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Sojourn times

C. Malchin , H. Daduna

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Bottleneck analysis of discrete time networks: Sojourn times

Christian Malchin Hans Daduna
Hamburg University
Department of Mathematics
20146 Hamburg
Germany

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Abstract

For a closed cycle of Bernoulli servers in discrete time with a single bottleneck we prove weak convergence of the suitably rescaled joint queue length vector for all nodes and weak convergence of the suitably rescaled cycle time when the network approaches heavy traffic regime. We further investigate asymptotic dependencies of the sojourn time vector and find unexpected behaviour of the covariance structure. A technical device for our proofs is a Harrison-type formula for arrival probabilities which is of independent interest.

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Keywords: sojourn times, cycle times, bottleneck, discrete time cyclic network, central limit theorem

1 Introduction

Bottleneck behaviour of closed product form queueing networks in continuous time is studied from the early days of the theory and dates back to the general model construction of Gordon and Newell [GN67]. It is well known that there are two essentially different pictures.
(i) All servers have the same load: Then the total population in system is shared equally by all nodes up to random fluctuations.
(ii) Differently loaded servers exist: Then bottlenecks occur, which in the simplest case with exactly one slowest server means that almost the whole population is queued up at this slowest server.

Similar pictures are observed in discrete time networks of queues. In the cyclic network case which we shall study in this paper this means that (i) all servers have the same service probability (the case of balanced machines), and (ii) if there is exactly one server with smallest service probability then this node evolves as a unique bottleneck. For a more detailed introduction into this class of models which emphasizes the differences of especially case (i) to the continuous time setting see [MD04].

Our focus in the present paper is the possibly more interesting case (ii): We consider cyclic discrete time networks of Bernoulli servers (geometrical nodes under first-come-first-served). Our main interest is in the detailed travel time behaviour of individual customers: The starting point is the steady state distribution of a customer's vector of successive sojourn times at the different nodes during a cycle. We further are interested in the customer's cycle time distribution. These distributions are in principle well known and given in the transform domain by their respective z-transforms (generating functions) [Dad01].
We transform the obtained formulae in a way that allows to prove weak convergence results for the customer's travel time behaviour when the bottleneck dominates the travel times – in an eventually dramatic way.
In continuous time this was studied under the heading of *influence of the slowest server* by Boxma [Box88], including qualitative characterization of the speed of convergence to total dominance. We study similar questions in Section 3 and are able to prove a sharper characterization (o-convergence instead of O-convergence).

In continuous time theory for closed cyclic product form networks with unique bottleneck the *usual interpretation* of the results obtained by Gordon and Newell [GN67] is that with increasing number of customers the bottleneck node is asymptotically approaching a Poissonian source for the network, while all the other nodes eventually form an open ergodic tandem system the behaviour of which is well understood: Local geometrical queue length distribution and independence over the nodes in steady state. While this is generally understood as a statement about the queue length description of the cycle, it seems to be rather obvious that a similar property should hold for cycle times and their asymptotic behaviour, respectively for the joint sojourn times of a customer during a cycle. In Section 4 we revisit this problem. Investigation of the asymptotic behaviour of (mixed) moments of sojourn time indicates that there occur some unexpected phenomena, which we discuss in detail: The covariances between a customer's sojourn times at the bottleneck and the other non-bottleneck nodes does not vanish in the limit. In contrast to this the *usual interpretation* states that in the limiting open tandem system the Poissonian source is independent from the service mechanism at the stations. This observation is the starting point to reprove the limit theorem for the joint sojourn time vector of a customer's visits at the non-bottleneck nodes, and support anew the *usual interpretation.*

Combining the results about the *influence of the slowest server* and about the *usual interpretation* of the limiting behaviour of non-bottleneck nodes and following the intuition suggested by these pictures we study in Section 5 jointly the asymptotics of the overall sojourn time vector. Our main result in Theorem 5.1 is a weak convergence limit theorem which in one coordinate looks like a central limit theorem, because rescaled sojourn times at the bottleneck are asymptotically normal, while at the other nodes we have without scaling convergence to geometric distributions. Furthermore the limiting distribution has independent coordinates.
This result and the *influence of the slowest server* observation suggest that there should be a standard normal limit of the scaled cycle times, which is proved in Theorem 5.2.

In Section 2 we provide the necessary prerequisites for our computations. The main device is a Harrison-type formula for norming constants in the discrete time setting. Such a formula was already obtained for the time stationary distribution by Pestin and Ramakrishnan [PR99]. We extend this formula and prove an analogue for the customer stationary distribution. The existence of a Harrison-type formula for the norming constant of the customer stationary distribution is surprising due to the rather strange form of these norming constants. We apply these formulas to sojourn time expressions found in the literature and utilize the so transformed expressions in our limit theorems.

Due to the occurrence of network applications which are based on a generic internal lattice structured time scale, networks in discrete time have found much interest in recent years in the literature. E.g. in the ATM protocol for high speed networks at least three relevant levels with different time scales are considered, call level, burst level, and cell level. The latter two levels can be modelled on a basis of generic discrete scales. For a discussion see the editorial introduction and several papers in the special volume [TGBT94]. A more recently evolved class of applications for such models is the analysis and control of Wireless Local Area Networks (WLANs), for more details see the relevant contributions in the Proceedings of the 18th International Teletraffic Congress (2003) [CLTG03], especially [SV03], [NSB$^+$03], [MKC03] and [KN03]. For surveys on discrete time network theory see [Woo94], [BK93], [CMP99], [Dad01], and the references cited there.
Our results contribute to better understanding the behaviour of such systems under heavy traffic conditions. Although we deal only with cyclic networks, our models are of relevance to transfer times in networks with general topology. The principle to transform problems in general networks into problems in linear systems is known as *principle of adjusted transfer rates* in continuous time theory, the application in discrete time is described in [Dad01][Section 4.1].

Conventions and notation:
$\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is the set of nonnegative integers. Empty sums are 0, empty products are 1. $\eta(n, m) := 1 - \delta(n, m)$ is the complementary Kronecker delta.

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2 Description of the system and prerequisites

We consider closed cyclic queueing networks with M nodes and N indistinguishable, clockwise circulating customers. The time scale for the system is $\mathbb{N}_0$. The servers at the nodes $Q[i], i = 1. \ldots, M$, are state independent Bernoulli servers, which are working independently, i. e. station $Q[i]$ is a single-server with service probability $p_i \in (0, 1)$, infinite waiting room and first-come-first-serve (FCFS) queueing discipline. If at time instant n at node $Q[i]$ a customer is in service, then his service ends at time instant $n+1$ with probability p_i. With probability $q_i := 1-p_i$ this customer will stay on at node $Q[i]$ requesting at least one more unit of service time. Customers being served at station $Q[i]$ jump instantaneously to node $Q[i + 1]$ (resp. to $Q[1]$ if a service ends at node $Q[M]$). If at some node at the same time an arrival and a departure occur we prescribe that the departure event takes place first (*late arrivals, departures first*, LA-DF). As a consequence, a customer arriving at the end of time slot $[n - 1, n)$ at $Q[i]$ has to stay there for at least until the end of time slot $[n, n + 1)$, that is, his sojourn time at any queue is one time unit at minimum. A sojourn time at $Q[i]$ is the sum of the waiting time at node $Q[i]$ and the subsequent service time there.

For station $Q[i]$ let $X_i(t)$ denote the queue length (= number of waiting customers + customer in service, if any) at time $t \in \mathbb{N}_0$. Then the joint queue length process $X = ((X_1(t), \ldots, X_M(t)) : t \in \mathbb{N}_0)$ is a discrete time Markov chain with state space

$$Z(M, N) := \{(n_1, \ldots, n_M) \in \mathbb{N}_0^M \mid n_1 + \cdots + n_M = N\}.$$

$(X_1(t), \ldots, X_M(t)) = (n_1, \ldots, n_M)$ indicates that n_i customers reside at time t at node $Q[i]$.
X is ergodic with unique stationary and limiting distribution [PR94]

$$\pi^{M,N}(n_1, \ldots, n_M) = G(M, N)^{-1} \prod_{i=1}^{M} \left(\frac{q_i}{p_i}\right)^{n_i} \left(\frac{1}{q_i}\right)^{\eta(0,n_i)}, \quad (n_1, \ldots, n_M) \in Z(M, N)$$

with normalizing constant

$$G(M, N) = \sum_{(n_1,\ldots,n_M)\in Z(M,N)} \prod_{i=1}^{M} \left(\frac{q_i}{p_i}\right)^{n_i} \left(\frac{1}{q_i}\right)^{\eta(0,n_i)}. \tag{1}$$

The investigations described in this paper are concerned mainly with customers' sojourn and cycle times. To determine these times we tag one customer to follow him on his itinerary through the cycle. To compute the joint sojourn times distribution of the tagged customer, the knowledge of the arrival distribution $\pi_1^{M,N}$ at arrival instants of the tagged customer at node $Q[1]$ is necessary. Let $\pi_k^{M,N}$ be defined as the conditional distribution of the other customers seen by the test customer jumping from node $Q[k-1]$ to node $Q[k]$ at his arrival instant at node $Q[k]$, given that such a transition of the tagged customer takes place ($k = 1$ means a jump from node $Q[M]$ to $Q[1]$).

Theorem 2.1. *[Dad96] Let $X = (X_n)_{n\in\mathbb{Z}}$ be the stationary continuation of $(X_n)_{n\in\mathbb{N}_0}$ under $\pi^{M,N}$. Further let $A(k)$ be the event* "At time instant $n = 0$ an arrival at node $Q[k]$ takes place"*. Then*

$$\begin{aligned} \pi_k^{M,N}(n_1, \ldots, n_m) &:= P\big(X_0 = (n_1, \ldots, n_{k-1}, n_k + 1, n_{k+1}, \ldots, n_m) \mid A(k)\big) \\ &= G_k(M, N)^{-1} \cdot \left(\frac{q_k}{p_k}\right)^{n_k} \prod_{\substack{i=1 \\ i\neq k}}^{M} \left(\frac{q_i}{p_i}\right)^{n_i} \left(\frac{1}{q_i}\right)^{\eta(0,n_i)} \end{aligned} \tag{2}$$

for all $(n_1, \ldots, n_M) \in Z(M, N-1)$.

The arrival probability $\pi_k^{M,N}$ depends on k, the number of the node where the arrival appears: the expression in (2) is asymmetric with respect to k and the arrival probability keeps trace of the arriving customer. Note that the value of the normalizing constant

$$G_k(M, N) = \sum_{\vec{n}\in Z(M,N-1)} \left(\frac{q_k}{p_k}\right)^{n_k} \prod_{\substack{i=1 \\ i\neq k}}^{M} \left(\frac{q_i}{p_i}\right)^{n_i} \left(\frac{1}{q_i}\right)^{\eta(0,n_i)} \tag{3}$$

is independent of the node number $k \in \underline{M}$, see [Dad01], lemma 7.3. The common value is denoted by $G_{\mathrm{A}}(M, N)$.

If necessary, we use the subscript (M, N) similarly as in $\pi^{M,N}$ and $\pi_k^{M,N}$ to indicate that the underlying distribution $P = P_{(M,N)}$ refers to M nodes and a total population of N circulating customers (including the test customer).

2.1 Norming constants

The structure of the normalizing constants in the steady state (1) and in the arrival distribution (3) occurs in a natural way. On the other hand from the continuous time analogue it is well known that computing norming constants is a difficult task, see, e.g., [BB80]. Because the norming constants in discrete time are even less smooth than in continuous time (see the Buzen-type formulae in [Dad01][Proposition 3.18 and 3.19]) similar problems arise.

A useful device to overcome at least some of the difficulties with the continuous time expressions are the formulae developed by Koenigsberg [Koe58] and Harrison [Har85a]. Surprisingly the more cumbersome discrete time constants admit Harrison-type formulae as well. These will be described next. For simplicity of the presentation we restrict our investigations to the case of either pairwise distinct probabilities or to the case of identical service probabilities at all nodes.

The first such result, see (4), for discrete time systems was derived by Pestien and Ramakrishnan for the steady state normalizing constant, see Theorem 4.1 of [PR94]. We shall provide a simplified proof in the appendix and derive similar expressions for (3) and, even more, for sojourn and cycle time distributions below.

We start with a useful lemma, which simplifies many of our later computations. The proof is given in the Appendix, see Section 6.1.

Lemma 2.2. *Let $x_1, \ldots, x_M$ be pairwise distinct complex numbers ($x_i \neq x_j$ for all $i, j \in \underline{M}$, $i \neq j$) and $x_i \neq 1$, for every $i \in \underline{M}$, $M \in \mathbb{N}$. Let $y_i := 1 - x_i$ and $N \in \mathbb{N}$. Then the following formulae hold.*

$$\sum_{n \in Z(M,N)} \prod_{i=1}^{M} \left(\frac{x_i}{y_i}\right)^{n_i} x_i^{\delta(0,n_i)-1} = \sum_{i=1}^{M} \frac{1}{x_i} \left(\frac{x_i}{y_i}\right)^N \prod_{\substack{j=1 \\ j \neq i}}^{M} \frac{y_j}{y_j - y_i} \tag{4}$$

and

$$\sum_{n \in Z(M,N)} \left(\frac{x_1}{y_1}\right)^{n_1} \prod_{i=2}^{M} \left(\frac{x_i}{y_i}\right)^{n_i} x_i^{\delta(0,n_i)-1} = \sum_{i=1}^{M} \left(\frac{x_i}{y_i}\right)^N \prod_{\substack{j=1 \\ j \neq i}}^{M} \frac{y_j}{y_j - y_i}. \tag{5}$$

From Lemma 2.2 we have as an immediate consequence

Corollary 2.3. *Let $M \in \mathbb{N}$, $0 < p_i < 1$, $q_i := 1 - p_i$, $p_i \neq p_j$ for all $i, j \in \underline{M}$, $i \neq j$ (pairwise distinct service probabilities). Then for $N \in \mathbb{N}$ the steady state normalizing constant is [PR94][Theorem 4.1]*

$$G(M,N) = \sum_{i=1}^{M} \left(\frac{1}{q_i}\right) \left(\frac{q_i}{p_i}\right)^N \prod_{\substack{j=1 \\ j \neq i}}^{M} \frac{p_j}{p_j - p_i} \tag{6}$$

and the normalizing constant of the arrival distribution is

$$G_{\mathrm{A}}(M, N+1) = \sum_{i=1}^{M} \left(\frac{q_i}{p_i}\right)^N \prod_{\substack{j=1 \\ j \neq i}}^{M} \frac{p_j}{p_j - p_i} \quad . \tag{7}$$

Formulae (6) and (7) are in the spirit of the formulae obtained by Koenigsberg [Koe58] and Harrison [Har85b] for the continuous time models. Additional care is needed here because of the differences in the expression caused by idle nodes, and of the different structure due to the arrival situation.

For pairwise distinct real numbers $p_i \in (0,1)$, $i \in \underline{M}$ and $q_i := 1 - p_i$ we define coefficients $C_{i,N}$ by

$$C_{i,N} := \frac{\left(\frac{q_i}{p_i}\right)^{N-1} \prod_{\substack{j=1\\ j\neq i}}^{M} \frac{p_j}{p_j - p_i}}{\sum_{i=1}^{M} \left(\frac{q_i}{p_i}\right)^{N-1} \prod_{\substack{j=1\\ j\neq i}}^{M} \frac{p_j}{p_j - p_i}} \ . \tag{8}$$

For notational convenience we assume henceforth $p_1 < p_2 < \cdots < p_M$ such that node $Q[1]$ is the *bottleneck*. (The steady state distribution and the joint sojourn time distribution is independent of the nodes' locations with respect to their service probabilities, see Theorem 2.5.)

Lemma 2.4. *The constants $C_{i,N}$ have the following properties:*

(i)

$$\sum_{i=1}^{M} C_{i,N} = 1 \ ,$$

(ii)

$$\lim_{N\to\infty} C_{1,N} = 1 \ ,$$

(iii)

$$\lim_{N\to\infty} N^r C_{k,N} \left(\frac{p_2}{p_1}\right)^N = 0, \quad k \neq 1, \quad r \in \mathbb{R}_+$$

and (in particular)

$$\lim_{N\to\infty} C_{k,N} = 0, \qquad \lim_{N\to\infty} N C_{k,N} = 0, \quad k \neq 1 \ ,$$

(iv)

$$C_{k,N} > 0 \textit{ for } k \textit{ odd and } C_{k,N} < 0 \textit{ for } k \textit{ even} \ .$$

Proof. Property (i) follows immediately from the definition. The ordering $0 < p_1 < p_2 < \cdots < p_M < 1$ implies

$$1 > \frac{p_1 q_2}{q_1 p_2} > \frac{p_1 q_3}{q_1 p_3} > \cdots > \frac{p_1 q_M}{q_1 p_M} > 0 \ .$$

Therefore

$$\lim_{N\to\infty} C_{1,N} = \lim_{N\to\infty} \frac{\prod_{\substack{j=1\\ j\neq 1}}^{M} \frac{p_j}{p_j - p_1}}{\prod_{j=2}^{M} \frac{p_j}{p_j - p_1} + \sum_{i=2}^{M} \left(\frac{p_1 q_i}{q_1 p_i}\right)^{N-1} \prod_{\substack{j=1\\ j\neq i}}^{M} \frac{p_j}{p_j - p_1}} = 1 \ .$$

In order to proof (iii) we write

$$N^r C_{k,N}\left(\frac{p_2}{p_1}\right)^N = N^r\left(\frac{p_2}{p_1}\right)^N \frac{\left(\frac{q_k}{p_k}\right)^{N-1}\prod_{\substack{j=1\\ j\neq k}}^{M}\frac{p_j}{p_j-p_k}}{\sum_{i=1}^{M}\left(\frac{q_i}{p_i}\right)^{N-1}\prod_{\substack{j=1\\ j\neq i}}^{M}\frac{p_j}{p_j-p_i}}$$

$$= \frac{N^r\left(\frac{p_2 q_k}{q_1 p_k}\right)^{N-1}\frac{p_2}{p_1}\prod_{\substack{j=2\\ j\neq k}}^{M}\frac{p_j}{p_j-p_k}}{\prod_{j=2}^{M}\frac{p_j}{p_j-p_1}+\sum_{i=2}^{M}\left(\frac{p_1 q_i}{q_1 p_i}\right)^{N-1}\prod_{\substack{j=1\\ j\neq i}}^{M}\frac{p_j}{p_j-p_i}} .$$

Taking into account $p_2 q_i/q_1 p_i < 1$ and $p_1 q_i/q_1 p_i < 1$ for every $i = 2, \ldots, M$ yields (iii).
Finally, by Corollary 2.3 the denominator of the rhs of (8) is equal to the normalizing constant $G_{\mathrm{A}}(M, N)$ and therefore positive. This is utilized to prove property (iv) by the alternating signs of

$$\left(\frac{q_i}{p_i}\right)^{N-1}\prod_{\substack{j=1\\ j\neq i}}^{M}\frac{p_j}{p_j-p_i} .$$

□

2.2 Joint sojourn times and cycle times

Let $(S_{1,N}, \ldots, S_{M,N})$ denote the vector of the sojourn times of the tagged customer at the successive nodes $Q[i], i = 1, \ldots, M$, of the cycle with N customers. The cycle is started when the tagged customer arrives at $Q[1]$ and ends when his next arrival occurs there. Then $S_N := S_{1,N} + \cdots + S_{M,N}$ is his cycle time. We always assume steady state conditions, i.e., when the tagged customer starts his cycle, he observes the other $N-1$ customers distributed according to $\pi_1^{M,N}$ from Theorem 2.1. The relevant known facts are as follows.

Theorem 2.5. *[Dad97] The generating function of the joint sojourn time vector $(S_{1,N}, \ldots, S_{M,N})$ in steady state with population size N is*

$$g^{(M,N)}(u_1, \ldots, u_M) = G_{\mathrm{A}}(M,N)^{-1}\sum_{(n_1,\ldots,n_M)\in Z(M,N-1)}\left(\frac{q_1}{p_1}\right)^{n_1}\left(\frac{p_1 u_1}{1-q_1 u_1}\right)^{n_1+1} \cdot\prod_{i=2}^{M}\left(\frac{q_i}{p_i}\right)^{n_i}(q_i u_i)^{\delta(0,n_i)-1}\left(\frac{p_i u_i}{1-q_i u_i}\right)^{n_i+1} . \tag{9}$$

The generating function of the cycle time distribution under this conditions is

$$h^{(M,N)}(u) \quad = \quad G_{\mathrm{A}}(M,N)^{-1} \sum_{(n_1,\ldots,n_M)\in Z(M,N-1)} \left(\frac{q_1}{p_1}\right)^{n_1} \left(\frac{p_1 u}{1-q_1 u}\right)^{n_1+1} \tag{10}$$
$$\cdot \prod_{i=2}^{M} \left(\frac{q_i}{p_i}\right)^{n_i} \left(\frac{p_i u}{1-q_i u}\right)^{n_i+1} (q_i u)^{\delta(0,n_i)-1}.$$

In the literature on continuous time linear networks it was remarked that the structure of the Laplace transform of the joint sojourn times and cycle time distribution resembles the structure of the norming constant. Due to the special role played by empty nodes and the asymmetry of the arrival distribution a similar observation is here not as obvious, but exists as we shall show below for the case of distinct service probabilities. Lemma 2.2 and Corollary 2.3 are the key results to explore the structure of joint sojourn times and cycle time in the spirit of Harrison-type expressions. Then the properties of the $C_{i,N}$ will enable us to compute asymptotics for the distributions.

Theorem 2.6. *Assume we have pairwise distinct service probabilities ($p_i \neq p_j$ for all $i,j \in \underline{M}$, $i \neq j$), a total of N customers circulating (including the test customer), and steady state conditions. Then the generating function of the cycle time distribution $S_N := S_{1,N} + \cdots + S_{M,N}$ is*

$$h^{(M,N)}(u) = \sum_{i=1}^{M} C_{i,N} \left(\frac{p_i u}{1-q_i u}\right)^{N} \quad , \quad u \in [0,1) \ . \tag{11}$$

Let

$$B := \{(v_1,\ldots,v_M) \in \mathbb{R}^M \mid \exists (i,j) \in \underline{M}^2,\ i \neq j \ : q_i v_i = q_j v_j\} \ . \tag{12}$$

The generating function of the joint sojourn times distribution on $[0,1)^M \setminus B$ is

$$g^{(M,N)}(u_1,\ldots,u_M) = \sum_{i=1}^{M} C_{i,N} \left(\frac{p_i u_i}{1-q_i u_i}\right)^{N} \prod_{\substack{j=1\\ j\neq i}}^{M} \left(\frac{q_i - q_j}{q_i u_i - q_j u_j} u_j\right) \ . \tag{13}$$

The Fourier transform of the joint sojourn times distribution is

$$\tilde{g}^{(M,N)}(x_1,\ldots,x_M) = \sum_{j=1}^{M} C_{j,N} \left(\frac{p_j e^{ix_j}}{1-q_j e^{ix_j}}\right)^{N} \prod_{\substack{l=1\\ l\neq j}}^{M} \left(\frac{q_j - q_l}{q_j e^{ix_j} - q_l e^{ix_l}} e^{ix_l}\right), \quad x \in \mathbb{R}^M \ . \tag{14}$$

Remark: Obviously the set B is the union of the planes $E_{i,j} := \{v \in \mathbb{R}^M \mid q_i v_i = q_j v_j\}$, $i \neq j$ and the point 1 is not contained in any of these (finitely many) planes. Consequently there is a neighbourhood U of 1 such that $U \cap B = \varnothing$ and the representation (13) of $g^{(M,N)}$ is valid in $U \cap [0,1)^M$.

Proof. Applying (5) of Lemma 2.2 to (10) yields

$$
\begin{aligned}
&\sum_{n\in Z(M,N-1)} \left(\frac{q_1}{p_1}\right)^{n_1} \left(\frac{p_1 u}{1-q_1 u}\right)^{n_1+1} \prod_{i=2}^{M} \left(\frac{q_i}{p_i}\right)^{n_i} \left(\frac{p_i u}{1-q_i u}\right)^{n_i+1} (q_i u)^{\delta(0,n_i)-1} \\
&= \left(\prod_{i=1}^{M} \frac{p_i u}{1-q_i u}\right) \sum_{n\in Z(M,N-1)} \left(\frac{q_1 u}{1-q_1 u}\right)^{n_1} \prod_{i=2}^{M} \left(\frac{q_i u}{1-q_i u}\right)^{n_i} (q_i u)^{\delta(0,n_i)-1} \\
&= \sum_{i=1}^{M} \left(\frac{q_i}{p_i}\right)^{N-1} \left(\frac{p_i u}{1-q_i u}\right)^{N} \prod_{\substack{j=1\\ j\neq i}}^{M} \frac{p_j}{p_j-p_i}.
\end{aligned}
$$

Using equation (7) of Corollary 2.3 leads to

$$
h^{(M,N)}(u) = \frac{\sum_{i=1}^{M} \left(\left(\frac{q_i}{p_i}\right)^{N-1} \prod_{\substack{j=1\\ j\neq i}}^{M} \frac{p_j}{p_j-p_i} \right) \left(\frac{p_i u}{1-q_i u}\right)^{N}}{\sum_{i=1}^{M} \left(\frac{q_i}{p_i}\right)^{N} \prod_{\substack{j=1\\ j\neq i}}^{M} \frac{p_j}{p_j-p_i}} = \sum_{i=1}^{M} C_{i,N} \left(\frac{p_i u_i}{1-q_i u_i}\right)^{N} .
$$

Applying (5) of Lemma 2.2 similarly to $g^{(M,N)}(u_1,\ldots,u_M)$ with $(u_1,\ldots,u_M) \in [0,1)^M \setminus B$ to (9) yields

$$
\begin{aligned}
&\sum_{n\in Z(M,N-1)} \left(\frac{q_1}{p_1}\right)^{n_1} \left(\frac{p_1 u_1}{1-q_1 u_1}\right)^{n_1+1} \prod_{i=2}^{M} \left(\frac{q_i}{p_i}\right)^{n_i} q_i^{\delta(0,n_i)-1} \left(\frac{p_i u_i}{1-q_i u_i}\right)^{n_i+1} u_i^{\delta(0,n_i)-1} \\
&= \left(\prod_{i=1}^{M} \frac{p_i u_i}{1-q_i u_i}\right) \sum_{n\in Z(M,N-1)} \left(\frac{q_1 u_1}{1-q_1 u_1}\right)^{n_1} \prod_{i=2}^{M} \left(\frac{q_i u_i}{1-q_i u_i}\right)^{n_i} (q_i u_i)^{\delta(0,n_i)-1} \\
&= \sum_{i=1}^{M} \left(\frac{q_i}{p_i}\right)^{N-1} \left(\frac{p_i u_i}{1-q_i u_i}\right)^{N} \prod_{\substack{j=1\\ j\neq i}}^{M} \frac{p_j u_j}{q_i u_i - q_j u_j}.
\end{aligned}
$$

And applying (5) of Lemma 2.2 to $\tilde{g}^{(M,N)}(x_1,\ldots,x_M)$ with $(x_1,\ldots,x_M) \in \mathbb{R}^M$ obtained in the form of (9) yields

$$
\begin{aligned}
&\sum_{n\in Z(M,N-1)} \left(\frac{q_1}{p_1}\right)^{n_1} \left(\frac{p_1 e^{ix_1}}{1-q_1 e^{ix_1}}\right)^{n_1+1} \prod_{j=2}^{M} \left(\frac{q_j}{p_j}\right)^{n_j} q_j^{\delta(0,n_j)-1} \left(\frac{p_j e^{ix_j}}{1-q_j e^{ix_j}}\right)^{n_j+1} \left(e^{ix_j}\right)^{\delta(0,n_j)-1} \\
&= \sum_{j=1}^{M} \left(\frac{q_j}{p_j}\right)^{N-1} \left(\frac{p_j e^{ix_j}}{1-q_j e^{ix_j}}\right)^{N} \prod_{\substack{k=1\\ k\neq j}}^{M} \frac{p_k e^{ix_k}}{q_j e^{ix_j} - q_k e^{ix_k}} .
\end{aligned}
$$

We only have to ensure that $q_1 e^{ix_1},\ldots,q_M e^{ix_M}$ are pairwise distinct for any choice of $(x_1,\ldots,x_M) \in \mathbb{R}^M$. But $q_k e^{kx_k} = q_l e^{ix_l}$ can hold only if $|q_k| = |q_l|$. Using equation (7) of Corollary 2.3 gives the desired results. □

Theorem 2.6 states that in the case of pairwise distinct service probabilities the cycle time distribution is a linear combination of negative binomial distributions convolved with a Dirac-distribution:

$$P^{S_N} = \sum_{i=1}^{M} C_{i,N} \left(\mathrm{Nb}(1,p_i) * \epsilon_1\right)^{*N} = \sum_{i=1}^{M} C_{i,N} \mathrm{Nb}(N,p_i) * \epsilon_N \quad . \tag{15}$$

Therefore the **mean cycle time** is

$$E(S_N) = \sum_{i=1}^{M} C_{i,N} N \frac{1}{p_i} \quad . \tag{16}$$

It should be noted, that $C_{i,N} N \frac{1}{p_i}$ is not the test customer's mean sojourn time at node $Q[i]$ (recall that some of the coefficients $C_{i,N}$ are negative). The mean sojourn time at node $Q[i]$ is given in equation (43) below.
Taking derivatives of the generating function, we easily find the **cycle time variance** (see page 33 in the Appendix)

$$\mathrm{Var}(S_N) = N \sum_{i=1}^{M} C_{i,N} \frac{q_i}{p_i^2} + N^2 \sum_{\substack{(i,j) \in \underline{M}^2 \\ i<j}} C_{i,N} C_{j,N} \left(\frac{1}{p_i} - \frac{1}{p_j} \right)^2 \quad . \tag{17}$$

We mention here an interesting observation which readily comes out from the definition of the $C_{i,N}$ and Corollary 2.3. We have for pairwise distinct service probabilities

$$\sum_{i=1}^{M} \frac{1}{p_i} C_{i,N} = \frac{G(M,N)}{G_{\mathrm{A}}(M,N)}$$

and therefore from (16)

$$E(S_N) = N \, \frac{G(M,N)}{G_{\mathrm{A}}(M,N)} \quad . \tag{18}$$

Denoting the network's *throughput* $N/E(S_N)$ by θ (which is the *local progress* of the network in the sense of ([PR99]) we obtain

Proposition 2.7. *In a closed cyclic M-station tandem with N customers, which is in equilibrium, the throughput θ is given by*

$$\theta = \frac{G_{\mathrm{A}}(M,N)}{G(M,N)} \quad . \tag{19}$$

So it comes out explicitly that the throughput is related to both the time stationary and the customer stationary behaviour of the network. In the continuous time setting this is hidden by the fact that the customer stationary normalizing constant equals the time stationary normalizing constant of an network with one customer less (see [CY01]).
The formula 19 is valid for the general case as well as can be seen by direct computation.

2.3 Sojourn times and cycle times: Densities

In the following a random vector $(S_1, \ldots, S_M)$ in $\mathbb{R}_+^M$, with generating function $g^{(M,N)}$, see (9) and (13) is interpreted as the family of successive sojourn times under steady-state conditions. Similarly a random variable S with generating function $h^{(M,N)}$, see (10) and (11) is interpreted as the steady-state cycle time.

The generating function $g^{(M,N)}$ of the joint distribution of the successive sojourn times can be inverted with a closed form expression for the corresponding pmf.
Recall that $Z(M,N) = \left\{(x_1, \ldots, x_M) \in \mathbb{N}_0^M \mid x_1 + \cdots + x_M = N\right\}$ and define

$$Z^\star(M,N) := \left\{(x_1, \ldots, x_M) \in \mathbb{N}^M \mid x_1 + \cdots + x_M = N\right\}$$

Lemma 2.8. *[MD04] The distribution of the successive sojourn times $(S_1, \ldots, S_M)$ experienced by the test-customer on his itinerary under steady-state conditions possesses the pmf $p^{(M,N)}$, given by*

$$p^{(M,N)}(s_1, \ldots, s_M) = G_{\mathrm{A}}(M,N)^{-1} \binom{s_1 + \cdots + s_M - 1}{N-1} \prod_{i=1}^{M} p_i q_i^{s_i - 1} \mathbf{1}[(s_1, \ldots, s_M) \in \mathbb{N}^M]. \tag{20}$$

The cycle time $S := S_1 + \cdots + S_M$ has the pmf $q^{(M,N)}$, given by

$$q^{(M,N)}(s) = G_{\mathrm{A}}(M,N)^{-1} \binom{s-1}{N-1} \sum_{(s_1, \ldots, s_M) \in Z^\star(M,s)} \prod_{i=1}^{M} p_i q_i^{s_i - 1}. \tag{21}$$

Note that $q^{(M,N)}(s) = 0$ for $s < \max(M,N)$: if $s < M$ holds, then $Z^\star(M,s) = \emptyset$, hence the empty sum is null by definition. If $s \geqq M$ but $s < N$ holds, then the binomial coefficient $\binom{s-1}{N-1}$ vanishes.

Utilizing the properties of the $C_{i,N}$ we discuss two special cases which show completely different behaviour.
For pairwise distinct service probabilities we have

$$q^{(M,N)}(s) = G_{\mathrm{A}}(M,N)^{-1} \binom{s-1}{N-1} \left(\prod_{i=1}^{M} p_i\right) \sum_{i=1}^{M} \frac{q_i^{s-1}}{\prod\limits_{\substack{j=1 \\ j \neq i}}^{M} (q_i - q_j)} \mathbf{1}[s \geqq M] \ . \tag{22}$$

Inserting the expression of Corollary 2.3 for $G_{\mathrm{A}}(M,N)$ yields

$$\begin{aligned} q^{(M,N)}(s) &= \frac{\binom{s-1}{N-1} \sum\limits_{i=1}^{M} p_i q_i^{s-1} \prod\limits_{\substack{j=1 \\ j \neq i}}^{M} \frac{p_j}{p_j - p_i} \mathbf{1}[s \geqq M]}{\sum\limits_{i=1}^{M} \left(\frac{q_i}{p_i}\right)^{N-1} \prod\limits_{\substack{j=1 \\ j \neq i}}^{M} \frac{p_j}{p_j - p_i}} \\ &= \sum_{i=1}^{M} C_{i,N} \left(\frac{p_i}{q_i}\right)^N \binom{s-1}{N-1} q_i^s \mathbf{1}[s \geqq M] \ . \end{aligned} \tag{23}$$

In the special case of balanced machines ($p_1 = \cdots = p_M =: p$) our technical lemmata are not applicable. By direct computation we have obtained a similar structural result in [MD04]:

$$
\begin{aligned}
G_A(M,N) &= \left(\frac{q}{p}\right)^{N-1} q^{1-M} \sum_{n\in Z(M,N-1)} q^{\delta(0,n_1)+\cdots+\delta(0,n_{M-1})} \\
&= \left(\frac{q}{p}\right)^{N-1} \sum_{l=0}^{M-1} \binom{M-1}{l} \binom{N-1}{M-1-l} \left(\frac{1}{q}\right)^{M-l-1} \qquad (24)
\end{aligned}
$$

leading to

$$
q^{(M,N)}(s) = \left(\frac{p}{q}\right)^N \frac{p^{M-1} q^s \binom{s-1}{N-1}\binom{s-1}{M-1}}{\sum_{l=0}^{M-1} \binom{M-1}{l}\binom{N-1}{M-1-l} q^l} \ . \qquad (25)
$$

As a by-product we get the identities

$$
G_{\mathrm{A}}(M,N) = \left(\frac{q}{p}\right)^{N-1} \sum_{l=0}^{M-1} \binom{M-1}{l}\binom{N-1}{M-1-l}\left(\frac{1}{q}\right)^{M-1-l} = \left(\frac{p}{q}\right)^M \sum_{s=M}^{\infty} q^s \binom{s-1}{N-1}\binom{s-1}{M-1} \qquad (26)
$$

and

$$
G_{\mathrm{A}}(M,N) = \sum_{i=1}^{M} \left(\frac{q_i}{p_i}\right)^{N-1} \prod_{\substack{j=1\\ j\neq i}}^{M} \frac{p_j}{p_j - p_i} = \sum_{s=M}^{\infty} \binom{s-1}{N-1} \sum_{i=1}^{M} p_i q_i^{s-1} \prod_{\substack{j=1\\ j\neq i}}^{M} \frac{p_j}{p_j - p_i} \ . \qquad (27)
$$

3 The influence of the slowest server

The strong influence of the slowest server on the performance of networks is already discussed by Gordon and Newell in their seminal paper [GN67].
We recall here that in discrete time the asymptotic bottleneck behaviour of the joint queue length vector process of the cycle is structural similar to the well known behaviour of continuous time systems. We restrict ourself to the case of exactly one bottleneck (slowest server) because this is the case of interest in the main part of the paper.

Theorem 3.1. *[MD04][Theorem 5.1] Let $(X_1, \ldots, X_M)$ be the joint queue-length process of a closed cyclic queueing system with M nodes and N indistinguishable customers in discrete time $\mathbb{N}_0$. The nodes are single-server state independent Bernoulli stations under FCFS with LA-DF regime. Let the service probability at queue $Q[i]$ be p_i, $i \in \underline{M}$ and assume $p_1 < \min(p_2, \ldots, p_M)$. Then the sequence $(\hat{\pi}^{(M,N)})_{N\in\mathbb{N}}$ of the $(M-1)$-dimensional marginal distributions of the queue-length at the stations $Q[2], \ldots, Q[M]$ under steady state conditions converges weakly to the equilibrium distribution of an open $(M-1)$ stage tandem with Bernoulli arrival process and service probabilities $p_2, \ldots, p_M$ respectively, that is*

$$
\lim_{N\to\infty} \hat{\pi}^{(M,N)}(n_2, \ldots, n_M) = \prod_{i=2}^{M} \left(1 - \frac{p_1}{p_i}\right)\left(\frac{p_1 q_i}{q_1 p_i}\right)^{n_i} \left(\frac{1}{q_i}\right)^{\eta(0,n_i)} , \quad \forall (n_2, \ldots, n_M) \in \mathbb{N}_0^{M-1}. \qquad (28)
$$

The arrival probability is p_1, the service probability of the slowest queue $Q[1]$ in the closed network.

The dramatic influence of the slowest server on the customers' cycle time distribution was discussed by Boxma [Box88]. He showed that for service rates $\mu_1 < \mu_2 < \cdots < \mu_M$ in continuous time holds

$$E\big(S_N\big) = N\mu_1^{-1}\big\{1 + O\big([\frac{\mu_1}{\mu_2}]^N\big)\big\}, \quad N \to \infty. \tag{29}$$

and

$$\mathrm{Var}\big(S_N\big) = N\mu_1^{-2}\big\{1 + O\big([\frac{\mu_1}{\mu_2}]^N\big)\big\}, \quad N \to \infty. \tag{30}$$

Our methods from Section 2.1 allow to prove stronger speed of convergence for the discrete time cycle.

Proposition 3.2. *Let $p_1 < p_2 < \ldots < p_M$. Then*

$$E(S_N) = N\frac{1}{p_1}\left\{1 + o\left(\left[\frac{p_1}{p_2}\right]^N\right)\right\} \tag{31}$$

and

$$\mathrm{Var}(S_N) = N\frac{q_1}{p_1^2}\left\{1 + o\left(\left[\frac{p_1}{p_2}\right]^N\right)\right\} \quad . \tag{32}$$

Proof. Using the the expression given in (16) for the mean cycle time we have (recall that $\sum_{i=1}^{M} C_{i,N} = 1$)

$$\left(\frac{p_2}{p_1}\right)^N \frac{E(S_N) - N\dfrac{1}{p_1}}{N\dfrac{1}{p_1}} = \left(\frac{p_2}{p_1}\right)^N \left(C_{1,N} - 1 + \sum_{i=2}^{M} C_{i,N}\frac{p_1}{p_i}\right) = \sum_{i=2}^{M} C_{i,N}\left(\frac{p_2}{p_1}\right)^N \frac{p_1 - p_i}{p_i} \quad .$$

Taking the limit gives (31) by observing property (iii) of Lemma 2.4. Similarly, with the expression given in (17) for the cycle time variance,

$$\begin{aligned}
&\left(\frac{p_2}{p_1}\right)^N \frac{\mathrm{Var}(S_N) - N\dfrac{q_1}{p_1^2}}{N\dfrac{q_1}{p_1^2}} \\
&= \left(\frac{p_2}{p_1}\right)^N \left\{C_{1,N} - 1 + \sum_{i=2}^{M} C_{i,N}\frac{p_1^2 q_i}{q_1 p_i^2} + N \sum_{\substack{(i,j)\in \underline{M}^2 \\ i<j}} C_{i,N}C_{j,N}\frac{p_1^2}{q_1}\left(\frac{1}{p_i} - \frac{1}{p_j}\right)^2\right\} \\
&= \sum_{i=2}^{M} C_{i,N}\left(\frac{p_2}{p_1}\right)^N \left(\frac{p_1^2 q_i}{q_1 p_i^2} - 1\right) + \sum_{\substack{(i,j)\in \underline{M}^2 \\ i<j}} N C_{i,N}C_{j,N}\left(\frac{p_2}{p_1}\right)^N \frac{p_1^2}{q_1}\left(\frac{1}{p_i} - \frac{1}{p_j}\right)^2 \quad .
\end{aligned} \tag{33}$$

Now (32) follows by taking the limit in (33) and utilizing properties (ii) and (iii) of Lemma 2.4. $\square$

Proposition 3.2 is the discrete time counterpart to Boxma's result.
The result of Proposition 3.2 is in the usual spirit of bottleneck analysis. But there is a result which seems to be completely counterintuitive in connection with the strong concentration on staying at node $Q[1]$ expressed by the bottleneck behaviour.
This is the following *semi-invariance property* which resembles the results for continuous time networks in [Kel83] and [MD05].

Theorem 3.3. *[MD04] In the cyclic network with N indistinguishable customers let $(S_1, \ldots, S_M)$ be the successive sojourn times at the FCFS state independent Bernoulli single-server nodes $Q[1], \ldots, Q[M]$ of a customer in steady-state.*
The conditional joint distribution $P_{(M,N)}^{(S_1,\ldots,S_M)|S}$ of the tagged customer's successive sojourn times, given his cycle time $S = S_1 + \cdots + S_M$, is distributed as an independent family $(X_1, \ldots, X_M)$, the components of which are geometrically distributed on $\mathbb{N}$ given their sum $X_1 + \cdots + X_M$.
That is, for $s \geqq \max(M, N)$, the pmf $p_{(M,N;s)}$ of $P_{(M,N)}^{(S_1,\ldots,S_M)|S=s}$ is

$$\begin{aligned} p_{(M,N;s)}(s_1, \ldots, s_M) &:= P_{(M,N)}(S_1 = s_1, \ldots, S_M = s_M \mid S = s) \\ &= \frac{\mathbf{1}[(s_1, \ldots, s_M) \in Z^\star(M, s)] \prod_{i=1}^{M} p_i q_i^{s_i - 1}}{\sum\limits_{(t_1,\ldots,t_M) \in Z^\star(M,s)} \prod_{i=1}^{M} p_i q_i^{t_i - 1}}. \end{aligned} \tag{34}$$

This result is striking. It means that looking at $p_{(M,N;s)}$ as a function of the total network population N, for feasible values s of the cycle time the conditional joint distribution is independent of the number of customers circulating: let $M \leqq s$. Then

$$p_{(M,N;s)} = p_{(M,1;s)}, \qquad N \leqq s.$$

If a feasible cycle time is given, the distribution of the successive sojourn times which the tagged customer experiences at the nodes $Q[1], \ldots, Q[M]$ of his cycle is the same, irrespective of whether there are many, few, or no other customers present, provided $N \leqq s$.
Saying it the other way round: if the tagged customer is the only customer cycling, his sojourn times coincide with his service times. If we compute in this trivial case the conditional distribution of the tagged customer's successive sojourn times, given his cycle time, then the theorem says that the result of the computation applies without any changes to *feasible* vectors and cycle time values for the case of further customers present in the system as well. So we can compute the (due to customer interactions) complicated conditional vector distribution for networks with large population sizes in a network where only one customer lives who has not to compete for network resources. This is unexpected and seems to be contrary to the intuition guided by the bottleneck behaviour of queueing networks as described above.

4 Heavy Traffic behaviour of sojourn times: Moments and the asymptotics of non-bottleneck nodes

Closed queueing systems in continuous time, consisting of single-server stations with exponentially distributed and queue independent service rates, a finite population of indistinguishable customers (jobs) and random routing, i. e. queueing-systems of the Gordon-Newell type show the well known *bottleneck behaviour*:
The slowest server becomes the bottleneck of the system. With increasing number of customers, the queue at the slowest server grows without bound, while the marginal distribution of the other node(s) remains finite, see [GN67].

In discrete time the cycle of section 2 shows such bottleneck behaviour as well, as was shown in (28) of Theorem 3.1. The *usual interpretation* of (28) is:
If in a closed cyclic queueing system with M nodes, with service probability p_i at queue $Q[i]$, $i \in \underline{M}$, and with $p_1 < \min(p_2, \dots, p_M)$, the number N of customers increases unboundedly, then the sequence $(\hat{\pi}^{(M,N)})_{N\in\mathbb{N}}$ of the $(M-1)$-dimensional marginal distributions of the queue-length at the stations $Q[2], \dots, Q[M]$ under steady state conditions converges weakly to the equilibrium distribution of an open $(M-1)$ stage tandem with Bernoulli arrival process (parameter p_1) and service probabilities $p_2, \dots, p_M$.
We now collect some information about the behaviour of moments of cycle times and sojourn times in this limiting process and restrict our investigation to the case of pairwise distinct service probabilities. The proof of following lemma 4.1] is given in the Appendix, page 41.

Lemma 4.1 (Asymptotic behaviour).

(i) Mean cycle time.
$$\lim_{N\to\infty} \frac{E(S_N)}{N} = \frac{1}{p_1} \quad .$$

(ii) Cycle time variance.
$$\lim_{N\to\infty} \frac{\mathrm{Var}\,(S_N)}{N} = \frac{q_1}{p_1^2} \quad .$$

(iii) Bottleneck mean sojourn time.
$$\lim_{N\to\infty} \frac{1}{N} E(S_{1,N}) = \frac{1}{p_1} \quad .$$

(iv) Non-bottleneck mean sojourn time.
$$\lim_{N\to\infty} E(S_{k,N}) = \frac{q_1}{q_1 - q_k} \quad .$$

(v) Bottleneck sojourn time variance.
$$\lim_{N\to\infty} \frac{1}{N} \mathrm{Var}(S_{1,N}) = \frac{q_1}{p_1^2} \quad .$$

(vi) *Non-bottleneck sojourn time variance.*

$$\lim_{N\to\infty} \mathrm{Var}(S_{k,N}) = \frac{q_1 q_k}{(q_1-q_k)^2} \quad .$$

(vii) *Covariance of sojourn times at bottleneck and some other station.*

$$\lim_{N\to\infty} \mathrm{Cov}\left(S_{1,N}, S_{l,N}\right) = \frac{q_1}{q_1-q_l}\frac{q_l}{q_l-q_1} \quad , \quad l \neq 1 \quad .$$

(viii) *Covariance of sojourn times at two non-bottleneck stations.*

$$\lim_{N\to\infty} \mathrm{Cov}\left(S_{k,N}, S_{l,N}\right) = 0 \quad , \quad k,l \neq 1 \quad .$$

(ix) *Correlation coefficient of sojourn times at bottleneck and some non-bottleneck node.*

$$\varrho(S_{1,N}, S_{l,N}) = \sqrt{\frac{q_l}{N}}\frac{p_1}{p_1-p_l} + o\left(N^{-\frac{1}{2}}\right) \quad .$$

(x) *Correlation coefficient of sojourn times at two non-bottleneck stations.*

$$\lim_{N\to\infty} \varrho(S_{k,N}, S_{l,N}) = 0, \quad k \neq l \quad .$$

The result (*vii*) in Lemma 4.1 on the asymptotic nonvanishing covariance between the bottleneck sojourn time and subsequent sojourn times at non-bottleneck nodes indicates that there is still something beside the *usual interpretation* sketched above. While in the *approximating open tandem* the (Bernoulli) arrival stream is independent from all service times, in the sequence of closed M-node cycles there persits some dependence between a customer's behaviour at the node $Q[1]$ at the subsequent nodes $Q[i], i = 2, \ldots, M.$,

Following from the *usual interpretation* the next Proposition 4.2 is part of the common knowledge on approximating networks. But due to (*vii*) in Lemma 4.1 a proof seems reasonable.

Let $(\tilde{S}_2, \ldots, \tilde{S}_M)$ be the family of successive sojourn times of a customer traversing the nodes $Q[2]$ to $Q[M]$ in an open $M-1$ station tandem, which is fed by an Bernoulli input stream, stochastically independent of the service processes, with parameter p_1. The nodes $Q[2]$ to $Q[M]$ are Bernoulli Servers with FCFS LA-DF service discipline with respective service probabilities $p_2, \ldots, p_M$.

The joint distribution of $(\tilde{S}_2, \ldots, \tilde{S}_M)$, given in terms of its generating function, is (see [Dad96, Dad97] and [Dad01], Corollary 4.9)

$$E_{\pi_{2,m}^{M-1}}\left[\prod_{j=2}^{M} \theta_j^{\tilde{S}_j}\right] = \prod_{j=2}^{M} \frac{1-\dfrac{q_j}{q_1}}{1-\dfrac{q_j}{q_1}u_j} u_j \quad |u_j| \le 1, \; j = 2, \ldots, M \quad .$$

So the individual sojourn times are independent and geometrically distributed with respective parameters

$$1 - \frac{q_j}{q_1}, \quad j = 2, \ldots, M.$$

This result is the analogue of Burke's and Reich's results on the independence of a customer's successive sojourn times in an open exponential tandem in continuous time (for a review see [BD90]).
Thinking of the limiting network under heavy traffic as an open $M-1$-station tandem with Bernoulli input with arrival probability p_1 (the bottleneck's service probability) is therefore supported by properties (iv), (vi) and (viii) of Lemma 4.1. Something more can be said.

Proposition 4.2. *The joint distribution of the family $(S_{2,N}, \dots, S_{M,N})$ of successive sojourn times of a customer traversing the nodes $Q[2]$ to $Q[M]$ in the closed M station tandem with N customers converges weakly to the distribution of the family $(\tilde{S}_2, \dots, \tilde{S}_M)$ of successive sojourn times of a customer traversing the open $M-1$ stage tandem with Bernoulli-p_1 input stream.*

Proof. The Fourier transform $\tilde{g}^{(N)}_{2,\dots,M}$ of the joint (marginal) sojourn time distribution of $(S_{2,N}, \dots, S_{M,N})$ is

$$\begin{aligned}
&\tilde{g}^{(N)}_{2,\dots,M}(x_2, x_3, \dots, x_M) = \tilde{g}^{(M,N)}(0, x_2, x_3, \dots, x_M) \\
&= C_{1,N} \prod_{j=2}^{M} \frac{q_1 - q_j}{q_1 - q_j e^{ix_j}} e^{ix_j} + \sum_{k=2}^{M} C_{k,N} \left(\frac{p_k e^{ix_k}}{1 - q_k e^{ix_k}} \right)^N \frac{q_1 - q_k}{q_1 - q_k e^{ix_k}} \prod_{\substack{j=2 \\ j \neq k}}^{M} \frac{q_k - q_j}{q_k e^{ix_k} - q_j e^{ix_j}} e^{ix_j} \ .
\end{aligned}$$

Now for every $k \in \underline{M}$

$$\left| \frac{p_k e^{ix_k}}{1 - q_k e^{ix_k}} \right| = \frac{p_k}{\sqrt{1 + q_k^2 - 2q_k \cos(x)}} = \frac{1}{\sqrt{1 + 2\frac{q_k}{p_k^2}\,(1 - \cos(x))}} \leqq 1 \ ,$$

and therefore from Lemma 2.4 (ii) and (iii)

$$\begin{aligned}
&\lim_{N \to \infty} \left| \sum_{k=2}^{M} C_{k,N} \left(\frac{p_k e^{ix_k}}{1 - q_k e^{ix_k}} \right)^N \frac{q_1 - q_k}{q_1 - q_k e^{ix_k}} \prod_{\substack{j=2 \\ j \neq k}}^{M} \frac{q_k - q_j}{q_k e^{ix_k} - q_j e^{ix_j}} e^{ix_j} \right| \\
&\leqq \sum_{k=2}^{M} \lim_{N \to \infty} |C_{k,N}| \cdot \left| \frac{q_1 - q_k}{q_1 - q_k e^{ix_k}} \prod_{\substack{j=2 \\ j \neq k}}^{M} \frac{q_k - q_j}{q_k e^{ix_k} - q_j e^{ix_j}} e^{ix_j} \right| = 0 \ .
\end{aligned}$$

Hence for any $x_2, \dots, , x_M \in \mathbb{R}$

$$\lim_{N \to \infty} \tilde{g}^{(N)}_{2,\dots,M}(x_2, x_3, \dots, x_M) = \lim_{N \to \infty} C_{1,N} \prod_{j=2}^{M} \frac{q_1 - q_j}{q_1 - q_j e^{ix_j}} e^{ix_j} = \prod_{j=2}^{M} \frac{1 - \dfrac{q_j}{q_1}}{1 - \dfrac{q_j}{q_1} e^{ix_j}} e^{ix_j} \ .$$

□

This is the (Fourier transform of the) distribution of the joint sojourn time distribution of a customer traversing an open $M-1$ station tandem with FCFS Bernoulli servers with the respective service probabilities $p_2, \dots, p_M$ fed by an Bernoulli input (arrival) stream, which is stochastically independent of the service processes and has arrival probability p_1.

On the other hand, recall that the limiting values of the covariance of any pair $(S_{1,N}, S_{l,N})$, i. e. bottleneck sojourn time versus any other sojourn time, is strictly negative (Lemma 4.1, (vii)):

$$\lim_{N\to\infty} \mathrm{Cov}\,(S_{1,N}, S_{l,N}) = -\frac{q_1 q_l}{(q_1-q_l)^2}, \quad l = 2,\ldots,M \ .$$

5 Weak convergence of the joint sojourn times distribution

Theorem 5.1. *We consider a sequence of M-station cyclic networks with nodes $Q[1], Q[2], \ldots, Q[M]$ which have pairwise distinct service probabilities $p_1, \ldots, p_M$ and where $Q[1]$ is the bottleneck i.e., $p_1 = \min\{p_j,\ j \in \underline{M}\}$). The networks' population size N increases unboundedly.*
Let $(S_{1,N}, S_{2,N}, \ldots, S_{M,N}), N \geq 1$, be the sequence of the joint sojourn times vectors of a test customer in equilibrium on his round trip and

$$T_{1,N} := \frac{S_{1,N} - E(S_{1,N})}{\sqrt{\mathrm{Var}(S_{1,N})}}.$$

the standardized sojourn time at the bottleneck. Then the sequence $(T_{1,N}, S_{2,N}, \ldots, S_{M,N})_{N\in\mathbb{N}}$ converges weakly to

$$N(0,1) \otimes \mathrm{Geo}_{(1)}\left(1-\frac{q_2}{q_1}\right) \otimes \cdots \otimes \mathrm{Geo}_{(1)}\left(1-\frac{q_M}{q_1}\right) \ . \tag{35}$$

Proof. By theorem 2.6 (14) the Fourier transform $\tilde{g}^{(M,N)}$ of of the joint sojourn times vector $(S_{1,N}, \ldots, S_{M,N})$ is

$$\begin{aligned}
&\tilde{g}^{(M,N)}(x_1,\ldots,x_M)\\
&= C_{1,N}\left(\frac{p_1 e^{ix_1}}{1-q_1 e^{ix_1}}\right)^N \prod_{k=2}^{M}\left(\frac{q_1-q_k}{q_1 e^{ix_1}-q_k e^{ix_k}} e^{ix_k}\right)\\
&\quad + \sum_{j=2}^{M} C_{j,N}\left(\frac{p_j e^{ix_j}}{1-q_j e^{ix_j}}\right)^N \left(\frac{q_j-q_1}{q_j e^{ix_j}-q_1 e^{ix_1}} e^{ix_1}\right) \prod_{\substack{k=2\\k\neq j}}^{M}\left(\frac{q_j-q_k}{q_j e^{ix_j}-q_k e^{ix_k}} e^{ix_k}\right) \ .
\end{aligned}$$

Therefore the Fourier Transform $\tilde{g}_N$ of $(T_{1,N}, S_{2,N}, \ldots, S_{M,N})$ is given by

$$\begin{aligned}
\tilde{g}_N(x) &= e^{-i\frac{x_1 E(S_{1,N})}{\sqrt{\mathrm{Var}(S_{1,N})}}}\, \tilde{g}^{(M,N)}\Big(\frac{x_1}{\sqrt{\mathrm{Var}(S_{1,N})}}, x_2, \ldots, x_M\Big)\\
&= C_{1,N}\left(\frac{p_1 e^{\frac{ix_1\left(1-\frac{E(S_{1,N})}{N}\right)}{\sqrt{\mathrm{Var}(S_{1,N})}}}}{1-q_1 e^{i\frac{x_1}{\sqrt{\mathrm{Var}(S_{1,N})}}}}\right)^N \prod_{k=2}^{M} \frac{(q_1-q_k)\, e^{ix_k}}{q_1 e^{i\frac{x_1}{\sqrt{\mathrm{Var}(S_{1,N})}}} - q_k e^{ix_k}}\\
&\quad + \sum_{j=2}^{M} C_{j,N} e^{-i\frac{x_1 E(S_{1,N})}{\sqrt{\mathrm{Var}(S_{1,N})}}} \left(\frac{p_j e^{ix_j}}{1-q_j e^{ix_j}}\right)^N \frac{(q_j-q_1)\, e^{i\frac{x_1}{\sqrt{\mathrm{Var}(S_{1,N})}}}}{q_j e^{ix_j} - q_1 e^{i\frac{x_1}{\sqrt{\mathrm{Var}(S_{1,N})}}}} \prod_{\substack{k=2\\k\neq j}}^{M}\left(\frac{q_j-q_k}{q_j e^{ix_j}-q_k e^{ix_k}} e^{ix_k}\right).
\end{aligned}$$

The mean bottleneck sojourn time and the variance of the bottleneck sojourn time are (see (43) and (44))

$$E(S_{1,N}) = C_{1,N} N \frac{1}{p_1} + \sum_{j=2}^{M} \left(\frac{q_j\, C_{j,N}}{q_j - q_1} - \frac{q_1\, C_{1,N}}{q_1 - q_j} \right)$$

and

$$\begin{aligned}
&\mathrm{Var}(S_{1,N}) \\
&= N^2 C_{1,N} \frac{1}{p_1^2} - N^2 C_{1,N}^2 \frac{1}{p_1^2} + N C_{1,N} \frac{q_1}{p_1^2} - 2 N C_{1,N} \sum_{r=2}^{M} \frac{1}{p_1} \frac{q_1}{q_1 - q_r} \\
&\quad + 2 C_{1,N}^2 N \sum_{r=2}^{M} \frac{1}{p_1} \frac{q_1}{q_1 - q_r} - 2 C_{1,N} N \sum_{i=2}^{M} C_{i,N} \frac{1}{p_1} \frac{q_i}{q_i - q_1} \\
&\quad + C_{1,N} \sum_{r=2}^{M} \frac{q_1 q_r}{(q_1 - q_r)^2} + C_{1,N} \sum_{r_1=2}^{M} \sum_{r_2=2}^{M} \frac{q_1}{q_1 - q_{r_1}} \frac{q_1}{q_1 - q_{r_2}} + \sum_{i=2}^{M} C_{i,N} \frac{q_i(q_i + q_1)}{(q_i - q_1)^2} \\
&\quad - \left(C_{1,N} \sum_{r=2}^{M} \frac{q_1}{q_1 - q_r} \right)^2 - \left(\sum_{i=2}^{M} C_{i,N} \frac{q_i}{q_i - q_1} \right)^2 + 2 \sum_{i=2}^{M} \sum_{r=2}^{M} C_{i,N} C_{1,N} \frac{q_i}{q_i - q_1} \frac{q_1}{q_1 - q_r} \quad .
\end{aligned}$$

We show

(i)

$$\lim_{N\to\infty} C_{1,N} \left(\frac{p_1 e^{\frac{i x_1 \left(1 - \frac{E(S_{1,N})}{N}\right)}{\sqrt{\mathrm{Var}(S_{1,N})}}}}{1 - q_1 e^{i \frac{x_1}{\sqrt{\mathrm{Var}(S_{1,N})}}}} \right)^N \prod_{k=2}^{M} \frac{(q_1 - q_k)\, e^{i x_k}}{q_1 e^{i \frac{x_1}{\sqrt{\mathrm{Var}(S_{1,N})}}} - q_k e^{i x_k}} = e^{-\frac{1}{2} x_1^2} \prod_{k=2}^{M} \frac{1 - \frac{q_k}{q_1}}{1 - \frac{q_k}{q_1} e^{i x_k}} e^{i x_k} \quad ;$$

(ii) For $j = 2, \ldots, M$

$$\lim_{N\to\infty} C_{j,N} e^{-i \frac{x_1 E(S_{1,N})}{\sqrt{\mathrm{Var}(S_{1,N})}}} \left(\frac{p_j e^{i x_j}}{1 - q_j e^{i x_j}} \right)^N \frac{(q_j - q_1)\, e^{i \frac{x_1}{\sqrt{\mathrm{Var}(S_{1,N})}}}}{q_j e^{i x_j} - q_1 e^{i \frac{x_1}{\sqrt{\mathrm{Var}(S_{1,N})}}}} \prod_{\substack{k=2 \\ k \neq j}}^{M} \left(\frac{q_j - q_k}{q_j e^{i x_j} - q_k e^{i x_k}} e^{i x_k} \right) = 0 \quad .$$

We abbreviate

$$\sigma_1(N) := \sqrt{\mathrm{Var}(S_{1,N})} \quad .$$

First, for every $k \in \underline{M}$

$$\left| \frac{p_k\, e^{\frac{i x \left(1 - \frac{E(S_{1,N})}{N}\right)}{\sigma_1(N)}}}{1 - q_k\, e^{\frac{i x}{\sigma_1(N)}}} \right| = \frac{1}{\sqrt{1 + 2 \frac{q_k}{p_k^2} \left(1 - \cos\left(\frac{x}{\sigma_1(N)} \right) \right)}} \leqq 1 \quad .$$

Utilizing $|u^n - v^n| \leqq n|u-v|$ (which holds for every $u, v \in \mathbb{C}$ with $|u| \leqq 1$, $v \leqq 1$ and $n \in \mathbb{N}$) (see [Kal02], Lemma 4.13) we have for N sufficiently large

$$\left| \left(\frac{p_1\, e^{\frac{ix\left(1-\frac{E(S_{1,N})}{N}\right)}{\sigma_1(N)}}}{1-q_1\, e^{\frac{ix}{\sigma_1(N)}}} \right)^N - \left(1-\frac{x^2}{2N}\right)^N \right| \leqq N \left| \frac{p_1\, e^{\frac{ix\left(1-\frac{E(S_{1,N})}{N}\right)}{\sigma_1(N)}}}{1-q_1\, e^{\frac{ix}{\sigma_1(N)}}} - \left(1-\frac{x^2}{2N}\right) \right|$$

$$= \quad N \left| \frac{p_1\, e^{\frac{ix\left(1-\frac{E(S_{1,N})}{N}\right)}{\sigma_1(N)}} - p_1 q_1\, e^{\frac{-ix\left(\frac{E(S_{1,N})}{N}\right)}{\sigma_1(N)}} - \left(p_1^2 + 2q_1\left(1-\cos\left(\frac{x}{\sigma_1(N)}\right)\right)\right)\left(1-\frac{x^2}{2N}\right)}{p_1^2 + 2q_1\left(1-\cos\left(\frac{x}{\sigma_1(N)}\right)\right)} \right| \ .$$

Because of $\lim\limits_{N\to\infty} \sigma_1(N) = \infty$ (Lemma 4.1, (v))

$$\lim_{N\to\infty} \left(p_1^2 + 2q_1\left(1-\cos\left(\frac{x}{\sigma_1(N)}\right)\right)\right) = p_1^2 \ .$$

Therefore it is sufficient to show

$$\lim_{N\to\infty} N \Bigg| p_1 \exp\left[\frac{ix\left(1-\frac{E(S_{1,N})}{N}\right)}{\sigma_1(N)}\right] - p_1 q_1 \exp\left[\frac{-ix\left(\frac{E(S_{1,N})}{N}\right)}{\sigma_1(N)}\right] - p_1^2 + \frac{x^2}{2N} p_1^2$$
$$- 2q_1\left(1-\cos\left(\frac{x}{\sigma_1(N)}\right)\right) + \frac{x^2}{2N} 2q_1\left(1-\cos\left(\frac{x}{\sigma_1(N)}\right)\right) \Bigg| = 0 \ .$$

Denoting

$$r_1(N) := \exp\left[\frac{ix\left(1-\frac{E(S_{1,N})}{N}\right)}{\sigma_1(N)}\right] - \sum_{k=0}^{2} \left(\frac{ix\left(1-\frac{E(S_{1,N})}{N}\right)}{\sigma_1(N)}\right)^k \frac{1}{k!}$$

and

$$r_2(N) := \exp\left[\frac{-ix\left(\frac{E(S_{1,N})}{N}\right)}{\sigma_1(N)}\right] - \sum_{k=0}^{2} \left(\frac{-ix\left(\frac{E(S_{1,N})}{N}\right)}{\sigma_1(N)}\right)^k \frac{1}{k!} \ ,$$

and using the triangular inequality we have

$$
\begin{aligned}
&\left|p_1 \exp\left[\frac{ix\left(1-\frac{E(S_{1,N})}{N}\right)}{\sigma_1(N)}\right] - p_1 q_1 \exp\left[\frac{-ix\left(\frac{E(S_{1,N})}{N}\right)}{\sigma_1(N)}\right] - p_1^2 + \frac{x^2}{2N}p_1^2\right. \\
&\qquad\qquad\left. -2q_1\left(1-\cos\left(\frac{x}{\sigma_1(N)}\right)\right) + \frac{x^2}{2N}2q_1\left(1-\cos\left(\frac{x}{\sigma_1(N)}\right)\right)\right| \\
&= \left|p_1 + p_1\frac{ix\left(1-\frac{E(S_{1,N})}{N}\right)}{\sigma_1(N)} - p_1\frac{x^2\left(1-\frac{E(S_{1,N})}{N}\right)^2}{2\sigma_1^2(N)} + p_1 r_1(N) - p_1 q_1 + p_1 q_1 \frac{ix\left(\frac{E(S_{1,N})}{N}\right)}{\sigma_1(N)}\right. \\
&\qquad + p_1 q_1 \frac{x^2\left(\frac{E(S_{1,N})}{N}\right)^2}{2\sigma_1^2(N)} - p_1 q_1 r_2(N) - p_1^2 - \frac{q_1 x^2}{\sigma_1^2(N)} + \frac{2q_1}{\sigma_1^3(N)}\sin\left(\frac{\xi_N}{\sigma_1(N)}\right) + \frac{x^2}{2N}p_1^2 \\
&\qquad\left. + \frac{q_1 x^4}{2N\sigma_1^2(N)} - \frac{1}{N}\frac{q_1 x^2}{\sigma_1^3(N)}\sin\left(\frac{\xi_N}{\sigma_1(N)}\right)\right| \\
&\leqq \left|p_1 - p_1 q_1 - p_1^2\right| + \left|p_1\frac{ix\left(1-\frac{E(S_{1,N})}{N}\right)}{\sigma_1(N)} + p_1 q_1 \frac{ix\left(\frac{E(S_{1,N})}{N}\right)}{\sigma_1(N)}\right| \\
&\qquad + \left|(-p_1)\frac{x^2\left(1-\frac{E(S_{1,N})}{N}\right)^2}{2\sigma_1^2(N)} + p_1 q_1 \frac{x^2\left(\frac{E(S_{1,N})}{N}\right)^2}{2\sigma_1^2(N)} - \frac{q_1 x^2}{\sigma_1^2(N)} + \frac{x^2}{2N}p_1^2\right| \\
&\qquad + |p_1 r_1(N)| + |p_1 q_1 r_2(N)| + \left|\frac{2q_1}{\sigma_1^2(N))\sigma_1(N)}\right| + \left|\frac{1}{N}\frac{q_1 x^2}{\sigma_1^2(N)\sigma_1(N)}\right| + \left|\frac{q_1 x^4}{2N\sigma_1^2(N)}\right| \ .
\end{aligned}
$$

We have used the Taylor expansion of $\exp(iy)$ and the Taylor expansion of order two of the real function

$$h_N(x) := 1 - \cos\left(\frac{x}{\sigma_1(N)}\right)$$

that is

$$h_N(x) = \frac{1}{2}\frac{x^2}{\sigma_1^2(N)} - \frac{1}{\sigma_1^3(N)}\sin\left(\frac{\xi_N}{\sigma_1(N)}\right)$$

for some $\xi_N \in (0, |x|),\ N \in \mathbb{N}$.
Now first

$$\left|p_1 - p_1 q_1 - p_1^2\right| = 0 \ .$$

Further

$$\left|p_1\frac{ix\left(1-\frac{E(S_{1,N})}{N}\right)}{\sigma_1(N)} + p_1 q_1 \frac{ix\left(\frac{E(S_{1,N})}{N}\right)}{\sigma_1(N)}\right| = \left|\frac{1 - p_1\frac{E(S_{1,N})}{N}}{\sigma_1(N)}|x||p_1|\right|$$

and

$$
\begin{aligned}
\lim_{N\to\infty} N\left|1-p_1\frac{1}{N}E(S_{1,N})\right| &= \lim_{N\to\infty} N\left|1-C_{1,N}-\frac{p_1}{N}\sum_{j=2}^{M}\left(\frac{q_j\,C_{j,N}}{q_j-q_1}-\frac{q_1\,C_{1,N}}{q_1-q_j}\right)\right| \\
&= \lim_{N\to\infty}\left|\sum_{j=2}^{M} NC_{j,N}-p_1\sum_{j=2}^{M}\left(\frac{q_j\,C_{j,N}}{q_j-q_1}-\frac{q_1\,C_{1,N}}{q_1-q_j}\right)\right| \\
&= \sum_{j=2}^{M}\frac{p_1q_1}{q_1-q_j} \ .
\end{aligned}
$$

Hence

$$
\lim_{N\to\infty} N\left|p_1\frac{ix\left(1-\frac{E(S_{1,N})}{N}\right)}{\sigma_1(N)}+p_1q_1\frac{ix\left(\frac{E(S_{1,N})}{N}\right)}{\sigma_1(N)}\right| = 0 \ .
$$

Furthermore

$$
\begin{aligned}
&\lim_{N\to\infty} N\left|-p_1\frac{x^2\left(1-\frac{E(S_{1,N})}{N}\right)^2}{2\sigma_1^2(N)}+p_1q_1\frac{x^2\left(\frac{E(S_{1,N})}{N}\right)^2}{2\sigma_1^2(N)}-\frac{1}{2}\frac{2q_1x^2}{\sigma_1^2(N)}+\frac{x^2}{2N}p_1^2\right| \\
&= \lim_{N\to\infty}\frac{1}{2}x^2\left|\frac{-p_1\left(1-\frac{E(S_{1,N})}{N}\right)^2+p_1q_1\left(\frac{E(S_{1,N})}{N}\right)^2-2q_1}{\frac{1}{N}\sigma_1^2(N)}+p_1^2\right| \\
&= \frac{1}{2}x^2\left|\frac{-p_1\left(1-\frac{1}{p_1}\right)^2+p_1q_1\left(\frac{1}{p_1}\right)^2-2q_1}{\frac{q_1}{p_1^2}}+p_1^2\right| = 0 \ . \qquad (36)
\end{aligned}
$$

For (36) we have used Lemma 4.1 (iii) and (v):

$$
\lim_{N\to\infty}\frac{1}{N}E(S_{1,N})=\frac{1}{p_1} \quad \text{and} \quad \lim_{N\to\infty}\frac{1}{N}\sigma_1^2(N)=\frac{q_1}{p_1^2} \ .
$$

To determine the limits

$$
\lim_{N\to\infty} N\,|r_k(N)| \ , \qquad k=1,2
$$

we observe that (see [Kal02], Lemma 4.14) for any $t\in\mathbb{R}$ and $n\in\mathbb{N}$

$$
\left|e^{it}-\sum_{k=0}^{n}\frac{(it)}{k!}\right| \leqq \frac{|t|^{n+1}}{(n+1)!} \ .
$$

Now, because of $\lim_{N\to\infty}\sigma_1(N)=\infty$,

$$
N\,|r_1(N)| \leqq \frac{\left|1-\frac{E(S_{1,N})}{N}\right|^3}{\frac{1}{N}\sigma_1^3(N)}\frac{|x|^3}{6} \underset{N\to\infty}{\longrightarrow} 0
$$

and

$$N\,|r_2(N)| \leqq \frac{\left|\frac{E(S_{1,N})}{N}\right|^3}{\frac{1}{N}\sigma_1^3(N)} \frac{|x|^3}{6} \underset{N\to\infty}{\longrightarrow} 0$$

Finally

$$\lim_{N\to\infty} N \left(\left|\frac{2q_1}{\sigma_1^3(N)}\right| + \left|\frac{1}{N}\frac{q_1 x^2}{\sigma_1^3(N)}\right| + \left|\frac{q_1 x^4}{2N\sigma_1^2(N)}\right| \right) = 0 \ .$$

This proofs (i). The proof of (ii) is straightforward:

$$\begin{aligned}
&\left| C_{j,N} e^{-i\frac{x_1 E(S_{1,N})}{\sqrt{\mathrm{Var}(S_{1,N})}}} \left(\frac{p_j e^{ix_j}}{1-q_j e^{ix_j}}\right)^N \frac{(q_j - q_1)\, e^{i\frac{x_1}{\sqrt{\mathrm{Var}(S_{1,N})}}}}{q_j e^{ix_j} - q_1 e^{i\frac{x_1}{\sqrt{\mathrm{Var}(S_{1,N})}}}} \prod_{\substack{k=2\\k\neq j}}^{M} \left(\frac{q_j - q_k}{q_j e^{ix_j} - q_k e^{ix_k}} e^{ix_k}\right) \right| \\
\leqq\ & \left| C_{j,N} \left(\frac{p_j e^{ix_j}}{1-q_j e^{ix_j}}\right)^N \right| \frac{|q_j - q_1|}{\left| q_j e^{ix_j} - q_1 e^{i\frac{x_1}{\sqrt{\mathrm{Var}(S_{1,N})}}} \right|} \prod_{\substack{k=2\\k\neq j}}^{M} \left| \frac{q_j - q_k}{q_j e^{ix_j} - q_k e^{ix_k}} \right| \\
\leqq\ & N\,|C_{j,N}| \frac{|q_j - q_1|}{\left| q_j e^{ix_j} - q_1 e^{i\frac{x_1}{\sqrt{\mathrm{Var}(S_{1,N})}}} \right|} \prod_{\substack{k=2\\k\neq j}}^{M} \left| \frac{q_j - q_k}{q_j e^{ix_j} - q_k e^{ix_k}} \right| \underset{N\to\infty}{\longrightarrow} 0 \ .
\end{aligned}$$

$\square$

Theorem 5.2. *Let S_N be the cycle time in equilibrium of a test customer in the closed M station tandem network of Theorem 5.1 with increasing population size N.*
Then the sequence of standardized cycle times

$$T_N := \frac{S_N - E(S_N)}{\sqrt{\mathrm{Var}(S_N)}}, \quad N \in \mathbb{N},$$

converges weakly to the Standard Normal Distribution $N(0,1)$.

Proof. We denote $\sigma(N) := \sqrt{\mathrm{Var}(S_N)}$. By theorem 2.6 (11) the cycle time Fourier transform $\tilde{h}_{S_N}(x)$ is

$$\tilde{h}_{S_N}(x) = \sum_{j=1}^{M} C_{j,N} \left(\frac{p_j\, e^{ix}}{1 - q_j\, e^{ix}}\right)^N \quad , \quad x \in \mathbb{R}.$$

Therefore the Fourier Transform $\tilde{h}_{T_N}$ of the normalized and centered sojourn time T_N is

$$\tilde{h}_{T_N}(x) = \exp\left[-\frac{ixE(S_N)}{\sigma(N)}\right] \cdot \tilde{h}_{S_N}\left(\frac{x}{\sigma(N)}\right), \quad x \in \mathbb{R}.$$

where the mean cycle time and the variance of the cycle time are (see (16) and (17))

$$E(S_N) = \sum_{j=1}^{M} C_{j,N} N \frac{1}{p_j}$$

and

$$\mathrm{Var}(S_N) = N \sum_{j=1}^{M} C_{j,N} \frac{q_j}{p_j^2} + N^2 \sum_{\substack{(j,k)\in \underline{M}^2 \\ j<k}} C_{j,N} C_{k,N} \left(\frac{1}{p_k} - \frac{1}{p_k} \right)^2 \quad .$$

Hence

$$\tilde{h}_{T_N}(x) = \sum_{j=1}^{M} C_{j,N} \left(\frac{p_j \, e^{\frac{ix\left(1-\sum_{k=1}^{M} C_{k,N}\frac{1}{p_k}\right)}{\sigma(N)}}}{1 - q_j \, e^{\frac{ix}{\sigma(N)}}} \right)^N \quad .$$

We show

(i)

$$\lim_{N\to\infty} C_{1,N} \left(\frac{p_1 \, e^{\frac{ix\left(1-\sum_{i=1}^{M} C_{i,N}\frac{1}{p_i}\right)}{\sigma(N)}}}{1 - q_1 \, e^{\frac{ix}{\sigma(N)}}} \right)^N = e^{-\frac{1}{2}x^2} \quad ;$$

(ii) For $j = 2, \ldots, M$

$$\lim_{N\to\infty} C_{j,N} \left(\frac{p_j \, e^{\frac{ix\left(1-\sum_{j=1}^{M} C_{j,N}\frac{1}{p_j}\right)}{\sigma(N)}}}{1 - q_j \, e^{\frac{ix}{\sigma(N)}}} \right)^N = 0 \quad .$$

First, for every $k \in \underline{M}$

$$\left| \frac{p_k \, e^{\frac{ix\left(1-\sum_{j=1}^{M} C_{j,N}\frac{1}{p_j}\right)}{\sigma(N)}}}{1 - q_k \, e^{\frac{ix}{\sigma(N)}}} \right| = \frac{1}{\sqrt{1 + 2\frac{q_k}{p_k^2}\left(1 - \cos\left(\frac{x}{\sigma(N)}\right)\right)}} \leqq 1 \quad . \tag{37}$$

Utilizing $|u^n - v^n| \leqq n|u-v|$ once more we have for N sufficiently large

$$\left| \left(\frac{p_1 \, e^{\dfrac{ix\left(1-\sum\limits_{j=1}^{M} C_{j,N}\frac{1}{p_j}\right)}{\sigma(N)}}}{1 - q_1 \, e^{\dfrac{ix}{\sigma(N)}}} \right)^N - \left(1 - \frac{x^2}{2N}\right)^N \right|$$

$$= \quad N \left| \frac{p_1 \, e^{\frac{ix\left(1-\sum\limits_{j=1}^{M} C_{j,N}\frac{1}{p_j}\right)}{\sigma(N)}} - p_1 q_1 \, e^{\frac{-ix\left(\sum\limits_{j=1}^{M} C_{j,N}\frac{1}{p_j}\right)}{\sigma(N)}} - \left(p_1^2 + 2q_1\left(1-\cos\left(\frac{x}{\sigma(N)}\right)\right)\right)\left(1-\frac{x^2}{2N}\right)}{p_1^2 + 2q_1\left(1-\cos\left(\frac{x}{\sigma(N)}\right)\right)} \right| \ .$$

Because of $\lim\limits_{N\to\infty} \mathrm{Var}\,(S_N) = \infty$ (see Proposition 3.2)

$$\lim_{N\to\infty} \left(p_1^2 + 2q_1\left(1-\cos\left(\frac{x}{\sigma(N)}\right)\right)\right) = p_1^2 \ .$$

Therefore it is sufficient to show

$$\lim_{N\to\infty} N \Bigg| p_1 \, \exp\left[\frac{ix\left(1-\sum\limits_{j=1}^{M} C_{j,N}\frac{1}{p_j}\right)}{\sigma(N)}\right] - p_1 q_1 \, \exp\left[\frac{-ix\left(\sum\limits_{j=1}^{M} C_{j,N}\frac{1}{p_j}\right)}{\sigma(N)}\right] - p_1^2 + \frac{x^2}{2N}p_1^2$$
$$- 2q_1\left(1-\cos\left(\frac{x}{\sigma(N)}\right)\right) + \frac{x^2}{2N}2q_1\left(1-\cos\left(\frac{x}{\sigma(N)}\right)\right)\Bigg| = 0 \ . \tag{38}$$

Denoting

$$r_3(N) := \exp\left[\frac{ix\left(1-\frac{E(S_N)}{N}\right)}{\sigma(N)}\right] - \sum_{k=0}^{2}\left(\frac{ix\left(1-\frac{E(S_N)}{N}\right)}{\sigma(N)}\right)^k \frac{1}{k!}$$

and

$$r_4(N) := \exp\left[\frac{-ix\left(\frac{E(S_N)}{N}\right)}{\sigma(N)}\right] - \sum_{k=0}^{2}\left(\frac{-ix\left(\frac{E(S_N)}{N}\right)}{\sigma(N)}\right)^k \frac{1}{k!} \ ,$$

and using the triangular inequality we have

$$\left| p_1\, e^{\frac{ix\left(1-\sum\limits_{j=1}^{M} C_{j,N}\frac{1}{p_j}\right)}{\sigma(N)}} - p_1 q_1\, e^{\frac{-ix\left(\sum\limits_{j=1}^{M} C_{j,N}\frac{1}{p_j}\right)}{\sigma(N)}} - p_1^2 + \frac{x^2}{2N}p_1^2 \right.$$

$$\left. -2q_1\left(1-\cos\left(\frac{x}{\sigma(N)}\right)\right) + \frac{x^2}{2N}2q_1\left(1-\cos\left(\frac{x}{\sigma(N)}\right)\right)\right|$$

$$= \left| p_1 + p_1\frac{ix\left(1-\sum\limits_{j=1}^{M} C_{j,N}\frac{1}{p_j}\right)}{\sigma(N)} - p_1\frac{x^2\left(1-\sum\limits_{j=1}^{M} C_{j,N}\frac{1}{p_j}\right)^2}{2\sigma^2(N)} + p_1 r_3(N) - p_1 q_1 \right.$$

$$+p_1q_1\frac{ix\left(\sum\limits_{j=1}^{M} C_{j,N}\frac{1}{p_j}\right)}{\sigma(N)} + p_1q_1\frac{x^2\left(\sum\limits_{j=1}^{M} C_{j,N}\frac{1}{p_j}\right)^2}{2\sigma^2(N)} - p_1q_1r_4(N) - p_1^2 - \frac{q_1x^2}{\sigma^2(N)}$$

$$\left. + \frac{2q_1}{\sigma^3(N)}\sin\left(\frac{\xi_N}{\sigma(N)}\right) + \frac{x^2}{2N}p_1^2 + \frac{q_1x^4}{2N\sigma^2(N)} - \frac{1}{N}\frac{q_1x^2}{\sigma^3(N)}\sin\left(\frac{\xi_N}{\sigma(N)}\right)\right|$$

$$\leqq \left|p_1 - p_1q_1 - p_1^2\right| + \left| p_1\frac{ix\left(1-\sum\limits_{j=1}^{M} C_{j,N}\frac{1}{p_j}\right)}{\sigma(N)} + p_1q_1\frac{ix\left(\sum\limits_{j=1}^{M} C_{j,N}\frac{1}{p_j}\right)}{\sigma(N)}\right|$$

$$+\left|(-p_1)\frac{x^2\left(1-\sum\limits_{j=1}^{M} C_{j,N}\frac{1}{p_j}\right)^2}{2\sigma^2(N)} + p_1q_1\frac{x^2\left(\sum\limits_{j=1}^{M} C_{j,N}\frac{1}{p_j}\right)^2}{2\sigma^2(N)} - \frac{q_1x^2}{\sigma^2(N)} + \frac{x^2}{2N}p_1^2\right|$$

$$+\left|p_1\sum_{k=3}^{\infty}\left(\frac{ix\left(1-\sum\limits_{j=1}^{M} C_{j,N}\frac{1}{p_j}\right)}{\sigma(N)}\right)^k\frac{1}{k!}\right| + \left|p_1q_1\sum_{k=3}^{\infty}\left(\frac{-ix\left(\sum\limits_{j=1}^{M} C_{j,N}\frac{1}{p_j}\right)}{\sigma(N)}\right)^k\frac{1}{k!}\right|$$

$$+\left|\frac{2q_1}{\sigma^3(N)}\right| + \left|\frac{1}{N}\frac{q_1x^2}{\sigma^3(N)}\right| + \left|\frac{q_1x^4}{2N\sigma^2(N)}\right|$$

We have used the Taylor series expansion of $\exp(iy)$ and the Taylor expansion of order two of the real function

$$h_N(x) := 1-\cos\left(\frac{x}{\sigma(N)}\right)$$

that is

$$h_N(x) = \frac{1}{2}\frac{x^2}{\sigma^2(N)} - \frac{1}{\sigma^3(N)}\sin\left(\frac{\xi_N}{\sigma(N)}\right)$$

for some $\xi_N \in (0,|x|),\ N \in \mathbb{N}$.
Now first

$$\left|p_1 - p_1 q_1 - p_1^2\right| = 0 \ \ .$$

Further

$$\begin{aligned}
&\lim_{N\to\infty} N\left|p_1 \frac{ix\left(1-\sum\limits_{j=1}^{M} C_{j,N}\frac{1}{p_j}\right)}{\sigma(N)} + p_1 q_1 \frac{ix\left(\sum\limits_{j=1}^{M} C_{j,N}\frac{1}{p_j}\right)}{\sigma(N)}\right| \\
&= \lim_{N\to\infty} |x|\frac{N}{\sigma(N)}\left|p_1 - (1-q_1)C_{1,N} - \sum_{j=2}^{M} C_{j,N}\frac{p_1}{p_j} + \sum_{j=2}^{M} C_{j,N}\frac{p_1 q_1}{p_j}\right| \\
&= \lim_{N\to\infty} |x|\frac{N}{\sigma(N)}\left|p_1 \sum_{j=2}^{M} C_{j,N} - \sum_{j=2}^{M} C_{j,N}\frac{p_1}{p_j} + \sum_{j=2}^{M} C_{j,N}\frac{p_1 q_1}{p_j}\right| \\
&= \lim_{N\to\infty} \frac{|x|}{\sigma(N)}\left|\sum_{j=2}^{M} N C_{j,N} p_1\left(1-\frac{p_1}{p_j}\right)\right| = 0 \ \ .
\end{aligned}$$

And

$$\begin{aligned}
&\lim_{N\to\infty} N\left|-p_1\frac{x^2\left(1-\sum\limits_{j=1}^{M} C_{j,N}\frac{1}{p_j}\right)^2}{2\sigma^2(N)} + p_1 q_1 \frac{x^2\left(\sum\limits_{j=1}^{M} C_{j,N}\frac{1}{p_j}\right)^2}{2\sigma^2(N)} - \frac{1}{2}\frac{2q_1 x^2}{\sigma^2(N)} + \frac{x^2}{2N}p_1^2\right| \\
&= \lim_{N\to\infty}\frac{1}{2}x^2\left|\frac{-p_1\left(1-\sum\limits_{j=1}^{M} C_{j,N}\frac{1}{p_j}\right)^2 + p_1 q_1\left(\sum\limits_{j=1}^{M} C_{j,N}\frac{1}{p_j}\right)^2 - 2q_1}{\frac{\sigma^2(N)}{N}} + p_1^2\right| \\
&= \frac{1}{2}x^2\left|\frac{-p_1\left(1-\frac{1}{p_1}\right)^2 + p_1 q_1\left(\frac{1}{p_1}\right)^2 - 2q_1}{\frac{q_1}{p_1^2}} + p_1^2\right| = 0 \ \ . \qquad (39)
\end{aligned}$$

For (39), we have used Lemma 4.1, (ii):

$$\lim_{N\to\infty}\frac{\sigma^2(N)}{N} = \frac{q_1}{p_1^2} \ \ .$$

Now, because of $\lim_{N\to\infty} \sigma(N) = \infty$,

$$N\,|r_3(N)| \leqq \frac{\left|1 - \frac{E(S_N)}{N}\right|^3}{\frac{1}{N}\sigma^3(N)} \frac{|x|^3}{6} \underset{N\to\infty}{\longrightarrow} 0$$

and

$$N\,|r_4(N)| \leqq \frac{\left|\frac{E(S_N)}{N}\right|^3}{\frac{1}{N}\sigma^3(N)} \frac{|x|^3}{6} \underset{N\to\infty}{\longrightarrow} 0 \ .$$

Finally

$$\lim_{N\to\infty} N\left(\left|\frac{2q_1}{\sigma^3(N)}\right| + \left|\frac{1}{N}\frac{q_1 x^2}{\sigma^3(N)}\right| + \left|\frac{q_1 x^4}{2N\sigma^2(N)}\right|\right) = 0 \ .$$

This proofs (38) (recall, that $\lim_{N\to\infty} C_{1,N} = 1$). From (37) we have for $k = 2, \ldots, M$,

$$\lim_{N\to\infty} N\left|C_{k,N}\left(\frac{p_k\, e^{\dfrac{ix\left(1-\sum_{j=1}^{M} C_{j,N}\frac{1}{p_j}\right)}{\sigma(N)}}}{1 - q_k\, e^{\dfrac{ix}{\sigma(N)}}}\right)^N\right| \leqq \lim_{N\to\infty} N\,|C_{k,N}| = 0 \ .$$

$\square$

6 Appendix

6.1 A useful lemma

In this section we prove Lemma 2.2, which relies on the following formulae.

Lemma 6.1. *For $M \geq 2$ and pairwise distinct complex numbers $y_1, \ldots, y_M \in \mathbb{C}$*

$$\sum_{i=1}^{M} \prod_{\substack{j=1\\ j\neq i}}^{M} \frac{1}{y_j - y_i} = 0 \tag{40}$$

or, equivalently,

$$\sum_{i=1}^{M-1} \prod_{\substack{j=1\\ j\neq i}}^{M-1} \frac{y_j - y_M}{y_j - y_i} = 1. \tag{41}$$

Proof. Consider

$$f(y_1, \ldots, y_{M-1}; x) := \sum_{i=1}^{M-1} \prod_{\substack{j=1 \\ j \neq i}}^{M-1} \frac{y_j - x}{y_j - y_i}$$

as a polynomial of degree $M-2$ in $x \in \mathbb{C}$. Then

$$f(y_1, \ldots, y_{M-1}; y_k) = \prod_{\substack{j=1 \\ j \neq k}}^{M-1} \frac{y_j - y_k}{y_j - y_k} + \sum_{\substack{i=1 \\ i \neq k}}^{M-1} \frac{y_k - y_k}{y_j - y_i} \prod_{\substack{j=1 \\ j \neq i,k}}^{M-1} \frac{y_j - y_k}{y_j - y_i} = 1$$

for $k = 1, \ldots, M-1$, hence $f(y_1, \ldots, y_{M-1}; x) = 1$ for any $x \in \mathbb{C}$. $\square$

We now are ready to prove Lemma 2.2:

Proof of Lemma 2.2. The proof of both equations is by induction with respect to M.
Equation (4): For $M = 1$ this is obviously satisfied (recall that empty products are defined to be

one). Now let $M \geqq 2$. Then

$$
\begin{aligned}
&\sum_{n\in Z(M,N)} \prod_{j=1}^{M} \left(\frac{x_j}{y_j}\right)^{n_j} x_j^{\delta(0,n_j)-1} \\
&= \sum_{n_M=0}^{N} \left[\sum_{n\in Z(M-1,N-n_M)} \prod_{j=1}^{M-1} \left(\frac{x_j}{y_j}\right)^{n_j} x_j^{\delta(0,n_j)-1} \right] \left(\frac{x_M}{y_M}\right)^{n_M} x_M^{\delta(0,n_M)-1} \\
&= \sum_{n\in Z(M-1,N)} \prod_{j=1}^{M-1} \left(\frac{x_j}{y_j}\right)^{n_j} x_j^{\delta(0,n_j)-1} + \sum_{n\in Z(M-1,0)} \left[\prod_{j=1}^{M-1} \left(\frac{x_j}{y_j}\right)^{n_j} x_j^{\delta(0,n_j)-1} \right] \left(\frac{x_M}{y_M}\right)^{N} \frac{1}{x_M} \\
&\quad + \sum_{n_M=1}^{N-1} \left[\sum_{n\in Z(M-1,N-n_M)} \prod_{j=1}^{M-1} \left(\frac{x_j}{y_j}\right)^{n_j} x_j^{\delta(0,n_j)-1} \right] \left(\frac{x_M}{y_M}\right)^{n_N} \frac{1}{x_M} \\
&= \sum_{i=1}^{M-1} \frac{1}{x_i} \left(\frac{x_i}{y_i}\right)^{N} \prod_{\substack{j=1\\ j\neq i}}^{M-1} \frac{y_j}{y_j-y_i} + \left(\frac{x_M}{y_M}\right)^{N} \frac{1}{x_M} \\
&\quad + \sum_{n_M=1}^{N-1} \left[\sum_{i=1}^{M-1} \frac{1}{x_i} \left(\frac{x_i}{y_i}\right)^{N-n_M} \prod_{\substack{j=1\\ j\neq i}}^{M-1} \frac{y_j}{y_j-y_i} \right] \left(\frac{x_M}{y_M}\right)^{n_N} \frac{1}{x_M} \\
&= \sum_{i=1}^{M-1} \frac{1}{x_i} \left(\frac{x_i}{y_i}\right)^{N} \prod_{\substack{j=1\\ j\neq i}}^{M-1} \frac{y_j}{y_j-y_i} + \left(\frac{x_M}{y_M}\right)^{N} \frac{1}{x_M} \\
&\quad + \sum_{i=1}^{M-1} \frac{1}{x_i x_M} \left(\prod_{\substack{j=1\\ j\neq i}}^{M-1} \frac{y_j}{y_j-y_i} \right) \left[\frac{y_i y_M}{y_M-y_i} \left(\left(\frac{x_i}{y_i}\right)^{N+1} - \left(\frac{x_M}{y_M}\right)^{N+1} \right) - \left(\frac{x_i}{y_i}\right)^{N} - \left(\frac{x_M}{y_M}\right)^{N} \right] \\
&= \sum_{i=1}^{M-1} \frac{1}{x_i} \left(\frac{x_i}{y_i}\right)^{N} \left(\prod_{\substack{j=1\\ j\neq i}}^{M-1} \frac{y_j}{y_j-y_i} \right) \left\{ 1 + \frac{1}{x_M} \left[\frac{y_i y_M}{y_M-y_i} \frac{x_i}{y_i} - 1 \right] \right\} \\
&\quad + \frac{1}{x_M} \left(\frac{x_M}{y_M}\right)^{N} \left[1 - \sum_{i=1}^{M-1} \frac{1}{x_i} \left(\prod_{\substack{j=1\\ j\neq i}}^{M-1} \frac{y_j}{y_j-y_i} \right) \left\{ \frac{y_i y_M}{y_M-y_i} \frac{x_M}{y_M} + 1 \right\} \right] \\
&= \sum_{i=1}^{M-1} \frac{1}{x_i} \left(\frac{x_i}{y_i}\right)^{N} \prod_{\substack{j=1\\ j\neq i}}^{M} \frac{y_j}{y_j-y_i} + \frac{1}{x_M} \left(\frac{x_M}{y_M}\right)^{N} \left(1 - \sum_{i=1}^{M-1} \prod_{\substack{j=1\\ j\neq i}}^{M} \frac{y_j}{y_j-y_i} \right) .
\end{aligned}
$$

Now we make use of equation (41), replacing there M by $M+1$ and taking $y_{M+1} := 0$, which leads to

$$
1 = \sum_{i=1}^{M} \prod_{\substack{j=1\\ j\neq i}}^{M} \frac{y_j}{y_j-y_i} = \sum_{i=1}^{M-1} \prod_{\substack{j=1\\ j\neq i}}^{M} \frac{y_j}{y_j-y_i} + \prod_{j=1}^{M-1} \frac{y_j}{y_j-y_i} .
$$

Hence

$$\sum_{i=1}^{M-1} \frac{1}{x_i} \left(\frac{x_i}{y_i}\right)^N \left(\prod_{\substack{j=1\\j\neq i}}^{M} \frac{y_j}{y_j - y_i} \right) + \frac{1}{x_M} \left(\frac{x_M}{y_M}\right)^N \left(1 - \sum_{i=1}^{M-1} \prod_{\substack{j=1\\j\neq i}}^{M} \frac{y_j}{y_j - y_i} \right) = \sum_{i=1}^{M} \frac{1}{x_i} \left(\frac{x_i}{y_i}\right)^N \left(\prod_{\substack{j=1\\j\neq i}}^{M} \frac{y_j}{y_j - y_i} \right) .$$

Equation (5): For $M = 1$ (and also for $M = 0$) this equation is obviously satisfied. Let us assume the validity of (5) for $M - 1$. Then

$$\begin{aligned}
&\sum_{n \in Z(M,N-1)} \left(\frac{x_1}{y_1}\right)^{n_1} \prod_{i=2}^{M} \left(\frac{x_i}{y_i}\right)^{n_i} {x_i}^{\delta(0,n_i)-1} \\
&= \sum_{n_M=0}^{N-1} \left(\sum_{n \in Z(M-1,N-1-n_M)} \left(\frac{x_1}{y_1}\right)^{n_1} \prod_{i=2}^{M-1} \left(\frac{x_i}{y_i}\right)^{n_i} {x_i}^{\delta(0,n_i)-1} \right) \left(\frac{x_M}{y_M}\right)^{n_M} {x_M}^{\delta(0,n_M)-1} \\
&= \sum_{n \in Z(M-1,N-1)} \left(\frac{x_1}{y_1}\right)^{n_1} \prod_{i=2}^{M-1} \left(\frac{x_i}{y_i}\right)^{n_i} {x_i}^{\delta(0,n_i)-1} \\
&\qquad + \sum_{n_M=1}^{N-1} \left(\sum_{n \in Z(M-1,N-1-n_M)} \left(\frac{x_1}{y_1}\right)^{n_1} \prod_{i=2}^{M-1} \left(\frac{x_i}{y_i}\right)^{n_i} {x_i}^{\delta(0,n_i)-1} \right) \left(\frac{x_M}{y_M}\right)^{n_M} \frac{1}{x_M} \\
&= \sum_{i=1}^{M-1} \left(\frac{x_i}{y_i}\right)^{N-1} \prod_{\substack{j=1\\j\neq i}}^{M-1} \frac{y_j}{y_j - y_i} + \sum_{n_M=1}^{N-1} \left(\sum_{i=1}^{M-1} \left(\frac{x_i}{y_i}\right)^{N-1-n_M} \prod_{\substack{j=1\\j\neq i}}^{M-1} \frac{y_j}{y_j - y_i} \right) \left(\frac{x_M}{y_M}\right)^{n_M} \frac{1}{x_M} \\
&= \sum_{i=1}^{M-1} \left(\frac{x_i}{y_i}\right)^{N-1} \prod_{\substack{j=1\\j\neq i}}^{M-1} \frac{y_j}{y_j - y_i} + \frac{1}{y_M} \sum_{i=1}^{M-1} \left(\prod_{\substack{j=1\\j\neq i}}^{M-1} \frac{y_j}{y_j - y_i} \right) \sum_{n_M=0}^{N-2} \left(\frac{x_i}{y_i}\right)^{N-2-n_M} \left(\frac{x_M}{y_M}\right)^{n_M} \\
&= \sum_{i=1}^{M-1} \left(\frac{x_i}{y_i}\right)^{N-1} \prod_{\substack{j=1\\j\neq i}}^{M-1} \frac{y_j}{y_j - y_i} + \sum_{i=1}^{M-1} \frac{y_i}{y_M} \left(\prod_{\substack{j=1\\j\neq i}}^{M} \frac{y_j}{y_j - y_i} \right) \left[\left(\frac{x_i}{y_i}\right)^{N-1} - \left(\frac{x_M}{y_M}\right)^{N-1} \right] \\
&= \sum_{i=1}^{M-1} \left(\frac{x_i}{y_i}\right)^{N-1} \left(\prod_{\substack{j=1\\j\neq i}}^{M-1} \frac{y_j}{y_j - y_i} \right) \left[1 + \frac{y_i}{y_M - y_i} \right] - \left(\frac{x_M}{y_M}\right)^{N-1} \sum_{i=1}^{M-1} \frac{y_i}{y_M} \prod_{\substack{j=1\\j\neq i}}^{M} \frac{y_j}{y_j - y_i} .
\end{aligned}$$

Now, using Lemma 6.1 once more,

$$\begin{aligned}
-\left(\frac{x_M}{y_M}\right)^{N-1} \sum_{i=1}^{M-1} \frac{y_i}{y_M} \prod_{\substack{j=1\\j\neq i}}^{M} \frac{y_j}{y_j - y_i} &= -\left(\frac{x_M}{y_M}\right)^{N-1} \left(\prod_{j=1}^{M-1} y_j \right) \sum_{i=1}^{M-1} \prod_{\substack{j=1\\j\neq i}}^{M} \frac{1}{y_j - y_i} \\
&= \left(\frac{x_M}{y_M}\right)^{N-1} \prod_{j=1}^{M-1} \frac{y_j}{y_j - y_i} .
\end{aligned}$$

Hence

$$\sum_{i=1}^{M-1}\left(\frac{x_i}{y_i}\right)^{N-1}\left(\prod_{\substack{j=1\\ j\neq i}}^{M-1}\frac{y_j}{y_j-y_i}\right)\left[1+\frac{y_i}{y_M-y_i}\right]-\left(\frac{x_M}{y_M}\right)^{N-1}\sum_{i=1}^{M-1}\frac{y_i}{y_M}\prod_{\substack{j=1\\ j\neq i}}^{M}\frac{y_j}{y_j-y_i}$$
$$=\sum_{i=1}^{M}\left(\frac{x_i}{y_i}\right)^{N-1}\prod_{\substack{j=1\\ j\neq i}}^{M}\frac{y_j}{y_j-y_i}\quad .$$

$\square$

6.2 The cycle time

Generating function of S_N (the cycle time with a total of N customers circulating, including the test customer, see Theorem 2.6 (11))

$$h^{(M,N)}(u)=\sum_{i=1}^{M}C_{i,N}\left(\frac{p_i u}{1-q_i u}\right)^N \quad .$$

First derivative of the cycle time generating function

$$\frac{d}{du}h^{(M,N)}(u)=\sum_{i=1}^{M}N\,C_{i,N}\left(\frac{p_i u}{1-q_i u}\right)^{N-1}\left\{\frac{p_i}{1-q_i u}+\left(\frac{p_i}{1-q_i u}\right)^2\frac{q_i}{p_i}u\right\} \quad .$$

Mean cycle time

$$E(S_N)=\sum_{i=1}^{M}N\,C_{i,N}\frac{1}{p_i} \quad .$$

Second derivative

$$\begin{aligned}\frac{d^2}{du^2}h^{(M,N)}(u) \;&=\; \sum_{i=1}^{M}N(N-1)\,C_{i,N}\left(\frac{p_i u}{1-q_i u}\right)^{N-2}\left\{\frac{p_i}{1-q_i u}+\left(\frac{p_i}{1-q_i u}\right)^2\frac{q_i}{p_i}u\right\}^2\\ &\quad+\sum_{i=1}^{M}N\,C_{i,N}\left(\frac{p_i u}{1-q_i u}\right)^{N-1}\left\{2\left(\frac{p_i}{1-q_i u}\right)^2\frac{q_i}{p_i}+2\left(\frac{p_i}{1-q_i u}\right)^3\left(\frac{q_i}{p_i}\right)^2u\right\} \quad .\end{aligned}$$

Therefore

$$E\left(S_N(S_N-1)\right)=\sum_{i=1}^{M}N(N-1)\,C_{i,N}\left(\frac{1}{p_i}\right)^2+\sum_{i=1}^{M}2N\,C_{i,N}\frac{q_i}{p_i^2} \quad .$$

Cycletime variance

$$
\begin{aligned}
\mathrm{Var}(S_N) &= \sum_{i=1}^{M} N(N-1)\, C_{i,N} \left(\frac{1}{p_i}\right)^2 + \sum_{i=1}^{M} 2N\, C_{i,N} \frac{q_i}{p_i^2} + \sum_{i=1}^{M} N\, C_{i,N} \frac{1}{p_i} - \left(\sum_{i=1}^{M} N\, C_{i,N} \frac{1}{p_i}\right)^2 \\
&= \sum_{i=1}^{M} N^2\, C_{i,N} \left(\frac{1}{p_i}\right)^2 - \sum_{i=1}^{M} N\, C_{i,N} \left(\frac{1}{p_i}\right)^2 + \sum_{i=1}^{M} 2N\, C_{i,N} \frac{q_i}{p_i^2} + \sum_{i=1}^{M} N\, C_{i,N} \frac{1}{p_i} \\
&\quad - \sum_{i=1}^{M} N^2\, C_{i,N}^2 \left(\frac{1}{p_i}\right)^2 - \sum_{\substack{i,j=1 \\ i \neq j}}^{M} N^2\, C_{i,N} C_{j,N} \frac{1}{p_i} \frac{1}{p_j} \quad .
\end{aligned}
$$

Collecting terms with N^2 we have

$$
\begin{aligned}
& N^2 \left(\sum_{i=1}^{M} C_{i,N} \left(\frac{1}{p_i}\right)^2 - \sum_{i=1}^{M} C_{i,N}^2 \left(\frac{1}{p_i}\right)^2 - \sum_{\substack{i,j=1 \\ i \neq j}}^{M} C_{i,N} C_{j,N} \frac{1}{p_i} \frac{1}{p_j} \right) \\
&= N^2 \left(\sum_{i=1}^{M} C_{i,N} \left(\frac{1}{p_i}\right)^2 - \sum_{i=1}^{M} C_{i,N} \left(1 - \sum_{\substack{j=1 \\ j \neq i}}^{M} C_{j,N}\right) \left(\frac{1}{p_i}\right)^2 - \sum_{\substack{i,j=1 \\ i \neq j}}^{M} C_{i,N} C_{j,N} \frac{1}{p_i} \frac{1}{p_j} \right) \\
&= N^2 \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{M} C_{i,N} C_{j,N} \left(\frac{1}{p_i}\right)^2 - \sum_{\substack{i,j=1 \\ i \neq j}}^{M} C_{i,N} C_{j,N} \frac{1}{p_i} \frac{1}{p_j} \right) \\
&= N^2 \left(\sum_{\substack{(i,j) \in \underline{M}^2 \\ i < j}} C_{i,N} C_{j,N} \left(\left(\frac{1}{p_i}\right)^2 + \left(\frac{1}{p_j}\right)^2 \right) - \sum_{\substack{(i,j) \in \underline{M}^2 \\ i < j}} 2 C_{i,N} C_{j,N} \frac{1}{p_i} \frac{1}{p_j} \right) \\
&= N^2 \sum_{\substack{(i,j) \in \underline{M}^2 \\ i < j}} C_{i,N} C_{j,N} \left(\frac{1}{p_i} - \frac{1}{p_j} \right)^2 \quad .
\end{aligned}
$$

Collecting terms with N we have

$$
\begin{aligned}
N \sum_{i=1}^{M} \left(-C_{i,N} \left(\frac{1}{p_i}\right)^2 + 2 C_{i,N} \frac{q_i}{p_i^2} + C_{i,N} \frac{1}{p_i} \right) &= N \sum_{i=1}^{M} C_{i,N} \left(\frac{1}{p_i}\right)^2 (-1 + 2q_i + p_i) \\
&= N \sum_{i=1}^{M} C_{i,N} \frac{q_i}{p_i^2} \quad ,
\end{aligned}
$$

hence

$$
\mathrm{Var}(S_N) = N \sum_{i=1}^{M} C_{i,N} \frac{q_i}{p_i^2} + N^2 \sum_{\substack{(i,j) \in \underline{M}^2 \\ i < j}} C_{i,N} C_{j,N} \left(\frac{1}{p_i} - \frac{1}{p_j} \right)^2 \quad .
$$

6.3 Sojourn time at node $Q[k]$

Looking at node $Q[k]$ separately, we get from (13) the generating function of $S_{k,N}$,

$$g_k^{(M,N)}(u_k) = C_{k,N}\left(\frac{p_k u_k}{1-q_k u_k}\right)^N \prod_{\substack{j=1\\j\neq k}}^{M}\left(\frac{q_k-q_j}{q_k u_k-q_j}\right) + \sum_{\substack{i=1\\i\neq k}}^{M} C_{i,N}\left(\frac{q_i-q_k}{q_i-q_k u_k}u_k\right) \quad .$$

This yields the **mean sojourn time at node** $Q[k]$

$$\begin{aligned} E(S_{k,N}) &= C_{k,N}N\frac{1}{p_k} - C_{k,N}\sum_{\substack{l=1\\l\neq k}}^{M}\frac{q_k}{q_k-q_l} + \sum_{\substack{l=1\\l\neq k}}^{M} C_{l,N}\frac{q_l}{q_l-q_k} &(42)\\ &= C_{k,N}N\frac{1}{p_k} + \sum_{\substack{l=1\\l\neq k}}^{M}\left(\frac{q_l C_{l,N}}{q_l-q_k} - \frac{q_k C_{k,N}}{q_k-q_l}\right) &(43)\end{aligned}$$

from the first derivative of the sojourn time generating function

$$\begin{aligned} &\frac{d}{du_k} g_k^{(M,N)}(u_k)\\ &= N\,C_{k,N}\left(\frac{p_k u_k}{1-q_k u_k}\right)^{N-1}\left\{\frac{p_k}{1-q_k u_k} + \frac{q_k}{p_k}\left(\frac{p_k}{1-q_k u_k}\right)^2 u_k\right\}\prod_{\substack{j=1\\j\neq k}}^{M}\left(\frac{q_k-q_j}{q_k u_k-q_j}\right)\\ &\quad - C_{k,N}\left(\frac{p_k u_k}{1-q_k u_k}\right)^N \sum_{\substack{r=1\\r\neq k}}^{M}\left(\prod_{\substack{j=1\\j\neq k}}^{M}\frac{q_k-q_j}{q_k u_k-q_j}\right)\left(\frac{q_k-q_r}{q_k u_k-q_r}\right)\frac{q_k}{q_k-q_r}\\ &\quad + \sum_{\substack{i=1\\i\neq k}}^{M} C_{i,N}\left(\frac{q_i-q_k}{q_i-q_k u_k} + u_k\left(\frac{q_i-q_k}{q_i-q_k u_k}\right)^2\frac{q_k}{q_i-q_k}\right) \quad .\end{aligned}$$

The **Variance of the sojourn time at node** $Q[k]$ is

$$
\begin{aligned}
&\mathrm{Var}(S_{k,N}) \\
= \quad & N^2 C_{k,N}\frac{1}{p_k^2} - N^2 C_{k,N}^2 \frac{1}{p_k^2} + N C_{k,N}\frac{q_k}{p_k^2} - 2N C_{k,N}\sum_{\substack{r=1\\ r\neq k}}^{M}\frac{1}{p_k}\frac{q_k}{q_k-q_r} \qquad (44)\\
&+ 2C_{k,N}^2 N \sum_{\substack{r=1\\ r\neq k}}^{M}\frac{1}{p_k}\frac{q_k}{q_k-q_r} - 2C_{k,N} N \sum_{\substack{i=1\\ i\neq k}}^{M} C_{i,N}\frac{1}{p_k}\frac{q_i}{q_i-q_k} \\
&+ C_{k,N}\sum_{\substack{r=1\\ r_1\neq k}}^{M}\frac{q_k q_r}{(q_k-q_r)^2} + C_{k,N}\sum_{\substack{r_1=1\\ r_1\neq k}}^{M}\sum_{\substack{r_2=1\\ r_2\neq k}}^{M}\frac{q_k}{q_k-q_{r_1}}\frac{q_k}{q_k-q_{r_2}} + \sum_{\substack{i=1\\ i\neq k}}^{M} C_{i,N}\frac{q_i(q_i+q_k)}{(q_i-q_k)^2} \\
&- \left(C_{k,N}\sum_{\substack{r=1\\ r\neq k}}^{M}\frac{q_k}{q_k-q_r}\right)^2 - \left(\sum_{\substack{i=1\\ i\neq k}}^{M} C_{i,N}\frac{q_i}{q_i-q_k}\right)^2 + 2\sum_{\substack{i=1\\ i\neq k}}^{M}\sum_{\substack{r=1\\ r\neq k}}^{M} C_{i,N}C_{k,N}\frac{q_i}{q_i-q_k}\frac{q_k}{q_k-q_r} \ .
\end{aligned}
$$

This is directly obtained by some tedious computations:

Second derivative of $g_k^{(M,N)}$

$$
\begin{aligned}
&\frac{d^2}{du_k^2} g_k^{(M,N)}(u_k) \\
&= \; N(N-1)C_{k,N}\left(\frac{p_k u_k}{1-q_k u_k}\right)^{N-2}\left\{\frac{p_k}{1-q_k u_k}+\frac{q_k}{p_k}\left(\frac{p_k}{1-q_k u_k}\right)^2 u_k\right\}^2 \prod_{\substack{j=1\\ j\neq k}}^{M}\left(\frac{q_k-q_j}{q_k u_k-q_j}\right) \\
&\qquad + 2NC_{k,N}\left(\frac{p_k u_k}{1-q_k u_k}\right)^{N-1}\left\{\frac{q_k}{p_k}\left(\frac{p_k}{1-q_k u_k}\right)^2+\left(\frac{q_k}{p_k}\right)^2 u_k\left(\frac{p_k}{1-q_k u_k}\right)^3\right\}\prod_{\substack{j=1\\ j\neq k}}^{M}\left(\frac{q_k-q_j}{q_k u_k-q_j}\right) \\
&\qquad - 2NC_{k,N}\sum_{\substack{r=1\\ r\neq k}}^{M}\left(\frac{p_k u_k}{1-q_k u_k}\right)^{N-1}\left\{\frac{p_k}{1-q_k u_k}+\frac{q_k}{p_k}\left(\frac{p_k}{1-q_k u_k}\right)^2 u_k\right\} \\
&\qquad\qquad\qquad \cdot\left(\prod_{\substack{j=1\\ j\neq k}}^{M}\frac{q_k-q_j}{q_k u_k-q_j}\right)\left(\frac{q_k-q_r}{q_k u_k-q_r}\right)\frac{q_k}{q_k-q_r} \\
&\qquad + C_{k,N}\left(\frac{p_k u_k}{1-q_k u_k}\right)^{N}\sum_{\substack{r_1=1\\ r_1\neq k}}^{M}\left\{\sum_{\substack{r_2=1\\ r_2\neq k}}^{M}\left(\prod_{\substack{j=1\\ j\neq k}}^{M}\frac{q_k-q_j}{q_k u_k-q_j}\right)\frac{q_k-q_{r_2}}{q_k u_k-q_{r_2}}\frac{q_k-q_{r_1}}{q_k u_k-q_{r_1}}\frac{q_k}{q_k-q_{r_1}}\frac{q_k}{q_k-q_{r_2}}\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. + \left(\prod_{\substack{j=1\\ j\neq k}}^{M}\frac{q_k-q_j}{q_k u_k-q_j}\right)\left(\frac{q_k-q_{r_1}}{q_k u_k-q_{r_1}}\right)^2\left(\frac{q_k}{q_k-q_{r_1}}\right)^2\right\} \\
&\qquad + 2\sum_{\substack{i=1\\ i\neq k}}^{M} C_{i,N}\left\{\left(\frac{q_i-q_k}{q_i-q_k u_k}\right)^3\left(\frac{q_k}{q_i-q_k}\right)^2 u_k+\left(\frac{q_i-q_k}{q_i-q_k u_k}\right)^2\frac{q_k}{q_i-q_k}\right\} \; .
\end{aligned}
$$

Hence

$$
\begin{aligned}
&E\left(S_{k,N}(S_{k,N}-1)\right) \\
&= \; N(N-1)C_{k,N}\left(\frac{1}{p_k}\right)^2 + 2NC_{k,N}\frac{q_k}{p_k^2} - 2NC_{k,N}\sum_{\substack{r=1\\ r\neq k}}^{M}\frac{1}{p_k}\frac{q_k}{q_k-q_r} \\
&\qquad + 2\sum_{\substack{i=1\\ i\neq k}}^{M} C_{i,N}\frac{q_i q_k}{(q_i-q_k)^2} + C_{k,N}\sum_{\substack{r_1=1\\ r_1\neq k}}^{M}\left\{\left(\frac{q_k}{q_k-q_{r_1}}\right)^2+\sum_{\substack{r_2=1\\ r_2\neq k}}^{M}\frac{q_k}{q_k-q_{r_1}}\frac{q_k}{q_k-q_{r_2}}\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
E\left(S_{k,N}^2\right) &= N^2 C_{k,N}\left(\frac{1}{p_k}\right)^2 - NC_{k,N}\left(\frac{1}{p_k}\right)^2 + 2NC_{k,N}\frac{q_k}{p_k^2} - 2NC_{k,N}\sum_{\substack{r=1\\ r\neq k}}^{M}\frac{1}{p_k}\frac{q_k}{q_k-q_r} \\
&\quad + 2\sum_{\substack{i=1\\ i\neq k}}^{M} C_{i,N}\frac{q_i q_k}{(q_i-q_k)^2} + C_{k,N}\sum_{\substack{r_1=1\\ r_1\neq k}}^{M}\left\{\left(\frac{q_k}{q_k-q_{r_1}}\right)^2 + \sum_{\substack{r_2=1\\ r_2\neq k}}^{M}\frac{q_k}{q_k-q_{r_1}}\frac{q_k}{q_k-q_{r_2}}\right\} \\
&\quad + N\,C_{k,N}\frac{1}{p_k} - C_{k,N}\sum_{\substack{r=1\\ r\neq k}}^{M}\frac{q_k}{q_k-q_r} + \sum_{\substack{i=1\\ i\neq k}}^{M} C_{i,N}\frac{q_i}{q_i-q_k} \quad . \\
&= N^2 C_{k,N}\left(\frac{1}{p_k}\right)^2 + NC_{k,N}\frac{q_k}{p_k^2} - 2NC_{k,N}\sum_{\substack{r=1\\ r\neq k}}^{M}\frac{1}{p_k}\frac{q_k}{q_k-q_r} + \sum_{\substack{i=1\\ i\neq k}}^{M} C_{i,N}\frac{q_i(q_i+q_k)}{(q_i-q_k)^2} \\
&\quad + C_{k,N}\sum_{\substack{r=1\\ r_1\neq k}}^{M}\frac{q_k q_r}{\left(q_k-q_r\right)^2} + C_{k,N}\sum_{\substack{r_1=1\\ r_1\neq k}}^{M}\sum_{\substack{r_2=1\\ r_2\neq k}}^{M}\frac{q_k}{q_k-q_{r_1}}\frac{q_k}{q_k-q_{r_2}} \quad .
\end{aligned}
$$

6.4 Sojourn times covariance

The **covariance of the sojourn time at the nodes** $Q[k]$ **and** $Q[l]$, $k, l \in \underline{M}$, is

$$
\begin{aligned}
&\mathrm{Cov}\,(S_{k,N}, S_{l,N}) \\
&= \sum_{\substack{i=1\\ i\neq k,l}}^{M} C_{i,N}\frac{q_i}{q_i-q_k}\frac{q_i}{q_i-q_l} - C_{l,N}\frac{q_l}{q_l-q_k}\sum_{\substack{r=1\\ r\neq k,l}}^{M}\frac{q_l}{q_l-q_r} - C_{k,N}\frac{q_k}{q_k-q_l}\sum_{\substack{r=1\\ r\neq k,l}}^{M}\frac{q_k}{q_k-q_r} \qquad (45)\\
&\quad - C_{k,N}\left(\frac{q_k}{q_k-q_l} + 2\frac{q_l}{q_k-q_l}\frac{q_k}{q_k-q_l}\right) - C_{l,N}\left(\frac{q_l}{q_l-q_k} + 2\frac{q_l}{q_l-q_k}\frac{q_k}{q_l-q_k}\right)\\
&\quad + NC_{k,N}\frac{1}{p_k}\frac{q_k}{q_k-q_l} + NC_{l,N}\frac{1}{p_l}\frac{q_l}{q_l-q_k}\\
&\quad - N^2 C_{k,N}C_{l,N}\frac{1}{p_k}\frac{1}{p_l} + NC_{k,N}C_{l,N}\sum_{\substack{r=1\\ r\neq l}}^{M}\frac{1}{p_k}\frac{q_l}{q_l-q_r} - NC_{k,N}\sum_{\substack{i=1\\ i\neq l}}^{M} C_{i,N}\frac{1}{p_k}\frac{q_i}{q_i-q_l}\\
&\quad + NC_{k,N}C_{l,N}\sum_{\substack{r=1\\ r\neq k}}^{M}\frac{q_k}{q_k-q_r}\frac{1}{p_l} - NC_{l,N}\frac{1}{p_l}\left(\sum_{\substack{i=1\\ i\neq k}}^{M} C_{i,N}\frac{q_i}{q_i-q_k}\right)\\
&\quad - C_{k,N}C_{l,N}\left(\sum_{\substack{r=1\\ r\neq k}}^{M}\frac{q_k}{q_k-q_r}\right)\left(\sum_{\substack{r=1\\ r\neq l}}^{M}\frac{q_l}{q_l-q_r}\right) + C_{k,N}\left(\sum_{\substack{r=1\\ r\neq k}}^{M}\frac{q_k}{q_k-q_r}\right)\left(\sum_{\substack{i=1\\ i\neq l}}^{M} C_{i,N}\frac{q_i}{q_i-q_l}\right)\\
&\quad + C_{l,N}\left(\sum_{\substack{i=1\\ i\neq k}}^{M} C_{i,N}\frac{q_i}{q_i-q_k}\right)\left(\sum_{\substack{r=1\\ r\neq l}}^{M}\frac{q_l}{q_l-q_r}\right) - \left(\sum_{\substack{i=1\\ i\neq k}}^{M} C_{i,N}\frac{q_i}{q_i-q_k}\right)\left(\sum_{\substack{i=1\\ i\neq l}}^{M} C_{i,N}\frac{q_i}{q_i-q_l}\right).
\end{aligned}
$$

This holds for the covariance between the bottleneck and any other station as well as for the covariance between two non bottleneck stations.
Again, direct but tedious computations suffice. We have the **generating function of** $(S_{k,N}, S_{l,N})$ (sojourn times at $Q[k]$ and $Q[l]$), see Theorem 2.6, (13)))

$$
\begin{aligned}
g_{k,l}^{(M,N)}(u_k,u_l) &= \sum_{\substack{i=1\\ i\neq k,l}}^{M} C_{i,N}\frac{q_i-q_k}{q_i-q_k u_k}u_k\frac{q_i-q_l}{q_i-q_l u_l}u_l \qquad (46)\\
&\quad + C_{l,N}\left(\frac{p_l u_l}{1-q_l u_l}\right)^N \frac{q_l-q_k}{q_l u_l - q_k u_k}u_k \prod_{\substack{j=1\\ j\neq k,l}}^{M}\frac{q_l-q_j}{q_l u_l - q_j}\\
&\quad + C_{k,N}\left(\frac{p_k u_k}{1-q_k u_k}\right)^N \frac{q_k-q_l}{q_k u_k - q_l u_l}u_l \prod_{\substack{j=1\\ j\neq k,l}}^{M}\frac{q_k-q_j}{q_k u_k - q_j}.
\end{aligned}
$$

Partial derivative with respect to u_k

$$
\begin{aligned}
&\frac{d}{du_k} g_{k,l}^{(M,N)}(u_k, u_l) \\
&= \sum_{\substack{i=1 \\ i\neq k,l}}^{M} C_{i,N} \left\{ \frac{q_i - q_k}{q_i - q_k u_k} + \left(\frac{q_i - q_k}{q_i - q_k u_k} \right)^2 \frac{q_k}{q_i - q_k} u_k \right\} \frac{q_i - q_l}{q_i - q_l u_l} u_l \\
&\quad + C_{l,N} \left(\frac{p_l u_l}{1 - q_l u_l} \right)^N \left\{ \frac{q_l - q_k}{q_l u_l - q_k u_k} + \left(\frac{q_l - q_k}{q_l u_l - q_k u_k} \right)^2 \frac{q_k}{q_l - q_k} u_k \right\} \prod_{\substack{j=1 \\ j\neq k,l}}^{M} \frac{q_l - q_j}{q_l u_l - q_j} \\
&\quad + N C_{k,N} \left(\frac{p_k u_k}{1 - q_k u_k} \right)^{N-1} \left\{ \frac{p_k}{1 - q_k u_k} + \frac{q_k}{p_k} \left(\frac{p_k}{1 - q_k u_k} \right)^2 u_k \right\} \frac{q_k - q_l}{q_k u_k - q_l u_l} u_l \prod_{\substack{j=1 \\ j\neq k,l}}^{M} \frac{q_k - q_j}{q_k u_k - q_j} \\
&\quad - C_{k,N} \left(\frac{p_k u_k}{1 - q_k u_k} \right)^N \left\{ \left(\frac{q_k - q_l}{q_k u_k - q_l u_l} \right)^2 \frac{q_k}{q_k - q_l} u_l \right\} \prod_{\substack{j=1 \\ j\neq k,l}}^{M} \frac{q_k - q_j}{q_k u_k - q_j} \\
&\quad - C_{k,N} \left(\frac{p_k u_k}{1 - q_k u_k} \right)^N \frac{q_k - q_l}{q_k u_k - q_l u_l} u_l \sum_{\substack{r=1 \\ r\neq k,l}}^{M} \left(\prod_{\substack{j=1 \\ j\neq k,l}}^{M} \frac{q_k - q_j}{q_k u_k - q_j} \right) \frac{q_k - q_r}{q_k u_k - q_r} \frac{q_k}{q_k - q_r} \quad .
\end{aligned} \tag{47}
$$

Mixed partial derivative of order two

$$
\begin{aligned}
&\frac{d^2}{du_l\,du_k} g_{k,l}^{(M,N)}(u_k,u_l) \\
&\quad = \sum_{\substack{i=1\\ i\neq k,l}}^{M} C_{i,N} \left\{ \frac{q_i-q_k}{q_i-q_k u_k} + \left(\frac{q_i-q_k}{q_i-q_k u_k}\right)^2 \frac{q_k}{q_i-q_k} u_k \right\} \left\{ \frac{q_i-q_l}{q_i-q_l u_l} + \left(\frac{q_i-q_l}{q_i-q_l u_l}\right)^2 \frac{q_l}{q_i-q_l} u_l \right\} \quad (48) \\
&\qquad + N C_{l,N} \left(\frac{p_l u_l}{1-q_l u_l}\right)^{N-1} \left\{ \frac{p_l}{1-q_l u_l} + \frac{q_l}{p_l}\left(\frac{p_l}{1-q_l u_l}\right)^2 u_l \right\} \\
&\qquad\qquad \cdot \left\{ \frac{q_l-q_k}{q_l u_l - q_k u_k} + \left(\frac{q_l-q_k}{q_l u_l - q_k u_k}\right)^2 \frac{q_k}{q_l-q_k} u_k \right\} \prod_{\substack{j=1\\ j\neq k,l}}^{M} \frac{q_l-q_j}{q_l u_l - q_j} \\
&\qquad - C_{l,N} \left(\frac{p_l u_l}{1-q_l u_l}\right)^{N} \left(\prod_{\substack{j=1\\ j\neq k,l}}^{M} \frac{q_l-q_j}{q_l u_l - q_j} \right) \\
&\qquad\qquad \cdot \left\{ \left(\frac{q_l-q_k}{q_l u_l - q_k u_k}\right)^2 \frac{q_l}{q_l-q_k} + 2\left(\frac{q_l-q_k}{q_l u_l - q_k u_k}\right)^3 \left(\frac{q_l}{q_l-q_k}\right)\left(\frac{q_k}{q_l-q_k}\right) u_k \right\} \\
&\qquad - C_{l,N} \left(\frac{p_l u_l}{1-q_l u_l}\right)^{N} \left(\sum_{\substack{r=1\\ r\neq k,l}}^{M} \frac{q_l-q_r}{q_l u_l - q_r}\frac{q_l}{q_l-q_r} \prod_{\substack{j=1\\ j\neq k,l}}^{M} \frac{q_l-q_j}{q_l u_l - q_j} \right) \\
&\qquad\qquad \cdot \left\{ \frac{q_l-q_k}{q_l u_l - q_k u_k} + \left(\frac{q_l-q_k}{q_l u_l - q_k u_k}\right)^2 \frac{q_k}{q_l-q_k} u_k \right\} \\
&\qquad + N C_{k,N} \left(\frac{p_k u_k}{1-q_k u_k}\right)^{N-1} \left\{ \frac{p_k}{1-q_k u_k} + \frac{q_k}{p_k}\left(\frac{p_k}{1-q_k u_k}\right)^2 u_k \right\} \\
&\qquad\qquad \cdot \left\{ \frac{q_k-q_l}{q_k u_k - q_l u_l} + \left(\frac{q_k-q_l}{q_k u_k - q_l u_l}\right)^2 \frac{q_l}{q_k-q_l} u_l \right\} \prod_{\substack{j=1\\ j\neq k,l}}^{M} \frac{q_k-q_j}{q_k u_k - q_j} \\
&\qquad - C_{k,N} \left(\frac{p_k u_k}{1-q_k u_k}\right)^{N} \left(\prod_{\substack{j=1\\ j\neq k,l}}^{M} \frac{q_k-q_j}{q_k u_k - q_j} \right) \\
&\qquad\qquad \cdot \left\{ \left(\frac{q_k-q_l}{q_k u_k - q_l u_l}\right)^2 \frac{q_k}{q_k-q_l} + 2\left(\frac{q_k-q_l}{q_k u_k - q_l u_l}\right)^3 \left(\frac{q_l}{q_k-q_l}\right)\left(\frac{q_k}{q_k-q_l}\right) u_l \right\} \\
&\qquad - C_{k,N} \left(\frac{p_k u_k}{1-q_k u_k}\right)^{N} \left\{ \frac{q_k-q_l}{q_k u_k - q_l u_l} + \left(\frac{q_k-q_l}{q_k u_k - q_l u_l}\right)^2 \frac{q_l}{q_k-q_l} u_l \right\} \\
&\qquad\qquad \cdot \sum_{\substack{r=1\\ r\neq k,l}}^{M} \left(\prod_{\substack{j=1\\ j\neq k,l}}^{M} \frac{q_k-q_j}{q_k u_k - q_j} \right) \frac{q_k-q_r}{q_k u_k - q_r}\frac{q_k}{q_k-q_r} \ .
\end{aligned}
$$

From (48) we obtain

$$
\begin{aligned}
&E\left(S_{k,N}\cdot S_{l,N}\right)\\
&= \sum_{\substack{i=1\\ i\neq k,l}}^{M} C_{i,N}\frac{q_i}{q_i-q_k}\frac{q_i}{q_i-q_l} - C_{l,N}\frac{q_l}{q_l-q_k}\sum_{\substack{r=1\\ r\neq k,l}}^{M}\frac{q_l}{q_l-q_r} - C_{k,N}\frac{q_k}{q_k-q_l}\sum_{\substack{r=1\\ r\neq k,l}}^{M}\frac{q_k}{q_k-q_r}\\
&\qquad - C_{k,N}\left(\frac{q_k}{q_k-q_l}+2\frac{q_l}{q_k-q_l}\frac{q_k}{q_k-q_l}\right) - C_{l,N}\left(\frac{q_l}{q_l-q_k}+2\frac{q_l}{q_l-q_k}\frac{q_k}{q_l-q_k}\right)\\
&\qquad + NC_{k,N}\frac{1}{p_k}\frac{q_k}{q_k-q_l} + NC_{l,N}\frac{1}{p_l}\frac{q_l}{q_l-q_k} \quad .
\end{aligned}
$$

From (43) we obtain

$$
\begin{aligned}
&E(S_{k,N})E(S_{l,N})\\
&= N^2 C_{k,N}C_{l,N}\frac{1}{p_k}\frac{1}{p_l} - NC_{k,N}C_{l,N}\sum_{\substack{r=1\\ r\neq l}}^{M}\frac{1}{p_k}\frac{q_l}{q_l-q_r} + NC_{k,N}\sum_{\substack{i=1\\ i\neq l}}^{M} C_{i,N}\frac{1}{p_k}\frac{q_i}{q_i-q_l}\\
&\quad - NC_{k,N}C_{l,N}\sum_{\substack{r=1\\ r\neq k}}^{M}\frac{q_k}{q_k-q_r}\frac{1}{p_l} + NC_{l,N}\frac{1}{p_l}\left(\sum_{\substack{i=1\\ i\neq k}}^{M} C_{i,N}\frac{q_i}{q_i-q_k}\right)\\
&\quad + C_{k,N}C_{l,N}\left(\sum_{\substack{r=1\\ r\neq k}}^{M}\frac{q_k}{q_k-q_r}\right)\left(\sum_{\substack{r=1\\ r\neq l}}^{M}\frac{q_l}{q_l-q_r}\right) - C_{k,N}\left(\sum_{\substack{r=1\\ r\neq k}}^{M}\frac{q_k}{q_k-q_r}\right)\left(\sum_{\substack{i=1\\ i\neq l}}^{M} C_{i,N}\frac{q_i}{q_i-q_l}\right)\\
&\quad - C_{l,N}\left(\sum_{\substack{i=1\\ i\neq k}}^{M} C_{i,N}\frac{q_i}{q_i-q_k}\right)\left(\sum_{\substack{r=1\\ r\neq l}}^{M}\frac{q_l}{q_l-q_r}\right) + \left(\sum_{\substack{i=1\\ i\neq k}}^{M} C_{i,N}\frac{q_i}{q_i-q_k}\right)\left(\sum_{\substack{i=1\\ i\neq l}}^{M} C_{i,N}\frac{q_i}{q_i-q_l}\right) \quad .
\end{aligned}
$$

6.5 Computation of the asymptotic moments

Proof of Lemma 4.1. Property (i) and (ii) are immediate consequences of Proposition 3.2. From Lemma 2.4 follows (iii) by

$$
\lim_{N\to\infty}\frac{E(S_{1,N})}{N} = \lim_{N\to\infty}\left(C_{1,N}\frac{1}{p_1} - \frac{1}{N}C_{1,N}\sum_{j=2}^{M}\frac{q_1}{q_1-q_j} + \frac{1}{N}\sum_{j=2}^{M}C_{j,N}\frac{q_j}{q_j-q_1}\right) = \frac{1}{p_1}
$$

and (v) by

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N}\mathrm{Var}(S_{1,N}) \\
&= \lim_{N\to\infty}\Bigg(N\frac{1}{p_1^2}C_{1,N}\left(1-C_{1,N}\right) + C_{1,N}\frac{q_1}{p_1^2} - 2C_{1,N}\sum_{\substack{r=1\\ r\neq 1}}^{M}\frac{1}{p_1}\frac{q_1}{q_1-q_r} \\
&\quad + 2C_{1,N}^2\sum_{\substack{r=1\\ r\neq 1}}^{M}\frac{1}{p_1}\frac{q_1}{q_1-q_r} - 2C_{1,N}\sum_{\substack{i=1\\ i\neq 1}}^{M}C_{i,N}\frac{1}{p_1}\frac{q_i}{q_i-q_1} \\
&\quad + \frac{1}{N}C_{1,N}\sum_{\substack{r=1\\ r_1\neq 1}}^{M}\frac{q_1 q_r}{(q_1-q_r)^2} + \frac{1}{N}C_{1,N}\sum_{\substack{r_1=1\\ r_1\neq 1}}^{M}\sum_{\substack{r_2=1\\ r_2\neq 1}}^{M}\frac{q_1}{q_1-q_{r_1}}\frac{q_1}{q_1-q_{r_2}} + \frac{1}{N}\sum_{\substack{i=1\\ i\neq 1}}^{M}C_{i,N}\frac{q_i(q_i+q_1)}{(q_i-q_1)^2} \\
&\quad - \frac{1}{N}\left(C_{1,N}\sum_{\substack{r=1\\ r\neq 1}}^{M}\frac{q_1}{q_1-q_r}\right)^2 - \frac{1}{N}\left(\sum_{\substack{i=1\\ i\neq 1}}^{M}C_{i,N}\frac{q_i}{q_i-q_1}\right)^2 \\
&\quad + 2\frac{1}{N}\sum_{\substack{i=1\\ i\neq 1}}^{M}\sum_{\substack{r=1\\ r\neq 1}}^{M}C_{i,N}C_{1,N}\frac{q_i}{q_i-q_1}\frac{q_1}{q_1-q_r}\Bigg) = \frac{q_1}{p_1^2} \quad .
\end{aligned}$$

From equation (43) we also get (iv) for $k \neq 1$

$$\lim_{N\to\infty} E(S_{k,N}) = \lim_{N\to\infty}\left(C_{k,N}N\frac{1}{p_k} - C_{k,N}\sum_{\substack{r=1\\ r\neq k}}^{M}\frac{q_k}{q_k-q_r} + \sum_{\substack{i=1\\ i\neq k}}^{M}C_{i,N}\frac{q_i}{q_i-q_k}\right) = \frac{q_1}{q_i-q_k} \quad .$$

(vi) Similarly from equation (44)

$$\lim_{N\to\infty}\mathrm{Var}(S_{k,N}) = \frac{q_1(q_1+q_k)}{(q_1-q_k)^2} - \left(\frac{q_1}{q_1-q_k}\right)^2 = \frac{q_1\, q_k}{(q_1-q_k)^2} \quad .$$

From equation (45)

$$
\begin{aligned}
&\lim_{N\to\infty} \operatorname{Cov}\left(S_{1,N}, S_{l,N}\right) \\
&= \lim_{N\to\infty} \left(C_{1,N} \left(\sum_{\substack{r=1\\ r\neq 1}}^{M} \frac{q_1}{q_1-q_r} \right) \left(\sum_{\substack{i=1\\ i\neq l}}^{M} C_{i,N} \frac{q_i}{q_i-q_l} \right) - C_{1,N} \frac{q_1}{q_1-q_l} \sum_{\substack{r=1\\ r\neq 1,l}}^{M} \frac{q_1}{q_1-q_r} \right. \\
&\qquad - C_{1,N} \left(\frac{q_1}{q_1-q_l} + 2\frac{q_l}{q_1-q_l}\frac{q_1}{q_1-q_l} \right) \\
&\qquad \left. + NC_{1,N} \frac{1}{p_1}\frac{q_1}{q_1-q_l} - NC_{1,N} \sum_{\substack{i=1\\ i\neq l}}^{M} C_{i,N} \frac{1}{p_1}\frac{q_i}{q_i-q_l} \right) \\
&= \frac{q_1}{q_1-q_l} \left(\sum_{\substack{r=1\\ r\neq 1}}^{M} \frac{q_1}{q_1-q_r} \right) - \frac{q_1}{q_1-q_l} \sum_{\substack{r=1\\ r\neq 1,l}}^{M} \frac{q_1}{q_1-q_r} - \left(\frac{q_1}{q_1-q_l} + 2\frac{q_l}{q_1-q_l}\frac{q_1}{q_1-q_l} \right) \\
&\qquad + \lim_{N\to\infty} \left(NC_{1,N} \frac{1}{p_1}\frac{q_1}{q_1-q_l} - NC_{1,N} \sum_{\substack{i=1\\ i\neq l}}^{M} C_{i,N} \frac{1}{p_1}\frac{q_i}{q_i-q_l} \right) .
\end{aligned}
$$

To determine the limiting behaviour of

$$
NC_{1,N} \frac{1}{p_1}\frac{q_1}{q_1-q_l} - NC_{1,N} \sum_{\substack{i=1\\ i\neq l}}^{M} C_{i,N} \frac{1}{p_1}\frac{q_i}{q_i-q_l}
$$

recall that $\sum_{i=1}^{M} C_{i,N} = 1$. Therefore

$$
\begin{aligned}
&NC_{1,N} \frac{1}{p_1}\frac{q_1}{q_1-q_l} - NC_{1,N} \sum_{\substack{i=1\\ i\neq l}}^{M} C_{i,N} \frac{1}{p_1}\frac{q_i}{q_i-q_l} \\
&= NC_{1,N} \frac{1}{p_1}\frac{q_1}{q_1-q_l} \left(1 - C_{1,N}\right) - NC_{1,N} \sum_{\substack{i=2\\ i\neq l}}^{M} C_{i,N} \frac{1}{p_1}\frac{q_i}{q_i-q_l} \\
&= C_{1,N} \frac{1}{p_1}\frac{q_1}{q_1-q_l} \left(\sum_{\substack{i=2\\ i\neq l}}^{M} NC_{i,N} \right) - C_{1,N} \sum_{\substack{i=2\\ i\neq l}}^{M} NC_{i,N} \frac{1}{p_1}\frac{q_i}{q_i-q_l} \\
&= C_{1,N} \frac{1}{p_1} \sum_{\substack{i=2\\ i\neq l}}^{M} NC_{i,N} \left(\frac{q_1}{q_1-q_l} - \frac{q_i}{q_i-q_l} \right)
\end{aligned}
$$

hence

$$
\lim_{N\to\infty} \operatorname{Cov}\left(S_{1,N}, S_{l,N}\right) = \frac{q_1}{q_1-q_l}\frac{q_l}{q_l-q_1} .
$$

Also from equation (45)

$$\lim_{N\to\infty} \operatorname{Cov}(S_{k,N}, S_{l,N})$$

$$= \lim_{N\to\infty}\left(\sum_{\substack{i=1\\ i\neq k,l}}^{M} C_{i,N}\frac{q_i}{q_i-q_k}\frac{q_i}{q_i-q_l} - \left(\sum_{\substack{i=1\\ i\neq k}}^{M} C_{i,N}\frac{q_i}{q_i-q_k}\right)\left(\sum_{\substack{i=1\\ i\neq l}}^{M} C_{i,N}\frac{q_i}{q_i-q_l}\right)\right) = 0 \ .$$

(ix) We have to show

$$\lim_{N\to\infty}\sqrt{N}\varrho(S_{1,N}, S_{l,N}) = \sqrt{q_l}\frac{p_1}{p_1-p_l} \ .$$

Using properties (vii), (v) and (vi) yields

$$\sqrt{N}\varrho(S_{1,N}, S_{l,N}) = \frac{\operatorname{Cov}(S_{1,N}, S_{l,N})}{\sqrt{\frac{\operatorname{Var}(S_{1,N})}{\sqrt{N}}\operatorname{Var}(S_{l,N})}} \underset{N\to\infty}{\longrightarrow} \frac{q_1}{q_1-q_l}\frac{q_l}{q_l-q_1}\frac{p_1}{\sqrt{q_1}}\frac{q_1-q_l}{\sqrt{q_1 q_l}} = \frac{\sqrt{q_l}p_1}{p_1-p_l} \ .$$

(x) Now $\varrho(S_{1,N}, S_{l,N}) \underset{N\to\infty}{\longrightarrow} 0$ is immediate from property (ix). And from properties (viii) and (vi)

$$\lim_{N\to\infty}\frac{\operatorname{Cov}(S_{1,N}, S_{l,N})}{\sqrt{\operatorname{Var}(S_{k,N})\operatorname{Var}(S_{l,N})}} = 0$$

for $k, l \leq 2$, $k \neq l$. $\square$

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