PROOF OF THE BANDWIDTH CONJECTURE OF BOLLOBÁS AND KOMLÓS

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Abstract. In this paper we prove the following conjecture by Bollobás and Komlós: *For every $\gamma > 0$ and integers $r \geq 1$ and Δ, there exists $\beta > 0$ with the following property. If G is a sufficiently large graph with n vertices and minimum degree at least $((r-1)/r + \gamma)n$ and H is an r-chromatic graph with n vertices, bandwidth at most βn and maximum degree at most Δ, then G contains a copy of H.*

Contents

1. Introduction and results

One of the fundamental results in extremal graph theory is the theorem by Erdős and Stone [17] which implies that any *fixed* graph H of chromatic number r is forced to appear as a subgraph in any sufficiently large graph G if the average degree of G is at least $(\frac{r-2}{r-1} + \gamma)n$, for an arbitrarily small but positive constant γ. In this paper we prove an analogue of this result for *spanning* subgraphs H which was conjectured by Bollobás and Komlós.

When trying to translate the Erdős–Stone theorem into a setting where the graphs H and G have the same number of vertices, then two changes are obviously necessary. First of all, the average degree condition must be replaced by one involving the minimum degree $\delta(G)$ of G, since we need (to be able to control) every single vertex of G. Also, for some graphs H it is clear that in this regime the lower bound has to be raised at least to $\delta(G) \geq \frac{r-1}{r}n$: simply consider the example where

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G is the complete r-partite graph with partition classes almost, but not exactly, of the same size (thus G has minimum degree almost $\frac{r-1}{r}n$) and let H be the spanning union of vertex disjoint r-cliques.

There are a number of results where a minimum degree of $\frac{r-1}{r}n$ is indeed sufficient to guarantee the existence of a certain spanning subgraph H. A well known example is Dirac's theorem [15]. It asserts that any graph G on n vertices with minimum degree $\delta(G) \geq n/2$ contains a Hamiltonian cycle. Another classical result of that type by Corrádi and Hajnal [11] states that every graph G with n vertices and $\delta(G) \geq 2n/3$ contains $\lfloor n/3 \rfloor$ vertex disjoint triangles. This was generalised by Hajnal and Szemerédi [18], who proved that every graph G with $\delta(G) \geq \frac{r-1}{r}n$ must contain a family of $\lfloor n/r \rfloor$ vertex disjoint cliques, each of size r.

A further extension of this theorem was suggested by Pósa (see, e.g., [16]) and Seymour [32], who conjectured that, at the same threshold $\delta(G) \geq \frac{r-1}{r}n$, such a graph G must in fact contain a copy of the $(r-1)$-st power of a Hamiltonian cycle (where the $(r-1)$-st power of an arbitrary graph is obtained by inserting an edge between every two vertices of distance at most $r-1$ in the original graph). This was proven in 1998 by Komlós, Sárközy, and Szemerédi [23] for sufficiently large n.

Recently, several other results of a similar flavour have been obtained which deal with a variety of spanning subgraphs H, such as, e.g., trees, F-factors, and planar graphs [3, 4, 5, 6, 12, 13, 21, 24, 25, 28, 29, 30, 33]. Thus, in an attempt to move away from results that concern only graphs H with a special, rigid structure, a naïve conjecture could be that $\delta(G) \geq (\frac{r-1}{r} + \gamma)n$ suffices to guarantee that G contains a spanning copy of any r-chromatic graph H of bounded maximum degree. However, the following simple example shows that this fails in general. Let H be a random bipartite graph with bounded maximum degree and partition classes of size $n/2$ each, and let G be the graph formed by two cliques of size $(1/2+\gamma)n$ each, which share exactly $2\gamma n$ vertices. It is then easy to see that G cannot contain a copy of H, since in H every vertex set X of size $(1/2-\gamma)n$ has more than $2\gamma n$ neighbours outside X.

One way to rule out such expansion properties for H is to restrict the *bandwidth* of H. A graph is said to have bandwidth at most b, if there exists a labelling of the vertices by numbers $1, \dots, n$, such that for every edge $\{i, j\}$ of the graph we have $|i-j| \leq b$. Bollobás and Komlós [20, Conjecture 16] conjectured that every r-chromatic graph on n vertices of bounded degree and bandwidth limited by $o(n)$, can be embedded into any graph G on n vertices with $\delta(G) \geq (\frac{r-1}{r} + \gamma)n$. In this paper we give a proof of this conjecture.

Theorem 1. *For all $r, \Delta \in \mathbb{N}$ and $\gamma > 0$, there exist constants $\beta > 0$ and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ the following holds.*

If H is an r-chromatic graph on n vertices with $\Delta(H) \leq \Delta$, and bandwidth at most βn and if G is a graph on n vertices with minimum degree $\delta(G) \geq (\frac{r-1}{r}+\gamma)n$, then G contains a copy of H.

The analogue of Theorem 1 for bipartite H was announced by Abbasi [1] in 1998, and a short proof based on our methods can be found in [19]. In [8], we proved the 3-chromatic case of this theorem.

Obviously, Hamiltonian cycles and their powers have constant bandwidth. In addition, Chung [9] proved that trees with constant maximum degree have bandwidth at most $O(n/\log n)$. Recently this result was extended to planar graphs, and more generally, to any hereditary class of bounded degree graphs with small separators. In fact, it can be shown that a hereditary class of bounded degree graphs has sublinear bandwidth if and only if it does not contain expanders of linear order [7].

These observations indicate that Theorem 1 can be regarded as a common generalisation of some of the results obtained earlier concerning the minimum degree threshold for containing certain spanning subgraphs. For $(r-1)$-st powers of Hamiltonian cycles H this is only true if r divides n, since otherwise $\chi(H) = r+1$. However, Theorem 2 below also includes those cases. Furthermore, note that for some of the earlier results, the additional term γn in the minimum degree condition is not needed (or can be replaced by a smaller term). In the general setting, however, this is not possible: Abbasi [2] showed that if $\gamma \to 0$ and $\Delta \to \infty$ then β must tend to 0 in Theorem 1. However, the bound on β coming from our proof is rather poor, having a tower-type dependence on $1/\gamma$.

Another question is what happens when we allow the maximum degree of H to grow with n. In [25], Komlós, Sárközy, and Szemerédi showed that every graph G with minimum degree at least $(\frac{1}{2}+\gamma)n$ must contain a copy of an arbitrary spanning tree H with maximum degree at most $cn/\log n$, and it may be worth while to try to extend Theorem 1 in a similar direction. The part in our proof which partitions H in such a way that it can be embedded into G does in fact not need an upper bound on $\Delta(H)$, see Lemma 8 below. However for the blow-up lemma, which we then apply to find the required embedding, $\Delta(H)$ needs to be bounded by a constant.

Finally we would like to address the rôle of the chromatic number in Theorem 1 again. In the same way that the Hamiltonian cycle on an odd number of vertices is forced as a spanning subgraph in any graph of minimum degree $\frac{1}{2}n$ (although it is 3- and not 2-chromatic), other $(r+1)$-chromatic graphs are forced already when $\delta(G) \geq (\frac{r-1}{r}+\gamma)n$. It seems that one important question here is whether all $r+1$ colours are needed by *many* vertices. For example, the *critical chromatic number* $\chi_{cr}(H)$ of a graph H is defined as $(\chi(H)-1)|V(H)|/(|V(H)|-\sigma)$, where σ denotes the minimum size of the smallest colour class in a colouring of H with $\chi(H)$ colours. Obviously, $\chi(H)-1 < \chi_{cr}(H) \leq \chi(H)$, with (approximate) equality for σ tending to 0 or $|V(H)|/\chi(H)$, respectively. This concept was introduced by Komlós [21], who proved that a minimum degree condition of $\delta(G) \geq (\chi_{cr}(H)-1)n/\chi_{cr}(H)$ suffices to find a family of disjoint copies of H covering all but εn vertices of G. Kühn and Osthus [27] further investigated this question and managed to determine for every H the corresponding minimum degree condition (up to an additive constant) for the containment of a spanning H-factor.

Our methods allow an extension of Theorem 1 that goes into a somewhat similar direction. Assume that the vertices of H are labelled $1,\dots,n$. For two positive integers x, y, an $(r+1)$-colouring $\sigma : V(H) \to \{0,\dots,r\}$ of H is said to be (x,y)-*zero free* with respect to such a labelling, if for each $t \in [n]$ there exists a t' with $t \leq t' \leq t+x$ such that $\sigma(u) \neq 0$ for all $u \in [t', t'+y]$. We also say that the interval $[t', t'+y]$ is *zero free*.

Theorem 2. *For all $r, \Delta \in \mathbb{N}$ and $\gamma > 0$, there exist constants $\beta > 0$ and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ the following holds.*

Let H be a graph with $\Delta(H) \leq \Delta$ whose vertices are labelled $1,\dots,n$ such that, with respect to this labelling, H has bandwidth at most βn, an $(r+1)$-colouring that is $(8r\beta n, 4r\beta n)$-zero free, and uses colour 0 for at most βn vertices in total.

If G is a graph on n vertices with minimum degree $\delta(G) \geq (\frac{r-1}{r} + \gamma)n$, then G contains a copy of H.

Obviously Theorem 2 implies Theorem 1, and the remaining part of this paper is devoted to the proof of Theorem 2.

2. Main lemmas and outline of the proof

In this section we introduce the central lemmas that are needed for the proof of our main theorem. Our emphasis in this section is to explain how they work together to give the proof of Theorem 2, which itself is then presented in full detail in the subsequent section, Section 3.

2.1. **The blow-up lemma.** One of the main ingredients to our proof is the so-called blow-up lemma [22], which is a powerful tool for embedding spanning graphs. For stating this lemma we first need to introduce the concept of an ε-regular pair. Let $G = (V, E)$ be a graph. For a vertex $v \in V$ we write $d_G(v) := |N_G(V)|$ for the degree of v in G. Let A, $B \subseteq V$ be disjoint vertex sets. We denote the number of edges with one end in A and the other end in B by $e(A, B)$. The ratio $d(A, B) := e(A, B)/(|A||B|)$ is called the *density* of (A, B).

Definition 3. *The pair (A, B) is* ε-regular*, if for all $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \geq \varepsilon|A|$ and $|B'| \geq \varepsilon|B|$ it is true that $|d(A, B) - d(A', B')| < \varepsilon$. An ε-regular pair (A, B) is called (ε, d)-regular, if it has density at least d.*

For a graph $G = (V, E)$ and a graph R_k on the vertex set $[k]$ we say that $V = V_1\dot\cup\cdots\dot\cup V_k$ is (ε, d)-regular on R_k *if (V_i, V_j) is (ε, d)-regular for every $\{i, j\} \in E(R_k)$. In this case R_k is also called a* reduced graph *for G.*

In addition we need the concept of a *super-regular pair*. Roughly speaking a regular pair is super-regular if every vertex has a sufficiently large degree.

Definition 4 (super-regular pair)**.** *Let ε, $d > 0$ and let (A, B) be an (ε, d)-regular pair. We say (A, B) is* (ε, d)-super-regular *if, in addition, every $v \in A$ has at least $d|B|$ neighbours in B and every $v \in B$ has at least $d|A|$ neighbours in A.*

Moreover, for a graph $G = (V, E)$ and a graph R_k on vertex set $[k]$ we say $V = V_1\dot\cup\cdots\dot\cup V_k$ is (ε, d)-super-regular on R_k *if (V_i, V_j) is (ε, d)-super-regular for every $\{i, j\} \in E(R_k)$.*

With this we are ready to state the blow-up lemma of Komlós, Sárközy, and Szemerédi [22] (see also [31] for an alternative proof). The simplest version of this lemma guarantees that bipartite spanning graphs of bounded degree can be embedded into sufficiently super-regular pairs.

Theorem 5 (Blow-up lemma [22])**.** *For every d, Δ, $c > 0$ and $k \in \mathbb{N}$ there exist constants $\varepsilon_{\mathrm{BL}} = \varepsilon_{\mathrm{BL}}(d, \Delta, c, k)$ and $\alpha_{\mathrm{BL}} = \alpha_{\mathrm{BL}}(d, \Delta, c, k)$ such that the following holds.*

Let n_1, n_2, $\ldots$, n_k be arbitrary positive integers, $0 < \varepsilon < \varepsilon_{\mathrm{BL}}$, and $G = (V_1\dot\cup V_2\dot\cup \ldots \dot\cup V_k, E)$ be a k-partite graph with $|V_i| = n_i$ for $i \in [k]$. Moreover, let S be a graph on vertex set $[k]$ such that $V_1\dot\cup \ldots \dot\cup V_k$ is (ε, d)-super-regular on S.

Suppose $H = (W_1\dot\cup W_2\dot\cup \ldots \dot\cup W_k, F)$ is a k-partite graph with $\Delta(H) \leq \Delta$ such that there exists a graph homomorphism $\phi\colon V(H) \to V(S)$ such that $|\phi^{-1}(i)| \leq n_i$ for every $i \in [k]$. Moreover, suppose that in each class W_i there is a set of at most

$\alpha_{\mathrm{BL}} \cdot \min_{j\in[k]} n_j$ special *vertices y, each of which is equipped with a set $C_y \subseteq V_i$ with $|C_y| \geq cn_i$.*

Then there is an embedding of H into G such that every special vertex y is mapped to a vertex in C_y.

We say that the special vertices y in Theorem 5 are *image restricted to C_y*.

2.2. **Preparing G and H for the blow-up lemma.** It remains to introduce the main lemmas that 'prepare' the graphs G and H for an application of the blow-up lemma. Before we can state these lemmas we will need a few more definitions.

Suppose that m and r are integers. Let C^r_m be the mr-vertex graph obtained from a path on m vertices by replacing every vertex by a clique of size r and replacing every edge by a complete bipartite graph minus a perfect matching. More precisely,

$$V(C^r_m) = [m] \times [r] \tag{1}$$

and

$$\{(i,j),(i',j')\} \in E(C^r_m) \qquad \text{iff} \qquad i = i' \quad \text{or} \quad |i-i'| = 1 \ \wedge \ j \neq j'. \tag{2}$$

Let K^r_m be the graph on vertex set $[m]\times[r]$ that is formed by the disjoint union of m complete graphs on r vertices. Then $K^r_m \subseteq C^r_m$ and we call the complete graph on vertices $(i,1),\dots,(i,r)$ the *i-th component of K^r_m* for $i \in [m]$. Note moreover, that $\sigma\colon\ [m]\times[r] \to [r]$ with

$$\sigma(i,j) := j \text{ for } i \in [m] \text{ and } j \in [r]$$

is a valid r-colouring of C^r_m. We will later consider vertex partitions $(V_{i,j})_{i\in[m],j\in[r]}$ that are (ε,d)-regular on C^r_m for some ε and d. Then we will also say, that $V_{i,j}$ has colour j.

For all $n,k,r \in \mathbb{N}$, we call an integer partition $(n_{i,j})_{i\in[k],j\in[r]}$ of n (with $n_{i,j} \in \mathbb{N}$ for all $i \in [k]$ and $j \in [r]$) *r-equitable*, if $|n_{i,j} - n_{i,j'}| \leq 1$ for all $i \in [k]$ and $j,j' \in [r]$.

We can now state (and then explain) our first main lemma which asserts a regular partition of the graph G with structural properties that will be suitable for embedding H into G.

Lemma 6 (Lemma for G)**.** *For all $r \in \mathbb{N}$ and $\gamma > 0$ there exist $d > 0$ and $\varepsilon_0 > 0$ such that for every positive $\varepsilon \leq \varepsilon_0$ there exist K_0 and $\xi_0 > 0$ such that for all $n \geq K_0$ and for every graph G on vertex set $[n]$ with $\delta(G) \geq ((r-1)/r + \gamma)n$ there exist $k \in \mathbb{N}\setminus\{0\}$ and a graph R^r_k on vertex set $[k]\times[r]$ with*

- *(R1)* $k \leq K_0$,
- *(R2)* $\delta(R^r_k) \geq ((r-1)/r + \gamma/2)kr$,
- *(R3)* $K^r_k \subseteq C^r_k \subseteq R^r_k$, *and*
- *(R4) there is an r-equitable integer partition $(m_{i,j})_{i\in[k],j\in[r]}$ of n with $m_{i,j} \geq (1-\varepsilon)n/(kr)$ such that the following holds.*

For every partition $(n_{i,j})_{i\in[k],j\in[r]}$ of n with $m_{i,j} - \xi_0 n \leq n_{i,j} \leq m_{i,j} + \xi_0 n$ there exists a partition $(V_{i,j})_{i\in[k],j\in[r]}$ of V with

- *(V1)* $|V_{i,j}| = n_{i,j}$,
- *(V2)* $(V_{i,j})_{i\in[k],j\in[r]}$ *is (ε,d)-regular on R^r_k, and*
- *(V3)* $(V_{i,j})_{i\in[k],j\in[r]}$ *is (ε,d)-super-regular on K^r_k.*

We give the proof of Lemma 6, which borrows ideas from [25], in Section 6.

For simplicity let us first assume $n_{i,j} = m_{i,j}$. In this case, Lemma 6 would guarantee a partition of the vertex set of G in such a way that the partition classes form many (super-)regular pairs, and that these pairs are organised in a sort of backbone, namely in the form of a C^r_k for the regular pairs, and, contained therein, a spanning family K^r_k of disjoint complete graphs for the super-regular pairs.

However, the lemma says more. When we come to the point (*R4*), the lemma 'has in mind' the partition we just described, but doesn't exhibit it. Instead, it only discloses the sizes $m_{i,j}$ and allows us to wish for small amendments: for every $i \in [k]$ and $j \in [r]$, we can now look at the value $m_{i,j}$ and ask for the size of the corresponding partition class to be adjusted to a new value $n_{i,j}$, differing from $m_{i,j}$ by at most $\xi_0 n$.

When proving Lemma 6, one thus needs to alter the partition by shifting a few vertices. Note that while (ε, d)-regularity is very robust towards such small alterations, (ε, d)-*super*-regularity is not, so this is where the main difficulty lies (cf. Proposition 14).

In [8] we proved the 3-chromatic case of Lemma 6. One central ingredient to the proof was the existence of the square of a Hamiltonian cycle in graphs of high minimum degree as asserted by the affirmative solution of the conjecture of Pósa (see, e.g. [23]) mentioned in the introduction. It was this square of a Hamiltonian cycle that allowed us to perform the alteration just described.

However, it turned out that the $(r-1)$-st power of a Hamiltonian cycle is not well connected enough to carry over these methods to the r-chromatic case. In order to deal with this problem we will first solve the following special case of Theorem 1, which asserts the copy of C^r_m in a graph of high minimum degree. Notice that C^r_m contains the $(r-1)$-st power of a Hamiltonian path on mr vertices and thus we now obtain a graph of a richer structure that will allow us to move vertices between partition classes.

Lemma 7 (backbone lemma). *For all integers $r \geq 1$ and positive constants γ and ε there exists $n_0 = n_0(r, \gamma, \varepsilon)$ such that for every $n \geq n_0$ the following holds. If G is an n-vertex graph with minimum degree $\delta(G) \geq ((r-1)/r + \gamma)n$, then G contains a copy of C^r_m with $rm \geq (1-\varepsilon)n$.*

Now we come to the second main lemma. It prepares the graph H so that it can be embedded into G. This is exactly the place where, given the values $m_{i,j}$, the new values $n_{i,j}$ in the setting described above are specified.

Lemma 8 (Lemma for H). *Let $r, k \geq 1$ be integers and let β, $\xi > 0$ satisfy $\beta \leq \xi^2/(3026r^3)$.*

Let H be a graph on n vertices with $\Delta(H) \leq \Delta$, and assume that H has a labelling of bandwidth at most βn and an $(r+1)$-colouring that is $(8r\beta n, 4r\beta n)$-zero free with respect to this labelling, and uses colour 0 for at most βn vertices in total. Let R^r_k be a graph with $V(R^r_k) = [k] \times [r]$ such that $\delta(R^r_k) > (r-1)k$ and $K^r_k \subseteq C^r_k \subseteq R^r_k$. Furthermore, suppose $(m_{i,j})_{i\in[k],j\in[r]}$ is an r-equitable integer partition of n with $m_{i,j} \geq 35\beta n$ for every $i \in [k]$ and $j \in [r]$.

Then there exists a mapping $f\colon V(H) \to [k] \times [r]$ and a set of special vertices $X \subseteq V(H)$ with the following properties

(*a*) $|X| \leq kr\xi n$,
(*b*) $m_{i,j} - \xi n \leq |W_{i,j}| := |f^{-1}(i,j)| \leq m_{i,j} + \xi n$ *for every* $i \in [k]$ *and* $j \in [r]$,

(*c*) *for every edge* $\{u,v\} \in E(H)$ *we have* $\{f(u), f(v)\} \in E(R^r_k)$*, and*
(*d*) *if* $\{u,v\} \in E(H)$ *and, moreover,* u *and* v *are both in* $V(H) \setminus X$*, then* $\{f(u), f(v)\} \in E(K^r_k)$.

In other words, Lemma 8 receives a graph H as input and, from Lemma 6, a reduced graph R^r_k (with $K^r_k \subseteq C^r_k \subseteq R^r_k$), an r-equitable partition $(m_{i,j})_{i\in[k],j\in[r]}$ of n, and a parameter ξ. Again we emphasise that this is all what Lemma 8 needs to know about G. It then provides us with a function f which maps the vertices of H onto the vertex set $[k] \times [r]$ of R^r_k in such a way that (i,j) with $i \in [k]$, $j \in [r]$ receives $n_{i,j} := |W_{i,j}|$ vertices from H, with $|n_{i,j} - m_{i,j}| \leq \xi n$. Although the vertex partition of G is not known exactly at this point, we already have its reduced graph R^r_k. Lemma 8 guarantees that the endpoints of an edge $\{u,v\}$ of H get mapped into vertices $f(u)$ and $f(v)$ of R^r_k, representing future partition classes $V_{f(u)}$ and $V_{f(v)}$ in G which will form a *super-regular* pair (see (*d*)) – except for those few edges with one or both endpoints in some small special set X. But even these edges will be mapped into pairs of classes in G that will form at least *regular* pairs (see (*c*)). Lemma 8 will then return the values $n_{i,j}$ to Lemma 6, which will finally produce a corresponding partition of the vertices of G.

If we consider the i-th component of K^r_k, then the blow-up lemma, Theorem 5, would immediately give us an embedding of

$$H[W_{i,1},\dots,W_{i,r}] \quad \text{into} \quad G[V_{i,1},\dots,V_{i,r}]$$

that takes care of all edges of $H[V(H) \setminus X]$.

Edges of H with one or both vertices in the special set X will need some special treatment. However, due to part (*a*) of Lemma 8 the size of X is quite small. In particular we will be able to ensure that $|X| \ll n/(kr)$. Our strategy will be first to find an embedding g of the vertices of X into $V(G)$ such that for every $y \in N_H(X) := \{y \in V(H) \setminus X \colon \exists\, xy \in E(H) \text{ with } x \in X\}$ the set $C_y := V_{f(y)} \cap \bigcap_{x\in N_H(y)\cap X} N_G(g(x))$ is sufficiently large. The following partial embedding lemma guarantees the existence of such an embedding g of X. Once we have applied it, we can complete the partial embedding g with the blow-up lemma, which will 'respect' the image restriction to C_y for every $y \in N_H(X)$.

Lemma 9 (Partial embedding lemma)**.** *For every integer* $\Delta \geq 2$ *and every* $d \in (0,1]$ *there exist constants* $c = c(\Delta, d)$ *and* $\varepsilon_{\rm PEL} = \varepsilon_{\rm PEL}(\Delta, d)$ *such that for every positive* $\varepsilon \leq \varepsilon_{\rm PEL}$ *the following is true.*

Let R^r_k *be a graph with* $V(R_k) = [k] \times [r]$ *and* G *be a graph on* n *vertices with* $V(G) = (V_{i,j})_{i\in[k],j\in[r]}$*, such that* $|V_{i,j}| \geq (1-\varepsilon_{\rm PEL})n/(kr)$ *for all* $i \in [k]$, $j \in [r]$ *and* $(V_{i,j})_{i\in[k],j\in[r]}$ *is* (ε, d)*-regular on* R^r_k*. Let, furthermore,* B *be a graph with* $V(B) = X \dot\cup Y$ *and* $f\colon V(B) \to V(R^r_k) = [k] \times [r]$ *be a mapping with* $\{f(b), f(b')\} \in E(R^r_k)$ *for all* $\{b, b'\} \in E(B)$.

If $|V(B)| \leq \varepsilon_{\rm PEL} n/(kr)$ *and* $\Delta(B) \leq \Delta$*, then there exists an injective mapping* $g\colon X \to V(G)$ *with* $g(x) \in V_{f(x)}$ *for all* $x \in X$ *such that for all* $y \in Y$ *there exist sets* $C_y \subseteq V_{f(y)} \setminus g(X)$ *such that*
(*i*) *if* $x, x' \in X$ *and* $\{x, x'\} \in E(B)$ *then* $\{g(x), g(x')\} \in E(G)$,
(*ii*) *for all* $y \in Y$ *we have* $C_y \subseteq N_G(g(x))$ *for all* $x \in N_B(y) \cap X$*, and*
(*iii*) $|C_y| \geq c|V_{f(y)}|$ *for every* $y \in Y$.

Such a lemma, in a slightly different context, was first obtained by Chvátal, Rödl, Szemerédi, and Trotter [10] (see also [14, Lemma 7.5.2]). The only difference between Lemma 9 and their embedding lemma is that we only embed *some* of the vertices of a given graph B into G and reserve sufficiently many places in G for a future embedding of the remaining vertices of B. For a proof of Lemma 9 we refer the reader to [8].

In the next section we give the precise details how Theorem 2 can be deduced from Lemma 6 and Lemma 8 following the outline discussed above.

3. Proof of Theorem 2

In this section we give the proof of Theorem 2 based on Theorem 5, Lemma 9, Lemma 6, and Lemma 8 from Section 2. In particular, we will use Lemma 6 for partitioning G, and Lemma 8 for assigning the vertices of H to the parts of G. For this, it will be necessary to split the application of Lemma 6 into two phases. The first phase is used to set up the parameters for Lemma 8. With this input, Lemma 8 then defines the sizes of the parts of G that are constructed during the execution of the second phase of Lemma 6.

Finally, H is embedded into G by using the blow-up lemma, Theorem 5, on the partition of G and by treating the special vertices $X \subseteq V(H)$ from Lemma 8 with the help of the partial embedding lemma, Lemma 9.

Proof of Theorem 2. Given r, Δ, and γ, let d and ε_0 be as asserted by Lemma 6 for input r and γ. Let $c = c(\Delta, d)$ and $\varepsilon_{\mathrm{PEL}} = \varepsilon_{\mathrm{PEL}}(\Delta, d)$ be as given by Lemma 9, and $\varepsilon_{\mathrm{BL}} = \varepsilon_{\mathrm{BL}}(d, \Delta, c, r)$ and $\alpha_{\mathrm{BL}} = \alpha_{\mathrm{BL}}(d, \Delta, c, r)$ as given by Theorem 5. Set

$$\varepsilon := \min\{\varepsilon_0, \varepsilon_{\mathrm{PEL}}/4, \varepsilon_{\mathrm{BL}}/2, d/4\}\,. \tag{3}$$

Then, Lemma 6 provides constants K_0 and ξ_0 for this ε. We define

$$\xi := \min\{\xi_0, 1/(4K_0), \varepsilon/(2K_0^2 r^2(\Delta+1)), \alpha_{\mathrm{BL}}/(2K_0^2 r^2(\Delta+1))\} \tag{4}$$

as well as $\beta := \min\{\xi^2/(3026r^3), (1-\varepsilon)/(35K_0 r)\}$ and $n_0 := K_0$, and consider arbitrary graphs H and G on $n \geq n_0$ vertices that meet the conditions of Theorem 2.

Applying Lemma 6 to G we get an integer k with $0 < k \leq K_0$, graphs $K_k^r \subseteq C_k^r \subseteq R_k^r$ on vertex set $[k] \times [r]$, and an r-equitable partition $(m_{i,j})_{i\in[k],j\in[r]}$ of n such that (*R1*)–(*R4*) are satisfied. Now all constants that appear in the proof are fixed. To summarize, this is how they are related:

$$\frac{1}{\Delta}, \gamma \gg d \gg \varepsilon \gg \frac{1}{K_0} \gg \xi \gg \beta, \qquad \text{as well as} \qquad c \gg \varepsilon \gg \alpha_{\mathrm{BL}}\,.$$

Before continuing with Lemma 6, we would like to apply the Lemma 8. Note that due to (*R4*) and the choice of β above, we have $m_{i,j} \geq (1-\varepsilon)n/(kr) \geq 35\beta n$ for every $i \in [k]$, $j \in [r]$. Consequently, for constants k, β, and ξ, graphs H and $K_k^r \subseteq C_k^r \subseteq R_k^r$, and the partition $(m_{i,j})_{i\in[k],j\in[r]}$ of n we can indeed apply Lemma 8. This yields a mapping $f\colon V(H) \to [k] \times [r]$ and a set of special vertices $X \subseteq V(H)$. These will be needed later. For the moment we are only interested in the sizes $n_{i,j} := |W_{i,j}| = |f^{-1}(i,j)|$ for $i \in [k]$ and $j \in [r]$. Condition (*b*) of Lemma 8 and the choice of $\xi \leq \xi_0$ in (4) imply that

$$m_{i,j} - \xi_0 n \leq m_{i,j} - \xi n \leq n_{i,j} \leq m_{i,j} + \xi n \leq m_{i,j} + \xi_0 n$$

for every $i \in [k]$, $j \in [r]$. Accordingly, we can continue with Lemma 6 to obtain a partition $V = (V_{i,j})_{i\in[k],j\in[r]}$ with $|V_{i,j}| = n_{i,j}$ that satisfies conditions (*V1*)–(*V3*) of Lemma 6. Note that

$$\begin{aligned} |V_{i,j}| = n_{i,j} \geq m_{i,j} - \xi n &\overset{(R4)}{\geq} (1-\varepsilon)\frac{n}{kr} - \xi n = (1-(\varepsilon+\xi kr))\frac{n}{kr} \\ &\overset{(3),(4)}{\geq} \left(1 - \frac{\varepsilon_{\mathrm{PEL}}}{2}\right)\frac{n}{kr} \geq \frac{1}{2}\frac{n}{kr}. \end{aligned} \tag{5}$$

Now, we have partitions $V(H) = (W_{i,j})_{i\in[k],j\in[r]}$ of H and $V(G) = (V_{i,j})_{i\in[k],j\in[r]}$ of G with $|W_{i,j}| = |V_{i,j}| = n_{i,j}$ for all $i \in [k]$, $j \in [r]$. We will build the embedding of H into G such that each vertex $v \in W_{i,j} \subseteq V(H)$ will be embedded into the corresponding set $V_{i,j} \subseteq V(G)$.

In order to embed the special vertices X of H in G, we use the partial embedding lemma (Lemma 9). We provide Lemma 9 with constants Δ and d, the graph R_k^r, the graph G with vertex partition $V(G) = (V_{i,j})_{i\in[k],j\in[r]}$, the graph $B := H[X\dot\cup Y]$ where $Y := N_H(X)$ consists of the neighbours of vertices of X outside X, and the mapping f restricted to $X\dot\cup Y$. By (*V2*) of Lemma 6 and (*c*) of Lemma 8, G and f fulfil the requirements of Lemma 9. Moreover, since $\Delta(B) \leq \Delta(H) \leq \Delta$

$$|V(B)| = |X| + |Y| \leq (\Delta+1)|X| \leq (\Delta+1)kr\xi n \overset{(4)}{\leq} \varepsilon\frac{n}{kr} \tag{6}$$

by (*a*) of Lemma 8. Accordingly, since $\varepsilon \leq \varepsilon_{\mathrm{PEL}}$ we can apply Lemma 9 to obtain an embedding g of the vertices in X together with sets C_y for every $y \in Y$ such that

$$C_y \subseteq V_{f(y)} \setminus g(X) \qquad \text{and} \qquad |C_y| \geq c|V_{f(y)}| \geq c|V_{f(y)} \setminus g(X)|\,.$$

The sets C_y will be used in the blow-up lemma for the image restriction of the vertices in $Y = N_H(X)$. We first check that there are not too many of these restrictions. Let $W'_{i,j} := W_{i,j} \setminus X$, $V'_{i,j} := V_{i,j} \setminus g(X)$ and $n'_{i,j} := |W'_{i,j}| = |V'_{i,j}|$ for each $i \in [k]$, $j \in [r]$. Observe that for any $i \in [k]$ and $j \in [r]$

$$|X| + |Y| \overset{(6)}{\leq} (\Delta+1)kr\xi n \overset{(4)}{\leq} \frac{\alpha_{\mathrm{BL}}}{2kr}n \overset{(5)}{\leq} \alpha_{\mathrm{BL}}n_{i,j},$$

and hence

$$|N_H(X)| = |Y| \leq \alpha_{\mathrm{BL}}n_{i,j} - |X| \leq \alpha_{\mathrm{BL}}(n_{i,j} - |X|) \leq \alpha_{\mathrm{BL}}n'_{i,j}$$

holds for any $i \in [k]$ and $j \in [r]$. Consequently, we have

$$|N_H(X)| = |Y| \leq \alpha_{\mathrm{BL}} \min_{i\in[k],j\in[r]} n'_{i,j}\,.$$

For any $i \in [k]$ we would like to apply the blow-up lemma, Theorem 5, and find an embedding of $H[W'_{i,1}\dot\cup\cdots\dot\cup W'_{i,r}]$ into $G[V'_{i,1}\dot\cup\cdots\dot\cup V'_{i,r}]$ in such a way that every $y \in N_H(X)$ will be embedded into C_y. It is easy to check that the respective necessary conditions are satisfied. Indeed, recall that by (*V3*) the pair $(V_{i,j}, V_{i,j'})$ is (ε, d)-super-regular for every $i \in [k]$, $j \neq j' \in [r]$ and that, by definition, $V'_{i,j} = V_{i,j} \setminus g(X)$. Hence it follows directly from the definition of a super-regular pair and (5), (6), and $\varepsilon \leq d/4$, that $(V'_{i,j}, V'_{i,j'})$ is $(2\varepsilon, d/2)$-super-regular with $\varepsilon \leq \varepsilon_{\mathrm{BL}}/2$ (see (3)).

Having applied the blow-up lemma for every $i \in [k]$, we have obtained a bijection

$$h\colon\ W'_{1,1}\dot\cup\cdots\dot\cup W'_{k,r} \to V'_{1,1}\dot\cup\cdots\dot\cup V'_{k,r} \text{ with } h(W'_{i,j}) = V'_{i,j} \text{ for every } i \in [k], j \in [r]$$

such that

$$\begin{aligned} h(y) \in C_y \quad \text{for every } y \in N_H(X) \\ \text{and} \quad H[W'_{1,1}\dot\cup\cdots\dot\cup W'_{k,r}] \subseteq G[h(W'_{1,1})\dot\cup\cdots\dot\cup h(W'_{k,r})]. \end{aligned} \tag{7}$$

Now we finish the proof by checking that the united embedding $\bar{h}\colon V(H) \to V(G)$ defined by

$$v \mapsto \bar{h}(v) := \begin{cases} h(v) & \text{if } v \in V(H) \setminus X \\ g(v) & \text{if } v \in X \end{cases}$$

is indeed an embedding of H into G. Let $e = \{u, v\}$ be an edge of H. We distinguish three cases.

If $u, v \in X$, then $\{\bar{h}(u), \bar{h}(v)\} = \{g(u), g(v)\}$, which is an edge in G since g is an embedding of $H[X]$ into G by the partial embedding lemma.

If $u \in X$ and $v \in V(H) \setminus X$, then $v \in N_H(u) \subseteq N_H(X)$, so we have $h(v) \in C_v \subseteq N_G(g(u))$ by (7) and part (*ii*) of Lemma 9, thus $\{\bar{h}(u), \bar{h}(v)\} = \{g(u), h(v)\} \in E(G)$.

If, finally, $u, v \in V(H) \setminus X$, then by part (*d*) of Lemma 8, $\{f(u), f(v)\} \in E(K_k^r)$. In other words, there exists an $i \in [k]$ such that

$$\{u, v\} \quad \text{is contained in} \quad H[W'_{i,1}\dot\cup\cdots\dot\cup W'_{i,r}]$$

and hence $\{\bar{h}(u), \bar{h}(v)\} = \{h(u), h(v)\} \in E(G)$ by (7). □

4. The regularity lemma

The proofs of the backbone lemma and the lemma for G, that will be presented in the subsequent sections, rely on the regularity lemma of Szemerédi [34].

The following is the so-called degree form of Szemerédi's regularity lemma (see, e.g., [26, Theorem 1.10]).

Theorem 10 (Regularity lemma)**.** *For every $\varepsilon > 0$ and every integer k_0 there is a $K_0 = K_0(\varepsilon, k_0)$ such that for every $d \in [0, 1]$ and for every graph G on at least K_0 vertices there exists a partition of $V(G)$ into $V_0, V_1, \ldots, V_k$ and a spanning subgraph G' of G such that the following holds:*

- (*i*) $k_0 \le k \le K_0$,
- (*ii*) $d_{G'}(x) > d_G(x) - (d + \varepsilon)|V(G)|$ *for all vertices* $x \in V(G)$,
- (*iii*) *for all* $i \ge 1$ *the induced subgraph* $G'[V_i]$ *is empty,*
- (*iv*) $|V_0| \le \varepsilon|V(G)|$,
- (*v*) $|V_1| = |V_2| = \cdots = |V_k|$,
- (*vi*) *for all* $1 \le i < j \le k$ *one of the following holds: either* (V_i, V_j) *is* (ε, d)*-regular or* $G'[V_i, V_j]$ *is empty.*

The sets V_i in Theorem 10 are called *clusters* and the set V_0 is the *exceptional set*. Notice that the partition $V_0\dot\cup V_1\dot\cup\cdots\dot\cup V_k$ provided by Theorem 10, induces the following reduced graph R for $V_1\dot\cup\cdots\dot\cup V_k$. The vertex set of R is $[k]$ and R has edges $\{i, j\}$ for $1 \le i, j \le k$ for exactly those pairs (V_i, V_j) that are (ε, d)-regular in G'. Thus, $\{i, j\}$ is an edge of R if and only if G' has an edge between V_i and V_j. We will also use the following simple corollary of Theorem 10 that states that a high minimum degree of G is inherited by the reduced graph R (see, e.g., [30, Proposition 9]).

Corollary 11. *For every $\gamma > 0$ there exist $d > 0$ and $\varepsilon_0 > 0$ such that for every $0 < \varepsilon \le \varepsilon_0$ and every integer k_0 there exists K_0 so that the following holds.*

For every $\nu \ge 0$, an application of Theorem 10 to a graph G of minimum degree at least $(\nu + \gamma)|V(G)|$ yields a partition $V_0, V_1, \dots, V_k$ of $V(G)$ and a subgraph G' of G so that additionally to properties (i)–(vi) the following holds:

(*vii*) *the reduced graph R has minimum degree at least $(\nu + \gamma/2)k$.*

Theorem 10 asserts a regular partition of G. However, for proving Lemma 6 we need to construct a partition that is also super-regular for certain cluster pairs. The following propositions indicate how to find such super-regular pairs in a regular partition. The first proposition implies that every (ε, d)-regular pair (A, B) contains a "large" super-regular sub-pair (A', B').

Proposition 12. *Let (A, B) be an (ε, d)-regular pair and B' be a subset of B of size at least $\varepsilon|B|$. Then there are at most $\varepsilon|A|$ vertices v in A with $|N(v) \cap B'| < (d - \varepsilon)|B'|$.*

Proof. Let $A' = \{v \in A\colon |N(v) \cap B'| < (d - \varepsilon)|B'|\}$ and assume to the contrary that $|A'| > \varepsilon|A|$. But then $d(A', B') < ((d - \varepsilon)|A'||B'|)/(|A'||B'|) = d - \varepsilon$ which is a contradiction since (A, B) is (ε, d)-regular. □

Repeating the last observation a fixed number of times, we obtain the following proposition, which we will later combine with Corollary 11.

Proposition 13. *With the notation of Corollary 11, let S be a subgraph of the reduced graph R with $\Delta(S) \le \Delta$. Then for each vertex i of S, the corresponding set V_i contains a subset V_i' of size $(1 - \varepsilon\Delta)|V_i|$ such that for every edge $\{i, j\} \in E(S)$ the pair (V_i', V_j') is $(\varepsilon/(1 - \varepsilon\Delta), d - \varepsilon\Delta)$-super-regular. Moreover, for every edge $\{i, j\}$ of the original reduced graph R, the pair (V_i', V_j') is still $(\varepsilon/(1 - \varepsilon\Delta), d - \varepsilon\Delta)$-regular.*

For the simple proof of Proposition 13 we refer to [30, Proposition 8]. We close this section with the following useful observation. It states that the notion of regularity is "robust" in view of small alterations of the respective vertex sets. A proof can be found in [8].

Proposition 14. *Let (A, B) be an (ε, d)-regular pair and let $(\hat{A}, \hat{B})$ be a pair such that $|\hat{A} \triangle A| \le \hat{\alpha}|\hat{A}|$ and $|\hat{B} \triangle B| \le \hat{\beta}|\hat{B}|$ for some $0 \le \hat{\alpha}, \hat{\beta} \le 1$. Then, $(\hat{A}, \hat{B})$ is an $(\hat{\varepsilon}, \hat{d})$-regular pair with*

$$\hat{\varepsilon} := \varepsilon + 3\big(\sqrt{\hat{\alpha}} + \sqrt{\hat{\beta}}\big) \quad \text{and} \quad \hat{d} := d - 2(\hat{\alpha} + \hat{\beta})\,.$$

If, moreover, (A, B) is (ε, d)-super-regular and each vertex v in $\hat{A}$ has at least $d|\hat{B}|$ neighbours in $\hat{B}$ and each vertex v in $\hat{B}$ has at least $d|\hat{A}|$ neighbours in $\hat{A}$, then $(\hat{A}, \hat{B})$ is $(\hat{\varepsilon}, \hat{d})$-super-regular with $\hat{\varepsilon}$ and $\hat{d}$ as above.

5. The backbone lemma

In this section we prove Lemma 7. The proof is a simple consequence of the aforementioned result of Komlós, Sárközy, and Szemerédi concerning the Pósa–Seymour conjecture [23]. Recall that we obtain the $(r - 1)$-st power of a graph by inserting edges between every two vertices of distance at most $r - 1$ in the original graph. For convenience we only consider the $(r - 1)$-st power of paths on ℓ vertices where ℓ is divisible by r. In what follows we denote by P_k^{r-1} the $(r-1)$-st power of a path on kr vertices, where

$$V(P_k^{r-1}) = [k] \times [r] \tag{8}$$

and

$$\{(s,t),(s',t')\} \in E(P_k^{r-1}) \qquad \text{iff} \qquad s = s' \quad \text{or} \quad s' = s+1 \ \wedge \ t' < t. \tag{9}$$

Theorem 15 (Komlós, Sárközy, and Szemerédi)**.** *For every $r \geq 2$ there exists k_0 such that every graph R on $kr \geq k_0$ vertices with minimum degree $\delta(R) \geq \frac{(r-1)}{r}kr = (r-1)k$ contains a copy of P_k^{r-1}.*

In fact in [23] the result was obtained for powers of Hamiltonian cycles (instead of paths) with no divisibility condition on the number of vertices of R. However, for our purposes the stated weaker version will suffice.

Note that P_m^{r-1} is a subgraph of the graph C_m^r defined in (1) and (2). On the other hand, there is an "equipartite" homomorphism from C_m^r to the $(r-1)$-st power of a Hamiltonian path.

Proposition 16. *Let $k \geq 1$ and $\ell \geq r \geq 1$. Let $C_{k\ell}^r$ be the graph defined in (1) and (2) and let P_k^{r-1} be the graph defined in (8) and (9). Then there exists a graph homomorphism $\phi\colon V(C_{k\ell}^r) \to V(P_k^{r-1})$ such that*

$$\ell - r < \left|\phi^{-1}\big((s,t)\big)\right| < \ell + r$$

for all $(s,t) \in [k] \times [r] = V(P_k^{r-1})$.

Proof. It is straight-forward to check that the following map

$$\phi\big((i,j)\big) = \left(\left\lceil \frac{\max\{i-j,0\}+1}{\ell} \right\rceil, j\right)$$

is a graph homomorphism from $C_{k\ell}^r$ to P_k^{r-1} with the desired property. □

Now Lemma 7 follows from a joint application of the regularity lemma (in form of Corollary 11), Theorem 15, Proposition 13, Proposition 16, Lemma 9, and the blow-up lemma (Theorem 5). More precisely, we first apply Corollary 11 to the graph G with $\delta(G) \geq ((r-1)/r + \gamma)n$ and infer that the corresponding reduced graph R satisfies $\delta(R) \geq ((r-1)/r)|V(R)|$. Consequently, Theorem 15 implies that $R \supseteq P_k^{r-1}$. Since $\Delta(P_k^{r-1}) \leq 3(r-1)$ we can, due to Proposition 13 applied with $S = P_k^{r-1}$, remove about $\varepsilon|V(G)|$ vertices from G such that edges of P_k^{r-1} correspond to super-regular pairs in the adjusted partition. Finally, due to Proposition 16 we can apply Lemma 9 and Theorem 5 and conclude that G contains an almost spanning copy of C_m^r. Below we give the technical details of this proof.

Proof of Lemma 7. For $r = 1$ the lemma is trivial, as C_m^1 is simply an independent set. Hence, let $r \geq 2$ and γ, $\varepsilon > 0$ be given. We apply Corollary 11 with γ and obtain constants d, $\varepsilon_0 > 0$. We set $d_{\mathrm{PEL}} = d/2$ and $\Delta_{\mathrm{PEL}} = 3(r-1)$ and get $\varepsilon_{\mathrm{PEL}}$ and c from Lemma 9. Then we set $d_{\mathrm{BL}} = d/4$ and $\Delta_{\mathrm{BL}} = r-1$, $k_{\mathrm{BL}} = r$ and $c_{\mathrm{BL}} = c/2$ and get $\varepsilon_{\mathrm{BL}}$ and α_{BL} from Theorem 5. We then set

$$\varepsilon_{\mathrm{RL}} := \frac{\min\{\varepsilon, \varepsilon_0, \varepsilon_{\mathrm{PEL}}, \varepsilon_{\mathrm{BL}}, d\}}{14(r-1)}\,.$$

Moreover, let k_0 be given by Theorem 15 for $r-1$ and set

$$k_1 := \max\{rk_0 + r, 4r/\gamma, 6r/\varepsilon\}\,.$$

Next, we apply Corollary 11 again with $\varepsilon_{\mathrm{RL}}$ and k_1 and obtain K_0. Finally, we let $n_0 = \lceil 6K_0^2 r^2(r+1)/(\varepsilon_{\mathrm{RL}}\alpha_{\mathrm{BL}})\rceil$. After we fixed all constants, we consider the input graph G on $n \ge n_0$ vertices from Lemma 7. We have $\delta(G) \ge ((r-1)/r+\gamma)n$. Consequently, Corollary 11, applied with γ, $\varepsilon_{\mathrm{RL}}$ and k_1 fixed above, yields an integer k', $k_1 \le k' \le K_0$, a partition $V_0\dot\cup V_1\dot\cup\dots\dot\cup V_{k'} = V(G)$ and a reduced graph R with vertex set $[k']$ such that properties (i)–(vi) of Theorem 10 and property (vii) of Corollary 11 hold. Without loss of generality, we may assume that $k' = kr$ for some integer $k \ge k_0$ (from Theorem 15), since otherwise, we simply unite V_0 with up to at most $r-1$ vertex classes V_i $(i > 0)$ and obtain an exceptional set V_0', which obeys

$$|V_0'| \le |V_0| + (r-1)\tfrac{n}{k'} \le \big(\varepsilon_{\mathrm{RL}} + \tfrac{r-1}{k_1}\big)n\,.$$

Note that, since $k' \ge k_1 \ge 4r/\gamma$ the resulting reduced graph R still satisfies

$$\delta(R) \ge ((r-1)/r+\gamma/2)k' - (r-1) \ge ((r-1)/r+\gamma/4)kr.$$

Moreover, as $k = \lfloor k'/r\rfloor \ge \lfloor k_1/r\rfloor \ge k_0$ (where k_0 came from Theorem 15), we infer by Theorem 15 that $P_k^{r-1} \subseteq R$.

We now apply Proposition 13 with $S = P_k^{r-1} \subseteq R$. This way we get an altered partition

$$V_0''\dot\cup V_1'\dot\cup\dots\dot\cup V_{kr}', \quad \text{where } V_0'' = V_0' \cup \dot{\bigcup_{i\in[kr]}} V_i \setminus V_i'\,.$$

Our choice of constants yields

$$\varepsilon_{\mathrm{RL}}/(1-3(r-1)\varepsilon_{\mathrm{RL}}) \le \varepsilon_{\mathrm{PEL}} \quad \text{and} \quad d - 3(r-1)\varepsilon_{\mathrm{RL}} > d/2$$

and hence, in view of Proposition 16 we can apply Lemma 9 (to $G = [V_1', \dots, V_{kr}']$ with $B = C_{k\ell}^r[X \cup N(X)]$ and X containing the $2(r+1)r(k-1)$ vertices with neighbours in two K_r's of the P_k^{r-1} under the homomorphism ϕ) followed by k applications of the blow-up lemma (with $S = K_r$), similar as in the proof of Theorem 2 in Section 3, where

$$\ell = \min_{i\in[kr]} |V_i'| - r\,.$$

Consequently, G contains a copy of C_m^r (with $m = k\ell$) on $rm = rk\ell$ vertices. Moreover, we have

$$\begin{aligned} rm &\ge n - |V_0'| - \sum_{i=1}^{kr}\big(|V_i| - (\min_{j\in[kr]}|V_j'| - r)\big) \\ &\ge n - \big(\varepsilon_{\mathrm{RL}} + \tfrac{r-1}{k_1}\big)n - 3(r-1)\varepsilon_{\mathrm{RL}}n - kr^2 \end{aligned}$$

and recalling that due the choice of the constants we have $\varepsilon_{\mathrm{RL}} < \varepsilon/(6(r-1))$, $k_1 \ge 6r/\varepsilon$, and $n \ge n_0 \ge 6K_0 r/\varepsilon \ge 6k'r/\varepsilon \ge 6kr^2/\varepsilon$ we infer

$$rm \ge (1-\varepsilon)n.$$

□

6. The lemma for G

The main ingredients for the proof of Lemma 6 are Szemerédi's regularity lemma, which provides a reduced graph R for G, and the backbone lemma which guarantees the copy of a C^r_k in R. This subgraph is sparse enough, so that we can transform the corresponding regular pairs into super-regular pairs. On the other hand, its structure is rich enough so that we can use it to develop a strategy for moving vertices between the clusters of R in order to adjust the sizes of these clusters.

We will first consider the special case of Lemma 6 that $n_{i,j} = m_{i,j}$ for all $i \in [k]$, $j \in [r]$. This is captured by the following proposition.

Proposition 17. *For all $r \in \mathbb{N}$ and $\gamma > 0$ there exist $d > 0$ and $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \le \varepsilon_0$ there exists K_0 such that for all $n \ge K_0 r$ and for every graph G on vertex set $[n]$ with $\delta(G) \ge ((r-1)/r + \gamma)n$ there exists $k \in \mathbb{N} \setminus \{0\}$, and a graph R^r_k on vertex set $[k] \times [r]$ with*

- (*R1*) $k \le K_0$,
- (*R2*) $\delta(R^r_k) \ge ((r-1)/r + \gamma/2)kr$,
- (*R3*) $K^r_k \subseteq C^r_k \subseteq R^r_k$, *and*
- (*R4*) *there is an r-equitable partition $(m_{i,j})_{i\in[k],j\in[r]}$ of n with the property $m_{i,j} \ge (1-\varepsilon)n/(kr)$ such that the following holds.*

There is a partition $(U_{i,j})_{i\in[k],j\in[r]}$ of V with

- (*U1*) $|U_{i,j}| = m_{i,j}$,
- (*U2*) $(U_{i,j})_{i\in[k],j\in[r]}$ *is (ε, d)-regular on R^r_k, and*
- (*U3*) $(U_{i,j})_{i\in[k],j\in[r]}$ *is (ε, d)-super-regular on K^r_k.*

Notice that once we have Proposition 17, the only thing that is left to be done when proving Lemma 6 is to show that the sizes of the classes $U_{i,j}$ can be slightly changed from $m_{i,j}$ to $n_{i,j}$ without "destroying" properties (*U2*) and (*U3*).

For the proof of Proposition 17 we proceed in three steps. From the regularity lemma we first obtain a partition $U'_0 \dot\cup U'_1 \dot\cup \cdots \dot\cup U'_{k'}$ of $V(G)$ with reduced graph R such that $K^r_k \subseteq C^r_k \subseteq R$. According to this occurrence, we will then rename the vertices of R from $[k']$ to $[k] \times [r]$ and thus obtain R^r_k. In a similar manner we rename the clusters in the partition. We then use Proposition 13 to get a new partition $U''_0 \dot\cup (U''_{i,j})_{i\in[r],j\in[k]}$ that is super-regular on K^r_k (and still regular on R^r_k). In a last step we distribute the vertices in U''_0 to the clusters $U''_{i,j}$ with $i \in [k]$ and $j \in [r]$, while maintaining the super-regularity. The partition obtained in this way will be the desired r-equitable partition $(U_{i,j})_{i\in[r],j\in[k]}$.

Proof of Proposition 17. We first fix all constants necessary for the proof. For $r = 1$ the Proposition holds trivially. Let $r \ge 2$ and $\gamma > 0$ be given. The regularity lemma in form of Corollary 11 applied with $\gamma' = \gamma$ yields positive constants d' and ε'_0. We fix the promised constants d and ε_0 for Proposition 17 by setting

$$d := \min\left\{\frac{d'}{3}, \frac{\gamma}{4}\right\} \quad \text{and} \quad \varepsilon_0 := \varepsilon'_0\,. \tag{10}$$

Now let some positive $\varepsilon \le \varepsilon_0$ be given, for which Proposition 17 asks us to define K_0. For that let k_0 be sufficiently large so that we can apply the backbone lemma, Lemma 7, with r, $\gamma/4$ and ε' to graphs R on $kr \ge rk_0$ vertices with minimum degree $\delta(R) \ge ((r-1)/r + \gamma/2)kr$. We then define some auxiliary constants ε' and

k'_0 by

$$\begin{aligned} \varepsilon' &:= \min\left\{\frac{\varepsilon^4}{(10(r+2))^4}, \left(\frac{d'}{24}\right)^2, \frac{\gamma^2}{4r^2(r+2)^2}\right\} \\ \text{and} \quad k'_0 &:= \max\left\{\frac{rk_0}{1-\varepsilon'}, \frac{8r}{\gamma}, \frac{2r}{\varepsilon'}, \frac{r(1+2\varepsilon')}{\varepsilon'(1-2\varepsilon')}\right\} + \frac{r}{1-\varepsilon'}\,. \end{aligned} \tag{11}$$

Let K'_0 be given by Corollary 11 applied with γ', ε', and k'_0. We finally set $K_0 := \lceil K'_0/r \rceil$ for Proposition 17. After we have defined K_0, let $G = (V, E)$ be a graph satisfying the assumptions of Proposition 17.

Since $\varepsilon' \le \varepsilon \le \varepsilon_0 = \varepsilon'_0$, by the choice of ε'_0 and d', Corollary 11 applied with input γ', ε', k'_0 and $\nu' := (r-1)/r$ yields a partition $U'_0 \dot\cup U'_1 \dot\cup \cdots \dot\cup U'_{k'} = V$ and a subgraph G' so that properties (i)–(vi) of Theorem 10 and (vii) from Corollary 11 hold. In particular, $k'_0 \le k' \le K'_0$, the set U'_0 is the exceptional set and there is a reduced graph R' on vertex set $[k']$ such that $U'_1 \dot\cup \cdots \dot\cup U'_{k'}$ is (ε', d')-regular on R' and such that $\delta(R') \ge ((r-1)/r + \gamma/2)k'$. Set

$$k := \left\lfloor \frac{(1-\varepsilon')k'}{r} \right\rfloor \ge \frac{(1-\varepsilon')k'}{r} - 1 \overset{(11)}{\ge} \frac{k'}{r(1+2\varepsilon')} \tag{12}$$

and let R be the graph induced by the vertices $[kr]$ in R'. Observe, that $kr \le k' \le K'_0 \le rK_0$. Therefore R satisfies property $(R1)$ of Proposition 17. Moreover, R is a reduced graph for $G[U'_1 \dot\cup \cdots \dot\cup U'_{kr}]$ with

$$|V(R)| = kr \ge (1-\varepsilon')k' - r \ge (1-\varepsilon')k'_0 - r \overset{(11)}{\ge} rk_0 \tag{13}$$

and

$$\delta(R) \ge \delta(R') - r \ge ((r-1)/r + \gamma/2)k' - r \overset{(11)}{\ge} ((r-1)/r + \gamma/4)kr.$$

Thus, we also have property $(R2)$. By (13) and the choice of k_0, Lemma 7 implies that $C^r_k \subseteq R$. According to this occurrence of C^r_k we will now rename the vertex set of R to $[k] \times [r]$ and call the resulting graph R^r_k. We clearly have $K^r_k \subseteq C^r_k \subseteq R^r_k$ and thus we get $(R3)$. In addition, we will also rename the clusters accordingly in order to obtain a vertex partition $\left(U'_{i,j}\right)_{i\in[k],j\in[r]}$. Let L' denote the size of the partition classes $U'_{i,j}$. Note that $|U'_0| \le \varepsilon' n$ and hence

$$(1-\varepsilon')n/k' \le |L'| \le n/k'. \tag{14}$$

Proposition 13 applied with $S := K^r_k$ and accordingly $\Delta(S) = r-1$ implies that for every $i \in [k]$, $j \in [r]$ there are subsets $U''_{i,j}$ of $U'_{i,j}$ of size

$$L'' := (1-(r-1)\varepsilon')L'\,,$$

such that $\left(U''_{i,j}\right)_{i\in[k],j\in[r]}$ is $(\varepsilon'/(1-(r-1)\varepsilon'), d'-(r-1)\varepsilon')$-regular on R^r_k, and $(\varepsilon'/(1-(r-1)\varepsilon'), d'-(r-1)\varepsilon')$-super-regular on K^r_k.

By (11) we have $\varepsilon'/(1-(r-1)\varepsilon') \le 2\varepsilon'$ and $d' - 2\varepsilon' \ge d'/2$. This implies that $\left(U''_{i,j}\right)_{i\in[k],j\in[r]}$ is $(2\varepsilon', d'/2)$-regular on R^r_k, and $(2\varepsilon', d'/2)$-super-regular on K^r_k. Moreover,

$$\begin{aligned} \frac{n}{kr} \ge L'' = (1-(r-1)\varepsilon')L' &\overset{(14)}{\ge} (1-(r-1)\varepsilon')(1-\varepsilon')\frac{n}{k'} \\ &\overset{(11)}{\ge} \frac{1-r\varepsilon'}{1+2\varepsilon'}\frac{n}{kr} \ge (1-(r+2)\varepsilon')\frac{n}{kr}. \end{aligned} \tag{15}$$

Now we collect all vertices from V not contained in $\big(U''_{i,j}\big)_{i\in[k],j\in[r]}$ in a set U''_0, i.e., let

$$U''_0 := V \setminus \bigcup_{(i,j)\in[k]\times[r]} U''_{i,j}.$$

It follows that

$$|U''_0| = n - \sum_{(i,j)\in[k]\times[r]} |U''_{i,j}| \overset{(15)}{\leq} n - (1-(r+2)\varepsilon')n = (r+2)\varepsilon' n. \tag{16}$$

In order to obtain the required partition of V with clusters $U_{i,j}$ for $i \in [k]$, $j \in [r]$ we will distribute the vertices in U''_0 to the clusters $U''_{i,j}$ so that the resulting partition is r-equitable and still (ε, d)-regular on R^r_k and (ε, d)-super-regular on K^r_k.

For this purpose, let u be a vertex in U''_0. The i-th component of K^r_k is called *u-friendly*, if u has at least $dn/(kr)$ neighbours in each of the clusters $U''_{i,j}$ with $j \in [r]$. We claim that each $u \in U''_0$ has at least γk u-friendly components. Indeed, assume for a contradiction that there were only $x < \gamma k$ u-friendly components for some u. Then, since u has less than $(r-1)L'' + dn/(kr)$ neighbours in clusters of components that are not u-friendly, we can argue that

$$\begin{aligned}
|N_G(u)| &< xrL'' + (k-x)\left((r-1)L'' + \frac{dn}{kr}\right) + |U''_0| \\
&= k(r-1)L'' + xL'' + (k-x)\frac{d}{kr}n + |U''_0| \\
&\overset{(15)}{<} k(r-1)\frac{n}{kr} + \gamma\frac{n}{r} + \frac{d}{r}n + (r+1)\varepsilon' n \\
&\overset{(10),(11)}{\leq} \frac{r-1}{r}n + \frac{\gamma}{r}n + \frac{\gamma}{2r}n + \frac{\gamma}{4}n \overset{r\geq 2}{\leq} \left(\frac{r-1}{r} + \gamma\right)n,
\end{aligned}$$

which is a contradiction.

In a first step we now assign the vertices $u \in U''_0$ as evenly as possible to u-friendly components of K^r_k. Since each vertex $u \in U''_0$ has at least γk u-friendly components, each component of K^r_k gets assigned at most $|U''_0|/(\gamma k)$ vertices.

Then in the second step, in each component we distribute the vertices that have been assigned to this component as evenly as possible among the r clusters of this component. It follows immediately that the resulting partition is r-equitable. Moreover, every cluster $U''_{i,j}$ with $i \in [k]$, $j \in [r]$ gains at most

$$\frac{|U''_0|}{\gamma k} \overset{(16)}{\leq} \frac{(r+2)\varepsilon' n}{\gamma k} \overset{(15)}{\leq} \frac{(r+2)\varepsilon' r}{\gamma(1-(r+2)\varepsilon')}L'' \overset{(11)}{\leq} 2r(r+2)\frac{\varepsilon'}{\gamma}|U''_{i,j}| \overset{(11)}{\leq} \sqrt{\varepsilon'}|U''_{i,j}| \tag{17}$$

vertices from U''_0 during this process. The resulting partition $(U_{i,j})_{i\in[k],j\in[r]}$ of V satisfies properties (*U1*)–(*U3*). Indeed, define

$$m_{i,j} := |U_{i,j}| \geq |U''_{i,j}| = L'' \overset{(15)}{\geq} (1-(r+2)\varepsilon')n/(kr) \geq (1-\varepsilon)n/(kr),$$

and note that for this choice (*R4*) and (*U1*) of Proposition 17 hold. Moreover, recall that $\big(U''_{i,j}\big)_{i\in[k],j\in[r]}$ is $(2\varepsilon', d'/2)$-regular on R^r_k and $(2\varepsilon', d'/2)$-super-regular on K^r_k. By (17), Proposition 14 with $\hat{\alpha} = \hat{\beta} = \sqrt{\varepsilon'}$ assures that $(U_{i,j})_{i\in[k],j\in[r]}$ is $(\hat{\varepsilon}, \hat{d})$-regular on R^r_k and $(\hat{\varepsilon}, \hat{d})$-super-regular on K^r_k, where

$$\hat{\varepsilon} := 2\varepsilon' + 6\sqrt[4]{\varepsilon'} \quad \text{and} \quad \hat{d} := \frac{d'}{2} - 4\sqrt{\varepsilon'}.$$

Since $2\varepsilon' + 6\sqrt[4]{\varepsilon'} \le \varepsilon$ and $d'/2 - 4\sqrt{\varepsilon'} \ge d'/3 \ge d$ by (10) and (11), this implies (*U2*) and (*U3*) and concludes the proof of Proposition 17. □

It remains to show how to deduce the lemma for G (Lemma 6) from Proposition 17. As mentioned earlier, we need to show that the sizes of the clusters can be slightly changed. In order to achieve this, we will develop a technique for adapting the cluster sizes step by step by moving one vertex at a time from one cluster to another cluster until each cluster has exactly the right number of vertices.

The problem that occurs here is the following. Although a pair remains almost as *regular* as before when a few vertices leave or enter a cluster, the property of being *super-regular* is not that robust: *every* vertex that is moved to a new cluster which is part of a super-regular component of K_k^r must make sure that it has sufficiently many neighbours inside the neighbouring clusters within the component.

For this, we will exploit the high minimum degree of R_k^r as well as the structure of C_k^r. The following two facts will allow us to move vertices between different clusters. The first observation will be useful to address imbalances within clusters of one colour class of C_k^r.

Fact 18. *Suppose that $(V_{i,j})_{i\in[k],j\in[r]}$ is a vertex partition that is (ε, d)-regular on C_k^r and satisfies $|V_{i,j}| \ge (1-\varepsilon)n/(kr)$ for all $i \in [k]$ and $j \in [r]$. Now, fix $i \in [k]$ and $j \in [r]$. Then, there are at least $(1-r\varepsilon)n/(kr)$ "good" vertices $v \in V_{i,j}$ that have at least $(d-2\varepsilon)n/(kr)$ neighbours in each set $V_{i',j'}$ with $i' \in \{i-1, i+1\}$ and $j' \in [r] \setminus \{j\}$.*

Proof. Note that (i,j) is connected to each of the (i',j') in C_k^r. Since the partition $(V_{i,j})_{i\in[k],j\in[r]}$ is (ε, d)-regular on C_k^r we can apply Proposition 12 with input ε, d, $A = V_{i,j}$, and $B = B' = V_{i',j'}$ for each $j' \in [r] \setminus \{j\}$. It follows that at least $|V_{i,j}| - (r-1)\varepsilon|V_{i,j}|$ vertices of $V_{i,j}$ have more than $(d-\varepsilon)|V_{i',j'}|$ neighbours in each $V_{i',j'}$. This implies the assertion of Fact 18, because

$$|V_{i,j}| - (r-1)\varepsilon|V_{i,j}| \ge (1-(r-1)\varepsilon)(1-\varepsilon)\frac{n}{kr} \ge (1-r\varepsilon)\frac{n}{kr} \tag{18}$$

and

$$(d-\varepsilon)|V_{i',j'}| \ge (d-\varepsilon)(1-\varepsilon)\frac{n}{kr} \ge (d-2\varepsilon)\frac{n}{kr}$$

□

Before we move on, let us quickly illustrate how Fact 18 is used in the proof of Lemma 6. For this purpose assume further that $(V_{i,j})_{i\in[k],j\in[r]}$ is also (ε, d)-super-regular on K_k^r. Now suppose that for some $i < i' \in [k]$ and $j \in [r]$ we would like to decrease the size of $V_{i,j}$ and increase the size of $V_{i',j}$. Then by Fact 18 there is some vertex v (in fact $(1-r\varepsilon)n/(kr)$ vertices) in $V_{i,j}$ which has "many" neighbours in each $V_{i+1,j'}$ with $j' \in [r] \setminus \{j\}$. Hence, we can move v from $V_{i,j}$ to $V_{i+1,j}$ without loosing the super-regularity on K_k^r. Repeating this process by moving a vertex from $V_{i+1,j}$ to $V_{i+2,j}$ and so on, we will eventually reach $V_{i',j}$ (see Figure 1). Observe that it is of course not necessarily the vertex $v \in V_{i,j}$ we started with, which is really moved all the way to $V_{i',j}$ during this process, but rather a sequence of vertices each moving one cluster further. The crucial thing to note is that whenever we move a vertex from one cluster to another, it still has many neighbours in the new neighbouring clusters within K_k^r. Therefore, after such a sequence of applications of Fact 18, we end up with a new partition $(V_{i,j})_{i\in[k],j\in[r]}$

with the following properties. The cardinality of $V_{i,j}$ decreased by one and $|V_{i',j}|$ increased by one. All other clusters do not change their size. Therefore such a sequence of moves, decreases the imbalances within clusters of colour j in C_k^r and we say that we *moved a vertex along colour class j of C_k^r from $V_{i,j}$ to $V_{i',j}$.*

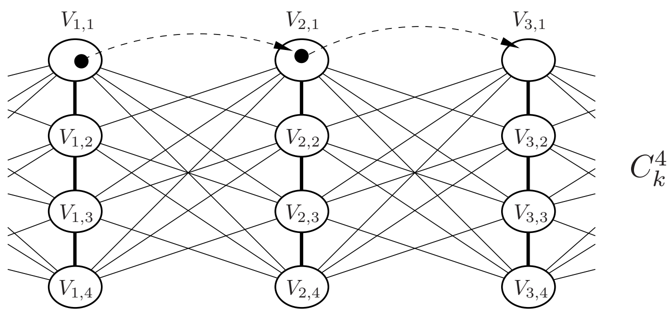

FIGURE 1. Moving a vertex from $V_{1,1}$ to $V_{3,1}$ along colour class 1 of C_k^4 and thus decreasing the size of $V_{1,1}$ and increasing the size of $V_{3,1}$.

The next simple fact allows to address imbalances across different colours. More precisely, it will be used for moving a vertex v from cluster $V_{i,j}$ to a cluster $V_{i^*,j'}$ with $j \neq j'$.

Fact 19. *Let R_k^r be a graph on vertex set $[k] \times [r]$ with $\delta(R_k^r) > (r-1)k$, and suppose that $(V_{i,j})_{i\in[k],j\in[r]}$ is a vertex partition that is (ε, d)-regular on R_k^r and satisfies $|V_{i,j}| \geq (1-\varepsilon)n/(kr)$ for all $i \in [k]$ and $j \in [r]$. Now, fix $i \in [k]$ and $j \in [r]$. Then there is an $i^* \in [k]$ such that for each $j' \in [r]$ the vertex (i^*, j') is a neighbour of (i,j) in R_k^r. Moreover, there are at least $(1-(r+1)\varepsilon)n/(kr)$ "good" vertices $v \in V_{i,j}$ that have at least $(d-2\varepsilon)n/(kr)$ neighbours in $V_{i^*,j'}$.*

Proof of Fact 19. Since $\delta(R_k^r) > (r-1)k$, there must be at least one component in K_k^r, say the i^*-th component, such that all r vertices of this component are adjacent to (i,j) in R_k^r. The existence of the vertices v follows similarly as in the proof of Fact 18. Indeed, by Proposition 12, there are at least

$$|V_{i,j}| - r\varepsilon|V_{i,j}| \geq (1-r\varepsilon)(1-\varepsilon)n/(kr) \geq (1-(r+1)\varepsilon)n/(kr)$$

such vertices (cf. (18)). □

The idea of the technique for adapting the cluster sizes now is as follows. We pick one cluster A that has too many vertices compared to the desired partition and one cluster B that has too few vertices at a time. If A and B have the same colour then we can move a vertex along C_k^r from A to B by repeatedly applying Fact 18. If A and B are of different colour on the other hand we can use Fact 19 in order to find a cluster C that has the same colour as B such that we can move a vertex from A to C. Then we can proceed as before and move a vertex along C_k^r from C to B. We repeat this process until every cluster has exactly the right size.

Proof of Lemma 6. We first fix the constants involved in the proof. Let r and $\gamma > 0$ be given by Lemma 6. For r and γ, Proposition 17 yields constants $d' > 0$ and

$\varepsilon_0' > 0$. For Lemma 6 we set

$$\varepsilon_0 := \min\{\varepsilon_0', d'/8\} \quad \text{and} \quad d := d'/2\,. \tag{19}$$

For given $\varepsilon \le \varepsilon_0$, we fix

$$\varepsilon' := \min\left\{\frac{1}{2(r+1)}, \frac{\varepsilon}{1+6\sqrt{2}}, \sqrt{\frac{d}{8}}\right\} \tag{20}$$

and note that $0 < \varepsilon' \le \varepsilon \le \varepsilon_0 \le \varepsilon_0'$. Therefore we can apply Proposition 17 with r, γ, and ε' to obtain K_0'. Finally, we define the constants K_0 and ξ_0 promised by Lemma 6 and set

$$K_0 := K_0' \qquad \text{and} \qquad \xi_0 := \left(\frac{\varepsilon'}{\sqrt{2}K_0 r}\right)^2\,. \tag{21}$$

Having fixed all the constants, let $G = (V, E)$ be a graph on $n \ge K_0$ vertices with $\delta(G) \ge ((r-1)/r + \gamma)n$. We now apply Proposition 17 with r, γ, and ε' to the input graph G and get a positive integer $k \le K_0'$, a graph R_k^r, and a partition $(U_{i,j})_{i\in[k],j\in[r]}$ of V so that (*R1*)–(*R4*) and (*U1*)–(*U3*) of Proposition 17 hold with ε replaced by ε' and d replaced by d'. Since $K_0 = K_0'$ and $\varepsilon \ge \varepsilon'$, this shows that k, R_k^r, and $m_{i,j} = |U_{i,j}|$ for all $i \in [k]$, $j \in [r]$ also satisfy properties (*R1*)–(*R4*) of Lemma 6.

It remains to prove the 'second part' of Lemma 6. For that let $(n_{i,j})_{i\in[k],j\in[r]}$ be an integer partition of $n = |V|$ satisfying $n_{i,j} = m_{i,j} \pm \xi_0 n$ for every $i \in [k]$, $j \in [r]$. Our goal is to modify the partition $(U_{i,j})_{i\in[k],j\in[r]}$ to obtain a partition $(V_{i,j})_{i\in[k],j\in[r]}$ of V that satisfies (*V1*)–(*V3*) for ε and d.

We initially set $V_{i,j} := U_{i,j}$ for all $i \in [k]$, $j \in [r]$. In the following, we will perform several steps to move vertices out of some clusters and into some other clusters. For this purpose we will use Facts 18 and 19. During this balancing process we will call a cluster $V_{i,j}$ *deficient*, if $|V_{i,j}| < n_{i,j}$, and *excessive*, if $|V_{i,j}| > n_{i,j}$. In the end we will neither have deficient clusters nor excessive clusters and thus obtain the desired partition.

As indicated earlier, one iteration of the balancing process is as follows. Choose an arbitrary excessive cluster $V_{i,j}$ and a deficient cluster $V_{i',j'}$. Note that there are deficient clusters as long as there are excessive clusters by definition, and vice versa. We distinguish two cases. If $j = j'$ we use Fact 18 for moving a vertex along colour class j of C_k^r from cluster $V_{i,j}$ to cluster $V_{i',j'}$. (We will argue below why the hypothesis of Fact 18 is satisfied.) Otherwise, we first apply Fact 19 to cluster $V_{i,j}$, which gives us an $i^* \in [k]$, so that we can move a vertex from cluster $V_{i,j}$ to $V_{i^*,j'}$. Then, we can proceed as in the previous case and move a vertex along colour class j' of C_k^r from cluster $V_{i^*,j'}$ to $V_{i',j'}$ with Fact 18.

In total at most

$$\sum_{i=1}^{k}\sum_{j=1}^{r} |n_{i,j} - m_{i,j}| \le kr\xi_0 n$$

iterations have to be performed in order to guarantee that $|V_{i,j}| = n_{i,j}$ for all $i \in [k]$ and $j \in [r]$. Moreover, in each iteration not more than one vertex gets moved out of each $V_{i,j}$ with $i \in [k]$, $j \in [r]$, and at most one vertex gets moved into each $V_{i,j}$. So, throughout the process we have

$$|U_{i,j} \triangle V_{i,j}| \le 2 \cdot kr\xi_0 n \overset{(21)}{\le} (\varepsilon')^2 \frac{n}{kr}, \tag{22}$$

for all $i \in [k]$, $j \in [r]$.

Note that since by (20) we have $(1-(r+1)\varepsilon')n/(kr) \geq (\varepsilon')^2 n/(kr)$, in every step the "moving" vertex v can be chosen from the set of $(1-(r+1)\varepsilon')n/(kr)$ "good" vertices guaranteed by Facts 18 and 19.

In addition it follows that

$$|V_{i,j}| \geq |U_{i,j}| - |U_{i,j} \triangle V_{i,j}| \overset{(R4),(22)}{\geq} \big(1-\varepsilon'-(\varepsilon')^2\big)\frac{n}{kr} \overset{(20)}{\geq} (1-\varepsilon)\frac{n}{kr} \qquad (23)$$

after the balancing process for all $i \in [k]$, $j \in [r]$. Recall that $(U_{i,j})_{i\in[k],j\in[r]}$ is (ε', d')-regular on R_k^r and (ε', d')-super-regular on K_k^r. Therefore, we can apply Proposition 14 with input ε', d', $A := U_{i,j}$, $\hat{A} := V_{i,j}$, and $B := U_{i',j'}$, $\hat{B} := V_{i',j'}$ for any neighbouring vertices (i,j) and (i',j') in R_k^r. For this, we set

$$\hat{\alpha} := \hat{\beta} := 2(\varepsilon')^2 \geq \frac{(\varepsilon')^2}{1-\varepsilon} \overset{(23),(22)}{\geq} \frac{|U_{p,q} \triangle V_{p,q}|}{|V_{p,q}|} \quad \text{for all } p \in [k] \text{ and } q \in [r]. \qquad (24)$$

Since

$$\hat{\varepsilon} = \varepsilon' + 3\big(\sqrt{\hat{\alpha}} + \sqrt{\hat{\beta}}\big) \overset{(24)}{=} \varepsilon' + 6\sqrt{2}\varepsilon' \overset{(20)}{\leq} \varepsilon$$

and

$$\hat{d} = d' - 2(\hat{\alpha} + \hat{\beta}) \overset{(24)}{=} d' - 8(\varepsilon')^2 \overset{(19),(20)}{\geq} d.$$

we deduce from Proposition 14 that $(V_{i,j})_{i\in[k],j\in[r]}$ remains (ε, d)-regular on R_k^r and, since we only moved "good" vertices, $(V_{i,j})_{i\in[k],j\in[r]}$ remains (ε, d)-super-regular on K_k^r throughout the entire process.

This, together with (23) and the assertions of Proposition 17, also justifies that the hypotheses of Facts 18 and 19 are satisfied and we could therefore indeed apply these facts throughout the entire balancing process. Therefore $(V_{i,j})_{i\in[k],j\in[r]}$ satisfies (*V1*)–(*V3*) and this concludes the proof of Lemma 6. □

7. The lemma for H

In order to prove the lemma for H (Lemma 8), we need to exhibit a mapping $f\colon V(H) \to [k] \times [r]$ with properties (*a*)–(*d*). Basically, we would like to use the fact that H is almost r-colourable, visit the vertices of H in the order of the bandwidth labelling and arrange that f maps the first vertices of colour 1 to $(1,1)$, the first vertices of colour 2 to $(1,2)$, the first vertices of colour 3 to $(1,3)$, and so on. Ignoring the vertices of colour 0, it would be ideal if in this way, at more or less the same moment, we would have dealt with $m_{1,1}$ vertices of colour 1, $m_{1,2}$ vertices of colour 2 and so on, since we could then move on and let f assign vertices to the next component of $K_k^r \subseteq C_k^r$.

However, the problem is that although the $m_{i,j}$ are r-equitable, i.e., almost identical, the colour classes of H may vary a lot in size. Therefore, the basic idea of our proof of Lemma 8 will be to find a recolouring of H with more or less balanced colour classes (besides colour 0).

We emphasise that everything in this section is completely elementary (i.e. it does not use any advanced machinery from the regularity method) but at times a bit technically cumbersome. Therefore we split it into a series of simple propositions.

Proposition 20. *Let $c_1, \dots, c_r \in \mathbb{R}$ be such that $c_1 \leq c_2 \leq \dots \leq c_r \leq c_1 + x$ and $c'_1, \dots, c'_r \in \mathbb{R}$ such that $c'_r \leq c'_{r-1} \leq \dots \leq c'_1 \leq c'_r + x$. If we set $c''_i := c_i + c'_i$ for* *all $i \in [r]$ then*

$$\max_i\{c''_i\} \leq \min_i\{c''_i\} + x.$$

Proof. It clearly suffices to show that $c_i + c'_i \leq c_j + c'_j + x$ for all $i, j \in [r]$. For $i \leq j$ this follows from $c_i \leq c_j$ and $c'_i \leq c'_r + x \leq c'_j + x$. Similarly, for $i > j$ we have that $c_i \leq c_1 + x \leq c_j + x$ and $c'_i \leq c'_j$. □

Now assume that the vertices of H are labelled $1, \dots, n$. Recall that for an $(r+1)$-colouring $\sigma : V(H) \to \{0, \dots, r\}$ of H an interval $[s, t] \subseteq [n]$ is called zero free, if $\sigma(u) \neq 0$ for all $u \in [s, t]$. Moreover, the colouring σ is called (x, y)-zero free on the interval $[a, b] \subseteq [n]$, if for each $t \in [n]$ there exists an interval $[t', t'+y] \subseteq [t, t+x+y]$ such that $[t', t'+y] \cap [a, b]$ is zero free.

The following proposition investigates under what conditions a colouring remains (x, y)-zero free when a few more vertices receive colour 0.

Proposition 21. *Assume that the vertices of H are labelled $1, \dots, n$. Let y be a positive integer, $a \in [n]$ and suppose that $\sigma : V(H) \to \{0, \dots, r\}$ is an $(r+1)$-colouring that is $(8y, y)$-zero free on $[n]$ as well as $(2y, y)$-zero free on $[a, n]$ with respect to this labelling.*

Let $a + 3y \leq b \leq a + 5y$ and suppose that σ' is another $(r+1)$-colouring that differs from σ in that some of the vertices in the interval $(b, b+y)$ now have colour 0, i.e., $(\sigma')^{-1}(0) \subseteq \sigma^{-1}(0) \cup (b, b+y)$.

Then σ' must still be $(8y, y)$-zero free on $[n]$ and $(2y, y)$-zero free on $[a+6y, n]$.

Proof. By definition $[b, b+y] \subseteq [a+3y, a+6y]$ and thus

$$\text{(i)} \quad \sigma'\big|_{[1,a+3y]} \equiv \sigma\big|_{[1,a+3y]} \qquad \text{and} \qquad \text{(ii)} \quad \sigma'\big|_{[b+y,n]} \equiv \sigma\big|_{[b+y,n]} . \tag{25}$$

First note that the second claim of the proposition is trivial, because $b+y \leq a+6y$ and part (ii) of (25) show that the fact that σ is $(2y, y)$-zero free on $[a, n]$ implies that σ' is $(2y, y)$-zero free on $[a+6y, n]$.

As for the first claim, we need to show that for every $t \in [n]$ there exists an interval $[t', t'+y] \subseteq [t, t+9y]$ which is zero-free under σ'. Here we need to distinguish several cases.

- $t < a - 6y$: By part (i) of (25) the assertion follows from the fact that σ is $(8y, y)$-zero free on $[n]$.
- $a - 6y \leq t < a$: The fact that σ is $(2y, y)$-zero free on $[a, n]$ implies (when applied to the vertex a) that there is a zero free interval $[t', t'+y] \subseteq [a, a+3y] \subseteq [t, t+9y]$ under σ. By part (i) of (25), $[t', t'+y]$ is also zero free under σ'.
- $a \leq t < b+y$: The fact that σ is $(2y, y)$-zero free on $[a, n]$ implies (when applied to the vertex $b+y$) that there is a zero free interval $[t', t'+y] \subseteq [b+y, b+4y] \subseteq [t, a+9y] \subseteq [t, t+9y]$ under σ. By part (ii) of (25), $[t', t'+y]$ is also zero free under σ'.
- $b + y \leq t$: Here the assertion follows because part (ii) of (25) shows that the fact that σ is $(8y, y)$-zero free on $[n]$ implies that σ' is $(8y, y)$-zero free on $[b+y, n]$.

□

Now we introduce the notion of *switching* two colours $l, l' \in [r]$ at some given vertex s, which will be essential to transform the given colouring of H into one that uses the colours $1, \dots, r$ in a more or less balanced manner. Basically, all vertices of colour l after s are coloured by l' and vice versa. In order to avoid adjacent vertices of the same colour, we use the bandwidth condition and colour vertices in the interval $s - \beta n, s + \beta n$ that previously had colour l with colour 0.

Proposition 22. *Assume that the vertices of H are labelled $1, \dots, n$ with bandwidth at most βn with respect to this labelling. Let $s \in [n]$ and suppose further that $\sigma\colon [n] \to \{0, \dots, r\}$ is a proper $(r+1)$-colouring of $V(H)$ such that $[s-2\beta n, s+2\beta n]$ is zero free.*

Then for any two colours $l, l' \in [r]$ the mapping $\sigma'\colon [n] \to \{0, \dots, r\}$ defined by

$$\sigma'(v) := \begin{cases} l & \text{if } \sigma(v) = l', s < v \\ l' & \text{if } \sigma(v) = l, s + \beta n < v \\ 0 & \text{if } \sigma(v) = l, s - \beta n \le v \le s + \beta n \\ \sigma(v) & \text{otherwise} \end{cases}$$

is a proper $(r+1)$*-colouring of H. (We will say that σ' is obtained from σ by an* $(l, l', \beta n)$-switch at vertex s.*)*

Note that we only introduced new vertices of colour 0 in the interval $[s - \beta n, s + \beta n]$ and that all these vertices are non-adjacent since they have colour l in σ.

Proof. Indeed, as σ' is derived from the proper colouring σ by interchanging the colours l and l' after the vertex s and introducing some new vertices of colour 0 in $[s - \beta n, s + \beta n]$, the only monochromatic edges that σ' could possibly yield are edges $\{u, v\}$ with either $u \le s$ and $s < v$ and $\{\sigma(u), \sigma(v)\} = \{l, l'\}$ or with $\sigma'(u) = \sigma'(v) = 0$. The second case is clearly ruled out by the facts that H has bandwidth at most βn, that $[s-2\beta n, s+2\beta n]$ is zero free under σ and that there are no edges between new vertices of colour 0. For the first case, since H has bandwidth at most βn, we must have that $u \in [s-\beta n, s]$ and $v \in [s+1, s+\beta n]$. But if $\sigma(u) = l$ and $\sigma(v) = l'$, then $\sigma'(u) = 0$ and $\sigma'(v) = l$. If $\sigma(u) = l'$ and $\sigma(v) = l$ on the other hand, then $\sigma'(u) = l'$ and $\sigma'(v) = 0$. Hence, σ' is a proper $(r+1)$-colouring. □

The next and final proposition is based on repeated applications of the three preceding ones and sums up what we have achieved so far. For that we need one more definition: For $x \in \mathbb{N}$, a colouring $\sigma\colon [n] \to \{0, \dots, r\}$ is called *x-balanced*, if for each interval $[a, b] \subseteq [n]$ and each $l \in [r]$, we have

$$\frac{b-a}{r} - x \le \left|\sigma^{-1}(l) \cap [a, b]\right| \le \frac{b-a}{r} + x.$$

Proposition 23. *Assume that the vertices of H are labelled $1, \dots, n$ with bandwidth at most βn and that H has an $(r+1)$-colouring that is $(8r\beta n, 4r\beta n)$-zero free with respect to this labelling, which uses at most βn vertices of colour 0 in total. Let ξ be a constant with $\beta < \xi^2/(48r)$ and assume that $1/\xi$ is an integer. Then there exists a proper $(r+1)$-colouring $\sigma\colon V(H) \to \{0, \dots, r\}$ that is $(32r\beta n, 4r\beta n)$-zero free and $5\xi n$-balanced.*

The idea of the proof is as follows. We cut H into pieces of length ξn and proceed by induction. Suppose that we have found a colouring that is zero free and balanced on the first p pieces. Then permute the colours on the remaining pieces such that the *largest* colour class of the union of pieces 1 to p has the same colour as the *smallest* colour class of the $(p+1)$-st piece, and vice versa (again, ignoring colour 0). Now glueing the colourings together (as in Proposition 22), the new colouring will be roughly as balanced on the first $p+1$ pieces (see Proposition 20) and as zero free (see Proposition 21) as the old one.

Proof. Suppose that H, β and ξ are given with the required properties. In the first part of the proof, we will prove the following statement by induction (on p): for all integers $p \in [1/\xi]$ there exists a proper $(r+1)$-colouring $\sigma_p\colon [n] \to \{0,\dots,r\}$ of the vertices of H with the following properties:

$$\sigma_p \text{ is } (32r\beta n, 4r\beta n)\text{-zero free on } [n], \tag{26}$$

$$\sigma_p \text{ is } (8r\beta n, 4r\beta n)\text{-zero free on } [p\xi n, n], \tag{27}$$

and for all $j \in [p]$

$$\max_{l\in[r]} \left\{|\sigma_p^{-1}(l) \cap [j\xi n]|\right\} \le \min_{l\in[r]} \left\{|\sigma_p^{-1}(l) \cap [j\xi n]|\right\} + \xi n + 24rj\beta n. \tag{28}$$

For $p = 1$, we let σ_1 be the original $(8r\beta n, 4r\beta n)$-zero free $(r+1)$-colouring of H. Hence (26), (27), and (28) hold trivially.

Next suppose that σ_p is given. For $i \in [r-1]$, we will fix colours l_i, l'_i and positions s_i and then obtain σ_{p+1} from σ_p by a series of $r-1$ appropriate $(l_i, l'_i, \beta n)$-switches at positions s_i. For this purpose recall that by induction (27) guarantees that σ_p is $(8r\beta n, 4r\beta n)$-zero free on the interval $[p\xi n, n]$. When applied to the vertex $t := p\xi n + 12r\beta n$, there exists a vertex $t' \in [p\xi n + 12r\beta n, p\xi n + 20r\beta n]$ such that $[t', t'+4r\beta n]$ is zero free. Now choose positions $s_1, \dots, s_{r-1}$ by letting $s_1 := t'+4\beta n$ and $s_i := s_{i-1} + 4\beta n$ for all $2 \le i \le r-1$. Thus

$$\begin{aligned} p\xi n + 12r\beta n \quad &\le \quad t' < s_1 - 2\beta n \le s_1 \le \dots \le s_{r-1} \le s_{r-1} + 2\beta n \\ &= \quad t' + 4(r-1)\beta n + 2\beta n \;<\; t' + 4r\beta n. \end{aligned} \tag{29}$$

Now let c_i be the number of vertices in $[p\xi n]$ with colour i under σ_p for $i \in [r]$ and suppose w.l.o.g. that $c_1 \le \dots \le c_r$. For some (not yet specified) colours l_i, l'_i we will obtain σ_{p+1} from σ_p by consecutive $(l_i, l'_i, \beta n)$-switches at s_i for all $i \in [r-1]$ and denote by c'_i the number of vertices in the interval

$$I := [t' + 4r\beta n, (p+1)\xi n].$$

which have colour i under σ_{p+1} for $i \in [r]$. Observe that by (29), all switches occur before the interval I, so since every permutation of the set $[r]$ can be written as the composition of at most $r-1$ transpositions, it is clear that we can choose the colours $l_1, l'_1, \dots, l_{r-1}, l'_{r-1} \in [r]$ such that $c'_r \le \dots \le c'_1$.

Again by (29) we have $[s_1 - 2\beta n, s_{r-1} + 2\beta n] \subseteq [t', t'+4r\beta n]$, which, by the choice of t', is zero free under σ_p at the beginning of the switches. Moreover, the switch at s_{i-1} introduces new vertices of colour 0 only in the interval $[s_{i-1} - \beta n, s_{i-1} + \beta n]$ which (by definition of the s_i) is disjoint from $[s_i - 2\beta n, s_{r-1} + 2\beta n]$.

Thus we can be sure that before we apply the switch at s_i, the interval $[s_i - 2\beta n, s_i + 2\beta n]$ is zero free. Hence we can apply Proposition 22 for each of the $r-1$ switches and obtain that σ_{p+1} is again a proper $(r+1)$-colouring of H.

It is now easy to check that σ_{p+1} satisfies the requirements (26), (27), and (28), with p replaced by $p+1$. Indeed, properties (26) and (27) follow by evoking Proposition 21 with $y := 4r\beta n$, $a := p\xi n$, and $b := t' \in [p\xi n + 12r\beta n, p\xi n + 20r\beta n]$. To prove (28), observe that as $\sigma_{p+1}(v) = \sigma_p(v)$ for all $v \le p\xi n$, we know by induction that (28) with σ_{p+1} in the place of σ_p still holds for all $j \le p$. Moreover, we have $|\sigma_{p+1}^{-1}(i) \cap [p\xi n]| = c_i$. Using that $c_1 \leq \cdots \leq c_r$ together with, again, (28) from the induction for $j = p$, we now have

$$\begin{aligned} |\sigma_{p+1}^{-1}(1) \cap [p\xi n]| &\leq |\sigma_{p+1}^{-1}(2) \cap [p\xi n]| \leq \cdots \leq |\sigma_{p+1}^{-1}(r) \cap [p\xi n]| \\ &\leq |\sigma_{p+1}^{-1}(1) \cap [p\xi n]| + \xi n + 24rp\beta n. \end{aligned}$$

On the other hand, we have $|\sigma_{p+1}^{-1}(i) \cap I| = c'_i$. Using that $c'_r \leq \cdots \leq c'_1$ and $|I| \leq \xi n \leq \xi n + 24rp\beta n$, we obtain

$$\begin{aligned} |\sigma_{p+1}^{-1}(r) \cap I| &\leq |\sigma_{p+1}^{-1}(r-1) \cap I| \leq \cdots \leq |\sigma_{p+1}^{-1}(1) \cap I| \\ &\leq |\sigma_{p+1}^{-1}(r) \cap I| + \xi n + 24rp\beta n. \end{aligned}$$

we can now apply Proposition 20 with $x := \xi n + 24rp\beta n$ to see that

$$\begin{aligned} &\max_{l \in [r]} \left\{ |\sigma_{p+1}^{-1}(l) \cap [(p+1)\xi n]| \right\} \\ &\quad \leq \min_{l \in [r]} \left\{ |\sigma_{p+1}^{-1}(l) \cap [(p+1)\xi n]| \right\} + \xi n + 24rp\beta n + \underbrace{\big|[p\xi n, (p+1)\xi n] \setminus I\big|}_{\leq t' + 4r\beta n - p\xi n \leq 24r\beta n}, \\ &\quad \leq \min_{l \in [r]} \left\{ |\sigma_{p+1}^{-1}(l) \cap [(p+1)\xi n]| \right\} + \xi n + 24r(p+1)\beta n \end{aligned}$$

which implies equation (28) for $j = p+1$ as well. This completes the inductive proof of statements (26), (27), and (28). Recall moreover that the switch at s_i introduces new vertices of colour 0 only in the interval $[s_i - \beta n, s_i + \beta n]$ for all $i \in [r-1]$. Therefore each of these switches introduces at most $2\beta n$ new vertices of colour 0. Since σ_1 has at most βn vertices of colour 0 it follows that σ_j colours at most $j(r-1)2\beta n + \beta n \leq 2rj\beta n$ vertices with 0.

For the second part of the proof, set $p := 1/\xi$ and consider the $(r+1)$-colouring $\sigma := \sigma_p$ whose existence we have proven in the first part. Recall that by (26) and (28) we know that

$$\sigma \text{ is } (32r\beta n, 4r\beta n)\text{-zero free on } [n] \tag{30}$$

and for all integers $1 \leq j \leq 1/\xi$

$$\max_{l \in [r]} \left\{ |\sigma^{-1}(l) \cap [j\xi n]| \right\} \leq \min_{l \in [r]} \left\{ |\sigma^{-1}(l) \cap [j\xi n]| \right\} + \xi n + 24rj\beta n. \tag{31}$$

It remains to prove that σ is $5\xi n$-balanced. Let i_+ and i_- be the colours in $[r]$ that are used most and least often in the interval $[j\xi n]$ by σ, respectively; and denote by c_{i_+} and c_{i_-} the number of vertices of colour i_+ and i_- in $[j\xi n]$, respectively. Set $\Lambda := \xi n + 24rj\beta n$ and rewrite property (31) as $c_{i_+} \leq c_{i_-} + \Lambda$. Thus, since σ uses at most $2rp\beta n$ vertices of colour 0 on $[j\xi n]$, we obtain that for all $l \in [r]$

$$\frac{j\xi n - 2rp\beta n}{r} - \Lambda \leq c_{i_+} - \Lambda \leq c_{i_-} \leq |\sigma^{-1}(l) \cap [j\xi n]| \leq c_{i_+} \leq c_{i_-} + \Lambda \leq \frac{j\xi n}{r} + \Lambda.$$

Since $\beta < \xi^2/(48r)$, we infer that for every $j \in [1/\xi]$

$$\frac{j\xi n}{r} - 2\xi n < |\sigma^{-1}(l) \cap [j\xi n]| < \frac{j\xi n}{r} + 2\xi n\,. \tag{32}$$

Now for an arbitrary interval $[a,b] \subseteq [n]$, we choose j, $j' \in [p]$ such that

$$a - \xi n \leq j\xi n \leq a \leq b \leq j'\xi n \leq b + \xi n\,.$$

This yields that

$$|\sigma^{-1}(l) \cap [(j+1)\xi n, (j'-1)\xi n]| \leq \ \big|\sigma^{-1}(l) \cap [a,b]\big| \ \leq |\sigma^{-1}(l) \cap [j\xi n, j'\xi n]|.$$

The lower bound is equal to

$$\begin{aligned}|\sigma^{-1}(l) \cap [(j'-1)\xi n]| &- |\sigma^{-1}(l) \cap [(j+1)\xi n]| \\ &\overset{(32)}{\geq} \left(\frac{(j'-1)\xi n}{r} - 2\xi n\right) - \left(\frac{(j+1)\xi n}{r} + 2\xi n + 1\right) \\ &\geq \left(\frac{b-\xi n}{r} - 2\xi n\right) - \left(\frac{a+\xi n}{r} + 2\xi n\right) - 1 \geq \frac{b-a}{r} - 5\xi n.\end{aligned}$$

Similarly, the upper bound equals

$$\begin{aligned}|\sigma^{-1}(l) \cap [j'\xi n]| &- |\sigma^{-1}(l) \cap [j\xi n)| \\ &\overset{(32)}{\leq} \left(\frac{j'\xi n}{r} + 2\xi n\right) - \left(\frac{j\xi n}{r} - 2\xi n - 1\right) \\ &\leq \left(\frac{b+\xi n}{r} + 2\xi n\right) - \left(\frac{a-\xi n}{r} - 2\xi n\right) + 1 \leq \frac{b-a}{r} + 5\xi n.\end{aligned}$$

Thus, σ is $5\xi n$-balanced, which completes the proof of Proposition 23. □

After these preparations, the proof of the lemma for H (Lemma 8) will be straightforward and the basic idea can be described as follows. We will take the $(r+1)$-colouring σ of H which is guaranteed by Proposition 23. Next we partition $V(H) = [n]$ into k intervals, where the i-th interval will have length roughly $m_{i,1} + \cdots + m_{i,r}$. In order to prove the lemma, define $f : V(H) \to V(R^r_k) = [k] \times [r]$ in such a way that it maps all vertices in the i-th interval with colour $j \neq 0$ to (i, j), i.e. the j-th vertex of the i-th component of K^r_k. Obviously the bandwidth condition implies that two adjacent vertices u, v will either lie in the same or in neighbouring intervals. If, for example, two adjacent vertices u, v both lie in the i-th interval, then $f(u)$ and $f(v)$ are connected by an edge in $E(K^r_k)$, as required by (d) in the lemma. If, on the other hand, u and v lie in neighbouring intervals, then $f(u)$ and $f(v)$ are vertices of different colours in neighbouring components of K^r_k, and as such connected by an edge of $E(C^r_k) \subseteq E(R^r_k)$ as needed by (c); and for this case we will need to define the set X to make sure that (d) will not be required here. Finally, a little more care is needed for the vertices that receive colour 0 by σ.

Proof of Lemma 8. Given r, k and β, let ξ, R^r_k and H be as required. Assume w.l.o.g. that the vertices of H are labelled $1, \dots, n$ with bandwidth at most βn and that H has an $(8r\beta n, 4r\beta n)$-zero free $(r+1)$-colouring with respect to this labelling. Set $\xi' = \xi/(11r)$, and note that $\beta \leq \xi^2/(3026r^3) < (\xi')^2/(48r)$. Therefore, by Proposition 23 with input β, ξ', and H, there is a $(32r\beta n, 4r\beta n)$-zero free and $5\xi' n$-balanced colouring $\sigma\colon V(H) \to \{0, \dots, r\}$ of H.

Observe that for each set of r vertices in R^r_k, the common neighbourhood of these vertices is nonempty, because $\delta(R^r_k) > (r-1)k$. It follows that for each $i \in [k]$ there exists a vertex $r_i \in V(R^r_k) = [k] \times [r]$ that is adjacent in R^r_k to each vertex of the i-th component of K^r_k:

$$\{r_i, (i,j)\} \in E(R^r_k) \qquad \forall j \in [r]. \tag{33}$$

The vertices r_i will be needed to construct the mapping f.

Given an r-equitable partition $(m_{i,j})_{i\in[k], j\in[r]}$ of n, set

$$M_i := \sum_{j\in[r]} m_{i,j}$$

for $i \in [k]$. Now let $t_0 := 0$ and $t_k := n$, and for every $i = 1, \dots, k-1$ choose a vertex

$$t_i \in \left[\sum_{i'=1}^{i} M_{i'}, \sum_{i'=1}^{i} M_{i'} + 33r\beta n \right]$$

such that σ is zero free on $[t_i - \beta n, t_i + \beta n]$. Indeed, such a t_i exists since σ is $(32r\beta n, 4r\beta n)$-zero free. We say that $(t_{i-1}, t_i]$ is the i-th interval of H. Vertices $v \in V(H)$ with $v \in [t_i - \beta n, t_i + \beta n]$ for some $i \in [k]$ are called *boundary vertices* of H. Observe that the choice of the t_i implies that boundary vertices are never assigned colour 0 by σ.

Using σ, we will now construct $f : V(H) \to [k] \times [r]$ and $X \subseteq V(H)$. For each $i \in [k]$, and each $v \in (t_{i-1}, t_i]$ in the i-th interval of H we set

$$f(v) := \begin{cases} r_i & \text{if } \sigma(v) = 0, \\ (i, \sigma(v)) & \text{otherwise,} \end{cases}$$

and

$$X := \big\{ v \in V(H) : \sigma(v) = 0 \big\} \cup \big\{ v \in V(H) : v \text{ is a boundary vertex} \big\}.$$

It remains to show that f and X satisfy properties (*a*)–(*d*) of Lemma 8.

Since σ is $5\xi' n$-balanced, $(n/r) - 5\xi' n \le |\sigma^{-1}(l)|$ for all $l \in [r]$. Consequently

$$|\{v \in [n] : \sigma(v) = 0\}| \le r \cdot 5\xi' n. \tag{34}$$

Moreover, there are exactly $k \cdot 2\beta n$ boundary vertices and so we can bound

$$|X| \le 5r\xi' n + 2k\beta n \le 6kr\xi' n \le kr\xi n,$$

which yields (*a*).

For (*b*), we need to estimate $|W_{i,j}|$, the number of vertices in H that are mapped by f to (i, j) for each $i \in [k]$ and $j \in [r]$. First, the number of vertices of colour 0 that are mapped to (i, j) can obviously be bounded from above by the bound in (34). Furthermore, the mapping f sends all vertices v in the i-th interval of H with $\sigma(v) = j \ne 0$ to (i, j), which are at most $(t_i - t_{i-1})/r + 5\xi' n$ vertices, because σ is $5\xi' n$-balanced. Thus, by the choice of t_{i-1} and t_i, and making use of the fact that $|m_{i,j} - M_i/r| \le 1$ (because the $m_{i,j}$ are known to be r-equitable), we can bound

$$|W_{i,j}| \le \frac{t_i - t_{i-1}}{r} + 5\xi' n + 5r\xi' n \le \frac{M_i + 33r\beta n}{r} + 10r\xi' n \le m_{i,j} + 11r\xi' n = m_{i,j} + \xi n.$$

Similarly, $|W_{i,j}| \ge m_{i,j} - \xi n$ and this shows (*b*).

Now, we turn to (*c*) and (*d*). For a vertex $u \in V(H)$, let $i(u)$ be the index in $[k]$ for which $u \in (t_{i(u)-1}, t_{i(u)}]$. Let $\{u, v\}$ be an edge of H. Since σ is a proper colouring, this implies that $\sigma(u) \ne \sigma(v)$.

We will first consider the case that u and v are in the same interval of H and not of colour 0, i.e. $i := i(u) = i(v)$ and $\sigma(u) \ne 0 \ne \sigma(v)$. By the definition of f, we have $f(u) = (i, \sigma(u))$ and $f(v) = (i, \sigma(v))$ and hence $\{f(u), f(v)\} \in E(K_k^r)$, which proves (*c*) and (*d*) for this case.

Next we consider the case where u and v are in the same interval $i = i(u) = i(v)$ of H and one of them, say u, has colour 0. Here, by definition of X, we do not need to worry about (*d*) and only need to verify (*c*). Indeed, $f(u) = r_i$ and $f(v) = (i, \sigma(v))$. Hence, by (33), $\{f(u), f(v)\} \in E(R_k^r)$.

It remains to consider the case where u and v are in different intervals of H. Then both of them are boundary vertices, because the bandwidth of H is at most βn, so again by definition of X, we only need to verify (*c*). Moreover, $\sigma(u) \neq 0 \neq \sigma(v)$ because by the choice of the t_i boundary vertices are never coloured with 0. Assume w.l.o.g. that $u < v$. It follows that $i(v) = i(u) + 1$ and so $f(u) = (i(u), \sigma(u))$ and $f(v) = (i(u) + 1, \sigma(v))$. This implies that $\{f(u), f(v)\} \in E(C_k^r) \subseteq E(R_k^r)$, which yields (*c*). □

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