $(\Pi_1, \mathscr{P}_1^+)$ and we let $\Pi_1^e \in \mathscr{H}$ be the copy extended by Π_1. If, on the other hand, $F_1 \in E(H)^+$, then we set $\Pi_1^e = F_1$. So in both cases we have $\Pi_1^e \in \mathscr{H}^+$. Let Π_2^e be defined similarly with respect to F_2.

If $\Pi_1^e = \Pi_2^e$, then F_1, F_2 belong to a common standard copy and the existence of f_1 follows from *(2)*. So we can henceforth assume $\Pi_1^e \neq \Pi_2^e$. Since $\mathscr{C}$ is a big cycle, each of the copies F_1, F_2 has at least one edge, and Lemma 7.3 implies $E(\Pi_1^e), E(\Pi_2^e) \neq \varnothing$. Using the unique m-wagon W^H of $\vec{H}$ we can thus form a big cycle $\Pi_1^e x \Pi_2^e W^H$ in $(H, \equiv_m^H, \mathscr{H}^+)$. Owing to *(4)* and Lemma 8.11 there exists an edge $f \in E(\Pi_1^e)$ passing through x.

In the special case $F_1 = f^+$ we can simply take $f_1 = f$ and otherwise we apply $\mathfrak{Girth}(\Pi_1, \equiv^{\Pi_1}, \mathscr{P}_1^+) > 1$ to the big cycle $F_1 x f^+ W^{\Pi_1}$, where W^{Π_1} denotes the unique m-wagon of $\vec{\Pi}_1$. Due to Lemma 8.11 we thus obtain the desired edge f_1. $\square$

12.3. **Amenable partite lemmata.** Both Proposition 10.14 and Proposition 10.15 assert that under certain inductive assumptions some karo principle holds. A commonality of their proofs is that they end with similar partite constructions that can be executed for roughly the same reasons. The main result of this subsection explains how this works. This involves the following concepts.

Definition 12.5. Let $\vec{g} \in \mathfrak{M}_\leqslant^\times$ be a nonempty nondecreasing sequence. Put $g = \inf(\vec{g})$, $m = |\vec{g}|$, and let $\vec{g}_\star \in \mathfrak{M}_\leqslant$ be obtained from $\vec{g}$ by removing its initial term g, so that $\vec{g} = (g) \circ \vec{g}_\star$.

(*a*) We say that Φ is a *Ramsey construction for $\vec{g}$-trains* if for every ordered f-partite train $\vec{F}$ of height m with $\mathfrak{girth}(\vec{F}) > \vec{g}$ and every number of colours r the train system $\Phi_r(\vec{F}) = (\vec{G}, \mathscr{G})$ is defined, $\vec{G}$ is a linear ordered f-partite train of height m with the same parameter as $\vec{F}$, the copies in $\mathscr{G}$ are strongly induced, and $\mathscr{G} \longrightarrow (\vec{F})_r$.

(*b*) A partite lemma Ξ is said to be *$\vec{g}$-amenable* if for every k-partite k-uniform train $\vec{F}$ of height m with $\mathfrak{girth}(\vec{F}) > \vec{g}$ and every number of colours r it generates a train system $\Xi_r(\vec{F}) = (\vec{H}, \mathscr{H})$ such that $\vec{H}$ has the same parameter as $\vec{F}$,

$$\mathscr{H} \longrightarrow (\vec{F})_r\,, \quad \mathfrak{Girth}(\vec{H}, \mathscr{H}^+) > \vec{g}_\star\,, \quad \text{and} \quad \mathrm{Girth}(H, \mathscr{H}^+) > (g, g)\,.$$

Lemma 12.6. *Let $\vec{g} \in \mathfrak{M}^{\times}_{\leqslant}$ be a nonempty nondecreasing sequence. If Φ denotes a Ramsey construction for $\vec{g}$-trains and the partite lemma Ξ is $\vec{g}$-amenable, then* $\mathrm{PC}(\Phi, \Xi)$ *exemplifies $\diamondsuit_{\vec{g}}$.*

Proof. Consider an arbitrary ordered f-partite train $\vec{F}$ with parameter $\vec{A}$ and $\mathfrak{girth}(\vec{F}) > \vec{g}$ as well as number of colours r, and construct the train system $\Phi_r(\vec{F}) = (\vec{G}, \mathscr{G})$. Since Φ is a Ramsey construction for $\vec{g}$-trains, we know that $\vec{G}$ is a linear ordered f-partite train with parameter $\vec{A}$, that the copies in $\mathscr{G}$ are strongly induced, and that $\mathscr{G} \longrightarrow (\vec{F})_r$. Without loss of generality we can assume that every edge of G belongs to at least one copy in $\mathscr{G}$.

As usual we let $\{e(1), \dots, e(N)\}$ enumerate the edges of the underlying hypergraph of $\vec{G}$. Now we run the partite construction, thereby creating a sequence $(\vec{\Pi}_\alpha, \mathscr{P}_\alpha, \psi_\alpha)_{\alpha \leqslant N}$ of train pictures. Picture zero can clearly be formed and by Fact 11.9 it has the property $\mathfrak{Girth}(\vec{\Pi}_0, \mathscr{P}_0^+) > \vec{g}$.

Now suppose that for some $\alpha \in [N]$ we have reached the picture $(\vec{\Pi}_{\alpha-1}, \mathscr{P}_{\alpha-1}, \psi_{\alpha-1})$ satisfying $\mathfrak{Girth}(\vec{\Pi}_{\alpha-1}, \mathscr{P}_{\alpha-1}^+) > \vec{g}$. Corollary 10.11 entails $\mathfrak{girth}(\vec{\Pi}_{\alpha-1}^{e(\alpha)}) > \vec{g}$ and the parameter $\vec{D}_\alpha$ of $\vec{\Pi}_{\alpha-1}^{e(\alpha)}$ has been described in Fact 12.2. By our assumptions on Ξ there exists a train system

$$\Xi_r\big(\vec{\Pi}_{\alpha-1}^{e(\alpha)}\big) = (\vec{H}_\alpha, \mathscr{H}_\alpha)$$

of height m with parameter $\vec{D}_\alpha$ that satisfies

$$\mathfrak{Girth}(\vec{H}_\alpha, \mathscr{H}_\alpha^+) > \vec{g}_\star \quad \text{and} \quad \mathrm{Girth}(H_\alpha, \mathscr{H}_\alpha^+) > (g, g)\,,$$

where $\vec{g} = (g) \circ \vec{g}_\star$. In view of Lemma 12.3 the amalgamation

$$(\vec{\Pi}_\alpha, \mathscr{P}_\alpha, \psi_\alpha) = (\vec{\Pi}_{\alpha-1}, \mathscr{P}_{\alpha-1}, \psi_{\alpha-1}) \ast (\vec{H}_\alpha, \mathscr{H}_\alpha)$$

provides the next train picture over $(\vec{G}, \mathscr{G})$ and in order to keep the construction going we need to check $\mathfrak{Girth}(\vec{\Pi}_\alpha, \mathscr{P}_\alpha^+) > \vec{g}$.

To this end we observe that the hypothesis $\mathfrak{Girth}(\vec{\Pi}_{\alpha-1}, \mathscr{P}_{\alpha-1}^+) > \vec{g}$ is equivalent to the conjunction of

$$\mathfrak{Girth}(\Pi_{\alpha-1}, \equiv_0^{\Pi_{\alpha-1}}, \mathscr{P}_{\alpha-1}^+) > g \quad \text{and} \quad \mathfrak{Girth}(\vec{\Pi}_{\alpha-1}, \mathscr{P}_{\alpha-1}^+) > \vec{g}_\star\,.$$

So Lemma 12.4 immediately yields

$$\mathfrak{Girth}(\vec{\Pi}_\alpha, \mathscr{P}_\alpha^+) > \vec{g}_\star \tag{12.3}$$

and Lemma 8.18 tells us $\mathrm{Girth}(\Pi_{\alpha-1}, \mathscr{P}_{\alpha-1}^+) > g$. Combined with $\mathrm{Girth}(H_\alpha, \mathscr{H}_\alpha^+) > (g, g)$ and Lemma 5.2 this leads to $\mathrm{Girth}(\Pi_\alpha, \mathscr{P}_\alpha^+) > g$, and a further application of Lemma 8.18 translates this back to $\mathfrak{Girth}(\vec{\Pi}_\alpha, \equiv_0^{\Pi_\alpha}, \mathscr{P}_\alpha^+) > g$. Together with (12.3) this establishes $\mathfrak{Girth}(\vec{\Pi}_\alpha, \mathscr{P}_\alpha^+) > \vec{g}$ and thus the construction can be continued.

Eventually, it produces a final picture $(\vec{\Pi}_N, \mathscr{P}_N, \psi_N)$ such that $\mathfrak{Girth}(\vec{\Pi}_N, \mathscr{P}_N^+) > \vec{g}$. Since $(G, \mathscr{G})$ has strongly induced copies, the same applies to $(\Pi_N, \mathscr{P}_N)$ (see Lemma 3.4

and its proof). So altogether, when viewed as an ordered f-partite structure the last picture is as required by $\diamondsuit_{\vec{g}}$. $\square$

12.4. **Girth resurrection.** Now we shall finally complete the proof of Proposition 10.14. So we consider an arbitrary nonempty nondecreasing sequence $\vec{g} \in \mathfrak{M}^{\times}_{\leqslant}$ and assume $\diamondsuit_{\vec{g}}$. Let Υ be the construction provided by Lemma 11.10. Evidently Υ is a Ramsey construction for $((2)\circ\vec{g})$-trains. So in view of Lemma 12.6 it only remains to exhibit an $((2)\circ\vec{g})$-amenable partite lemma.

By restricting our attention to unordered k-partite k-uniform trains we can regard Υ as a partite lemma for trains and in this manner $\Xi = \mathrm{PC}(\Upsilon, \Upsilon)$ becomes a partite lemma as well. It will turn out that Ξ has the required amenability property.

Lemma 12.7. *For every k-partite k-uniform train $\vec{F}$ of height $m+1$ with $\mathfrak{girth}(\vec{F}) > (2)\circ\vec{g}$ and every number of colours r the train system $\Xi_r(\vec{F}) = (\vec{H}, \mathscr{H})$ is defined. Moreover, the train $\vec{H}$ has the same parameter as $\vec{F}$, the copies in $\mathscr{H}$ are strongly induced, their intersections are clean, and $\mathfrak{Girth}(\vec{H}, \mathscr{H}^+) > \vec{g}$.*

Proof. Owing to Lemma 11.10 the k-partite k-uniform train system $\Upsilon_r(\vec{F}) = (\vec{G}, \mathscr{G})$ satisfies $\mathscr{G} \longrightarrow (\vec{F})_r$, its copies are strongly induced, and the parameters of $\vec{F}$ and $\vec{G}$ are the same. Without loss of generality we can assume that every edge of G belongs to some copy in $\mathscr{G}$.

Let $E(G) = \{e(1), \dots, e(N)\}$ enumerate the edges of G. We will create a sequence $(\vec{\Pi}_\alpha, \mathscr{P}_\alpha, \psi_\alpha)_{\alpha \leqslant N}$ of train pictures over $(\vec{G}, \mathscr{G})$ by means of the partite construction method. Clearly, picture zero $(\vec{\Pi}_0, \mathscr{P}_0, \psi_0)$ exists and it has strongly induced copies with clean intersections. Moreover, Fact 11.9 yields $\mathfrak{Girth}(\vec{\Pi}_0, \mathscr{P}_0^+) > (2)\circ\vec{g}$, which implies, in particular, $\mathfrak{Girth}(\vec{\Pi}_0, \mathscr{P}_0^+) > \vec{g}$.

Now suppose inductively that for some positive $\alpha \leqslant N$ we have just constructed a train picture $(\vec{\Pi}_{\alpha-1}, \mathscr{P}_{\alpha-1}, \psi_{\alpha-1})$ such that the copies in $\mathscr{P}_{\alpha-1}$ are strongly induced, their intersections are clean, and $\mathfrak{Girth}(\vec{\Pi}_{\alpha-1}, \mathscr{P}^+_{\alpha-1}) > \vec{g}$.

Combined with Lemma 8.17 and the linearity of $\Pi_{\alpha-1}$ this shows, in particular, that $\mathfrak{girth}(\vec{\Pi}^{e(\alpha)}_{\alpha-1}) > (2)\circ\vec{g}$ and thus Lemma 11.10 yields a k-partite k-uniform train system

$$\Upsilon_r(\vec{\Pi}^{e(\alpha)}_{\alpha-1}) = (\vec{H}_\alpha, \mathscr{H}_\alpha)$$

with the correct parameter satisfying $\mathfrak{Girth}(\vec{H}_\alpha, \mathscr{H}_\alpha) > \vec{g}$. Due to Lemma 12.3 the next train picture

$$(\vec{\Pi}_\alpha, \mathscr{P}_\alpha, \psi_\alpha) = (\vec{\Pi}_{\alpha-1}, \mathscr{P}_{\alpha-1}, \psi_{\alpha-1}) * (\vec{H}_\alpha, \mathscr{H}_\alpha)$$

exists. By Lemma 3.4 the copies in $\mathscr{P}_\alpha$ are again strongly induced and their intersections are clean. Moreover, Lemma 12.4 tells us $\mathfrak{Girth}(\vec{\Pi}_\alpha, \mathscr{P}^+_\alpha) > \vec{g}$. Thus the construction goes on and the final train picture we eventually reach has the desired properties. $\square$

By Lemma 4.13 the systems $(\vec{H}, \mathscr{H})$ produced by Ξ satisfy $\mathrm{Girth}(H, \mathscr{H}) > (2, 2)$ and, consequently, Ξ is indeed $((2) \circ \vec{g})$-amenable. Thus the discussion at the beginning of this subsection shows that the construction $\mathrm{PC}(\Upsilon, \Xi)$ exemplifies $\diamondsuit_{(2)\circ\vec{g}}$. This concludes the proof of Proposition 10.14. For later reference we formulate a straightforward consequence of the results we currently have.

Corollary 12.8. *For every $\vec{g} \in \mathfrak{M}^\times$ there exists a Ramsey construction for $\vec{g}$-trains.*

Proof. Due to Lemma 10.13 and Proposition 10.14 we have $\diamondsuit_{(2)^m}$ for every positive integer m. Clearly, every construction exemplifying $\diamondsuit_{(2)^m}$ for $m = |\vec{g}|$ is a Ramsey construction for $\vec{g}$-trains. □

12.5. **Revisability.** The remainder of this section is devoted to the proof of Proposition 10.15. To this end we fix a (possibly empty) nondecreasing sequence $\vec{g}$ as well as an integer g such that $\inf(\vec{g}) > g \geqslant 2$. Let ℓ be the length of $\vec{g}$ and write $\vec{g} = (g_1, \dots, g_\ell)$. We assume that for every positive integer m there exists a construction Ψ^m exemplifying the principle $\diamondsuit_{(g)^m\circ\vec{g}}$. Our deduction of $\diamondsuit_{(g+1)\circ\vec{g}}$ consists of three steps.

- First, we explain how the constructions Ψ^m are going to be used.
- Second, we perform a "diagonal" partite construction in order to derive a $((g+1)\circ\vec{g})$-amenable partite lemma.
- Third, we conclude $\diamondsuit_{(g+1)\circ\vec{g}}$.

The given constructions Ψ^m will only be used as partite lemmata, i.e., we shall only apply them to unordered k-partite k-uniform trains. More precisely, each Ψ^m will be reinterpreted as a partite lemma applicable to certain trains $\vec{F}$ of height $\ell + 1$ that will be called m-revisable. Here is the definition of this concept.

Definition 12.9. A k-partite k-uniform train $\vec{F} = (F, \equiv_0, \dots, \equiv_{\ell+1})$ of height $\ell + 1$ with index set I and parameter $\vec{A} = (A_1, \dots, A_{\ell+1})$ is said to be *m-revisable* for a positive integer m if there exist equivalence relations $\sim_1, \dots, \sim_{m-1}$ on $E(F)$ and an m-tuple $\vec{B} = (B_1, \dots, B_m) \in \wp(A_1)^m$ such that

- $\vec{F}_\bullet = (F, \equiv_0, \sim_1, \dots, \sim_{m-1}, \equiv_1, \dots, \equiv_{\ell+1})$ is a train of height $\ell + m$ with parameter $\vec{B} \circ (A_2, \dots, A_{\ell+1})$;
- $\mathfrak{girth}(\vec{F}_\bullet) > (g)^m \circ \vec{g}$;
- and $|B_\mu| \leqslant 1$ for every $\mu \in [m]$.

In this situation we call the pair $(\vec{F}_\bullet, \vec{B})$ an *m-revision* of $\vec{F}$.

We illustrate this notion with an easy example that will later be applied to the constituents of a train picture zero.

Fact 12.10. *Let $\vec{F}$ be a k-partite k-uniform train of height $\ell+1$. If its underlying hypergraph is a matching, possibly together with some isolated vertices, then $\vec{F}$ is 1-revisable.*

Proof. For every $\lambda \in [\ell+1]$ any two distinct $(\lambda-1)$-wagons of $\vec{F}$ are vertex-disjoint. Thus we have $\mathfrak{girth}(\vec{F}) > (g) \circ \vec{g}$ and $(\vec{F}, (\varnothing))$ is a 1-revision of $\vec{F}$. □

Here is a necessary condition for revisability that could be shown to be quite far from being sufficient.

Fact 12.11. *If a k-partite k-uniform train of height $\ell+1$ is m-revisable for some positive integer m, then $\mathfrak{girth}(\vec{F}) > (g+1) \circ \vec{g}$.*

Proof. Let $\vec{A}$, $\vec{B}$, and $\vec{F}_\bullet$ be as in Definition 12.9. We need to prove that for every $\lambda \in [\ell+1]$ and every λ-wagon W of $\vec{F}$ the $(\lambda-1)$-wagons in W form a set system whose girth exceeds the λ^{th} entry of $(g+1) \circ \vec{g}$. For $\lambda \neq 1$ this follows immediately from $\mathfrak{girth}(\vec{F}_\bullet) > (g)^m \circ \vec{g}$.

Thus it remains to show $\mathrm{girth}(W) > g+1$ for every 1-wagon W of $\vec{F}$. To this end we notice that by restricting the first $m+1$ equivalence relations of $\vec{F}_\bullet$ to $E(W)$ we obtain a train $\overrightarrow{W} = (W, \equiv_0^W, \sim_1^W, \dots, \sim_{m-1}^W, \equiv_1^W)$ of height m. We know $\mathfrak{girth}(\overrightarrow{W}) > (g)^m$ and the parameter of $\overrightarrow{W}$ is $\vec{B} = (B_1, \dots, B_m)$. As we required $|B_\mu| \leqslant 1$ for every $\mu \in [m]$, all assumptions of Lemma 10.9 are satisfied and we have indeed $\mathrm{girth}(W) > g+1$. □

Given an m-revisable k-partite k-uniform train $\vec{F}$ as well as a number of colours r we can take an arbitrary m-revision $(\vec{F}_\bullet, \vec{B})$ of $\vec{F}$ and construct the train system

$$\Psi_r^m(\vec{F}_\bullet) = (H, \equiv_0^H, \sim_1^H, \dots, \sim_{m-1}^H, \equiv_1^H, \dots, \equiv_{\ell+1}^H, \mathscr{H}_\bullet)\,. \tag{12.4}$$

The outcome does not depend on $\vec{F}$ alone, for there could be many distinct m-revisions of $\vec{F}$. This ambiguity, however, is irrelevant to the main concern of this subsection and it will be convenient to write

$$\Psi_r^m(\vec{F}) = (\vec{H}, \mathscr{H})\,,$$

where $\vec{H} = (H, \equiv_0^H, \dots, \equiv_{\ell+1}^H)$ and the copies in $\mathscr{H} \subseteq \binom{\vec{H}}{\vec{F}}$ are obtained from the copies of $\vec{F}_\bullet$ in $\mathscr{H}_\bullet$ by forgetting the $\sim$-relations.

Lemma 12.12. *If $\vec{F}$ denotes an m-revisable k-partite k-uniform train of height $\ell+1$ with parameter $\vec{A}$, and r is a number of colours, then the train system $\Psi_r^m(\vec{F}) = (\vec{H}, \mathscr{H})$ has again the parameter $\vec{A}$ and satisfies*

$$\mathscr{H} \longrightarrow (\vec{F})_r\,, \quad \mathfrak{girth}(\vec{H}) > (g+1) \circ \vec{g}, \quad \textit{and} \quad \mathfrak{Girth}(\vec{H}, \mathscr{H}^+) > (g) \circ \vec{g}\,.$$

Proof. We keep using the notation from Definition 12.9 and (12.4). As a first step we shall show that $\vec{A}$ parametrises $\vec{H}$. Due to $\diamondsuit_{(g)^m \circ \vec{g}}$ the train

$$\vec{H}_\bullet = (H, \equiv_0^H, \sim_1^H, \dots, \sim_{m-1}^H, \equiv_1^H, \dots, \equiv_{\ell+1}^H)$$

has the same parameter as $\vec{F}_\bullet$, i.e., $\vec{B} \circ (A_2, \dots, A_{\ell+1})$. Setting for simplicity $\sim_0^H = \equiv_0^H$ and $\sim_m^H = \equiv_1^H$ this means that for any two edges $e, e' \in E(H)$ the following two statements hold.

(*1*) If $\mu \in [m]$, $e \sim_\mu^H e'$, and $e \not\sim_{\mu-1}^H e'$, then $e \cap e' \subseteq V_{B_\mu}(H)$.

(*2*) If $\lambda \in [2, \ell+1]$, $e \equiv^H_\lambda e'$, and $e \not\equiv^H_{\lambda-1} e'$, then $e \cap e' \subseteq V_{A_\lambda}(H)$.

In view of (*2*) it only remains to be shown that all edges $e, e' \in E(H)$ with $e \equiv^H_1 e'$ and $e \not\equiv^H_0 e'$ satisfy $e \cap e' \subseteq V_{A_1}(H)$. Due to $e \sim^H_m e'$ there exists a smallest integer $\mu \in [0, m]$ such that $e \sim^H_\mu e'$. Since $e \not\sim^H_0 e'$, we have $\mu > 0$, and thus the minimality of μ yields $e \not\sim^H_{\mu-1} e'$. So (*1*) reveals $e \cap e' \subseteq V_{B_\mu}(H)$ and because of $B_\mu \subseteq A_1$ the desired inclusion $e \cap e' \subseteq V_{A_1}(H)$ follows.

This concludes our discussion of parameters and we proceed with the three displayed properties of the train system $(\vec{H}, \mathscr{H})$. As the train construction Ψ^m exemplifies the principle $\diamondsuit_{(g)^m \circ \vec{g}}$, the partition relation $\mathscr{H} \longrightarrow (\vec{F})_r$ is clear.

Moreover, we have

$$\mathfrak{Girth}(\vec{H}_\bullet, \mathscr{H}_\bullet^+) > (g)^m \circ \vec{g}\,. \tag{12.5}$$

According to Definition 10.10 this implies

- $\mathfrak{Girth}(H, \equiv^H_0, \mathscr{H}^+) > g$,
- $\mathfrak{Girth}(H, \equiv^H_\lambda, \mathscr{H}^+) > g_\lambda$ for every $\lambda \in [\ell]$,
- and $\mathfrak{Girth}(H, \equiv^H_{\ell+1}, \mathscr{H}^+) > 1$,

which in turn yields $\mathfrak{Girth}(\vec{H}, \mathscr{H}^+) > (g) \circ \vec{g}$. By Corollary 10.11 and (12.5) we have $\mathfrak{girth}(\vec{H}_\bullet) > (g)^m \circ \vec{g}$. So $(\vec{H}_\bullet, \vec{B})$ is an m-revision of $\vec{H}$ and $\mathfrak{girth}(\vec{H}) > (g+1) \circ \vec{g}$ follows from Fact 12.11. $\square$

This concludes the first step of our proof of Proposition 10.15, i.e., the conversion of the given $\diamondsuit$-principles into partite lemmata Ψ^m applicable to certain trains $\vec{F}$ of height $\ell+1$.

12.6. **Train constituents.** Recall that our initial motivation for introducing trains was the observation that partite constructions produce trains; moreover trains seem to offer a chance to handle hypergraphs of the next larger girth in the context of an argument by induction. In some sense, this subsection is the place of the whole article where all these hopes materialise.

Lemma 12.13. *There exists a $((g+1) \circ \vec{g})$-amenable partite lemma Ξ.*

This partite lemma Ξ will be obtained by a diagonal variation of the partite construction method. Vertically we use an arbitrary Ramsey construction Φ for $((g+1) \circ \vec{g})$-trains (see Corollary 12.8). Horizontally the basic observation is that it was never written into stone that one has to employ the very same partite lemma in each stage of the iterative procedure. Rather, it is reasonable to adjust to the increasing complexity of the arising pictures by using more and more sophisticated partite lemmata as the construction progresses. In fact, we plan to use Ψ^m when we need a partite lemma for the m^{th} time, so a suggestive notation for the construction we are about to describe could be $\Xi = \mathrm{PC}\big(\Phi, (\Psi^m)_{m \in \mathbb{N}}\big)$. Here is the picturesque statement we shall iterate.

Lemma 12.14. *Let* $(\vec{G}, \mathscr{G})$ *be a* k*-partite* k*-uniform train system of height* $\ell+1$ *with parameter* $\vec{A}$ *whose underlying hypergraph* G *is linear. Suppose further that*

$$(\vec{\Sigma}, \mathscr{Q}, \psi_\Sigma) = (\vec{\Pi}, \mathscr{P}, \psi_\Pi) \ast (\vec{H}, \mathscr{H})$$

holds for two train pictures $(\vec{\Pi}, \mathscr{P}, \psi_\Pi)$, $(\vec{\Sigma}, \mathscr{Q}, \psi_\Sigma)$ *of height* $\ell+1$ *with parameter* $\vec{A}$ *and a* k*-partite* k*-uniform train system* $(\vec{H}, \mathscr{H})$ *of height* $\ell+1$ *such that*

$$\mathfrak{Girth}(\vec{\Sigma}, \mathscr{Q}^+) > \vec{g} \quad \text{and} \quad \mathrm{Girth}(H, \mathscr{H}^+) > g\,.$$

If this amalgamation occurs over the edge $e \in E(G)$,

- *another edge* $e_\star \in E(G)$ *is distinct to* e,
- *and the constituent* $\vec{\Pi}^{e_\star}$ *is* α*-revisable for some* $\alpha \in \mathbb{N}$,

then $\vec{\Sigma}^{e_\star}$ *is* $(\alpha+1)$*-revisable.*

Proof. Write $\vec{A} = (A_1, \dots, A_{\ell+1})$ and recall that according to Fact 12.2 the parameter $\vec{D} = (D_1, \dots, D_{\ell+1})$ of the constituent $\vec{\Pi}^{e_\star}$ is given by $D_\lambda = e_\star \cap V_{A_\lambda}(G)$ for every $\lambda \in [\ell+1]$. Let $(\vec{\Pi}^{e_\star}_\bullet, \vec{B})$ be an α-revision of $\vec{\Pi}^{e_\star} = (\Pi^{e_\star}, \equiv^\Pi_0, \dots, \equiv^\Pi_{\ell+1})$, where

$$\vec{\Pi}^{e_\star}_\bullet = (\Pi^{e_\star}, \equiv^\Pi_0, \sim^\Pi_1, \dots, \sim^\Pi_{\alpha-1}, \equiv^\Pi_1, \dots, \equiv^\Pi_{\ell+1}) \quad \text{and} \quad \vec{B} = (B_1, \dots, B_\alpha) \in \wp(D_1)^\alpha\,.$$

Our task is to exhibit an $(\alpha+1)$-revision of the new constituent

$$\vec{\Sigma}^{e_\star} = (\Sigma^{e_\star}, \equiv^\Sigma_0, \dots, \equiv^\Sigma_{\ell+1})\,.$$

For every standard copy $\vec{\Pi}_\circ$ of $\vec{\Pi}$ in $\vec{\Sigma}$ we can copy the above revision of $\vec{\Pi}^{e_\star}$ onto the constituent $\vec{\Pi}^{e_\star}_\circ$, thus getting a train

$$(\Pi^{e_\star}_\circ, \equiv^{\Pi_\circ}_0, \sim^{\Pi_\circ}_1, \dots, \sim^{\Pi_\circ}_{\alpha-1}, \equiv^{\Pi_\circ}_1, \dots, \equiv^{\Pi_\circ}_{\ell+1})\,. \tag{12.6}$$

It will be convenient to set $\sim^{\Pi_\circ}_0 = \equiv^{\Pi_\circ}_0$ and $\sim^{\Pi_\circ}_\alpha = \equiv^{\Pi_\circ}_1$ for every such standard copy. The edge set of the new constituent $\Sigma^{e_\star}$ is the disjoint union of the edge sets of all $\Pi^{e_\star}_\circ$ as $\Pi_\circ$ varies over the standard copies of $\vec{\Pi}$ in Σ. We can thus define a quasitrain

$$\vec{\Sigma}^{e_\star}_\bullet = (\Sigma^{e_\star}, \equiv^\Sigma_0, \sim^\Sigma_1, \dots, \sim^\Sigma_\alpha, \equiv^\Sigma_1, \dots, \equiv^\Sigma_{\ell+1})$$

of height $\ell+\alpha+1$ by declaring for every $\mu \in [\alpha]$ and any two edges $e', e'' \in E(\Sigma^{e_\star})$ that the statement $e' \sim^\Sigma_\mu e''$ means: there is a common standard copy $\Pi_\circ$ containing e', e'', and $e' \sim^{\Pi_\circ}_\mu e''$ holds. Next we define a set $B_{\alpha+1}$ by setting

$$B_{\alpha+1} = \begin{cases} e \cap e_\star & \text{if } e \equiv^G_1 e_\star \\ \varnothing & \text{if } e \not\equiv^G_1 e_\star. \end{cases}$$

We shall eventually show that $\big(\vec{\Sigma}^{e_\star}_\bullet, \vec{B} \circ (B_{\alpha+1})\big)$ is an $(\alpha+1)$-revision of $\vec{\Sigma}^{e_\star}$.

Claim 12.15. *We have* $B_{\alpha+1} \subseteq D_1$ *and* $|B_{\alpha+1}| \leqslant 1$.

Proof. If $B_{\alpha+1} = \varnothing$ both assertions are clear, so we may assume $B_{\alpha+1} = e \cap e_\star$ and $e \equiv_1^G e_\star$ from now on. The linearity of G implies $|B_{\alpha+1}| \leqslant 1$. We also know $e \cap e_\star \subseteq V_{A_1}(G)$, because $\vec{G}$ has the parameter $\vec{A}$. Thus we have indeed

$$B_{\alpha+1} = e \cap e_\star \subseteq e_\star \cap V_{A_1}(G) = D_1 \,. \qquad \square$$

Claim 12.16. *The sequence $\vec{B} \circ (B_{\alpha+1}, D_2, \dots, D_{\ell+1})$ parametrises $\vec{\Sigma}_\bullet^{e_\star}$.*

Proof. For every standard copy $\vec{\Pi}_\circ$ of $\vec{\Pi}$ in $\vec{\Sigma}$ the train (12.6) has the same parameter as $\vec{\Pi}_\bullet^{e_\star}$, i.e., $\vec{B} \circ (D_2, \dots, D_{\ell+1})$ and, therefore, the statements involving the sets $B_1, \dots, B_\alpha$ hold. Similarly, the claims on $D_2, \dots, D_{\ell+1}$ follow from $\vec{A}$ being the parameter of the entire train $\vec{\Sigma}$ and Fact 12.2.

So it remains to be shown that if two edges $e', e'' \in E(\Sigma^{e_\star})$ satisfy $e' \equiv_1^\Sigma e''$ but $e' \not\sim_\alpha^\Sigma e''$, then $e' \cap e'' \in V_{B_{\alpha+1}}(\Sigma^{e_\star})$. Notice that this can only happen if e' and e'' are in distinct standard copies of $\vec{\Pi}$. So in view of Lemma 7.3(*c*)(*ii*) there needs to exist, in particular, an edge $e_0 \in E(H)$ satisfying $e' \equiv_1^\Sigma e_0 \equiv_1^\Sigma e''$. Now Definition 12.1(*iii*) discloses $e = \psi_\Sigma(e_0) \equiv_1^G \psi_\Sigma(e') = e_\star$, which in turn leads to $B_{\alpha+1} = e \cap e_\star$. Since ψ_Σ projects all vertices of $e' \cap e''$, if there exist any, into $e \cap e_\star$, this proves that $e' \cap e'' \in V_{B_{\alpha+1}}(\Sigma^{e_\star})$ is indeed true. $\square$

Now it remains to be shown that $\mathfrak{girth}(\vec{\Sigma}_\bullet^{e_\star}) > (g)^{\alpha+1} \circ \vec{g}$. Most parts of this claim are straightforward consequences of $\mathfrak{girth}(\vec{\Pi}_\bullet^{e_\star}) > (g)^\alpha \circ \vec{g}$ and of $\mathfrak{Girth}(\vec{\Sigma}, \mathscr{Q}^+) > \vec{g}$. In fact, the only part requiring some attention is that if W denotes an $(\alpha+1)$-wagon of $\vec{\Sigma}_\bullet^{e_\star}$, i.e., a wagon of $(\Sigma^{e_\star}, \equiv_1^\Sigma)$, then $\mathfrak{girth}(W, \sim_\alpha^W) > g$.

Assume contrariwise that for some $n \in [2, g]$ there is an n-cycle

$$\mathscr{C} = W_1 v_1 \dots W_n v_n \,,$$

where $W_1, \dots, W_n$ denote wagons of $(W, \sim_\alpha^W)$ and $v_1, \dots, v_n$ are vertices. Due to the definition of $\sim_\alpha^\Sigma$ there exists for every $t \in \mathbb{Z}/n\mathbb{Z}$ a standard copy $\vec{\Pi}_t$ of $\vec{\Pi}$ such that the wagon W_t is entirely contained in $\vec{\Pi}_t^{e_\star}$. Moreover, $W_t \ne W_{t+1}$ yields $\vec{\Pi}_t \ne \vec{\Pi}_{t+1}$ and due to $v_t \in V(W_t) \cap V(W_{t+1})$ we have $v_t \in V(H)$. We thus arrive at a cycle of copies

$$\mathscr{D} = \Pi_1^e v_1 \dots \Pi_n^e v_n$$

in $(H, \mathscr{H}^+)$ whose length is n and all of whose connectors are vertices sitting on the same music line of Σ (namely the line projected to the unique vertex in $e \cap e_\star$). In particular, $\mathscr{D}$ is tidy and none of its copies is collapsible, meaning that $\mathscr{D}$ has no master copy. But because of $\mathrm{ord}(\mathscr{D}) = n \leqslant g$ this contradicts $\mathrm{Girth}(H, \mathscr{H}^+) > g$. $\square$

Proof of Lemma 12.13. Suppose that $\vec{F}$ is a k-partite k-uniform train of height $\ell+1$ with

$$\mathfrak{girth}(F) > (g+1) \circ \vec{g} \tag{12.7}$$

and parameter $\vec{A}$, and that r is a number of colours. Pick a Ramsey construction Φ for $((g+1)\circ\vec{g})$-trains (cf. Corollary 12.8) and set $\Phi_r(\vec{F}) = (\vec{G}, \mathscr{G})$. Recall that $\vec{G}$ is a linear train of height $\ell+1$ with parameter $\vec{A}$ and that the copies in $\mathscr{G}$ are strongly induced. Without loss of generality we can assume that all edges of G belong to some copy in $\mathscr{G}$.

Let $E(G) = \{e(1), \ldots, e(N)\}$ enumerate the edges of G. We intend to construct recursively a sequence of train pictures $(\vec{\Pi}_\alpha, \mathscr{P}_\alpha, \psi_\alpha)_{0\leqslant\alpha\leqslant N}$ over $(\vec{G}, \mathscr{G})$ that starts with picture zero. This is to be done in such a way that for every $\alpha \in [0, N]$ the α^{th} picture has the properties

(*a*) $\mathrm{Girth}(\Pi_\alpha, \mathscr{P}_\alpha^+) > (g+1, g+1)$;
(*b*) $\mathfrak{Girth}(\vec{\Pi}_\alpha, \mathscr{P}_\alpha^+) > \vec{g}$;
(*c*) and for every $\beta \in (\alpha, N]$ the constituent $\vec{\Pi}_\alpha^{e(\beta)}$ is $(\alpha+1)$-revisable.

Before we can move any further we need to check that picture zero has these properties for $\alpha = 0$. Fact 11.9 and (12.7) imply $\mathfrak{Girth}(\vec{\Pi}_0, \mathscr{P}_0^+) > (g+1)\circ\vec{g}$, which entails (*b*) immediately and in view of Lemma 8.18 clause (*a*) follows as well. Furthermore, Fact 12.10 yields (*c*).

Now let any positive $\alpha \leqslant N$ be given for which we have already managed to reach a train picture $(\vec{\Pi}_{\alpha-1}, \mathscr{P}_{\alpha-1}, \psi_{\alpha-1})$ satisfying (*a*), (*b*), and (*c*) for $\alpha-1$ instead of α. As a consequence of the last condition we know that its constituent $\Pi_{\alpha-1}^{e(\alpha)}$ is α-revisable. So Lemma 12.12 provides a train system

$$\Psi_r^\alpha\big(\vec{\Pi}_{\alpha-1}^{e(\alpha)}\big) = (\vec{H}_\alpha, \mathscr{H}_\alpha)$$

with the correct parameter satisfying

$$\mathscr{H}_\alpha \longrightarrow \big(\vec{\Pi}_{\alpha-1}^{e(\alpha)}\big)_r, \quad \mathfrak{girth}(\vec{H}_\alpha) > (g+1)\circ\vec{g}, \quad \text{and} \quad \mathfrak{Girth}(\vec{H}_\alpha, \mathscr{H}_\alpha^+) > (g)\circ\vec{g}\,.$$

In view of Lemma 12.3 the quasitrain picture

$$(\vec{\Pi}_\alpha, \mathscr{P}_\alpha, \psi_\alpha) = (\vec{\Pi}_{\alpha-1}, \mathscr{P}_{\alpha-1}, \psi_{\alpha-1}) \ast (\vec{H}_\alpha, \mathscr{H}_\alpha)$$

is actually a train picture over $(\vec{G}, \mathscr{G})$. We need to check that it has the properties (*a*)–(*c*).

Observe first that $\mathfrak{girth}(\vec{H}_\alpha) > (g+1)\circ\vec{g}$, our assumption $\inf(\vec{g}) > g$, and Lemma 10.8 imply

$$\mathrm{girth}(H_\alpha) > g+1\,. \tag{12.8}$$

Moreover, $\mathfrak{Girth}(\vec{H}_\alpha, \mathscr{H}_\alpha^+) > (g)\circ\vec{g}$ contains the information $\mathfrak{Girth}(H_\alpha, \equiv_0^{H_\alpha}, \mathscr{H}_\alpha^+) > g$, which in view of Lemma 8.18 leads to $\mathrm{Girth}(H_\alpha, \mathscr{H}_\alpha^+) > g$. Together with our induction hypothesis $\mathrm{Girth}(\Pi_{\alpha-1}, \mathscr{P}_{\alpha-1}^+) > (g+1, g+1)$ and (12.8) this allows us to apply Lemma 5.8 with $g+1$ here in place of g there. This proves (*a*).

Clause (*b*) follows from Lemma 12.4. For the verification of (*c*) we consider an arbitrary integer $\beta \in (\alpha, N]$. We already know that $\Pi_{\alpha-1}^{e(\beta)}$ is α-revisable and we want to deduce the $(\alpha+1)$-revisability of $\Pi_\alpha^{e(\beta)}$ with the help of Lemma 12.14. Its assumptions $\mathfrak{Girth}(\vec{\Pi}_\alpha, \mathscr{P}_\alpha^+) > \vec{g}$ and $\mathrm{Girth}(H_\alpha, \mathscr{H}_\alpha^+) > g$ have already been established and $e(\beta) \neq e(\alpha)$ is clear. Thereby (*c*) has been confirmed as well and the partite construction continues.

Eventually we arrive at a final picture $(\vec{\Pi}_N, \mathscr{P}_N, \psi_N)$, which due to (*a*) and (*b*) satisfies

$$\mathrm{Girth}(\Pi_N, \mathscr{P}_N^+) > (g+1, g+1) \quad \text{as well as} \quad \mathfrak{Girth}(\vec{\Pi}_N, \mathscr{P}_N^+) > \vec{g}\,.$$

Thus the partite lemma Ξ defined by $\Xi_r(\vec{F}) = (\vec{\Pi}_N, \mathscr{P}_N^+)$ is $((g+1)\circ\vec{g})$-amenable. □

It is now immediate from Lemma 12.6 that the construction $\mathrm{PC}(\Phi, \Xi)$, where Φ again denotes a Ramsey construction for $((g+1)\circ\vec{g})$-trains, exemplifies $\diamondsuit_{(g+1)\circ\vec{g}}$. This concludes the proof of Proposition 10.15. In the light of our earlier work Corollary 10.18, Theorem 10.19, and Theorem 1.4 have thereby been demonstrated as well (see Summary 10.20).

§13. Paradise

This section is concerned with the proofs of Theorem 1.7 and Theorem 1.6. To this end we need a connection between Girth and forests of copies, which is established in §13.1. The main result there (Lemma 13.11) asserts that large Girth forces small sets of copies to be forests. In §13.2 this lemma will allow us to analyse a partite construction utilising the construction $\Omega^{(g)}$ provided by Theorem 10.19 as a partite lemma. As a result, we shall prove Theorem 1.7 in a very strong form (Theorem 13.12). The deduction of Theorem 1.6 will then be straightforward.

13.1. **Forests of copies.** Let us recall that forests of copies were introduced in Definition 1.5 as being certain systems $\mathscr{N} \subseteq \binom{H}{F}$ consisting of mutually isomorphic copies.

Marginally generalising the underlying setup, we shall henceforth say for a linear system of hypergraphs $(H, \mathscr{N})$ that $\mathscr{N}$ is a *forest of copies* if one can write $\mathscr{N} = \{F_1, \dots, F_{|\mathscr{N}|}\}$ in such a way that for every $j \in [2, |\mathscr{N}|]$ the set $z_j = V(F_j) \cap \big(\bigcup_{i<j} V(F_i)\big)$ is either an edge in $E(F_j) \cap \big(\bigcup_{i<j} E(F_i)\big)$ or it consists of at most one vertex. In this situation, $(F_1, \dots, F_{|\mathscr{N}|})$ is called an *admissible enumeration* of $\mathscr{N}$. It may be helpful to point out that forests of copies are thereby allowed to contain both edge copies and real copies.

Most results in this subsection can be regarded as appropriate adaptations of well-known facts on ordinary forests—a notion briefly recapitulated immediately before Theorem 1.1. To distinguish such forests from forests of copies we shall call them *edge forests* in the discussion that follows. Here is a list of four standard facts on edge forests.

(*i*) Every subset of an edge forest is again an edge forest.

(*ii*) The result of gluing two edge forests together along a vertex or an edge is again an edge forest.

(*iii*) If a hypergraph $H = (V, E)$ satisfies $\mathrm{girth}(H) > |E|$, then E is an edge forest.

(*iv*) Every edge forest consisting of at least two edges has at least two leaves (belonging to different edges).

Now we already saw in the introduction that forests of copies are not closed under taking subsets, meaning that the naïve generalisation of (*i*) fails (see Figure 1.1). Nevertheless, we shall show in Lemma 13.4 that deleting edge copies preserves being a forest of copies. In Lemma 13.5 and Lemma 13.6 we shall then see that (*ii*) generalises in the obvious way. Statement (*iii*) remains valid if we replace lower case girth by capital Girth (see Lemma 13.11). Finally, when dealing with (*iv*) we shall work with the following analogue of edges acting as "leaves".

Definition 13.1. Given a forest of copies $\mathscr{N}$ and a copy $F_\star \in \mathscr{N}$ we say that $F_\star$ is *terminal in* $\mathscr{N}$ if there exists an admissible enumeration $(F_1, \dots, F_{|\mathscr{N}|})$ of $\mathscr{N}$ such that $F_\star = F_{|\mathscr{N}|}$.

For instance, if $\mathscr{N}$ denotes a forest consisting of two copies, then both enumerations of $\mathscr{N}$ are admissible and, consequently, both copies in $\mathscr{N}$ are terminal. Now the variant of (*iv*) we have been alluding to reads as follows.

Lemma 13.2. *Every forest of at least two copies possesses at least two terminal copies.*

In the argument that follows and at several other occasions occurring in the rest of this section it will be convenient to employ the following notation. Given a set of hypergraphs $\mathscr{N}$, we shall write $V(\mathscr{N})$ for $\bigcup_{F_\star \in \mathscr{N}} V(F_\star)$ and, similarly, $E(\mathscr{N})$ will abbreviate $\bigcup_{F_\star \in \mathscr{N}} E(F_\star)$.

Proof of Lemma 13.2. We argue by induction on the size $|\mathscr{N}|$ of the forest of copies under consideration. The base case $|\mathscr{N}| = 2$ has already been studied. For the induction step we suppose that $\mathscr{N}$ is a forest consisting of $n \geqslant 3$ copies and that every forest of copies $\mathscr{N}'$ with $|\mathscr{N}'| = n - 1$ has at least two terminal copies.

Since $\mathscr{N}$ admits an admissible enumeration, it possesses some terminal copy $F_\star$. Notice that the set

$$z_\star = V(F_\star) \cap V(\mathscr{N} \smallsetminus \{F_\star\})$$

is either an edge in $E(F_\star) \cap E(\mathscr{N} \smallsetminus \{F_\star\})$ or it consists of at most one vertex. Therefore there exists a copy $F_\circ \in \mathscr{N} \smallsetminus \{F_\star\}$ such that

- $z_\star \subseteq V(F_\circ)$
- and if $z_\star$ is an edge, then $z_\star \in E(F_\circ)$.

The terminality of $F_\star$ implies that $\mathscr{N} \smallsetminus \{F_\star\}$ is a forest of copies and the induction hypothesis yields an admissible enumeration $(F_1, \dots, F_{n-1})$ of $\mathscr{N} \smallsetminus \{F_\star\}$ whose terminal copy is distinct from $F_\circ$. We contend that F_{n-1} is terminal in $\mathscr{N}$ as well and that the enumeration

$$(F_1, \dots, F_{n-2}, F_\star, F_{n-1}) \tag{13.1}$$

exemplifies this fact. The claims that we need to check concerning the copies $F_2, \ldots, F_{n-2}$ follow immediately from the admissibility of $(F_1, \ldots, F_{n-1})$, and the claim on $F_\star$ is a consequence of $F_\circ \in \{F_1, \ldots, F_{n-2}\}$. It remains to verify that the set

$$z^+ = V(F_{n-1}) \cap V\big(\{F_1, \ldots, F_{n-2}, F_\star\}\big)$$

is either an edge in $E(F_{n-1}) \cap E\big(\{F_1, \ldots, F_{n-2}, F_\star\}\big)$ or that it consists of at most one vertex. Due to the admissibility of $(F_1, \ldots, F_{n-1})$ we know that the set

$$z^- = V(F_{n-1}) \cap V\big(\{F_1, \ldots, F_{n-2}\}\big)$$

has analogous properties and thus it suffices to prove $z^+ = z^-$. Because of

$$V(F_{n-1}) \cap V(F_\star) \subseteq V(F_{n-1}) \cap z_\star \subseteq V(F_{n-1}) \cap V(F_\circ) \subseteq z^-$$

and

$$z^+ = z^- \cup \big(V(F_{n-1}) \cap V(F_\star)\big)$$

this is indeed the case. Altogether we have thereby proved that the enumeration (13.1) is admissible. Hence $F_\star$ and F_{n-1} are two terminal copies of $\mathscr{N}$, and the induction step is complete. □

The concept "dual" to a terminal copy is that of an *initial copy*, by which we mean a copy in a forest that is capable of standing in the first position of an admissible enumeration.

Lemma 13.3. *In a forest of copies every copy is initial.*

Proof. Again we argue by induction on the size of the forest under consideration. In the base case, when the forest consists of at most two copies, every enumeration is admissible and, therefore, every copy is initial.

For the induction step we look at a forest of copies $\mathscr{N}$ with $n = |\mathscr{N}| \geqslant 3$ and at an arbitrary copy $F_1 \in \mathscr{N}$. Lemma 13.2 discloses that $\mathscr{N}$ has a terminal copy $F_n \neq F_1$. Now $\mathscr{N}' = \mathscr{N} \smallsetminus \{F_n\}$ is again a forest of copies and due to the induction hypothesis F_1 is an initial copy of $\mathscr{N}'$, i.e., there is an admissible enumeration $(F_1, \ldots, F_{n-1})$ of $\mathscr{N}'$ that starts with F_1. Clearly $(F_1, \ldots, F_n)$ is an admissible enumeration of $\mathscr{N}$ and, consequently, F_1 is indeed an initial copy of $\mathscr{N}$. □

Next we address "subforests" of copies obtained by the removal of edge copies.

Lemma 13.4. *The result of deleting an edge copy from a forest of copies is again a forest of copies.*

Proof. Let $\mathscr{N}$ be a forest of copies containing some edge copy e^+. We are to prove that $\mathscr{N}' = \mathscr{N} \smallsetminus \{e^+\}$ is a forest of copies as well. To avoid trivialities we may suppose that

$\mathscr{N}' \neq \varnothing$. If $e \in E(\mathscr{N}')$ we choose a copy $F_1 \in \mathscr{N}'$ satisfying $e \in E(F_1)$ and otherwise we let $F_1 \in \mathscr{N}'$ be arbitrary. By Lemma 13.3 there is an admissible enumeration

$$(F_1, \ldots, F_i, e^+, F_{i+1}, \ldots, F_{|\mathscr{N}'|})$$

of $\mathscr{N}$ starting with F_1 and one checks immediately that the enumeration $(F_1, \ldots, F_{|\mathscr{N}'|})$ of $\mathscr{N}'$ obtained by deleting e^+ is admissible. □

Roughly speaking, the lemma that follows asserts that if we glue two forests of copies $\mathscr{A}$ and $\mathscr{B}$ together at a vertex, then the result is again a forest of copies.

Lemma 13.5. *Suppose that $(H, \mathscr{N})$ is a linear system of hypergraphs admitting a partition $\mathscr{N} = \mathscr{A} \charfusion[\mathbin]{\cup}{\cdot} \mathscr{B}$ such that $|V(\mathscr{A}) \cap V(\mathscr{B})| \leqslant 1$. If both $\mathscr{A}$ and $\mathscr{B}$ are forests of copies, then so is $\mathscr{N}$.*

Proof. The case $\mathscr{B} = \varnothing$ being clear we may suppose that $\mathscr{B}$ contains at least one copy. Since the set $z = V(\mathscr{A}) \cap V(\mathscr{B})$ consists of at most one vertex, there exists a copy $F'_1 \in \mathscr{B}$ such that $z \subseteq V(F'_1)$. Now Lemma 13.3 shows that $\mathscr{B}$ has an admissible enumeration $(F'_1, \ldots, F'_{|\mathscr{B}|})$ starting with F'_1. It is not hard to check that if $(F_1, \ldots, F_{|\mathscr{A}|})$ is an arbitrary admissible enumeration of $\mathscr{A}$, then the enumeration $(F_1, \ldots, F_{|\mathscr{A}|}, F'_1, \ldots, F'_{|\mathscr{B}|})$ of $\mathscr{N}$ is admissible as well. □

There is a similar statement addressing the case that the glueing occurs in an entire edge of the underlying hypergraph.

Lemma 13.6. *Let $(H, \mathscr{N})$ be a linear system of hypergraphs and let $\mathscr{N} = \mathscr{A} \charfusion[\mathbin]{\cup}{\cdot} \mathscr{B}$ be a partition such that $V(\mathscr{A}) \cap V(\mathscr{B}) \subseteq e$ holds for some edge e of H. If both $\mathscr{A} \cup \{e^+\}$ and $\mathscr{B} \cup \{e^+\}$ are forests of copies, then so is $\mathscr{N}$.*

Proof. Take an arbitrary admissible enumeration $(F_1, \ldots, F_a)$ of $\mathscr{A} \cup \{e^+\}$, as well as an admissible enumeration $(e^+, F'_2, \ldots, F'_b)$ of $\mathscr{B} \cup \{e^+\}$, where $a = |\mathscr{A} \cup \{e^+\}|$ and $b = |\mathscr{B} \cup \{e^+\}|$, respectively. Now $(F_1, \ldots, F_a, F'_2, \ldots, F'_b)$ is an admissible enumeration of $\mathscr{N} \cup \{e^+\}$. If $e^+ \in \mathscr{N}$ this proves that $\mathscr{N}$ is indeed a forest of copies and in case $e^+ \notin \mathscr{N}$ Lemma 13.4 leads us to the same conclusion. □

An iterative application of the next result will allow us to relate forests of copies to Girth.

Lemma 13.7. *Given a linear system of hypergraphs $(H, \mathscr{N})$ with $\mathrm{Girth}(H, \mathscr{N}^+) > |\mathscr{N}| \geqslant 2$ there exists a copy $F_\star \in \mathscr{N}$ such that the set*

$$z_\star = V(F_\star) \cap V(\mathscr{N} \smallsetminus \{F_\star\}) \tag{13.2}$$

is either an edge in $E(F_\star) \cap E(\mathscr{N} \smallsetminus \{F_\star\})$ or it consists of at most one vertex.

Proof. Assume for the sake of contradiction that no such copy $F_\star$ exists. We aim at building a tidy cycle of copies in $(H, \mathscr{N}^+)$ consisting of at most $|\mathscr{N}|$ distinct copies, which does not possess a master copy. Looking at the potential connectors of such a cycle, we call a vertex $x \in V(H)$ *useful* if there exist distinct copies $F', F'' \in \mathscr{N}$ such that $x \in V(F') \cap V(F'')$. Similarly, an edge $e \in E(H)$ is said to be *useful* if there are distinct copies $F', F'' \in \mathscr{N}$ satisfying $e \in E(F') \cap E(F'')$. For instance, for every copy $F_\star \in \mathscr{N}$ all vertices belonging to the corresponding set $z_\star$ defined in (13.2) are useful.

Claim 13.8. *For every useful vertex x there are two distinct copies $F'_x, F''_x \in \mathscr{N}$ such that*
 (i) $x \in V(F'_x) \cap V(F''_x)$;
 (ii) and each of F'_x, F''_x has at most one useful edge containing x.

Proof. Given x we consider an auxiliary set system S_x with vertex set

$$V(S_x) = \big\{F_\star \in \mathscr{N} : x \in V(F_\star)\big\}\,.$$

For every useful edge e containing x the set $\varphi_e = \big\{F_\star \in V(S_x) : e \in E(F_\star)\big\}$ has at least the size 2. Thus we can define the edge set of our set system S_x by

$$E(S_x) = \big\{\varphi_e : e \text{ is useful and } x \in e\big\}\,.$$

We are to prove that S_x has at least two vertices whose degree is at most one. Due to $v(S_x) \geqslant 2$ the failure of this statement would imply that S_x contains some cycle $\varphi_{e(1)}F_1 \dots \varphi_{e(n)}F_n$ (in the sense of Definition 1.3). But then $F_1e(2)\dots F_ne(1)$ is a tidy cycle of copies in $(H, \mathscr{N})$ all of whose connectors are edges, which in view of

$$\mathrm{Girth}(H, \mathscr{N}^+) > |\mathscr{N}| \geqslant v(S_x) \geqslant n$$

contradicts Lemma 4.15. □

Next we shall construct a cycle of copies in $(H, \mathscr{N})$ which has only vertex connectors and some further special property. We commence by examining desirable subsequences of the form $q'F_\star q''$ of the envisaged cycle. Given a copy $F_\star \in \mathscr{N}$ and distinct useful vertices $q', q'' \in V(F_\star)$ we say that $F_\star$ *is secure between q' and q''* provided the following statement holds: If there is an edge f satisfying $q', q'' \in f \in E(F_\star)$, then this edge is not useful. The intuition here is that it is impossible to collapse copies that occur securely between their neighbouring connectors.

Claim 13.9. *If $F_\star \in \mathscr{N}$ and $q' \in V(F_\star)$ is useful, then there exists a copy $F_{\star\star} \in \mathscr{N} \smallsetminus \{F_\star\}$ satisfying $q' \in V(F_{\star\star})$ together with another useful vertex $q'' \in V(F_{\star\star})$ such that $F_{\star\star}$ is secure between q' and q''.*

Proof. Let $F'_{q'}, F''_{q'} \in \mathscr{N}$ be the copies Claim 13.8 provides for $x = q'$, choose $F_{\star\star} \in \{F'_{q'}, F''_{q'}\}$ such that $F_{\star\star} \neq F_\star$, and observe that Claim 13.8(*i*) ensures $q' \in V(F_{\star\star})$. Let the set $z^{\star\star}$

be defined with respect to $F_{\star\star}$ as in (13.2). The failure of our lemma entails $|z^{\star\star}| \geqslant 2$ and thus there exists a vertex $q^\circ \in z^{\star\star}$ distinct from q'. If $F_{\star\star}$ is secure between q' and q°, we can just take $q'' = q^\circ$, so suppose from now on that this is not the case. This means that there exists a useful edge f satisfying $q', q^\circ \in f \in E(F_{\star\star})$.

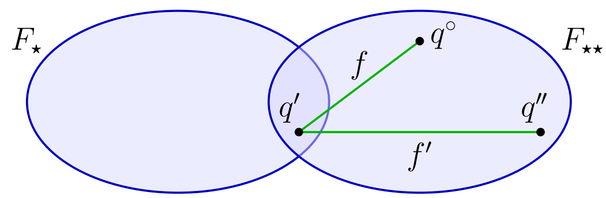


FIGURE 13.1. Security of $F_{\star\star}$

Clearly $f \subseteq z^{\star\star}$ and by appealing to the failure of our lemma again we learn $z^{\star\star} \neq f$. Hence there exists a vertex $q'' \in z^{\star\star} \smallsetminus f$ and it suffices to prove that $F_{\star\star}$ is secure between q' and q'' (see Figure 13.1). If this were not the case, there had to exist a useful edge f' such that $q', q'' \in f' \in E(F_{\star\star})$. Now $q'' \in f' \smallsetminus f$ yields $f \neq f'$ and, therefore, $F_{\star\star}$ violates clause *(ii)* of Claim 13.8. This contradiction concludes the proof of Claim 13.9. □

Let us call a cycle of copies $F_1 q_1 \dots F_n q_n$ in $(H, \mathscr{N})$ or $(H, \mathscr{N}^+)$ *special* if

- the copies $F_1, \dots, F_n$ are distinct,
- the connectors $q_1, \dots, q_n$ are vertices,
- and with at most one exception every copy F_i is secure between q_{i-1} and q_i.

Claim 13.10. *There exist a special cycle of copies in $(H, \mathscr{N})$.*

Proof. An iterative application of Claim 13.9 allows us to construct an infinite sequence

$$F_1 q_1 F_2 q_2 \dots$$

consisting of copies $F_1, F_2, \dots \in \mathscr{N}$ and useful vertices $q_1, q_2, \dots$, such that for every $i \in \mathbb{N}$

(1) the vertex q_i belongs to $V(F_i) \cap V(F_{i+1})$,
(2) the copies F_i, F_{i+1} are distinct,
(3) and F_{i+1} is secure between q_i and q_{i+1}.

Indeed, let $F_1 \in \mathscr{N}$ be arbitrary and let $q_1 \in V(F_1)$ be a useful vertex. If for some natural number m we have already constructed the initial segment $F_1 q_1 \dots F_m q_m$ of our infinite sequence, we apply Claim 13.9 to (F_m, q_m) here in place of $(F_\star, q')$ there, thus obtaining a copy F_{m+1} and a useful vertex q_{m+1} that allow us to continue.

Since $\mathscr{N}$ is finite, there exists some $n \in \mathbb{N}$ such that $F_{n+1} \in \{F_1, \dots, F_n\}$. If n denotes the least such natural number, then $F_1, \dots, F_n$ are distinct and, hence, there is a unique index $i \in [n]$ such that $F_i = F_{n+1}$. For notational simplicity we may suppose that $i = 1$.

Since *(2)* implies $n \geqslant 2$, the cyclic sequence

$$\mathscr{C} = F_1 q_1 \dots F_n q_n$$

has all properties of a cycle of copies except that we do not know whether its connectors are distinct. Moreover, *(3)* tells us that F_1 is the only copy in $\mathscr{C}$ that might be insecure.

Thus if the connectors of $\mathscr{C}$ happen to be distinct, then $\mathscr{C}$ is the desired special cycle of copies. If $q_1, \dots, q_n$ fail to be distinct, we take a pair (r, s) of indices with $1 \leqslant r < s \leqslant n$, $q_r = q_s$, and, subject to this, such that $s - r$ is minimal. Now one checks easily that $F_{r+1} q_{r+1} \dots F_s q_s$ is a special cycle of copies. □

Throughout the rest of the proof we consider a special cycle of copies $\mathscr{C} = F_1 q_1 \dots F_n q_n$ in the extended system $(H, \mathscr{N}^+)$ whose length n is minimal. Claim 13.10 discloses $n \leqslant |\mathscr{N}|$ and, consequently, we have $\mathrm{ord}(\mathscr{C}) < \mathrm{Girth}(H, \mathscr{N}^+)$. In other words,

(a) either $\mathscr{C}$ fails to be tidy

(b) or $\mathscr{C}$ has a master copy.

We shall show that both alternatives lead to a contradiction. Let us deal with case *(a)* first. Since $\mathscr{C}$ has no edge connectors, this is only possible if *(T2)* fails for some edge $f \in E(H)$. By symmetry we can suppose that $1 \in M(f)$. Write $M(f) = \{i(1), \dots, i(m)\}$ with $1 = i(1) < i(2) < \dots i(m) \leqslant n$ and $m \geqslant 2$. We observe that the cyclic sequences

$$\mathscr{D}_\mu = f^+ q_{i(\mu)} F_{i(\mu)+1} q_{i(\mu)+1} \dots F_{i(\mu+1)} q_{i(\mu+1)} \qquad \text{for } \mu \in [m-1]$$

$$\text{and} \qquad \mathscr{D}_m = f^+ q_{i(m)} F_{i(m)+1} q_{i(m)+1} \dots F_1 q_1$$

are shorter than $\mathscr{C}$. Indeed, this is clear for $m \geqslant 3$ and in case $m = 2$ it follows from the fact that the two members of $M(f)$ cannot be consecutive in $\mathbb{Z}/n\mathbb{Z}$.

We aim at showing that at least one of $\mathscr{D}_1, \dots, \mathscr{D}_m$ contradicts the minimality of n. Notice that the newly inserted edge copy f^+ can cause trouble in two different ways. First, it might be insecure in all of $\mathscr{D}_1, \dots, \mathscr{D}_m$ and, second, it might happen that one of $\mathscr{D}_1, \dots, \mathscr{D}_m$ contains f^+ twice. However, since $\mathscr{C}$ itself is special, at most one of $\mathscr{D}_1, \dots, \mathscr{D}_m$ contains an insecure appearance of one of $F_1, \dots, F_n$. Thus at most two among $\mathscr{D}_1, \dots, \mathscr{D}_m$ can fail to contradict the minimality of n, one due to containing two insecure copies and the (potential) other one due to containing f^+ twice.

In other words, the only case where we are not done yet occurs if $m = |M(f)| = 2$ and f^+ is among $F_1, \dots, F_n$. As $f^+ = F_i$ implies $i-1, i \in M(f)$, this case requires $M(f) = \{i-1, i\}$, which contradicts the choice of f. Altogether, we have thereby shown that *(a)* is indeed impossible, i.e., that our minimal special cycle of copies $\mathscr{C}$ in $(H, \mathscr{N}^+)$ is tidy.

Therefore *(b)* holds, i.e., $\mathscr{C}$ has a master copy. By symmetry we may suppose that F_1 is a master copy of $\mathscr{C}$ and that the family of edges $\{f_i \in E(F_1) \colon i \in [2, n]\}$ exemplifies this fact. For every $i \in [2, n]$ Fact 4.11 tells us $f_i \in E(F_i) \cap E(F_1)$, wherefore the edge f_i

is useful. Combined with $q_{i-1}, q_i \in f_i$ this implies that F_i fails to be secure between q_{i-1} and q_i. Thus despite being special $\mathscr{C}$ contains at least $n-1$ insecure copies, which is only possible if $n = 2$. But now the useful edge f_2 exemplifies that F_1 is insecure as well and we have obtained the final contradiction that rules out option *(b)* and thereby concludes the proof of Lemma 13.7. $\square$

Lemma 13.11. *Let $(H, \mathscr{N})$ be a linear system of hypergraphs. If* $\mathrm{Girth}(H, \mathscr{N}^+) > |\mathscr{N}|$, *then $\mathscr{N}$ is a forest of copies.*

Proof. We argue by induction on $n = |\mathscr{N}|$, the base case $n = 1$ being clear. In the induction step we have to deal with a linear system $(H, \mathscr{N})$ consisting of $n \geqslant 2$ copies that satisfies $\mathrm{Girth}(H, \mathscr{N}^+) > |\mathscr{N}|$. Let $F_\star \in \mathscr{N}$ be a copy obtained by applying Lemma 13.7 to $\mathscr{N}$ and set $\mathscr{N}_\star = \mathscr{N} \smallsetminus \{F_\star\}$. Since $\mathrm{Girth}(H, \mathscr{N}_\star^+) > |\mathscr{N}| > |\mathscr{N}_\star|$, the induction hypothesis shows that $\mathscr{N}_\star$ is a forest of copies. If $(F_1, \dots, F_{n-1})$ is an admissible enumeration of $\mathscr{N}_\star$, then $(F_1, \dots, F_{n-1}, F_\star)$ is the desired admissible enumeration of $\mathscr{N}$. $\square$

13.2. **The final partite construction.** We shall prove the following strong form of Theorem 1.7 alluded to in the introduction.

Theorem 13.12. *Let a hypergraph F and a natural number $g \geqslant 2$ satisfy* $\mathrm{girth}(F) > g$. *If $r, n \geqslant 2$ are two further natural numbers, then there exists a linear system of hypergraphs $(H, \mathscr{H})$ such that*

- $\mathscr{H} \longrightarrow (F)_r$
- *and for every $\mathscr{N} \subseteq \mathscr{H}^+$ with $|\mathscr{N}| \in [2, n]$ there exists some $\mathscr{X} \subseteq \mathscr{H}$ for which $\mathscr{N} \cup \mathscr{X}$ is a forest of copies and $|\mathscr{X}| \leqslant \frac{|\mathscr{N}|-2}{g-1}$.*

Before we embark on the proof of this result we would like to point out that for $n \leqslant g$ it follows quickly from a pair of statements that have been obtained earlier. Indeed, the construction $\Omega^{(g)}$ (see Theorem 10.19) delivers a linear system of hypergraphs $(H, \mathscr{H})$ such that $\mathscr{H} \longrightarrow (F)_r$ and $\mathrm{Girth}(H, \mathscr{H}^+) > g$. Now for every $\mathscr{N} \subseteq \mathscr{H}^+$ with $|\mathscr{N}| \in [2, g]$ we have $\mathrm{Girth}(H, \mathscr{N}^+) > g \geqslant |\mathscr{N}|$ and owing to Lemma 13.11 this implies that $\mathscr{N}$ is a forest of copies. In other words, the second bullet holds for $\mathscr{X} = \varnothing$, as desired.

The general case of Theorem 13.12 will be proved by means of a further application of the partite construction method. This will involve pictures of the following kind.

Definition 13.13. Given integers $g, n \geqslant 2$ a picture $(\Pi, \mathscr{P}, \psi)$ is said to be (g, n)-*certified* if each of its constituents Π^e satisfies $\mathrm{girth}(\Pi^e) > n$ and for every $\mathscr{N} \subseteq \mathscr{P}^+$ with $|\mathscr{N}| \in [2, n]$ there exists some $\mathscr{X} \subseteq \mathscr{P}$ such that $\mathscr{N} \cup \mathscr{X}$ is a forest of copies and $|\mathscr{X}| \leqslant \frac{|\mathscr{N}|-2}{g-1}$.

Let us show first that being certified is a property we can expect any reasonable picture zero to have.

Lemma 13.14. *Suppose that a hypergraph F and an integer $g \geqslant 2$ satisfy* $\mathrm{girth}(F) > g$. *If $(G, \mathscr{G})$ denotes any linear system of hypergraphs with $\mathscr{G} \subseteq \binom{G}{F}$, then the picture zero over this system is (g, n)-certified for every integer $n \geqslant 2$.*

Proof. Let $(\Pi, \mathscr{P}, \psi)$ denote the picture zero under discussion. Since all of its constituents are matchings, their girth is greater than n for every $n \geqslant 2$. Proceeding with the second property, we suppose that any $\mathscr{N} \subseteq \mathscr{P}^+$ with $|\mathscr{N}| \geqslant 2$ is given. For every copy $F_\star \in \mathscr{P}$ we set $\mathscr{N}(F_\star) = \mathscr{N} \cap \big(E^+(F_\star) \cup \{F_\star\}\big)$. Since Π is the disjoint union of the copies in $\mathscr{P}$, this stipulation yields a partition $\mathscr{N} = \dot{\bigcup}_{F_\star \in \mathscr{P}} \mathscr{N}(F_\star)$.

There are two sufficient conditions ensuring that a partition class $\mathscr{N}(F_\star)$ is a forest of copies. First, if $F_\star \in N(F_\star)$, then every enumeration of $\mathscr{N}(F_\star)$ starting with $F_\star$ itself is admissible. Second, if $F_\star \notin \mathscr{N}(F_\star)$ and $|\mathscr{N}(F_\star)| \leqslant g$, then $\mathscr{N}(F_\star)$ is a forest of edge copies due to $\mathrm{girth}(F_\star) > g$. For these reasons, the set

$$\mathscr{X} = \big\{F_\star \in \mathscr{P} \colon |\mathscr{N}(F_\star)| \geqslant g + 1\big\}$$

has the property that $\mathscr{N} \cup \mathscr{X}$ is a forest of copies.

So it remains to prove $|\mathscr{X}| \leqslant \frac{|\mathscr{N}|-2}{g-1}$. The obvious bound $|\mathscr{X}| \leqslant |\mathscr{N}|/(g+1)$ rewrites as $(g-1)|\mathscr{X}| + 2(|\mathscr{X}| - 1) \leqslant |\mathscr{N}| - 2$ and provided that $\mathscr{X}$ is nonempty the desired upper bound on $|\mathscr{X}|$ follows. Moreover, if $\mathscr{X}$ is empty, then we just need to appeal to $|\mathscr{N}| \geqslant 2$. □

The picturesque lemma appropriate for the present context reads as follows.

Lemma 13.15. *Suppose that $g, n \geqslant 2$ are integers and that*

$$(\Sigma, \mathscr{Q}, \psi_\Sigma) = (\Pi, \mathscr{P}, \psi_\Pi) \ast (H, \mathscr{H}) \tag{13.3}$$

holds for two pictures $(\Pi, \mathscr{P}, \psi_\Pi)$ and $(\Sigma, \mathscr{Q}, \psi_\Sigma)$ over a linear system of hypergraphs $(G, \mathscr{G})$ and a linear k-partite k-uniform system of hypergraphs $(H, \mathscr{H})$. If

- *$(\Pi, \mathscr{P}, \psi_\Pi)$ is (g, n)-certified*
- *and* $\mathrm{Girth}(H, \mathscr{H}^+) > n$,

then $(\Sigma, \mathscr{Q}, \psi_\Sigma)$ is (g, n)-certified as well.

Proof. The demand on the girth of the constituents of $(\Sigma, \mathscr{Q}, \psi_\Sigma)$ follows easily from the corresponding property of the constituents of $(\Pi, \mathscr{P}, \psi_\Pi)$ combined with $\mathrm{Girth}(H, \mathscr{H}^+) > n$ and the linearity of the vertical hypergraph G. So it remains to deal with the forest extension property.

We shall say that $(\mathscr{N}, \varphi)$ is a *good pair* if $\mathscr{N} \subseteq \mathscr{Q}^+$ and $\varphi \colon \mathscr{N} \longrightarrow \mathscr{H}^+$ is a map such that for every copy $F_\star \in \mathscr{N}$ one of the following two cases occurs:

- Either $F_\star \in E^+(H)$ is an edge copy and $\varphi(F_\star) = F_\star$
- or $\varphi(F_\star) = \Pi^e_\star \in \mathscr{H}$ and the standard copy $(\Pi_\star, \mathscr{P}_\star)$ extending $\Pi^e_\star$ satisfies $F_\star \in \mathscr{P}^+_\star$.

Notice that for every $\mathscr{N} \subseteq \mathscr{Q}^+$ there is a map $\varphi\colon \mathscr{N} \longrightarrow \mathscr{H}^+$ such that $(\mathscr{N}, \varphi)$ is a good pair. But for clarity we would like to remark that φ does not need to be uniquely determined by $\mathscr{N}$, since for $f^+ \in \mathscr{N} \cap E^+(H)$ there may be several legitimate choices for $\varphi(f^+)$.

A good pair $(\mathscr{X}, \xi)$ is said to *resolve* another good pair $(\mathscr{N}, \varphi)$ if

(*i*) $\mathscr{X} \subseteq \mathscr{Q}$,
(*ii*) $\xi[\mathscr{X}] \subseteq \varphi[\mathscr{N}]$,
(*iii*) and $\mathscr{N} \cup \mathscr{X}$ is a forest of copies.

We shall prove the following strengthening of our claim: Every good pair $(\mathscr{N}, \varphi)$ such that $|\mathscr{N}| \in [2, n]$ is resolved by another good pair $(\mathscr{X}, \xi)$ satisfying $|\mathscr{X}| \leqslant \frac{|\mathscr{N}|-2}{g-1}$.

Arguing indirectly we fix a good pair $(\mathscr{N}, \varphi)$ with $|\mathscr{N}| \in [2, n]$

(*1*) that is not resolved by any good pair $(\mathscr{X}, \xi)$ with $|\mathscr{X}| \leqslant \frac{|\mathscr{N}|-2}{g-1}$
(*2*) and subject to this in such a way that $\big|\varphi[\mathscr{N}]\big|$ minimal.

Assume first that $\big|\varphi[\mathscr{N}]\big| \leqslant 1$. Because of $|\mathscr{N}| \geqslant 2$ this is only possible if there exists some standard copy $(\Pi_\star, \mathscr{P}_\star)$ such that $\mathscr{N} \subseteq \mathscr{P}_\star^+$ and $\varphi[\mathscr{N}] = \{\Pi_\star^e\}$, where $\Pi_\star$ extends $\Pi_\star^e$. As the picture $(\Pi, \mathscr{P}, \psi_\Pi)$ is (g, n)-certified, there exists a set $\mathscr{X} \subseteq \mathscr{P}_\star \subseteq \mathscr{Q}$ such that $\mathscr{N} \cup \mathscr{X}$ is a forest of copies and $|\mathscr{X}| \leqslant \frac{|\mathscr{N}|-2}{g-1}$. Let ξ be the unique map from $\mathscr{X}$ to $\{\Pi_\star^e\}$. Now the good pair $(\mathscr{X}, \xi)$ resolves $(\mathscr{N}, \varphi)$ and thus it contradicts (*1*). This argument proves

$$\big|\varphi[\mathscr{N}]\big| \geqslant 2\,. \tag{13.4}$$

Since $|\varphi[\mathscr{N}]| \leqslant |\mathscr{N}| \leqslant n < \mathrm{Girth}(H, \mathscr{H}^+)$, Lemma 13.11 tells us that $\varphi[\mathscr{N}]$ is a forest of copies. We choose a terminal copy $\Pi_\star^e$ in this forest, partition $\mathscr{N}$ into the sets

$$\mathscr{A} = \varphi^{-1}(\Pi_\star^e) \quad \text{and} \quad \mathscr{B} = \mathscr{N} \smallsetminus \mathscr{A}\,,$$

and remark that (13.4) implies

$$1 \leqslant \big|\varphi[\mathscr{A}]\big|, \big|\varphi[\mathscr{B}]\big| < \big|\varphi[\mathscr{N}]\big|\,. \tag{13.5}$$

Due to $\varphi[\mathscr{A}] = \{\Pi_\star^e\}$ and $\varphi[\mathscr{B}] = \varphi[\mathscr{N}] \smallsetminus \{\Pi_\star^e\}$ the terminal choice of $\Pi_\star^e$ guarantees that the set $x = V(\varphi[A]) \cap V(\varphi[\mathscr{B}])$ is either an edge in $E(\varphi[A]) \cap E(\varphi[\mathscr{B}])$ or it consists of at most one vertex. We begin with the former case, as it illustrates better how the upper bound $\frac{|\mathscr{N}|-2}{g-1}$ arises in the proof.

First Case. $x \in E(\varphi[A]) \cap E(\varphi[\mathscr{B}])$.

Let us consider the sets $\mathscr{A}' = \mathscr{A} \cup \{x^+\}$ and $\mathscr{B}' = \mathscr{B} \cup \{x^+\}$. We claim that there are maps $\alpha\colon \mathscr{A}' \longrightarrow \mathscr{H}^+$ and $\beta\colon \mathscr{B}' \longrightarrow \mathscr{H}^+$ such that

$$(\mathscr{A}', \alpha) \text{ and } (\mathscr{B}', \beta) \text{ are good pairs, } \alpha[\mathscr{A}'] = \varphi[\mathscr{A}], \text{ and } \beta[\mathscr{B}'] = \varphi[\mathscr{B}]\,. \tag{13.6}$$

Concerning the existence of α, we observe that in case $x^+ \in \mathscr{A}$ we may just take the restriction of φ to $\mathscr{A}$. If $x^+ \notin \mathscr{A}$ we additionally need to specify an appropriate value of $\alpha(x^+)$ in $\varphi[\mathscr{A}]$, which is possible because of $x \in E(\varphi[\mathscr{A}])$. Thus the desired map α does indeed exist and we may argue similarly with respect to β. Thereby (13.6) is proved.

Next we seek to establish

$$|\mathscr{A}'|, |\mathscr{B}'| \in [2, n]\,. \tag{13.7}$$

Since (13.5) implies $\mathscr{A}, \mathscr{B} \neq \mathscr{N}$, the upper bounds follow from $|\mathscr{N}| \leqslant n$. Assume towards a contradiction that $|\mathscr{A}'| \leqslant 1$. By $\mathscr{A} \neq \varnothing$ this is only possible if $\mathscr{A} = \{x^+\}$, whence $\mathscr{B}' = \mathscr{N}$. Now $(\mathscr{N}, \beta)$ is a good pair, which by (13.5) and (13.6) satisfies $\big|\beta[\mathscr{N}]\big| < \big|\varphi[\mathscr{N}]\big|$. So by the minimality demand *(2)* there is a good pair $(\mathscr{X}, \xi)$ resolving $(\mathscr{N}, \beta)$ and satisfying $|\mathscr{X}| \leqslant \frac{|\mathscr{N}|-2}{g-1}$. In particular, $(\mathscr{X}, \xi)$ resolves $(\mathscr{N}, \varphi)$, contrary to *(1)*. This whole argument shows $|\mathscr{A}'| \geqslant 2$ and in a similar fashion one confirms $|\mathscr{B}'| \geqslant 2$ as well. Thereby (13.7) is proved.

For all these reasons *(2)* tells us that the good pairs $(\mathscr{A}', \alpha)$ and $(\mathscr{B}', \beta)$ are resolved by certain good pairs $(\mathscr{Y}, \upsilon)$ and $(\mathscr{Z}, \zeta)$ that satisfy

$$|\mathscr{Y}| \leqslant \frac{|\mathscr{A}'|-2}{g-1} \quad \text{ and } \quad |\mathscr{Z}| \leqslant \frac{|\mathscr{B}'|-2}{g-1}\,, \tag{13.8}$$

respectively. Set $\mathscr{X} = \mathscr{Y} \cup \mathscr{Z}$ and let $\xi \subseteq \upsilon \cup \zeta$ be a map from $\mathscr{X}$ to $\mathscr{H}^+$. We contend that

$$\text{the good pair } (\mathscr{X}, \xi) \text{ resolves } (\mathscr{N}, \varphi)\,. \tag{13.9}$$

The demand *(i)* is clear and *(ii)* follows from

$$\xi[\mathscr{X}] \subseteq \upsilon[\mathscr{Y}] \cup \zeta[\mathscr{Z}] \overset{(ii)}{\subseteq} \alpha[\mathscr{A}'] \cup \beta[\mathscr{B}'] \overset{(13.6)}{=} \varphi[\mathscr{A}] \cup \varphi[\mathscr{B}] = \varphi[\mathscr{N}]\,.$$

For the verification of *(iii)* we want to apply Lemma 13.6 to $\mathscr{A} \cup \mathscr{Y}$, $\mathscr{B} \cup \mathscr{Z}$, and x here in place of $\mathscr{A}$, $\mathscr{B}$, and e there. As the assumption that $(\mathscr{A} \cup \mathscr{Y}) \cup \{x^+\} = \mathscr{A}' \cup \mathscr{Y}$ and $(\mathscr{B} \cup \mathscr{Z}) \cup \{x^+\} = \mathscr{B}' \cup \mathscr{Z}$ need to be forests of copies are satisfied by our choice of the good pairs $(\mathscr{Y}, \upsilon)$ and $(\mathscr{Z}, \zeta)$, it remains to check that

$$V(\mathscr{A} \cup \mathscr{Y}) \cap V(\mathscr{B} \cup \mathscr{Z}) \subseteq x\,. \tag{13.10}$$

Towards the proof of this inclusion we observe that $V(F_\star) \cap V(H) \subseteq V\big(\varphi(F_\star)\big)$ holds for every $F_\star \in \mathscr{A}$, whence $V(\mathscr{A}) \cap V(H) \subseteq V(\varphi[\mathscr{A}])$. Similarly one has

$$V(\mathscr{Y}) \cap V(H) \subseteq V(\upsilon[\mathscr{Y}]) \overset{(ii)}{\subseteq} V(\alpha[\mathscr{A}']) \overset{(13.6)}{=} V(\varphi[\mathscr{A}])$$

and both statements together yield $V(\mathscr{A} \cup \mathscr{Y}) \cap V(H) \subseteq V(\varphi[\mathscr{A}])$. Proceeding similarly with $\mathscr{B} \cup \mathscr{Z}$ and combining the results we infer

$$\big(V(\mathscr{A} \cup \mathscr{Y}) \cap V(\mathscr{B} \cup \mathscr{Z})\big) \cap V(H) \subseteq V(\varphi[\mathscr{A}]) \cap V(\varphi[\mathscr{B}]) = x\,. \tag{13.11}$$

Moreover, the definition of the partite amalgamation (13.3) entails $V(F_\star) \cap V(F_{\star\star}) \subseteq V(H)$ for all $F_\star \in \mathscr{A} \cup \mathscr{Y}$ and $F_{\star\star} \in \mathscr{B} \cup \mathscr{Z}$ and for this reason we have

$$V(\mathscr{A} \cup \mathscr{Y}) \cap V(\mathscr{B} \cup \mathscr{Z}) \subseteq V(H)\,.$$

Together with (13.11) this establishes (13.10) and thus concludes the proof of (13.9).

However, in the light of

$$|\mathscr{X}| \leqslant |\mathscr{Y}| + |\mathscr{Z}| \overset{(13.8)}{\leqslant} \frac{|\mathscr{A}|-1}{g-1} + \frac{|\mathscr{B}|-1}{g-1} = \frac{|\mathscr{N}|-2}{g-1}$$

this contradicts *(1)*. In other words, the case that x is an edge is impossible.

Second Case. $|x| \leqslant 1$.

We argue similar as in the first case. The pairs $(\mathscr{A}, \varphi \restriction \mathscr{A})$ and $(\mathscr{B}, \varphi \restriction \mathscr{B})$ are good and the main point is that they are resolved by certain good pairs $(\mathscr{Y}, \upsilon)$ and $(\mathscr{Z}, \zeta)$ such that

$$|\mathscr{Y}| \leqslant \frac{|\mathscr{A}|-1}{g-1} \quad \text{and} \quad |\mathscr{Z}| \leqslant \frac{|\mathscr{B}|-1}{g-1}\,, \tag{13.12}$$

respectively. In fact, if $|\mathscr{A}| \geqslant 2$ we can argue exactly as in the first case in order to obtain such a good pair $(\mathscr{Y}, \upsilon)$ satisfying the stronger estimate $|\mathscr{Y}| \leqslant \frac{|\mathscr{A}|-2}{g-1}$. Moreover, in case $|\mathscr{A}| = 1$ we can just take $\mathscr{Y} = \upsilon = \varnothing$. The claim on $(\mathscr{Z}, \zeta)$ is proved in the same way.

Again we set $\mathscr{X} = \mathscr{Y} \cup \mathscr{Z}$ and take a map $\xi \subseteq \upsilon \cup \zeta$ from $\mathscr{X}$ to $\mathscr{H}^+$. We remark that (13.10) is still valid in the present case and the only change we need to make when concluding that $\mathscr{N} \cup \mathscr{X}$ is a forest of copies is that this time we need to appeal to Lemma 13.5. Apart from these small modifications, the proof that the good pair $(\mathscr{X}, \xi)$ resolves $(\mathscr{N}, \varphi)$ is still the same. Finally, (13.12) implies $|\mathscr{X}| \leqslant \frac{|\mathscr{N}|-2}{g-1}$, meaning that again we reach a contradiction to the choice of $(\mathscr{N}, \varphi)$ in *(1)*. □

After these preparations we can quickly establish the main result of this subsection.

Proof of Theorem 13.12. Let Φ be any linear Ramsey construction and let Ξ be a partite lemma applicable to k-partite k-uniform hypergraphs F with $\mathrm{girth}(F) > n$ that delivers linear systems of hypergraphs $(H, \mathscr{H})$ satisfying $\mathrm{Girth}(H, \mathscr{H}^+) > n$. We recall that the existence of such a partite lemma is an immediate consequence of Theorem 10.19.

Let us attempt to perform the partite construction $\mathrm{PC}(\Phi, \Xi)_r(F) = (H, \mathscr{H})$. By Lemma 13.14 picture zero is (g, n)-certified and an iterative application of Lemma 13.15 discloses that all further pictures are well-defined and, likewise, (g, n)-certified. In particular, the final picture is well-defined and the fact that it is (g, n)-certified implies that it has the desired property. □

Finally, we deduce the last statement announced in the introduction.

Proof of Theorem 1.6. Given a linear k-uniform hypergraph F and $r, n \in \mathbb{N}$ we apply Theorem 1.7 with $n' = \binom{n}{k}$ instead of n, thus obtaining some linear system $(H, \mathscr{H})$. Without loss of generality we can assume that $E(H) = E(\mathscr{H})$, for deleting edges from H that belong to none of the copies in $\mathscr{H}$ cannot influence whether the system $(H, \mathscr{H})$ satisfies the conclusion of Theorem 1.7.

In order to show that H has the desired property we consider any set $X \subseteq E(H)$ whose size it at most n. For every edge e of H contained in X we fix some copy $F_e \in \mathscr{H}$ such that $e \in E(F_e)$. Since there are at most $\binom{|X|}{k}$ such edges, the set

$$\mathscr{N}^- = \left\{F_e \in \mathscr{H} : e \in E(H) \text{ and } e \subseteq X\right\}$$

has at most the size n'. In the special case that $|\mathscr{N}^-| \leqslant 1$ we can set $\mathscr{N} = \mathscr{N}^-$ and are done. Otherwise the conclusion of Theorem 1.7 yields a set $\mathscr{X} \subseteq \mathscr{H}$ such that $\mathscr{N} = \mathscr{N}^- \cup \mathscr{X}$ is a forest of copies with the desired property. □

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