SPARSE PARTITION UNIVERSAL GRAPHS FOR GRAPHS OF BOUNDED DEGREE

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Dedicated to the memory of Professor Richard H. Schelp

ABSTRACT. In 1983, Chvátal, Trotter and the two senior authors proved that for any Δ there exists a constant B such that, for any n, any 2-colouring of the edges of the complete graph K_N with $N \geq Bn$ vertices yields a monochromatic copy of any graph H that has n vertices and maximum degree Δ. We prove that the complete graph may be replaced by a sparser graph G that has N vertices and $O(N^{2-1/\Delta} \log^{1/\Delta} N)$ edges, with $N = \lceil B'n \rceil$ for some constant B' that depends only on Δ. Consequently, the so called *size-Ramsey number* of any H with n vertices and maximum degree Δ is $O(n^{2-1/\Delta} \log^{1/\Delta} n)$. Our approach is based on random graphs; in fact, we show that the classical Erdős–Rényi random graph with the numerical parameters above satisfies a stronger partition property with high probability, namely, that any 2-colouring of its edges contains a monochromatic *universal graph* for the class of graphs on n vertices and maximum degree Δ.

The main tool in our proof is the regularity method, adapted to a suitable sparse setting. The novel ingredient developed here is an embedding strategy that allows one to embed bounded degree graphs of linear order in certain pseudorandom graphs. Crucial to our proof is the fact that regularity is typically inherited at a scale that is much finer than the scale at which it is assumed.

1. INTRODUCTION AND RESULTS

The regularity method has proved to be a powerful tool in asymptotic combinatorics. Regular decompositions of graphs and hypergraphs reveal much of the structure of such objects, and have been fundamental in approaching diverse problems in the area (see [26, 29]). The regularity method for *dense graphs* is the best developed direction in this line of research, with a long history of applications and such surprising tools as the blow-up lemma [27, 28]. Thanks to recent advances [18, 30, 34], one is now able to apply the regularity method to *hypergraphs*; for instance, one may now give a fully combinatorial proof of theorems such as the Furstenberg–Katznelson theorem [15] on the existence of homothetic copies of finite configurations in dense subsets of the integer lattice, generalizing [35] to arbitrary dimensions (see, e.g., [31]). The regularity method for *sparse graphs* is,

however, still under development: it appears that even the embedding lemma for graphs of constant size has not been proved in its full generality or strength (see, e.g., [17, 23, 25]). In this paper, we contribute to the development of the regularity method for sparse graphs, providing an embedding strategy for large graphs of bounded degree in the sparse setting. As an application, we prove a numerical result in Ramsey theory: we prove an upper bound for a variant of the Ramsey number for graphs of bounded degree (for numbers in Ramsey theory, see [19]).

For graphs G and H, write $G \longrightarrow H$ if every 2-colouring of the edges of G contains a monochromatic copy of H. Erdős, Faudree, Rousseau and Schelp [13] considered the question of how few edges G may have if $G \longrightarrow H$. Following [13] we denote the *size-Ramsey number* $\widehat{r}(H) = \min\{e(G)\colon G \longrightarrow H\}$, where $e(G)$ denotes the cardinality of the edge set of G.

For example $\widehat{r}(K_{1,n}) = 2n - 1$ for the star $K_{1,n}$ on $n + 1$ vertices. In [6] Beck disproved a conjecture of Erdős [12] and showed that $\widehat{r}(P_n) \leq 900n$. More generally, it follows from the result of Friedman and Pippenger [14] that the size-Ramsey number of bounded degree trees grows linearly with the size of the tree (for further results in this direction, see [7, 20]). Moreover, it was proved in [21] that cycles also have linear size-Ramsey numbers. Beck asked in [7] if $\widehat{r}(H)$ is always linear in the number of vertices of H for graphs H of bounded degree. This was disproved by Rödl and Szemerédi [33], who proved that there is a constant $c > 0$ such that there are graphs H of order n with maximum degree three for which $\widehat{r}(H) \geq n \log^c n$. These authors also conjectured that, for every $\Delta \geq 3$, there exists $\varepsilon = \varepsilon(\Delta) > 0$ such that

$$n^{1+\varepsilon} \leq \widehat{r}_{\Delta,n} := \max\{\widehat{r}(H)\colon H \in \mathcal{H}_{\Delta,n}\} \leq n^{2-\varepsilon}, \tag{1}$$

where $\mathcal{H}_{\Delta,n}$ is the class of all n-vertex graphs with maximum degree at most Δ, up to isomorphism. In this paper, we prove the upper bound conjectured in (1).

In fact, our proof method yields a stronger result. Let us say that a graph is $\mathcal{H}_{\Delta,n}$-*universal* if it contains every member of $\mathcal{H}_{\Delta,n}$ as a subgraph. Furthermore, let us say that a graph is *partition universal for the class of graphs* $\mathcal{H}_{\Delta,n}$ if any 2-colouring of its edges contains a monochromatic $\mathcal{H}_{\Delta,n}$-universal graph. We shall establish for every Δ the existence of a graph G with $O(n^{2-1/\Delta} \log^{1/\Delta} n)$ edges that is partition universal for $\mathcal{H}_{\Delta,n}$.

Theorem 1. *For every $\Delta \geq 2$ there exist constants B and C such that for every n and N satisfying $N \geq Bn$ there exists a graph G on N vertices and at most $CN^{2-1/\Delta} \log^{1/\Delta} N$ edges that is is partition universal for $\mathcal{H}_{\Delta,n}$. In particular, $G \longrightarrow H$ for every $H \in \mathcal{H}_{\Delta,n}$.*

Remark 2.   (i) As observed in [2], one can show that the number of edges in any $\mathcal{H}_{\Delta,n}$-universal graph is $\Omega(n^{2-2/\Delta})$ and, hence, the exponent $2 - 1/\Delta$ of N in Theorem 1 cannot be reduced to $2 - 2/\Delta - \varepsilon$ for any given $\varepsilon > 0$. For completeness, let us quickly see how to obtain this lower bound on the number of edges M in an $\mathcal{H}_{\Delta,n}$-universal graph G. Let us suppose first that $n\Delta$ is even. Note that we must have

$$\binom{M}{n\Delta/2} \geq \frac{1}{n!}L_{\Delta,n}, \tag{2}$$

where $L_{\Delta,n}$ denotes the number of labeled graphs on n vertices that are Δ-regular. Bender and Canfield [8] showed that, for any fixed Δ, as $n \to \infty$

with $n\Delta$ even, we have

$$L_{\Delta,n} = (1 + o(1))\sqrt{2}e^{-(\Delta^2-1)/4}\left(\frac{\Delta^{\Delta/2}}{e^{\Delta/2}\Delta!}\right)^n n^{n\Delta/2}.$$

Therefore, for $n\Delta$ even, $L_{\Delta,n} = \Omega(c^n n^{n\Delta/2})$ for a constant $c = c(\Delta)$. Combining this with (2), we see that $(2eM/n\Delta)^{n\Delta/2} \geq \binom{M}{n\Delta/2} \geq L_{\Delta,n}/n! = \Omega(c^n n^{n\Delta/2}/n^n)$, whence $M = \Omega(n^{2-2/\Delta})$, as required. If $n\Delta$ is odd, simply observe that an $\mathcal{H}_{\Delta,n}$-universal graph is also $\mathcal{H}_{\Delta,n-1}$-universal.

   We mention that a recent, remarkable result of Alon and Capalbo [4] confirms the existence of $\mathcal{H}_{\Delta,n}$-universal graphs with $O(n^{2-2/\Delta})$ edges (see also [2, 1, 3].

(ii) A weaker version of Theorem 1, with $|E(G)| = N^{2-1/2\Delta+o(1)}$, was proved earlier by Kohayakawa, Rödl, and Szemerédi (unpublished).

Let $G(N, p)$ be the standard random graph on N vertices, with all the edges present with probability p, independently of one another (see [9, 22] for the theory of random graphs). To prove Theorem 1, we shall show that $G(N, p)$ with an appropriate choice of $p = p(N)$ has the required properties with high probability.

Theorem 3. *For every $\Delta \geq 2$ there exist constants B and C for which the following holds. Let $N = \lceil Bn \rceil$ and $p = p(N) = C(\log N/N)^{1/\Delta}$. Then*

$$\lim_{n\to\infty} \mathbb{P}\left(G(N, p) \text{ is partition universal for } \mathcal{H}_{\Delta,n}\right) = 1. \tag{3}$$

Remark.   (i) In Theorem 1, we have restricted ourselves to the 2-colour case for simplicity. One may easily prove the same result for more than two colours (the constants B and C would then depend on both Δ and on the number of colours). Similarly, Theorem 3 holds as stated for any fixed number of colours, that is, we may generalize the notion of partition universality to any fixed number of colours r and prove the same result (the constant C would then depend on both Δ and r).

(ii) Theorem 1 follows from Theorem 3. In the remainder of this paper, we focus our attention on the proof of Theorem 3.

The main tool in our proof of Theorem 3 is the regularity method, adapted to the appropriate sparse and random setting. The key novel ingredient in our approach is an embedding strategy that allows one to embed bounded degree graphs of linear order in suitably pseudorandom graphs (see the proof of Lemma 19). Crucial in the proof is a rather surprising phenomenon, namely, the fact that regularity is typically inherited at a scale that is much finer than the scale at which it is assumed. This phenomenon was first spelt out in full in [24], but we use an improved version proved in [16].

Organization. This paper is organized as follows. In Section 2 we recall some basic facts about regularity, including the results on inheritance of regularity proved in [16] (see Section 2.1). In Section 3.3, the results on the hereditary nature of regularity, in the form that is required here, are derived from the results quoted in Section 2.1. Other relevant results on random graphs are given in Sections 3.1 and 3.2. The proof of Theorem 3 is given in Section 4. We conclude with some remarks and open problems in Section 5.

2. The sparse regularity lemma

Let $G = (V, E)$ be a graph. Suppose $0 < p \leq 1$, $\eta > 0$ and $K > 1$. For two disjoint subsets X, Y of V, we let $e_G(X, Y)$ be the number of edges of G with one endpoint in X and the other endpoint in Y. Furthermore, we let

$$d_{G,p}(X, Y) = \frac{e_G(X, Y)}{p|X||Y|},$$

which we refer to as the *p-density* of the pair (X, Y). We say that G is an (η, K)-*bounded* graph with respect to density p if for all pairwise disjoint sets X, $Y \subseteq V$ with $|X|$, $|Y| \geq \eta|V|$, we have

$$e_G(X, Y) \leq Kp|X||Y|.$$

For $\varepsilon > 0$ fixed and X, $Y \subseteq V$, $X \cap Y = \emptyset$, we say that the pair (X, Y) is (ε, p)-*regular* if for all $X' \subseteq X$ and $Y' \subseteq Y$ with

$$|X'| \geq \varepsilon|X| \quad \text{and} \quad |Y'| \geq \varepsilon|Y|,$$

we have

$$|d_{G,p}(X, Y) - d_{G,p}(X', Y')| \leq \varepsilon.$$

Note that for $p = 1$ we get the well-known definition of ε-regularity [36].

Let $\dot{\bigcup}_{j=0}^t V_j$ be a partition of V. We call V_0 the *exceptional class*. This partition is called (ε, t)-*equitable* if $|V_0| \leq \varepsilon|V|$ and $|V_1| = \cdots = |V_t|$.

We say that an (ε, t)-equitable partition $\dot{\bigcup}_{j=0}^t V_j$ of V is (ε, p)-*regular* if all but at most $\varepsilon\binom{t}{2}$ pairs (V_i, V_j), $1 \leq i < j \leq k$, are (ε, p)-regular. Now we state a variant of the Szemerédi's regularity lemma [36] for sparse graphs, which was observed independently by Kohayakawa and Rödl (see, e.g., [23, 25]).

Theorem 4 (Sparse regularity lemma). *For any $\varepsilon > 0$, $K > 1$, and $t_0 \geq 1$, there exist constants T_0, η, and N_0 such that any graph G with at least N_0 vertices that is (η, K)-bounded with respect to density $0 < p \leq 1$ admits an (ε, t)-equitable (ε, p)-regular partition of its vertex set with $t_0 \leq t \leq T_0$.* $\qquad\square$

2.1. **The hereditary nature of sparse regularity.** We shall also use the fact that ε-regularity is typically inherited on "small" (sublinear) subsets. This was essentially observed for the classical notion of (dense) regular pairs by Duke and Rödl [11] and for sparse regular pairs in [16, 24]. Here we shall use a result from [16] regarding the hereditary nature of (ε, α, p)-*denseness* (or "one sided-regularity").

Definition 5. *Let $\alpha, \varepsilon > 0$, and $0 < p \leq 1$ be given and let $G = (V, E)$ be a graph. For sets X, $Y \subseteq V$, $X \cap Y = \emptyset$, we say that the pair (X, Y) is (ε, α, p)-dense if for all $X' \subseteq X$ and $Y' \subseteq Y$ with $|X'| \geq \varepsilon|X|$ and $|Y'| \geq \varepsilon|Y|$, we have*

$$d_{G,p}(X', Y') \geq \alpha - \varepsilon.$$

It follows immediately from the definition that (ε, α, p)-denseness is inherited on large sets, i.e., that for a (ε, α, p)-dense pair (X, Y) and any sets $X' \subseteq X$ and $Y' \subseteq Y$ with $|X'| \geq \mu|X|$ and $|Y'| \geq \mu|Y|$ the pair (X', Y') is $(\varepsilon/\mu, \alpha, p)$-dense. The following result from [16] states that this "denseness-property" is even inherited on randomly chosen subsets of much smaller size with overwhelming probability.

Theorem 6 ([16, Theorem 3.6]). *For every $\alpha, \beta > 0$ and $\varepsilon' > 0$, there exist $\varepsilon_0 = \varepsilon_0(\alpha, \beta, \varepsilon') > 0$ and $L = L(\alpha, \varepsilon')$ such that, for any $0 < \varepsilon \leq \varepsilon_0$ and $0 <$*

$p < 1$, if (X, Y) is an (ε, α, p)-dense pair in a graph G, then the number of sets $X' \subseteq X$ with $|X'| = w \geq L/p$ such that (X', Y) is an $(\varepsilon', \alpha, p)$-dense pair is at least $(1 - \beta^w)\binom{|X|}{w}$. $\hfill\square$

The following is a direct consequence of Theorem 6, which we obtain by applying it first to X and then to Y.

Corollary 7 ([16, Corollary 3.8]). *For every α, $\beta > 0$ and $\varepsilon' > 0$, there exist $\varepsilon_0 = \varepsilon_0(\alpha, \beta, \varepsilon') > 0$ and $L = L(\alpha, \varepsilon')$ such that, for any $0 < \varepsilon \leq \varepsilon_0$ and $0 < p < 1$, every (ε, α, p)-dense pair (X, Y) in a graph G has the following property: the number of pairs (X', Y') of sets with $X' \subseteq X$ and $Y' \subseteq Y$ with $|X'| = w_1 \geq L/p$ and $|Y'| = w_2 \geq L/p$ and such that (X', Y') is an $(\varepsilon', \alpha, p)$-dense pair is at least $(1 - \beta^{\min\{w_1, w_2\}})\binom{|X|}{w_1}\binom{|Y|}{w_2}$.* $\hfill\square$

3. PROPERTIES OF RANDOM GRAPHS

In this section we shall verify a few properties of random graphs that will be useful for the proof of Theorem 3.

3.1. **Uniform edge distribution.** We start with a well known fact, which follows easily from the properties of the binomial distribution, concerning the edge distribution of $G(N, p)$.

Definition 8. *For an integer N and $0 < p \leq 1$ we define the family of graphs $\mathscr{U}_{N,p}$ on $[N] = \{1, \ldots, N\}$ with uniform edge distribution by*

$$\mathscr{U}_{N,p} := \Big\{ G \colon V(G) = [N] \text{ and } \forall U, W \subseteq V(G) \text{ with } U \cap W = \emptyset, |U| \geq \tfrac{N}{\log N},$$

$$\text{and } |W| \geq \tfrac{N}{\log N} \text{ we have } e_G(U, W) = (1 \pm \tfrac{1}{\log N})p|U||W| \Big\}.$$

The following proposition follows directly from the Chernoff bound for binomially distributed random variables.

In Proposition 9 below and in the remainder of this paper, $o(1)$ denotes a function that tends to 0 as $N \to \infty$. We also use the symbols $\ll$ and $\gg$; e.g., we write $f(N) \ll g(N)$ to mean that $f(N)/g(N) \to 0$ as $N \to \infty$.

Proposition 9. *If $p = p(N) \gg (\log N)^4/N$, then $\mathbb{P}(G(N, p) \in \mathscr{U}_{N,p}) = 1 - o(1)$.* $\hfill\square$

3.2. **Congestion property of neighbourhoods.** For a graph $G = (V, E)$ and an integer $k \geq 1$, we define the auxiliary, bipartite graph $\Gamma(k, G) = (\binom{V}{k} \dot\cup V, E_{\Gamma(k,G)})$ by

$$(K, v) \in E_{\Gamma(k,G)} \quad \Longleftrightarrow \quad \{w, v\} \in E(G) \text{ for all } w \in K. \tag{4}$$

Proposition 11, given below, states that if G is the random graph $G(N, p)$, then the graph $\Gamma(k, G)$ has no "dense patches". More precisely, we consider the following property.

Definition 10. *Let integers N and $k \geq 1$ and reals $\xi > 0$ and $0 < p \leq 1$ be given. We say that a graph $G = (V, E)$ with $V = [N]$ has the congestion property $\mathscr{C}_{N,p}^k(\xi)$ if for every $U \subseteq V$ and every family $\mathcal{F}_k \subseteq \binom{V \setminus U}{k}$ of pairwise disjoint k-sets with*

    (i) $|\mathcal{F}_k| \leq \xi N$ and

    (ii) $|U| \leq |\mathcal{F}_k|$

we have

$$e_{\Gamma(k,G)}(\mathcal{F}_k, U) \leq p^k |\mathcal{F}_k||U| + 6\xi N p^k |\mathcal{F}_k| \,. \tag{5}$$

We show that for appropriate p the random graph $G(N, p)$ asymptotically almost surely has property $\mathscr{C}_{N,p}^k(\xi)$.

Proposition 11. *For every integer $k \geq 1$ and real $\xi > 0$, there exists $C > 1$ such that if $p > C(\log N/N)^{1/k}$, then $\mathbb{P}(G(N, p) \in \mathscr{C}_{N,p}^k(\xi)) = 1 - o(1)$.*

Since $(\log N/N)^{1/\Delta} \geq (\log N/N)^{1/k}$ for $1 \leq k \leq \Delta$ we obtain the following corollary.

Corollary 12. *For every integer $\Delta \geq 1$ and real $\xi > 0$, there exists $C > 1$ such that if $p > C(\log N/N)^{1/\Delta}$, then $\mathbb{P}(G(N, p) \in \bigcap_{k=1}^{\Delta} \mathscr{C}_{N,p}^k(\xi)) = 1 - o(1)$.* □

Proof of Proposition 11. For given k and ξ we let C be a constant satisfying

$$C^k > k/\xi \,.$$

Let $\mathcal{F}_k$ and U satisfy (i) and (ii) of Definition 10. We consider two cases depending on the size of $\mathcal{F}_k$.

Case 1 ($|\mathcal{F}_k| \geq N/\log N$). Observe that for fixed $\mathcal{F}_k$ and U the edges of $\Gamma[\mathcal{F}_k, U] = \Gamma(k, G(N, p))[\mathcal{F}_k, U]$ appear independently with probability p^k. Thus $e_\Gamma(\mathcal{F}_k, U)$ is a binomial random variable with distribution $\mathrm{Bi}(|\mathcal{F}_k||U|, p^k)$. From Chernoff's inequality

$$\mathbb{P}(X \geq \mathbb{E}X + t) \leq \exp(-t)$$

for a binomial random variable X and $t \geq 6\mathbb{E}X$ (see e.g. [22, Corollary 2.4]), we infer

$$\mathbb{P}\big(e_\Gamma(\mathcal{F}_k, U) > p^k|\mathcal{F}_k||U| + 6\xi N p^k |\mathcal{F}_k|\big) \leq \exp\big(-6\xi N p^k |\mathcal{F}_k|\big) \,,$$

since we have $|U| \leq |\mathcal{F}_k| \leq \xi N$ from (i) and (ii) of Definition 10.

Moreover, the number of choices for $\mathcal{F}_k$ (satisfying the assumptions of this case) is at most $\sum_{f=N/\log N}^{\xi N} N^{kf}$ and the number of choices for the set U is at most 2^N. Since

$$\sum_{f=N/\log N}^{\xi N} N^{kf} 2^N \exp(-6\xi N p^k f) \to 0$$

as $N \to \infty$ follows from the choice of $C^k > k/\xi$ and $p > C(\log N/N)^{1/k}$, the proposition is established in this case.

Case 2 ($|\mathcal{F}_k| < N/\log N$). The analysis in this case is very similar to the first. Instead of Chernoff's inequality we use that if X is a binomial random variable $X \sim \mathrm{Bi}(M, q)$ then

$$\mathbb{P}(X \geq t) \leq q^t \binom{M}{t} \leq \left(\frac{eqM}{t}\right)^t \,.$$

Consequently,

$$\mathbb{P}\big(e_\Gamma(\mathcal{F}_k, U) \geq p^k|\mathcal{F}_k||U| + 6\xi N p^k |\mathcal{F}_k|\big) \leq \mathbb{P}\big(e_\Gamma(\mathcal{F}_k, U) \geq 6\xi N p^k |\mathcal{F}_k|\big)$$

$$\leq \left(\frac{e|U|}{6\xi N}\right)^{6\xi N p^k |\mathcal{F}_k|} \leq \exp\big(-6\xi N p^k |\mathcal{F}_k| \log(2\xi N/|U|)\big) \,.$$

In this case, the number of choices for the pair $(\mathcal{F}_k, U)$ is at most

$$\sum_{f=1}^{N/\log N} \sum_{u=1}^{f} N^{kf} \binom{N}{u}.$$

Consequently, from the union bound we infer that the probability that there exists a family $\mathcal{F}_k$ and a set U with $|U| \leq |\mathcal{F}_k| < N/\log N$ such that $e_\Gamma(\mathcal{F}_k, U) \geq p^k|\mathcal{F}_k||U| + 6\xi N p^k|\mathcal{F}_k|$ is at most

$$\sum_{f=1}^{N/\log N} \sum_{u=1}^{f} \exp\left(kf\log N + u\log(\mathrm{e}N/u) - 6\xi N p^k f \log(2\xi N/u)\right) \to 0,$$

as $N \to \infty$ since $p^k N \gg \log N/\log\log N$.

This concludes the proof of Proposition 11. $\qquad\square$

3.3. **Hereditary nature of (ε, α, p)-denseness.** In this section we shall show that in the random graph $G(N, p)$ all sufficiently large (not necessarily induced) 3-partite subgraphs, say with vertex set $X \dot\cup Y \dot\cup Z$, in which all the three pairs (X, Y), (X, Z) and (Y, Z) are (ε, α, p)-dense, have the following property: The (ε, α, p)-denseness of the pair (Y, Z) is "typically" inherited on the one-sided neighbourhood $(N(x) \cap Y, Z)$ as well as on the two-sided neighbourhood $(N(x) \cap Y, N(x) \cap Z)$ for $x \in X$. Below we introduce classes $\mathcal{B}_p^{\mathrm{I}}$ and $\mathcal{B}_p^{\mathrm{II}}$ of "bad" tripartite graphs, which fail to have the above one-sided and two-sided property (for similar concepts see [24]).

Definition 13. *Let integers m_1, m_2, and m_3 and reals $\alpha, \varepsilon', \varepsilon, \mu > 0$, and $0 < p \leq 1$ be given.*

(I) *Let $\mathcal{B}_p^{\mathrm{I}}(m_1, m_2, m_3, \alpha, \varepsilon', \varepsilon, \mu)$ be the family of tripartite graphs with vertex set $X \dot\cup Y \dot\cup Z$, where $|X| = m_1$, $|Y| = m_2$, and $|Z| = m_3$, satisfying*
  (a) *(X, Y) and (Y, Z) are (ε, α, p)-dense pairs and*
  (b) *there exists $X' \subseteq X$ with $|X'| \geq \mu|X|$ such that for every $x \in X'$ the pair $(N(x) \cap Y, Z)$ is not $(\varepsilon', \alpha, p)$-dense.*

(II) *Let $\mathcal{B}_p^{\mathrm{II}}(m_1, m_2, m_3, \alpha, \varepsilon', \varepsilon, \mu)$ be the family of tripartite graphs with vertex set $X \dot\cup Y \dot\cup Z$, where $|X| = m_1$, $|Y| = m_2$, and $|Z| = m_3$, satisfying*
  (a) *(X, Y), (X, Z), and (Y, Z) are (ε, α, p)-dense pairs and*
  (b) *there exists $X' \subseteq X$ with $|X'| \geq \mu|X|$ such that for every $x \in X'$ the pair $(N(x) \cap Y, N(x) \cap Z)$ is not $(\varepsilon', \alpha, p)$-dense.*

Next we define the family of graph $\mathscr{D}_{N,p}^\Delta$. This family consist of those graphs G with vertex set $[N]$ which contain no graph from and $\mathcal{B}_p^{\mathrm{I}}$ no graph $\mathcal{B}_p^{\mathrm{II}}$ as a not necessarily induced subgraph.

Definition 14. *For integers N and $\Delta \geq 2$ and reals α, γ, ε', ε, $\mu > 0$ and $0 < p \leq 1$ we say that a graph $G = (V, E)$ with $V = [N]$ has the denseness property $\mathscr{D}_{N,p}^\Delta(\gamma, \alpha, \varepsilon', \varepsilon, \mu)$, if G contains no member from*

$$\mathcal{B}_p^{\mathrm{I}}(m_1^{\mathrm{I}}, m_2^{\mathrm{I}}, m_3^{\mathrm{I}}, \alpha, \varepsilon', \varepsilon, \mu) \cup \mathcal{B}_p^{\mathrm{II}}(m_1^{\mathrm{II}}, m_2^{\mathrm{II}}, m_3^{\mathrm{II}}, \alpha, \varepsilon', \varepsilon, \mu)$$

with $m_1^{\mathrm{I}}, m_3^{\mathrm{I}} \geq \gamma p^{\Delta-1} N$ and $m_2^{\mathrm{I}}, m_1^{\mathrm{II}}, m_2^{\mathrm{II}}, m_3^{\mathrm{II}} \geq \gamma p^{\Delta-2} N$ as a (not necessarily induced) subgraph.

The following proposition is the main result of this section. It asserts that with high probability the random graph $G(N, p)$ enjoys the property $\mathscr{D}_{N,p}^\Delta$ when $p \gg (\log N/N)^{1/\Delta}$.

Proposition 15. *For every integer $\Delta \geq 2$ and positive reals α, ε', and μ there exists*

$$\varepsilon = \varepsilon(\Delta, \alpha, \varepsilon', \mu) > 0 \tag{6}$$

such that for every $\gamma > 0$ there exists $C(\Delta, \alpha, \varepsilon', \mu, \gamma) > 1$ such that if $p > C(\log N/N)^{1/\Delta}$, then

$$\mathbb{P}(G(N, p) \in \mathscr{D}_{N,p}^{\Delta}(\gamma, \alpha, \varepsilon', \varepsilon, \mu)) = 1 - o(1).$$

Before we turn to the proof of Proposition 15, we deduce the following corollary.

Corollary 16. *For all integers Δ, $\widetilde{\Delta} \geq 2$ and all reals α, μ, γ, and $\varepsilon^* > 0$, there exist $C > 1$ and $\varepsilon_0, \ldots, \varepsilon_{\widetilde{\Delta}}$ satisfying $0 < \varepsilon_0 \leq \cdots \leq \varepsilon_{\widetilde{\Delta}} \leq \varepsilon^*$ such that if $p > C(\log N/N)^{1/\Delta}$, then $\mathbb{P}\big(G(N, p) \in \bigcap_{k=1}^{\widetilde{\Delta}} \mathscr{D}_{N,p}^{\Delta}(\gamma, \alpha, \varepsilon', \varepsilon, \mu)\big) = 1 - o(1)$.*

Proof. Let Δ, $\widetilde{\Delta} \geq 2$ and α, μ, γ, and $\varepsilon^* > 0$ be given. We appeal $\widetilde{\Delta}$-times to Proposition 15 to define $\varepsilon_0, \ldots, \varepsilon_{\widetilde{\Delta}}$. In fact, we set $\varepsilon_{\widetilde{\Delta}} = \varepsilon^*$ and for $\widetilde{\Delta} > k \geq 1$ let ε_{k-1} be recursively defined, by

$$\varepsilon_{k-1} = \min \left\{ \varepsilon(\Delta, \alpha, \varepsilon_k, \mu), \, \varepsilon_k \right\},$$

where $\varepsilon(\Delta, \alpha, \varepsilon_k, \mu)$ is the given by Proposition 15. Finally, let C be the maximum of all $C(\Delta, \alpha, \varepsilon_k, \mu, \gamma)$ for $k = 1, \ldots, \widetilde{\Delta}$. Owing to the choice of C and ε_{k-1} for $k \in [\widetilde{\Delta}]$, Proposition 15 yields that a.a.s. $G(N, p) \in \mathscr{D}_{N,p}^{\Delta}(\gamma, \alpha, \varepsilon_k, \varepsilon_{k-1}, \mu)$ for every $k \in [\widetilde{\Delta}]$ for $p > C(\log N/N)^{1/\Delta}$. $\square$

We first verify Proposition 15 for the special case in which $m_1^{\mathrm{I}} = pm_2^{\mathrm{I}} = m_3^{\mathrm{I}}$ and $m_1^{\mathrm{II}} = m_2^{\mathrm{II}} = m_3^{\mathrm{II}}$. (Strictly speaking, we should write, say, $\lfloor pm_2^{\mathrm{I}} \rfloor$, because m_1^{I} is an integer. However, throughout this paper we omit floor and ceiling signs, whenever they have no significant effect on the arguments.)

To deal with the special case specified above, we consider the families of graphs $\mathcal{B}_p^{\mathrm{I}}(m, \alpha, \varepsilon', \varepsilon, \mu)$ and $\mathcal{B}_p^{\mathrm{II}}(m, \alpha, \varepsilon', \varepsilon, \mu)$ for $m \in \mathbb{N}$ and $\alpha, \varepsilon', \varepsilon, \mu > 0$ defined as

$$\mathcal{B}_p^{\mathrm{I}}(m, \alpha, \varepsilon', \varepsilon, \mu) = \mathcal{B}_p^{\mathrm{I}}(pm, m, pm, \alpha, \varepsilon', \varepsilon, \mu)$$

and

$$\mathcal{B}_p^{\mathrm{II}}(m, \alpha, \varepsilon', \varepsilon, \mu) = \mathcal{B}_p^{\mathrm{II}}(m, m, m, \alpha, \varepsilon', \varepsilon, \mu).$$

Similarly, for integers N and Δ and positive reals $\alpha, \gamma, \varepsilon', \varepsilon, \mu > 0$ and $0 < p \leq 1$, we say that a graph $G = (V, E)$ with $V = [N]$ has property $\widehat{\mathscr{D}}_{N,p}^{\Delta}(\gamma, \alpha, \varepsilon', \varepsilon, \mu)$ if G contains no member from $\mathcal{B}_p^{\mathrm{I}}(m, \alpha, \varepsilon', \varepsilon, \mu) \cup \mathcal{B}_p^{\mathrm{II}}(m, \alpha, \varepsilon', \varepsilon, \mu)$ with $m = \gamma p^{\Delta-2} N$ as a (not necessarily induced) subgraph. Next we prove that $G(N, p)$ has property $\widehat{\mathscr{D}}_{N,p}^{\Delta}(\gamma, \alpha, \varepsilon', \varepsilon, \mu)$ with high probability.

Proposition 17. *For an integer $\Delta \geq 2$ and $\alpha, \varepsilon', \mu \in (0, 1]$ there exists $\varepsilon > 0$ such that for every $\gamma \in (0, 1]$ there exists $C \geq 1$ such that if $p > C(\log N/N)^{1/\Delta}$, then $\mathbb{P}(G(N, p) \in \widehat{\mathscr{D}}_{N,p}^{\Delta}(\gamma, \alpha, \varepsilon', \varepsilon, \mu)) = 1 - o(1)$.*

Next we verify Proposition 17. This will be followed by a reduction of Proposition 15 to Proposition 17 (see Claim 18 below).

Proof of Proposition 17. Let Δ, α, ε', and μ be given. We set

$$\beta = \left(\frac{1}{4\mathrm{e}}\right)^{4/(\alpha\mu)} \frac{\alpha^2}{4\mathrm{e}^2}$$

and let ε_1 and L_1 be given by Theorem 6 and let ε_2 and L_2 be given by Corollary 7 applied with α, β, and ε'. We fix

$$\varepsilon = \min\{\alpha/2\,,\mu/4\,,\varepsilon_1,\varepsilon_2\}\,,$$

and for every $\gamma > 0$ we set

$$C = \left(\frac{4}{\gamma}\right)^{1/\Delta} \tag{7}$$

and let N be sufficiently large.

First we show that a.a.s. $G(N,p)$ contains no graph from

$$\mathcal{B}_p^{\mathrm{I}}(m,\alpha,\varepsilon',\varepsilon,\mu) = \mathcal{B}_p^{\mathrm{I}}(m,pm,m,\alpha,\varepsilon',\varepsilon,\mu)\,.$$

Suppose $T = (X\dot\cup Y\dot\cup Z, E_T)$ is a tripartite graph from $\mathcal{B}_p^{\mathrm{I}}(m,\alpha,\varepsilon',\varepsilon,\mu)$. We will show that such a graph T is unlikely to appear in $G(N,p)$. Because of the assumption on T, the bipartite subgraphs $T[X,Y]$ and $T[Y,Z]$ of T contain at least $(\alpha-\varepsilon)p^2m^2$ edges each. Furthermore, there is a set $X' \subseteq X$ with $|X'| \geq \mu|X|$ such that for every $x \in X'$ the pair $(N_T(x) \cap Y, Z)$ is not (ε',α,p)-dense. Set

$$X'' = \{x \in X'\colon\ |N_T(x) \cap Y| \geq \alpha p^2 m/2\}\,.$$

From the (ε,α,p)-denseness of $T[X,Y]$ we infer that

$$|X''| \geq (1 - \varepsilon/\mu)|X'| \geq |X'|/2 \geq \mu pm/2\,.$$

Without loss of generality we may assume that

$$|X''| = \frac{1}{2}\mu pm\,. \tag{8}$$

Fix $x \in X''$. An easy averaging argument shows that there is a set $Y_x' \subseteq N_T(x) \cap Y$ of size precisely $\varepsilon'\alpha pm/2$ such that $d_{T,p}(Y_x', Z) < \alpha - \varepsilon'$. Now let Y_x be such that $Y_x' \subseteq Y_x \subseteq N_T(x) \cap Y$ with $|Y_x| = \alpha pm/2$. Then, clearly, $T[Y_x, Z]$ is not (ε',α,p)-dense. We may thus find a family of sets $\{Y_x\colon\ x \in X''\}$ such that (Y_x, Z) is not (ε',α,p)-dense. We shall show that such a configuration is unlikely to occur in $G(N,p)$.

Indeed we can fix the sets X'', Y, Z and the edges of the bipartite graph $T[Y,Z]$ in at most

$$\sum_{t \geq (\alpha-\varepsilon)pm^2} \binom{N}{m}\binom{N}{pm}^2\binom{pm^2}{t}$$

ways. Note that for sufficiently large N we have

$$p^\Delta N > C^\Delta \log N \geq \frac{2\max\{L_1,L_2\}}{\alpha\gamma}. \tag{9}$$

Moreover, owing to the definition of $\widehat{\mathscr{D}}_{N,p}^\Delta(\gamma,\alpha,\varepsilon',\varepsilon,\mu)$ we have $m = \gamma p^{\Delta-2}N$. Consequently, we have $w := \alpha pm/2 \geq L_1/p$ and, hence, we can apply Theorem 6 to $T[Y,Z]$ and infer that there are at most

$$\left(\beta^{\alpha pm/2}\binom{m}{\alpha pm/2}\right)^{\mu pm/2}$$

possibilities for choosing the sets Y_x for $x \in X''$. Combining the two estimates above, in view of 8 we infer that the probability that $T[X'',Y,Z]$ appears in $G(N,p)$

is bounded from above by

$$\sum_{t \geq (\alpha - \varepsilon)p^2 m^2} \binom{N}{m} \binom{N}{pm}^2 \binom{pm^2}{t} p^t \times \left(\beta^{\alpha pm/2} \binom{m}{\alpha pm/2} \right)^{\mu pm/2} p^{\mu \alpha p^2 m^2/4}$$

$$\leq \sum_{t \geq (\alpha - \varepsilon)p^2 m^2} N^{3m} \left(\frac{p^2 m^2 \mathrm{e}}{t} \right)^t \times \left(\beta \frac{2\mathrm{e}}{\alpha} \right)^{\mu \alpha p^2 m^2/4}$$

$$\leq m^2 N^{3m} \left(\mathrm{e} \left(\frac{2\mathrm{e}}{\alpha} \right)^{\mu \alpha/4} \beta^{\mu \alpha/4} \right)^{p^2 m^2},$$

where, for the last inequality, we used the fact that the function $f(t) = (p^2 m^2 \mathrm{e}/t)^t$ is maximized for $t = p^2 m^2$.

Finally, we note that the right-hand side of the last inequality tends to 0 as $N \to \infty$. In fact, it follows from the choice of β that $\mathrm{e}(2\mathrm{e}/\alpha)^{\mu\alpha/4}\beta^{\mu\alpha/4} \leq 1/4$ and, moreover, $p^2 m^2 \geq 4m \log N$ since $p > C(\log N/N)^{1/\Delta}$, $m = \gamma p^{\Delta-2} N$ and C is chosen as in (10). Consequently, a.a.s. $G(N, p)$ contains no graph from $\mathcal{B}_p^{\mathrm{I}}(m, \alpha, \varepsilon', \varepsilon, \mu)$ as a subgraph.

It is left to show that a.a.s. $G(N, p)$ also contains no graph from

$$\mathcal{B}_p^{\mathrm{II}}(m, \alpha, \varepsilon', \varepsilon, \mu) = \mathcal{B}_p^{\mathrm{II}}(m, m, m, \alpha, \varepsilon', \varepsilon, \mu)$$

as a subgraph. The proof of this is almost identical to the $\mathcal{B}_p^{\mathrm{I}}$. However, owing to the different size of the vertex class Y (now $|Y| = m$ and not pm) some calculations will change. (In fact, for this case we could weaken the assumption on p and only $p > CN^{-1/\Delta}$ is required.)

Suppose $T = (X \dot\cup Y \dot\cup Z, E_T)$ is a tripartite graph from $\mathcal{B}_p^{\mathrm{II}}(m, \alpha, \varepsilon', \varepsilon, \mu)$. We shall again find a subgraph of T that is unlikely to appear in $G(N, p)$. Because of the assumption on T, the bipartite subgraphs $T[X, Y]$, $T[X, Z]$, and $T[Y, Z]$ of T contain at least $(\alpha - \varepsilon)pm^2$ edges each. Furthermore, there is a set $X' \subseteq X$ with $|X'| \geq \mu|X|$ such that for every $x \in X'$ the pair $(N_T(x) \cap Y, N_T(x) \cap Z)$ is not $(\varepsilon', \alpha, p)$-dense. Set

$$X'' = \{x \in X' \colon |N_T(x) \cap Y| \geq \alpha pm/2 \text{ and } |N_T(x) \cap Z| \geq \alpha pm/2\}.$$

From the (ε, α, p)-denseness of $T[X, Y]$ and $T[X, Z]$ we infer that

$$|X''| \geq (1 - 2\varepsilon/\mu)|X'| \geq |X'|/2 \geq \mu m/2.$$

Without loss of generality we may assume that $\mu m/2$ is an integer and that we have $|X''| = \mu m/2$.

Fix $x \in X''$. An easy averaging argument shows that there are sets $Y'_x \subseteq N_T(x) \cap Y$ and $Z'_x \subseteq N_T(x) \cap Z$ of size precisely $\varepsilon' \alpha pm/2$ each such that $d_{T,p}(Y'_x, Z'_x) < \alpha - \varepsilon'$. Now let Y_x and Z_x be such that $Y'_x \subseteq Y_x \subseteq N_T(x) \cap Y$ and $Z'_x \subseteq Z_x \subseteq N_T(x) \cap Z$ and $|Y_x| = |Z_x| = \alpha pm/2$. Then, clearly, $T[Y_x, Z_x]$ is not $(\varepsilon', \alpha, p)$-dense. We may thus find a family of pairs $\{(Y_x, Z_x) \colon x \in X''\}$ that are not $(\varepsilon', \alpha, p)$-dense. We shall show that such a configuration is unlikely to occur in $G(N, p)$.

Indeed we can fix the sets X'', Y, Z and the edges of the bipartite graph $T[Y, Z]$ in at most

$$\sum_{t \geq (\alpha - \varepsilon)pm^2} \binom{N}{m}^3 \binom{m^2}{t}$$

ways. Since $m = \gamma p^{\Delta-2} N$ (see the definition of $\widehat{\mathscr{D}}_{N,p}^{\Delta}(\gamma, \alpha, \varepsilon', \varepsilon, \mu)$) we again infer from (9) that $\alpha pm/2 \geq L_2/p$. Hence, we can apply Corollary 7 to $T[Y, Z]$ and infer that there are at most

$$\left(\beta^{\alpha pm/2} \binom{m}{\alpha pm/2}^2 \right)^{\mu m/2}$$

possibilities for choosing all pairs (Y_x, Z_x) for $x \in X''$. Combining the two estimates above we infer that the probability that such a configuration appears in $G(N, p)$ is bounded from above by

$$\sum_{t \geq (\alpha-\varepsilon)pm^2} \binom{N}{m}^3 \binom{m^2}{t} p^t \times \left(\beta^{\alpha pm/2} \binom{m}{\alpha pm/2}^2 \right)^{\mu m/2} p^{\mu \alpha pm^2/2}$$

$$\leq \sum_{t \geq (\alpha-\varepsilon)pm^2} N^{3m} \left(\frac{pm^2 \mathrm{e}}{t} \right)^t \times \left(\sqrt{\beta} \frac{2\mathrm{e}}{\alpha} \right)^{\mu \alpha pm^2/2}$$

$$\leq m^2 N^{3m} \left(\mathrm{e} \left(\frac{2\mathrm{e}}{\alpha} \right)^{\mu \alpha/2} \beta^{\mu \alpha/4} \right)^{pm^2},$$

where, for the last inequality, we used the fact that the function $f(t) = (pm^2 \mathrm{e}/t)^t$ is maximized for $t = pm^2$.

Finally, we observe that the right-hand side of the last inequality tends to 0 as $N \to \infty$, since $\mathrm{e}(2\mathrm{e}/\alpha)^{\mu \alpha/2} \beta^{\mu \alpha/4} = 1/4$ (owing to the choice of β) and $pm^2 \gg m \log N$ (owing to the choice of $p > C(\log N/N)^{1/\Delta}$ and $m = \gamma p^{\Delta-2} N$). $\square$

We now deduce Proposition 15 from Proposition 17.

Proof of Proposition 15. In order to prove Proposition 15 we need to strengthen Proposition 17 and consider the families $\mathcal{B}_p^{\mathrm{I}}$ and $\mathcal{B}_p^{\mathrm{II}}$ with more general parameters m_1, m_2, and m_3. We shall show that, perhaps surprisingly, this more general statement follows from the "weaker" Proposition 17. Indeed, roughly speaking, we show that each "bad" tripartite graph $T \in \mathcal{B}_p^{\mathrm{II}}(m_1, m_2, m_3, \alpha, \varepsilon', \varepsilon, \mu)$ with arbitrary m_1, m_2, $m_3 \geq m$ contains a subgraph $\widehat{T} \in \mathcal{B}_p^{\mathrm{II}}(m, \alpha, \varepsilon'/2, \widehat{\varepsilon}, \mu/4)$ for some appropriate $\widehat{\varepsilon}$. The following deterministic statement makes this precise.

Claim 18. *For an integer $\Delta \geq 2$ and positive reals α, ε', μ, and $\widehat{\varepsilon}$ there exists $\varepsilon > 0$ such that for every $\gamma > 0$ there exists $C > 1$ and N_0 such that if $N \geq N_0$ and $p > C(\log N/N)^{1/\Delta}$, then every tripartite graph $T = (X \dot\cup Y \dot\cup Z, E_T) \in \mathcal{B}_p^{\mathrm{II}}(m_1, m_2, m_3, \alpha, \varepsilon', \varepsilon, \mu)$ with $\min\{m_1, m_2, m_3\} \geq m = \gamma p^{\Delta-2} N$ contains a subgraph $\widehat{T} \in \mathcal{B}_p^{\mathrm{II}}(m, \alpha, \varepsilon'/2, \widehat{\varepsilon}, \mu/4)$.*

The same claim holds for $\mathcal{B}_p^{\mathrm{I}}$ (and, in fact, the proof is a little simpler), but we only focus on $\mathcal{B}_p^{\mathrm{II}}$ here. Before we prove Claim 18, we note that that claim, combined with Proposition 17, yields Proposition 15, as Proposition 17 guarantees that with probability $1 - o(1)$ the random graph $G(N, p)$ contains no such $\widehat{T}$ from $\mathcal{B}_p^{\mathrm{I}}(m, \alpha, \varepsilon'/2, \widehat{\varepsilon}, \mu/4) \cup \mathcal{B}_p^{\mathrm{II}}(m, \alpha, \varepsilon'/2, \widehat{\varepsilon}, \mu/4)$. $\square$

Proof of Claim 18. Let $\Delta \geq 2$ and α, ε', μ, and $\widehat{\varepsilon}$ be given. Next we appeal to Corollary 7. Let the functions $\varepsilon_0(\cdot, \cdot, \cdot)$ and $L(\cdot, \cdot)$ be given by Corollary 7.

Set $\beta = 1/2$, $\delta = \varepsilon'/8$ and

$$\varepsilon = \min\{\varepsilon_0(\alpha, \beta, \widehat{\varepsilon}), \alpha/2, \mu/4\}\,.$$

Now for any given γ let

$$L' = \max\left\{L(\alpha, \widehat{\varepsilon}), \frac{640\mathrm{e}}{(\delta\varepsilon')^2\alpha}\right\} \quad \text{and} \quad C \geq \frac{L'}{\gamma}\,. \tag{10}$$

Let N be sufficiently large and $T = (X\dot\cup Y\dot\cup Z, E_T) \in \mathcal{B}_p^{\mathrm{II}}(m_1, m_2, m_3, \alpha, \varepsilon', \varepsilon, \mu)$ be given. Hence, there exists a set $X' \subseteq X$ with $|X'| \geq \mu|X|$ such that for every vertex $x \in X'$ the pair $(N_T(x) \cap Y, N_T(x) \cap Z)$ is not $(\varepsilon', \alpha, p)$-dense. We consider the set

$$X'' = \{x \in X'\colon |N_T(x) \cap Y| \geq \alpha pm_2/2 \text{ and } |N_T(x) \cap Z| \geq \alpha pm_3/2\}\,.$$

Owing to the choice of $\varepsilon \leq \mu/4$, we infer from the (ε, α, p)-denseness of $T[X, Y]$ and $T[X, Z]$ that

$$|X''| \geq \mu m_1/2\,.$$

Let each of $\widehat{X} \in \binom{X}{m}$, $\widehat{Y} \in \binom{Y}{m}$, and $\widehat{Z} \in \binom{Z}{m}$ be chosen uniformly at random and let $\widehat{T} = T[\widehat{X}, \widehat{Y}, \widehat{Z}]$. We shall show that with positive probability $\widehat{T}$ is from $\mathcal{B}_p^{\mathrm{II}}(m, \alpha, \varepsilon'/2, \widehat{\varepsilon}, \mu/4)$.

By Corollary 7, with probability at least $1 - \beta^m$ each of the pairs $(\widehat{X}, \widehat{Y})$, $(\widehat{X}, \widehat{Z})$, and $(\widehat{Y}, \widehat{Z})$ is $(\widehat{\varepsilon}, \alpha, p)$-dense. Consequently, with probability at least $1 - 3\beta^m$ we have

$$(\widehat{X}, \widehat{Y}), (\widehat{X}, \widehat{Z}), \text{ and } (\widehat{Y}, \widehat{Z}) \text{ are } (\widehat{\varepsilon}, \alpha, p)\text{-dense}, \tag{11}$$

which is property (a) of part (II) in Definition 13. Below we shall verify that property (b) also holds with high probability.

For the set $\widehat{X}'' = \widehat{X} \cap X''$, the concentration of the hypergeometric distribution tells us that, with probability at least $1 - 2\exp(-\mu m/24)$,

$$|\widehat{X}''| \geq \frac{1}{4}\mu m\,. \tag{12}$$

Similarly, with probability at least $1 - 4|\widehat{X}''|\exp(-\delta^2\alpha pm/6)$, we have, for every $x \in \widehat{X}''$, that

$$|N_{\widehat{T}}(x) \cap \widehat{Y}| = (1 \pm \delta)\frac{|N_T(x) \cap Y|}{m_2}m \geq \frac{1}{2}(1 - \delta)\alpha pm \tag{13}$$

and

$$|N_{\widehat{T}}(x) \cap \widehat{Z}| = (1 \pm \delta)\frac{|N_T(x) \cap Z|}{m_3}m \geq \frac{1}{2}(1 - \delta)\alpha pm. \tag{14}$$

Recall that for every vertex $x \in \widehat{X}'' \subseteq X'$ there exist a set $Y_x \subseteq N_T(x) \cap Y$ and a set $Z_x \subseteq N_T(x) \cap Z$ such that

$$Y_x \geq \varepsilon'|N_T(x) \cap Y| \geq \varepsilon'\alpha pm_2/2 \quad \text{and} \quad Z_x \geq \varepsilon'|N_T(x) \cap Z| \geq \varepsilon'\alpha pm_3/2\,, \tag{15}$$

and

$$d_{T,p}(Y_x, Z_x) < \alpha - \varepsilon'. \tag{16}$$

As before, applying the concentration of the hypergeometric distribution, we obtain that, with probability at least $1 - 4|\widehat{X}''|\exp(-\delta^2\varepsilon'\alpha pm/6)$, we have, for every vertex $x \in \widehat{X}''$, that

$$|Y_x \cap \widehat{Y}| = (1 \pm \delta)\frac{|Y_x|}{m_2}m \geq \frac{1}{2}(1 - \delta)\varepsilon'\alpha pm \tag{17}$$

and

$$|Z_x \cap \widehat{Z}| = (1 \pm \delta)\frac{|Z_x|}{m_3}m \geq \frac{1}{2}(1-\delta)\varepsilon'\alpha pm. \tag{18}$$

Below we shall show that given (13), (14), (17), and (18) hold, with probability at least $1 - 2/N^2$, for any given $x \in \widehat{X}''$, the pair $(N_{\widehat{T}}(x) \cap \widehat{Y}, N_{\widehat{T}}(x) \cap \widehat{Z})$ is not $(\varepsilon'/2, \alpha, p)$-dense. Summing the failure probabilities $2/N^2$ over all choices of $x \in \widehat{X}''$ and adding the failure probabilities for (11), (12), (13), (14), (17), and (18) it follows that with positive probability $\widehat{T} = T[\widehat{X}, \widehat{Y}, \widehat{Z}] \in \mathcal{B}_p^{\mathrm{II}}(m, \alpha, \varepsilon'/2, \widehat{\varepsilon}, \mu/4)$.

Fix $x \in \widehat{X}''$. Below, we may and shall assume that (13), (14), (17), and (18) hold. For any integer ζ with

$$(1-\delta)|Z_x|\frac{m}{m_3} \leq \zeta \leq (1+\delta)|Z_x|\frac{m}{m_3}\,,$$

we shall consider the conditional space in which $|Z_x \cap \widehat{Z}| = \zeta$. To remind ourselves of this conditioning, we shall write $\mathbb{P}_\zeta$ and $\mathbb{E}_\zeta$ to denote the probability and the expectation in this space.

For all $y \in Y_x$, let $Z_x(y) = N_T(y) \cap Z_x$. We have

$$\mathbb{E}_\zeta(|Z_x(y) \cap \widehat{Z}|) = |Z_x(y)|\frac{|Z_x \cap \widehat{Z}|}{|Z_x|} = \frac{|Z_x(y)|}{|Z_x|}\zeta\,.$$

Suppose now that $|Z_x(y)| \geq (\varepsilon'/20\mathrm{e})p|Z_x|$. Then, owing to the concentration of the hypergeometric distribution (see, e.g., [22, Theorem 2.10]), we have

$$\mathbb{P}_\zeta\left(|Z_x(y) \cap \widehat{Z}| \geq (1+\delta)\frac{|Z_x(y)|}{|Z_x|}\zeta\right) \leq 2\exp\left(-\frac{1}{3}\delta^2\frac{|Z_x(y)|}{|Z_x|}\zeta\right)$$

$$\leq 2\exp\left(-\frac{1}{3}\delta^2\frac{\varepsilon'}{20\mathrm{e}}p\zeta\right). \tag{19}$$

Consider now the case in which $|Z_x(y)| < (\varepsilon'/20\mathrm{e})p|Z_x|$. Then, owing to standard estimates for the hypergeometric distribution (see, e.g., [24, Lemma 10]), we have

$$\mathbb{P}_\zeta\left(|Z_x(y) \cap \widehat{Z}| \geq \frac{|Z_x(y)|}{|Z_x|}\zeta + \frac{\varepsilon'}{10}p\zeta\right) \leq \mathbb{P}_\zeta\left(|Z_x(y) \cap \widehat{Z}| \geq \frac{\varepsilon'}{10}p\zeta\right)$$

$$\leq \left(\frac{\mathrm{e}}{(\varepsilon'/10)p\zeta}\frac{|Z_x(y)|}{|Z_x|}\zeta\right)^{(\varepsilon'/10)p\zeta}$$

$$\leq \left(\frac{\mathrm{e}}{(\varepsilon'/10)p}(\varepsilon'/20\mathrm{e})p\right)^{(\varepsilon'/10)p\zeta}$$

$$= \left(\frac{1}{2}\right)^{(\varepsilon'/10)p\zeta}. \tag{20}$$

Let us note that, if $|Z_x(y)| < (\varepsilon'/20\mathrm{e})p|Z_x|$, then

$$\frac{|Z_x(y)|}{|Z_x|}\zeta + \frac{\varepsilon'}{10}p\zeta \leq \frac{\varepsilon'}{20\mathrm{e}}p\zeta + \frac{\varepsilon'}{10}p\zeta \leq \frac{1}{8}\varepsilon'p\zeta. \tag{21}$$

Moreover, since

$$\zeta \geq (1-\delta)|Z_x|\frac{m}{m_3} \overset{(15)}{\geq} (1-\delta)\frac{1}{2}\varepsilon'\alpha pm \geq \frac{1}{4}\varepsilon'\alpha pm\,.$$

and since

$$p^2 m \geq \gamma p^\Delta N > \gamma C\log N \geq L'\log N\,, \tag{22}$$

we can further bound the probabilities in (19) and (20) by

$$\max\left\{(2^{-\varepsilon'p\zeta/10},\, 2\exp\left(-\delta^2\varepsilon'p\zeta/(60e)\right)\right\} \overset{(10)}{\le} \frac{1}{N^2}\,. \tag{23}$$

Consequently we have, with probability at least $1 - 1/N^2$, that

$$e(Y_x, Z_x \cap \widehat{Z}) \overset{(21)}{\le} \sum_{y \in Y_x}(1+\delta)\frac{|Z_x(y)|}{|Z_x|}\zeta + \sum_{y \in Y_x}\frac{1}{8}\varepsilon'p\zeta$$

$$= (1+\delta)\frac{\zeta}{|Z_x|}\sum_{y \in Y_x}|Z_x(y)| + \frac{1}{8}\varepsilon'p\zeta|Y_x|$$

$$\overset{(16)}{\le} (1+\delta)\frac{\zeta}{|Z_x|}(\alpha - \varepsilon')p|Y_x||Z_x| + \frac{1}{8}\varepsilon'p\zeta|Y_x|,$$

whence, recalling that $|Z_x \cap \widehat{Z}| = \zeta$ and $\delta = \varepsilon'/8$,

$$d_{T,p}(Y_x, Z_x \cap \widehat{Z}) \le (1+\delta)(\alpha - \varepsilon') + \frac{1}{8}\varepsilon' \le \alpha - \frac{3}{4}\varepsilon'. \tag{24}$$

Note that the size of Y_x played no rôle in the argument above. Hence, we can repeat the same argument with Y_x replaced with $Z_x \cap \widehat{Z}$ and with Z_x replaced with Y_x and with (16) replaced by (24). This way we obtain that, with probability $1 - 2/N^2$,

$$d_{T,p}(Y_x \cap \widehat{Y}, Z_x \cap \widehat{Z}) \le \alpha - \frac{1}{2}\varepsilon'.$$

This concludes the proof of Claim 18. $\qquad\qquad\square$

4. Ramsey universal graphs

4.1. **Proof of the main result.** In this section we prove Theorem 3, namely, we show that for $p = p(N) \ge C(\log N/N)^{1/\Delta}$ the random graph $G(N, p)$ is partition universal for $\mathcal{H}_{\Delta,n}$ for n of the form $\lfloor cN \rfloor$ for some $c > 0$. In view of the results from Section 3 this follows directly from the following deterministic statement.

Lemma 19. *For every $\Delta \ge 2$ there exist $\widetilde{\Delta} \ge 2$ and positive constants μ, α, ε^*, ξ, and $\gamma > 0$ and $B > 1$ and n_0 such that for every $\varepsilon_0, \ldots, \varepsilon_{\widetilde{\Delta}}$ satisfying $0 < \varepsilon_0 \le \cdots \le \varepsilon_{\widetilde{\Delta}} \le \varepsilon^*$ and for every $n \ge n_0$ the following holds. If $G = (V, E)$ is a graph on $V = [N]$, where $N \ge Bn$, such that for some $0 < p \le 1$ we have*

    (i) $G \in \mathcal{U}_{N,p}$,
    (ii) $G \in \mathscr{C}^k_{N,p}(\xi)$ *for every $k = 1, \ldots, \Delta$, and*
    (iii) $G \in \mathscr{D}^\Delta_{N,p}(\gamma, \alpha, \varepsilon_k, \varepsilon_{k-1}, \mu)$ *for every $k = 1, \ldots, \widetilde{\Delta}$,*

then G is partition universal for $\mathcal{H}_{\Delta,n}$.

Before we prove Lemma 19, we deduce Theorem 3 from it.

Proof of Theorem 3. Let $\Delta \ge 2$ be given by Theorem 3. For this Δ Lemma 19 yields constants $\widetilde{\Delta} \ge 2$ and μ, α, ε^*, ξ, $\gamma > 0$ and $B > 1$ and n_0.

Next we will show that there exists a C such that for $p > C(\log N/N)^{1/\Delta}$ the random graph $G(N, p)$ satisfies a.a.s. the assumptions (i)–(iii) of Lemma 19. This, however, is guaranteed by Proposition 9 for property (i), by Corollary 12 for property (ii), and by Corollary 16 for property (iii).

Consequently, Lemma 19 asserts that a.a.s. $G(N, p)$ is partition universal for $\mathcal{H}_{\Delta,n}$ as long as $N \ge Bn$, which is the conclusion of Theorem 3. $\qquad\square$

4.2. **Proof of the main technical lemma.** In this section we prove the main technical lemma, Lemma 19. The proof follows the strategy in the proof of Chvátal et al. in [10], but includes ideas from [5] and [32], and is based on the sparse regularity lemma.

Proof of Lemma 19. The proof consists of four parts. In the first part we fix all constants needed in the proof. In the second part we consider the given graph G along with a fixed 2-colouring of its edges. We have to show that G contains a monochromatic $\mathcal{H}_{\Delta,n}$-universal graph. In other words, we have to embed every graph $H \in \mathcal{H}_{\Delta,n}$ into one of the two monochromatic subgraphs of G. To that end, we first prepare the graph G and here the sparse regularity lemma will be the key tool. In the third part we shall prepare a given graph $H \in \mathcal{H}_{\Delta,n}$ for the embedding. In the last part we then embed H into a monochromatic subgraph of G.

Constants. Let $\Delta \geq 2$ be an integer. We first fix

$$\widetilde{\Delta} = \Delta^4 + 2\Delta + 1$$

and we set

$$r = R(\widetilde{\Delta}, \widetilde{\Delta}),$$

where $R(\widetilde{\Delta}, \widetilde{\Delta})$ is the Ramsey number that guarantees that every 2-colouring of the edges of the complete graph K_r yields a monochromatic copy of $K_{\widetilde{\Delta}}$. Next we define the constants μ, α, ε^*, ξ, γ, B, and n_0 of Lemma 19. First we set

$$\mu = \frac{1}{4\Delta^2} \quad \text{and} \quad \alpha = \frac{1}{3}, \tag{25}$$

and

$$\varepsilon^* = \frac{1}{12\widetilde{\Delta}}. \tag{26}$$

Next we set

$$\varepsilon = \min\left\{\frac{\varepsilon_0}{2}, \frac{1}{2(r-1)}\right\}, \quad K = 2, \quad \text{and} \quad t_0 = 2r \tag{27}$$

and let T_0, η, and N_0 be the constants guaranteed by the sparse regularity lemma, Theorem 4, for ε, K, and t_0 given above. Finally, we set

$$\xi = \frac{1}{7 \cdot 4^{\Delta+1} \cdot T_0}, \qquad \gamma = \frac{1-\varepsilon}{4^{\Delta-1}T_0}, \qquad B = \frac{1}{\xi}, \tag{28}$$

and

$$n_0 = \max\left\{\frac{N_0}{B}, \frac{1}{\eta^2}, \frac{T_0^2}{\varepsilon}, 2^{4/\varepsilon_0}, e^{1/\eta}\right\}. \tag{29}$$

This concludes the definition of the constants involved in the proof of Lemma 19.

Preparing G. Now let $\varepsilon_0, \ldots, \varepsilon_{\widetilde{\Delta}}$ satisfy

$$0 < \varepsilon_0 \leq \cdots \leq \varepsilon_{\widetilde{\Delta}} \leq \varepsilon^* \overset{(26)}{=} \frac{1}{12\widetilde{\Delta}} \tag{30}$$

and let $n \geq n_0$ be given. Let $G = (V, E)$ be a graph on $V = [N]$, where $N \geq Bn \geq N_0$, satisfies assumptions (i)–(iii) of Lemma 19 for some $0 < p \leq 1$. We fix an arbitrary colouring of the edges $E = E_R \dot\cup E_B$ of G with two colours, say red and blue, and let $G_R = (V, E_R)$ and $G_B = (V, E_B)$ be the corresponding monochromatic subgraphs. We have to show that one of G_R or G_B will contain every H in $\mathcal{H}_{\Delta,n}$. To that end, first use the sparse regularity lemma to "locate" an appropriate "regular" subgraph in either G_R or G_B.

More precisely, we apply the regularity lemma with $\varepsilon = \min\{\varepsilon_0/2, 1/(r-1)\}$, $K = 2$, $t_0 = 2r$, and p to G_R. Note that, owing to property (i) of Lemma 19 (see Definition 8), the graph G is $(1/\log N, 1 + 1/\log N)$-bounded. Since $G_R \subseteq G$, $1/\log N \leq 1$, and $N/\log N \leq \eta N$ (because of the choice of n_0 in (29)) we infer that indeed G_R is (η, K)-bounded (see (27)). Consequently, Theorem 4 yields an (ε, t)-equitable (ε, p)-regular partition $V_0 \dot\cup V_1 \dot\cup \ldots V_t$ of V in G_R with $t_0 \leq t \leq T_0$.

We consider an auxiliary graph A with vertex set $[t] = \{1, \ldots, t\}$ and $\{i, j\}$ being an edge if and only if the pair (V_i, V_j) is (ε, p)-regular for G_R. Since the partition $V_0 \dot\cup V_1 \dot\cup \ldots V_t$ is (ε, p)-regular in G_R, at most $\varepsilon \binom{t}{2} \leq \frac{1}{2(r-1)} \binom{t}{2} < (r-1) \binom{t/(r-1)}{2}$ of the pairs of the auxiliary graph are missing and hence, by Turán's theorem, A contains a clique K_r with r vertices. In other words, there exists an index set $I_r = \{i_1, \ldots, i_r\} \subseteq [t]$ such that (V_i, V_j) is (ε, p)-regular for G_R for all $\{i, j\} \in \binom{I_r}{2}$. Moreover, since $G \in \mathscr{U}_{N,p}$ and since $1/\log N \leq \varepsilon N/T_0$ (see (29)) it follows directly from the definition of (ε, p)-regularity that (V_i, V_j) is $(\varepsilon + 2/\log N, p)$-regular for the graph G_B. Because of (27) and (29), we have $\varepsilon + 2/\log N \leq \varepsilon_0/2 + \varepsilon_0/2$ and, hence, (V_i, V_j) is (ε_0, p)-regular for G_R and for G_B for all $\{i, j\} \in \binom{I_r}{2}$.

Next we colour the edges of the clique $K_r \subseteq A$ red and blue. We colour an edge $\{i, j\} \in \binom{I_r}{2}$ red if $d_{G_R, p}(V_i, V_j) \geq d_{G_B, p}(V_i, V_j)$ and blue otherwise. Note that, again from the fact that $G \in \mathscr{U}_{N,p}$ and $1/\log N \leq N/T_0$ we infer that $d_{G_R, p}(V_i, V_j) + d_{G_B, p}(V_i, V_j) \geq 1 - 1/\log N$ and, therefore,

$$\max\{d_{G_R, p}(V_i, V_j), d_{G_B, p}(V_i, V_j)\} \geq \frac{1}{2} - \frac{1}{2\log N} \geq \frac{1}{3}$$

for every $\{i, j\} \in \binom{[t]}{2}$.

Because of the choice of $r \geq R(\widetilde{\Delta}, \widetilde{\Delta})$ there exists a monochromatic clique $K_{\widetilde{\Delta}} \subseteq K_r \subseteq A$ on $\widetilde{\Delta}$ vertices. Let $J \subseteq I_r$ be the vertex set of the monochromatic clique $K_{\widetilde{\Delta}}$. Summarizing, the above ensures the existence of a set $J \subseteq [t]$ of cardinality $\widetilde{\Delta}$ such that either

$$(V_i, V_j) \text{ is } (\varepsilon_0, p)\text{-regular for } G_R \text{ and } d_{G_R, p}(V_i, V_j) \geq 1/3 \text{ for all } \{i, j\} \in \binom{J}{2} \quad (31)$$

or the same statement holds for G_B. Without loss of generality we assume that (31) holds and we shall show that G_R induced on $\bigcup_{i \in J} V_i$ will contain any H from $\mathcal{H}_{\Delta, n}$.

Preparing H.   Fix some $H = (W, F) \in \mathcal{H}_{\Delta, n}$. We consider the third power $H^3 = (W, F^3)$ of H, i.e., $\{w, w'\} \in F^3$ if and only if $w \neq w'$ and there exists a w-w'-path with at most three edges in H. Since $\Delta(H) \leq \Delta$ we have

$$\Delta(H^3) \leq \Delta + \Delta(\Delta - 1) + \Delta(\Delta - 1)^2 = \Delta^3 - \Delta^2 + \Delta$$

and consequently $\chi(H^3) \leq \Delta^3 - \Delta^2 + \Delta + 1$. Fix a $(\Delta^3 - \Delta^2 + \Delta + 1)$-vertex colouring f of H^3 with colours $1, \ldots, \Delta^3 - \Delta^2 + \Delta + 1$. This way we obtain a partition of W into $\Delta^3 - \Delta^2 + \Delta + 1$ classes such that if two vertices w and w' are elements of the same class, then their distance in H is at least four; in particular, there are no edges between $N_H(w)$ and $N_H(w')$. We now refine the partition induced by the colour classes of f according to the "left-degrees" of the vertices. More precisely, we say two vertices w and w' are equivalent if $f(w) = f(w')$ and

$$\left| N_H(w) \cap \{x \in W \colon f(x) < f(w)\} \right| = \left| N_H(w') \cap \{x \in W \colon f(x) < f(w')\} \right|,$$

i.e., w and w' are equivalent if they have the same colour in f and the same number of neighbours with colours of smaller number. Clearly, this equivalence

relation partitions W into at most $(\Delta^3 - \Delta^2 + \Delta + 1)(\Delta + 1) = \widetilde{\Delta}$ classes. Denote the partition classes by $W_1, \ldots, W_{\widetilde{\Delta}}$ (allowing empty classes if necessary) and let $g \colon W \to [\widetilde{\Delta}]$ be the corresponding partition function, i.e.,

$$g(w) = j \quad \text{if and only if} \quad w \in W_j.$$

Thus, if $g(w) = g(w')$, then $\big|N_H(w) \cap \{x \in W \colon g(x) < g(w)\}\big| = \big|N_H(w') \cap \{x \in W \colon g(x) < g(w')\}\big|$. For an integer $\ell \leq g(w)$ we denote by

$$\mathrm{ldeg}_g^\ell(w) := \big|N_H(w) \cap \{x \in W \colon g(x) \leq \ell\}\big|$$

the *left-degree of w with respect to g and ℓ*.

Embedding of H into G. After the preparation of G and H we are able to embed H into G_R. We may relabel the vertex classes V_i of G_R with $i \in J$ and assume $J = [\widetilde{\Delta}]$. We proceed inductively and embed the vertex class W_ℓ into V_ℓ one at a time, for $\ell = 1, \ldots, \widetilde{\Delta}$. To this end, we verify the following statement $(\mathcal{S}_\ell)$ for $\ell = 0, \ldots, \widetilde{\Delta}$.

$(\mathcal{S}_\ell)$ There exists a partial embedding φ_ℓ of $H[\bigcup_{j=1}^\ell W_j]$ into $G_R[\bigcup_{j=1}^\ell V_j]$ such that for every $z \in \bigcup_{j=\ell+1}^{\widetilde{\Delta}} W_j$ there exists a *candidate set* $C_\ell(z) \subseteq V(G)$ given by
   (a) $C_\ell(z) = \bigcap \{N_{G_R}(\varphi_\ell(x)) \colon x \in N_H(z) \text{ and } g(x) \leq \ell\} \cap V_{g(z)}$,
   satisfying
   (b) $|C_\ell(z)| \geq (p/4)^{\mathrm{ldeg}_g^\ell(z)} m$, where $m = |V_{g(z)}| \geq (1 - \varepsilon)N/t$, and
   (c) for every edge $\{z, z'\} \in F = E(H)$ with $g(z), g(z') > \ell$ the pair $\big(C_\ell(z), C_\ell(z')\big)$ is $(\varepsilon_\ell, 1/3, p)$-dense in G_R.

Remark. In what follows we shall use the following convention. Vertices from G_R will be denoted by v and vertices from H will be usually named w. However, since the embedding of H into G will be divided into $\widetilde{\Delta}$ rounds, we shall find it convenient to distinguish among the vertices of H. We shall use the letter x for vertices that have already been embedded, the letter y for vertices that will be embedded in the current round, while z will denote vertices that we shall embed at a later step.

Statement $(\mathcal{S}_\ell)$ ensures the existence of a partial embedding of the first ℓ classes $W_1, \ldots, W_\ell$ of H such that for every unembedded vertex z there exists a candidate set $C_\ell(z)$ that is not too small (see part (b)). Moreover, if we embed z into its candidate set, then its image will be adjacent to all vertices $\varphi_\ell(x)$ with $x \in (W_1 \cup \cdots \cup W_\ell) \cap N_H(z)$ (see part (a)). The last property, part (c), concerns the edges of H both endvertices of which have not yet been embedded: those edges are such that the candidate sets of their endvertices induce (ε, α, p)-dense pairs. This property will be crucial for the inductive proof.

Before we verify $(\mathcal{S}_\ell)$ for $\ell = 0, \ldots, \widetilde{\Delta}$ by induction on ℓ we note that $(\mathcal{S}_{\widetilde{\Delta}})$ implies that H can be embedded into G_R. Since H was an arbitrary graph from $\mathcal{H}_{\Delta,n}$ and we fixed an arbitrary colouring of the edges of G, this implies $G \longrightarrow H$ for every $H \in \mathcal{H}_{\Delta,n}$. Consequently, verifying $(\mathcal{S}_\ell)$ yields the proof of Lemma 19.

Basis of the induction: $\ell = 0$. We first verify $(\mathcal{S}_0)$. In this case φ_0 is the empty mapping and for every $z \in W$ we have, according to (a), $C_0(z) = V_{g(z)}$, as there is no vertex $x \in N_H(z)$ with $g(x) \leq 0$. Also, property (b) holds by definition of $C_0(z)$ for every $z \in W$. Finally, property (c) follows from the property that (V_i, V_j) is

(ε_0, p)-regular for G_R and, consequently, $\big(C_0(z), C_0(z')\big)$ is $(\varepsilon_0, 1/3, p)$-dense in G_R for every edge $\{z, z'\}$ of H (see (31)).

Induction step: $\ell \to \ell + 1$. For the inductive step, we suppose that $\ell < \widetilde{\Delta}$ and assume that statement $(\mathcal{S}_\ell)$ holds; we have to construct $\varphi_{\ell+1}$ with the required properties. Our strategy is as follows. In the first step, we find for every $y \in W_{\ell+1}$ an appropriate subset $C(y) \subseteq C_\ell(y)$ of the candidate set such that if $\varphi_{\ell+1}(y)$ is chosen from $C(y)$, then the new candidate set $C_{\ell+1}(z) := C_\ell(z) \cap N_{G_R}(\varphi_{\ell+1}(y))$ of every "right-neighbour" z of y will not shrink too much and property (c) will continue to hold.

Note, however, that in general $|C(y)| \leq |C_\ell(y)| = o(N) \ll |W_{\ell+1}|$ (if $\mathrm{ldeg}_g^\ell \geq 1$) and, hence, we cannot "blindly" select $\varphi_{\ell+1}(y)$ from $C(y)$. Instead, in the second step, we shall verify Hall's condition to find a system of distinct representatives for the family $\{C(y) \colon y \in W_{\ell+1}\}$ and we let $\varphi_{\ell+1}(y)$ be the representative of $C(y)$. (A similar idea was used in [5, 32].) We now give the details of those two steps.

For the first step, fix $y \in W_{\ell+1}$ and set

$$N_H^{\ell+1}(y) := \{z \in N_H(y) \colon g(z) > \ell + 1\}.$$

A vertex $v \in C_\ell(y)$ will be "bad" (i.e., we shall not select v for $C(y)$) if there exists a vertex $z \in N_H^{\ell+1}(y)$ for which the set $N_{G_R}(v) \cap C_\ell(z)$ violates condition (b) or (c) of $(\mathcal{S}_{\ell+1})$ and, hence, it cannot play the rôle of $C_{\ell+1}(z)$.

We first prepare for (b) of $(\mathcal{S}_{\ell+1})$. Fix a vertex $z \in N_H^{\ell+1}(y)$. Since $(C_\ell(y), C_\ell(z))$ is an $(\varepsilon_\ell, 1/3, p)$-dense pair, there exist at most $\varepsilon_\ell |C_\ell(y)| \leq \varepsilon_{\widetilde{\Delta}} |C_\ell(y)|$ vertices v in $C_\ell(y)$ such that

$$|N_{G_R}(v) \cap C_\ell(z)| < \big(d_{G_R, p}(C_\ell(y), C_\ell(z)) - \varepsilon_{\widetilde{\Delta}}\big) p |C_\ell(y)|.$$

Repeating the above for all $z \in N_H^{\ell+1}(y)$, we infer from (a) and (b) of $(\mathcal{S}_\ell)$, that there are at most $\Delta \varepsilon_{\widetilde{\Delta}} |C_\ell(y)|$ vertices $v \in C_\ell(y)$ such that the following fails to be true for some $z \in N_H^{\ell+1}(y)$:

$$|N_{G_R}(v) \cap C_\ell(z)| \geq \big(d_{G_R, p}(C_\ell(y), C_\ell(z)) - \varepsilon_{\widetilde{\Delta}}\big) p |C_\ell(z)|$$
$$\overset{(b),(c)}{\geq} \left(\frac{1}{3} - \varepsilon_\ell\right) p \left(\frac{p}{4}\right)^{\mathrm{ldeg}_g^\ell(z)} |V_{g(z)}| \overset{(30)}{\geq} \left(\frac{p}{4}\right)^{\mathrm{ldeg}_g^{\ell+1}(z)} |V_{g(z)}|. \quad (32)$$

For property (c) of $(\mathcal{S}_{\ell+1})$, we fix an edge $e = \{z, z'\}$ with $g(z), g(z') > \ell+1$ and with at least one end vertex in $N_H^{\ell+1}(y)$. There are at most $\Delta(\Delta - 1) < \Delta^2$ such edges. Note that if both vertices z and z' are neighbours of y, i.e., $z, z' \in N_H^{\ell+1}(y)$, then

$$\max\big\{\mathrm{ldeg}_g^\ell(y), \mathrm{ldeg}_g^\ell(z), \mathrm{ldeg}_g^\ell(z')\big\} \leq \Delta - 2,$$

since all three vertices y, z, and z' have at least two neighbours in $W_{\ell+1} \cup \cdots \cup W_{\widetilde{\Delta}}$. From property (b) of $(\mathcal{S}_\ell)$ we infer

$$\min\Big\{|C_\ell(y)|, |C_\ell(z)|, |C_\ell(z')|\Big\}$$
$$\geq \left(\frac{p}{4}\right)^{\max\{\mathrm{ldeg}_g^\ell(y), \mathrm{ldeg}_g^\ell(z), \mathrm{ldeg}_g^\ell(z')\}} (1 - \varepsilon)\frac{N}{T_0} \overset{(28)}{\geq} \gamma p^{\Delta-2} N.$$

Recall that $\alpha = 1/3$ (see (25)). Hence $G_R \subseteq G$ and $G \in \mathscr{D}_{N,p}^\Delta(\gamma, \alpha, \varepsilon_{\ell+1}, \varepsilon_\ell, \mu)$ imply that there are at most $\mu |C_\ell(y)|$ vertices v contained in $C_\ell(y)$ such that the pair $(N_{G_R}(v) \cap C_\ell(z), N_{G_R}(v) \cap C_\ell(z'))$ fails to be $(\varepsilon_{\ell+1}, 1/3, p)$-dense.

If, on the other hand, say, only $z \in N_H^{\ell+1}(y)$ and $z' \notin N_H^{\ell+1}(y)$, then

$$\max\{\mathrm{ldeg}_g^\ell(y), \mathrm{ldeg}_g^\ell(z')\} \leq \Delta - 1 \quad \text{and} \quad \mathrm{ldeg}_g^\ell(z) \leq \Delta - 2.$$

Consequently, (similarly as above)

$$\min\left\{|C_\ell(y)|, |C_\ell(z')|\right\} \geq \gamma p^{\Delta-1} N \quad \text{and} \quad |C_\ell(z)| \geq \gamma p^{\Delta-2} N$$

and we can appeal to the fact that $G \in \mathscr{D}_{N,p}^\Delta(\gamma, \alpha, \varepsilon_{\ell+1}, \varepsilon_\ell, \mu)$ to infer that there are at most $\mu|C_\ell(y)|$ vertices $v \in C_\ell(y)$ such that $(N_{G_R}(v) \cap C_\ell(z), C_\ell(z'))$ fails to be $(\varepsilon_{\ell+1}, 1/3, p)$-dense. For a given $v \in C_\ell(y)$, let $\widehat{C}_\ell(z) = C_\ell(z) \cap N_{G_R}(v)$ if $z \in N_H^{\ell+1}(y)$ and $\widehat{C}_\ell(z) = C_\ell(z)$ if $z \notin N_H^{\ell+1}(y)$, and define $\widehat{C}_\ell(z')$ analogously.

Summarizing the above we infer that there are at least

$$(1 - \Delta\varepsilon_{\widetilde{\Delta}} - \Delta^2\mu)|C_\ell(y)| \tag{33}$$

vertices $v \in C_\ell(y)$ such that

    (b') $|N_{G_R}(v) \cap C_\ell(z)| \geq (p/4)^{\mathrm{ldeg}_g^{\ell+1}(z)}|V_{g(z)}|$ for every $z \in N_H^{\ell+1}(y)$ (see (32))
       and
    (c') $(\widehat{C}_\ell(z), \widehat{C}_\ell(z'))$ is $(\varepsilon_{\ell+1}, 1/3, p)$-dense for all edges $\{z, z'\}$ of H with $g(z)$, $g(z') > \ell + 1$ and $\{z, z'\} \cap N_H^{\ell+1}(y) \neq \emptyset$.

Let $C(y)$ be the set of those vertices v from $C_\ell(y)$ satisfying properties (b') and (c') above. Recall that $\mathrm{ldeg}_g^\ell(y) = \mathrm{ldeg}_g^\ell(y')$ for all $y, y' \in W_{\ell+1}$ and set

$$k = \mathrm{ldeg}_g^\ell(y) \text{ for some } y \in W_{\ell+1}.$$

Since $y \in W_{\ell+1}$ was arbitrary, we infer from (33), the choice of μ in (25) combined with $\varepsilon_{\widetilde{\Delta}} \leq \varepsilon^* = (12\widetilde{\Delta})^{-1}$ (see (30)) and property (b) of $(\mathcal{S}_\ell)$ that

$$|C(y)| \geq (1 - \Delta\varepsilon_{\widetilde{\Delta}} - \Delta^2\mu)|C_\ell(y)|$$

$$\geq (1 - \Delta\varepsilon_{\widetilde{\Delta}} - \Delta^2\mu)\left(\frac{p}{4}\right)^k (1-\varepsilon)\frac{N}{T_0} \geq \frac{1}{4^{k+1}}p^k\frac{N}{T_0}. \tag{34}$$

We now turn to the aforementioned second part of the inductive step. Here we ensure the existence of a system of distinct representatives for the set system $(C(y))_{y \in W_{\ell+1}}$. We shall appeal to Hall's condition and show that for every $Y \subseteq W_{\ell+1}$ we have

$$|Y| \leq \left|\bigcup_{y \in Y} C(y)\right|. \tag{35}$$

Because of (34), assertion (35) holds for all sets Y with $1 \leq |Y| \leq 4^{-k-1}p^k N/T_0$.

Thus, consider a set $Y \subseteq W_{\ell+1}$ with $|Y| > 4^{-k-1}p^k N/T_0$. For every $y \in W_{\ell+1}$ we have $\mathrm{ldeg}_g^\ell(y) = k$. Hence, we have a k-tuple $K(y) = \{u_1(y), \ldots, u_k(y)\}$ of already embedded vertices of H such that $K(y) = N_H(y) \setminus N_H^{\ell+1}(y)$. Note that for two distinct vertices $y, y' \in W_{\ell+1}$ the sets $K(y)$ and $K(y')$ are disjoint. This follows from the fact that the distance in H between y and y' is at least four and if $K(y) \cap K(y') \neq \emptyset$, then this distance would be at most two. Consequently, the sets of already embedded vertices $\varphi_\ell(K(y))$ and $\varphi_\ell(K(y'))$ are disjoint as well and, therefore, $\mathcal{F}_k = \{\varphi_\ell(K(y)) : y \in W_{\ell+1}\} \subseteq \binom{V}{k}$ is a family of pairwise disjoint k-sets in V. Moreover,

$$C(y) \subseteq \bigcap_{v \in \varphi(K(y))} N_{G_R}(v) \subseteq \bigcap_{v \in \varphi(K(y))} N_G(v).$$

Let

$$U = \bigcup_{y \in Y} C(y) \subseteq V_{\ell+1}\,.$$

Note that

$$\mathcal{F}_k \subseteq \binom{V_1 \dot\cup \ldots \dot\cup V_\ell}{k} \subseteq \binom{V \setminus U}{k}\,.$$

Suppose for a contradiction that

$$|U| < |Y| = |\mathcal{F}_k|. \tag{36}$$

We now use property (ii) of Lemma 19, namely, $G \in \mathscr{C}_{N,p}^k(\xi)$ applied for $\mathcal{F}_k$ and U. We deduce that

$$e_{\Gamma(k,G)}(\mathcal{F}_k, U) \le p^k |\mathcal{F}_k||U| + 6\xi N p^k |\mathcal{F}_k|\,.$$

On the other hand, because of (34), we have

$$e_{\Gamma(k,G)}(\mathcal{F}_k, U) \ge \frac{1}{4^{k+1}} p^k \frac{N}{T_0} |\mathcal{F}_k|.$$

Combining the last two inequalities we infer

$$\left| \bigcup_{y \in Y} C(y) \right| = |U| \ge \left(\frac{1}{4^{k+1}} \frac{1}{T_0} - 6\xi \right) N \overset{(28)}{\ge} \xi N \ge \xi B n \overset{(28)}{=} n \ge |W_{\ell+1}| \ge |Y|,$$

which contradicts (36). This contradiction shows that (36) does not hold, that is, Hall's condition (35) does hold. Hence, there exists a system of representatives for $(C(y))_{y \in W_{\ell+1}}$, i.e., an injective mapping $\psi \colon W_{\ell+1} \to \bigcup_{y \in W_{\ell+1}} C(y)$ such that $\psi(y) \in C(y)$ for every $y \in W_{\ell+1}$.

Finally, we extend φ_ℓ and define $C_{\ell+1}(z)$ for $z \in \bigcup_{j=\ell+2}^{\widetilde{\Delta}} W_j$. For that we set

$$\varphi_{\ell+1}(w) = \begin{cases} \varphi_\ell(w)\,, & \text{if } w \in \bigcup_{j=1}^{\ell} W_j, \\ \psi(w)\,, & \text{if } w \in W_{\ell+1}\,. \end{cases}$$

Note that every $z \in \bigcup_{j=\ell+2}^{\widetilde{\Delta}} W_j$ has at most one neighbour in $W_{\ell+1}$, as otherwise there would be two vertices y and $y' \in W_{\ell+1}$ with distance at most 2 in H, which contradicts the fact that g and f are valid vertex colourings of H^3. Consequently, for every $z \in \bigcup_{j=\ell+2}^{\widetilde{\Delta}} W_j$ we can set

$$C_{\ell+1}(z) = \begin{cases} C_\ell(z)\,, & \text{if } N_H(z) \cap W_{\ell+1} = \emptyset, \\ C_\ell(z) \cap N_{G_R}(\varphi_{\ell+1}(y))\,, & \text{if } N_H(z) \cap W_{\ell+1} = \{y\}. \end{cases}$$

In what follows we show that $\varphi_{\ell+1}$ and $C_{\ell+1}(z)$ for every $z \in \bigcup_{j=\ell+2}^{\widetilde{\Delta}} W_j$ have the desired properties and validate $(\mathcal{S}_{\ell+1})$.

First of all, from (a) of $(\mathcal{S}_\ell)$, combined with $\varphi_{\ell+1}(y) \in C(y) \subseteq C_\ell(y)$ for every $y \in W_{\ell+1}$ and the property that $\{\varphi_{\ell+1}(y) \colon y \in W_{\ell+1}\}$ is a system of distinct representatives, we infer that $\varphi_{\ell+1}$ is indeed a partial embedding of $H[\bigcup_{j=1}^{\ell+1} W_j]$.

Next we shall verify properties (a) and (b) of $(\mathcal{S}_{\ell+1})$. So let $z \in \bigcup_{j=\ell+2}^{\widetilde{\Delta}} W_j$ be fixed. If $N_H(z) \cap W_{\ell+1} = \emptyset$, then $C_{\ell+1}(z) = C_\ell(z)$, $\mathrm{ldeg}_g^{\ell+1}(z) = \mathrm{ldeg}_g^{\ell}(z)$, which yields (a) and (b) of $(\mathcal{S}_{\ell+1})$ for that case. If, on the other hand, $N_H(z) \cap W_{\ell+1} \ne \emptyset$, then there exists a unique neighbour $y \in W_{\ell+1}$ of H (owing to the fact that g is a refinement of a valid vertex colouring of H^3). Because of the definition of $C_{\ell+1}(z) = C_\ell(z) \cap N_{G_R}(\varphi_{\ell+1}(y))$ part (a) of $(\mathcal{S}_{\ell+1})$ follows in this case. Moreover,

since $\varphi_{\ell+1}(y) \in C(y)$, we infer directly from (b') that (b) of $(\mathcal{S}_{\ell+1})$ is satisfied in this case.

Finally, we verify property (c) of $(\mathcal{S}_{\ell+1})$. Let $\{z, z'\}$ be an edge of H with $z, z' \in \bigcup_{j=\ell+2}^{\widetilde{\Delta}} W_j$. We consider three cases, depending on the size of $N_H(z) \cap W_{\ell+1}$ and of $N_H(z') \cap W_{\ell+1}$. If $N_H(z) \cap W_{\ell+1} = \emptyset$ and $N_H(z') \cap W_{\ell+1} = \emptyset$, then part (c) of $(\mathcal{S}_{\ell+1})$ follows directly from part (c) of $(\mathcal{S}_\ell)$ and $\varepsilon_{\ell+1} \geq \varepsilon_\ell$, combined with $C_{\ell+1}(z) = C_\ell(z), C_{\ell+1}(z') = C_\ell(z')$. If $N_H(z) \cap W_{\ell+1} = \{y\}$ and $N_H(z') \cap W_{\ell+1} = \emptyset$, then (c) of $(\mathcal{S}_{\ell+1})$ follows from (c') and the definition of $C_{\ell+1}(z)$ and $C_{\ell+1}(z')$. If $N_H(z) \cap W_{\ell+1} = \{y\}$ and $N_H(z') \cap W_{\ell+1} = \{y'\}$, then $y = y'$, as otherwise there would be a y-y'-path in H with three edges, i.e., $\{y, y'\}$ would be an edge in H^3, which would imply that $g(y) \neq g(y')$. Consequently, (c) of $(\mathcal{S}_{\ell+1})$ follows from (c') and the definition of $C_{\ell+1}(z)$ and $C_{\ell+1}(z')$.

We have therefore verified (a)–(c) of $(\mathcal{S}_{\ell+1})$, thus concluding the induction step. The proof of Lemma 19 follows by induction. $\square$

5. Concluding remarks

Theorem 1 asserts the existence of a partition universal graph G for the class of graphs $\mathcal{H}_{\Delta,n}$ with G having $O(n^{2-1/\Delta} \log^{1/\Delta} n)$ edges. We believe it would be rather interesting to decide whether one can substantially improve on this upper bound. In particular, we believe that bringing this bound down to a bound of the form $O(n^{2-1/\Delta-\varepsilon})$ for some $\varepsilon > 0$ would require a completely new idea. The only lower bound that we know is of the form $\Omega(n^{2-2/\Delta})$ (see Remark 2(i)).

Our proof of Theorem 1 is heavily based on random graphs, and we do not know how to prove this result or anything numerically similar by constructive means. In particular, for instance, we do not know whether (N, d, λ)-graphs with reasonable parameters are partition universal for $\mathcal{H}_{\Delta,n}$.

Another interesting question is whether one can prove Theorem 1 without the regularity method.

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