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Abstract

We consider a closed queueing network, consisting of two FCFS single server queues in series: a queue with general service times and a queue with exponential service times. A fixed number N of customers cycles through this network. We determine the joint sojourn time distribution of a tagged customer in, first, the general queue and, then, the exponential queue. Subsequently, we indicate how the approach towards this closed system also allows us to study the joint sojourn time distribution of a tagged customer in the equivalent *open* two-queue system, consisting of FCFS single server queues with general and exponential service times, respectively, in the case that the input process to the first queue is a Poisson process.
Keywords: closed cyclic queue, tandem queue, sojourn time

1 Introduction

In this paper we consider a closed queueing network, consisting of two FCFS single server queues Q_G and Q_M in series. Here Q_G denotes a queue with generally distributed service times, and Q_M a queue with exponentially distributed service times. A fixed number N of customers cycles through this network, alternatingly visiting Q_G and Q_M. We determine the joint sojourn time distribution of a tagged customer in, first, Q_G and, then, Q_M. Subsequently, we indicate how the approach towards this closed system also allows us to study the joint sojourn time distribution of a tagged customer in the equivalent *open* two-queue system, consisting of Q_G and Q_M, in the case that the input process to Q_G is a Poisson process.

Early results on sojourn times in open tandem queues are due to Reich [24, 25]. Using reversibility, Reich showed that the successive sojourn times of a tagged customer along a series of $M/M/1$ queues are independent and exponentially distributed. Burke [8] proved that the sojourn times are even independent if the first and last of these queues are multiserver queues ($M/M/c$). At the end of the seventies, various authors investigated to what extent this independence of successive sojourn times remains true in a path of an open product-form network. A key condition turned out to be that the path should be *overtake-free*; see in particular Walrand and Varaiya [29]. After that seminal paper for sojourn times on overtake-free paths in open networks, attention shifted to closed networks. Starting-point was a paper of Chow [10] on a two-node closed system consisting of two exponential FCFS single server queues. He proved that the cycle time distribution is a mixture of two Erlang distributions. Boxma and Donk [3] generalized his result by computing the joint sojourn time distribution in Chow's model, and Schassberger and Daduna [26] obtained the cycle time distribution for a closed J-node tandem system, $J \geq 2$. Boxma, Kelly and Konheim

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[7] subsequently derived the joint distribution of the successive sojourn times of a tagged customer along the J queues. Daduna [12] had also derived the passage time distribution for an overtake-free path in a closed single-server Gordon-Newell network; Kelly and Pollett [20] extended this further by deriving the joint distribution of a tagged customer's sojourn times along such a path. A survey of these results may be found in [6]; its Theorem 2.4 contains a unified formulation of the above-mentioned results for the joint sojourn time LST (Laplace-Stieltjes transform) along a so-called quasi overtake-free path in an (open or closed) product-form network. This LST is shown to exhibit a product form w.r.t. the underlying product form of the joint queue length distribution at jump epochs. Slight generalizations of that Theorem are given in [21] and [15]. More recent developments can be found in the work of Zazanis [31] who investigated the internal structure of sojourn time distributions in closed exponential cycles. Related to our investigation of successive sojourn times in open tandem systems is the work of Karpelevitch and Kreinin [19], who investigated the joint distribution of a test customer's *waiting times* in an open exponential two-station tandem queue.

In [4], attention is shifted to a *non-product-form* closed two-queue system, consisting of two FCFS single server queues Q_G and Q_M in series, the service times in Q_G (Q_M) being generally (resp. exponentially) distributed. In that paper, the joint distribution of the successive sojourn times of a tagged customer in *first* Q_M and *then* Q_G was obtained, by studying the transient behaviour of an $M/G/1$ queue. It was also pointed out that, unfortunately, this does not solve the problem of obtaining the joint distribution of the successive sojourn times of a tagged customer in *first* Q_G and *then* Q_M. One can easily show that these two joint distributions are in general not the same, due to different correlation structures (unless both service time distributions are exponential). In particular, in the case of deterministic service at Q_G, the successive sojourn times at (first) this Q_D and (then) Q_M are independent, whereas they are negatively correlated at (first) Q_M and (then) Q_D [4]. Daduna [13, 14] obtained the cycle time distribution for the case of Q_G followed by Q_M, and Boxma and Donk [3] derived two approximations for the joint sojourn time distribution (Q_G followed by Q_M), but the problem of obtaining an exact expression for the joint sojourn time distribution remained open.

Another direction of research is described in [1] by Ayhan, Palmowski, and Schlegel who determine the asymptotic tail behaviour of cycle time and waiting time distributions in a cyclic queue under the assumption that at least one of the servers has subexponential service times.

Due to the lack of exact results, even in this simple framework, several approximations are developed by many researchers for these and more complex systems. A survey which reports literature up to around 1990 on that topic is Section 3 in [6]. A more recent survey with emphasis on numerical computation of sojourn time quantiles is compiled by Harrison and Knottenbelt [17]. Another way to overcome the lack of explicit results on sojourn time distributions is to use heavy traffic limiting results. In the closed cyclic queue this means that bottleneck analysis is performed, which is even in productform networks of value, due to computational problems when large populations are considered, see [6][Section II.7], and more recently [16]. For non exponential service times, see [6][Section III.7], and more recently [23]. More recent book sections on diffusion approximations for general closed networks via functional central limit theorems are [9][Section 7.10], [22][Section 6.2], and the survey [30]. In the heavy traffic analysis described there, one of the usual measures of interest is actual workloads, which in case of FCFS is the actual waiting time of a customer and therefore related to our investigation.

The first goal of the present paper is to revisit the problem of [3] and to obtain the exact expression for the joint sojourn time distribution in a cycle Q_G followed by Q_M. Our second goal is to indicate how the joint sojourn time distribution of the open tandem queue, consisting of Q_G followed by Q_M, can be obtained. Here we exploit ideas that we develop for the closed cyclic case.

We thus solve two long-standing open problems for two of the most elementary non product-form queueing networks.

The paper is organized as follows. In Section 2 we present a model description of the two-queue closed network, and we review results from [5] regarding the joint distribution of queue length and residual service time in Q_G at arrival instants to that queue. Subsequently we express the LST of the joint sojourn time distribution of a tagged customer at Q_G and Q_M into the former joint queue length and residual service time distribution, and an unknown function $\psi_{\cdot,\cdot}(\cdot,\cdot)$. Section 3 is devoted to the determination of that function ψ. In Section 4 we obtain the two-dimensional generating function of $\psi_{k,h}(\cdot,\cdot)$. This not only helps us in tackling the sojourn time problem for the closed two-queue system; in Section 5 we show how it also can be used to determine the LST of the joint distribution of the sojourn times of a tagged customer in the *open* tandem queue consisting of Q_G (with Poisson input) and Q_M.

2 Analysis

Let us first describe the model under consideration in more detail. We consider a closed two-queue queueing network with N customers. Q_G is a FCFS single server queue; the service times at Q_G are independent, identically distributed random variables $B_1, B_2, \dots$ with distribution $B(\cdot)$ and LST $\beta(\cdot)$. Q_M is a FCFS single server queue; the service times at Q_M are independent, exponentially distributed with mean $1/\mu$. Customers who are served in Q_G (Q_M) immediately enter Q_M (Q_G). All service times at the two queues are assumed to be independent. It is well-known, and easily seen, that Q_G behaves exactly like the finite-capacity $M/G/1-N$ queue with arrival rate μ and service time distribution $B(\cdot)$; define its traffic load by $\rho := \mu \mathbb{E} B_1$.

Consider the arrival of a tagged customer C at Q_G. Let Z denote the number of customers found by C at Q_G, and let R be the residual service time of the customer in service, if any. Since Q_G behaves like an $M/G/1-N$ queue, $\mathbb{P}(Z=0)$ and $\mathbb{P}(Z=k, R<t)$, $k=1,\dots,N-1$ are the probabilities given in (4.11) and (4.13) of [5], which were based on results in Section III.6.3 of [11]. These probabilities are specified as follows. First introduce, for an $M/G/1-N$ queue with arrival rate μ and service time distribution $B(\cdot)$, the joint steady-state distribution of number of customers X and *past* service time V of the customer in service (cf. Cohen [11], Section III.6.3):

$$R_0 := \mathbb{P}(X=0) = [1 + \frac{\rho}{2\pi\iota}\int_{D_\omega} \frac{1}{\beta(\mu(1-\omega))-\omega}\frac{\mathrm{d}\omega}{\omega^{N-1}}]^{-1},$$

$$R_k := \mathbb{P}(X=k) = \frac{R_0}{2\pi\iota}\int_{D_\omega} \frac{1-\beta(\mu(1-\omega))}{\beta(\mu(1-\omega))-\omega}\frac{\mathrm{d}\omega}{\omega^{k}}, \quad k=1,\dots,N-1,$$

$$R_N := \mathbb{P}(X=N) = \frac{R_0}{2\pi\iota}\int_{D_\omega} \frac{1}{\beta(\mu(1-\omega))-\omega}[\rho - \frac{1-\beta(\mu(1-\omega))}{1-\omega}]\frac{\mathrm{d}\omega}{\omega^{N-1}},$$

$$R_k(\eta)\mathrm{d}\eta := \mathbb{P}(X=k, V\in(\eta,\eta+\mathrm{d}\eta)) = \frac{R_0\mu(1-B(\eta))\mathrm{d}\eta}{2\pi\iota}\int_{D_\omega} \frac{(1-\omega)\mathrm{e}^{-\mu(1-\omega)\eta}}{\beta(\mu(1-\omega))-\omega}\frac{\mathrm{d}\omega}{\omega^{k}},$$
$$k=1,\dots,N-1, \quad \eta>0,$$

$$R_N(\eta)\mathrm{d}\eta := \mathbb{P}(X=N, V\in(\eta,\eta+\mathrm{d}\eta)) = \frac{R_0\mu(1-B(\eta))\mathrm{d}\eta}{2\pi\iota}\int_{D_\omega} \frac{1-\mathrm{e}^{-\mu(1-\omega)\eta}}{\beta(\mu(1-\omega))-\omega}\frac{\mathrm{d}\omega}{\omega^{N-1}}, \quad \eta>0. \tag{2.1}$$

Here D_ω is a circle with center at zero and radius ω, $|\omega| < \gamma$, with γ the zero of $p - \beta(\mu(1-p))$ which is smallest in absolute value (the integral $\int_{D_\omega}$ is a contour integral, sometimes also indicated by $\oint$). Then (4.11) and (4.13) of [5] read:

$$\mathbb{P}(Z=0) = \frac{R_0}{1-R_N}, \tag{2.2}$$

$$\mathbb{E}[\mathrm{e}^{-sR}(Z=k)] = \mathbb{P}(Z=0)\frac{1}{2\pi\iota}\int_{D_\omega}\frac{(1-\omega)}{\beta(\mu(1-\omega))-\omega}\frac{\beta(\mu(1-\omega))-\beta(s)}{(s/\mu)+\omega-1}\frac{\mathrm{d}\omega}{\omega^k},$$
$$k=1,\dots,N-1, \quad \mathrm{Re}\ s \geq 0.$$

In fact, a closer inspection of (4.13) of [5] reveals that, for $k=1,\dots,N-1$,

$$\mathbb{P}(Z=k, R\in(t,t+\mathrm{d}t)) = \mathbb{P}(Z=0)\frac{1}{2\pi\iota}\int_{D_\omega}\frac{(1-\omega)}{\beta(\mu(1-\omega))-\omega}[\int_{\eta=0}^{\infty}\mathrm{e}^{-\mu(1-\omega)\eta}\mathrm{d}_t B(t+\eta)\mathrm{d}\eta]\frac{\mathrm{d}\omega}{\omega^k}. \tag{2.3}$$

Our approach to determining the LST $\mathbb{E}[\mathrm{e}^{-\omega_G S_G-\omega_M S_M}]$, where S_G and S_M are the successive sojourn times of the tagged customer C at Q_G and Q_M, is the following.
(i) We condition on (Z,R), as seen by C upon his arrival at Q_G.
(ii) Consider the case $Z=0$, so Q_G is empty; it is straightforward to calculate $\mathbb{E}[\mathrm{e}^{-\omega_G S_G-\omega_M S_M}|Z=0]$.
(iii) Consider the case $Z=k>0$, so that Q_M holds $N-k-1$ customers. Look ahead for an amount of time $R=t$, after which the customer in service moves to Q_M.
(iv) Define $\psi_{k,h}(\omega_G,\omega_M)$, $k,h\geq 0$, as the LST of the joint distribution of the remaining sojourn time of C in Q_G and the subsequent sojourn time in Q_M, given that a new service starts right now in Q_G and that at this epoch C sees k other customers before him in Q_G and h in Q_M.
(v) Observe that the LST of the joint distribution of the sojourn times S_G and S_M of C in, successively, Q_G and Q_M is expressed in $\psi_{k,h}(\omega_G,\omega_M)$, $k,h\geq 0$:

$$\begin{aligned}
&\mathbb{E}[\mathrm{e}^{-\omega_G S_G-\omega_M S_M}] = \mathbb{P}(Z=0)\psi_{0,N-1}(\omega_G,\omega_M)\\
+\ &\int_0^\infty \mathrm{e}^{-\omega_G t}\psi_{N-2,1}(\omega_G,\omega_M)\mathrm{d}\mathbb{P}(Z=N-1,R<t)\\
+\ &\sum_{k=1}^{N-2}\int_0^\infty \mathrm{e}^{-\omega_G t}\left\{\sum_{l=0}^{N-k-2}\mathrm{e}^{-\mu t}\frac{(\mu t)^l}{l!}\psi_{k-1,N-k-l}(\omega_G,\omega_M)\right.\\
+\ &\left.\sum_{l=N-k-1}^{\infty}\mathrm{e}^{-\mu t}\frac{(\mu t)^l}{l!}\psi_{k-1,1}(\omega_G,\omega_M)\right\}\mathrm{d}\mathbb{P}(Z=k,R<t).
\end{aligned} \tag{2.4}$$

(vi) In order to determine $\mathbb{E}[\mathrm{e}^{-\omega_G S_G-\omega_M S_M}]$ we finally need to determine the functions $\psi_{k,h}(\omega_G,\omega_M)$. We shall do this in two different ways. In Section 3 we present a recursive approach, expressing $\psi_{k,h}(\omega_G,\omega_M)$ in terms only involving $\psi_{k-1,j}(\omega_G,\omega_M)$ and known terms. We thus eventually arrive at $\psi_{0,j}(\omega_G,\omega_M)$, terms which are easily calculated explicitly. As an alternative approach, in Section 4 we derive an expression for the double generating function $A(x,y,\omega_G,\omega_M)$ $:= \sum_{k=0}^\infty\sum_{h=1}^\infty x^k y^h\psi_{k,h}(\omega_G,\omega_M)$ (notice that $h\geq 1$ for all $\psi_{k,h}(\omega_G,\omega_M)$ terms in (2.4), except for the degenerate case $N=1$). This double generating function uniquely determines all needed $\psi_{k,h}(\omega_G,\omega_M)$. This alternative approach is also the key to analyzing the *open* tandem queue consisting of Q_G followed by Q_M.

We notice, that the LSTs defined in (iv) above determine via $\psi_{k,n}(\omega,\omega)$, $k,n \geq 0$, the LST of C's residual travel time through both queues when C stands at the end of the line of Q_G, given that a new service starts right now in Q_G and that at this epoch he sees k other customers before him in Q_G and n in Q_M.

We conclude that for the case of Q_G being an exponential server as well, with $B = \exp(\alpha)$ (and setting $\mu \to \beta$), we have

$$\psi_{k,n}(\omega,\omega) = \mathbb{E}(e^{-\omega S_{k,n}}),$$

where $S_{k,n}$ is a test customer's travel time through two exponential queues given he sees k other customers before him in the first queue and n customers in the second queue. The LST of S_k is expressed explicitly via complex combinatorial terms in formula (1) in [28].

3 An algorithmic approach

In this section we present an algorithmic, recursive, approach to the determination of the $\psi_{k,h}(\omega_G,\omega_M)$.

One may easily verify that

$$\psi_{0,0}(\omega_G,\omega_M) = \beta(\omega_G)\frac{\mu}{\mu+\omega_M}, \tag{3.1}$$

and that, for $h = 1,2,\ldots,$

$$\begin{aligned}\psi_{0,h}(\omega_G,\omega_M) &= \int_0^\infty \mathrm{e}^{-\omega_G t}\left\{\sum_{l=0}^{h-1}\mathrm{e}^{-\mu t}\frac{(\mu t)^l}{l!}\left(\frac{\mu}{\mu+\omega_M}\right)^{h-l+1}\right.\\ &\quad+\left.\sum_{l=h}^{\infty}\mathrm{e}^{-\mu t}\frac{(\mu t)^l}{l!}\frac{\mu}{\mu+\omega_M}\right\}\mathrm{d}B(t). \end{aligned}\tag{3.2}$$

Now consider the case $k > 0, h = 0$:

$$\psi_{k,0}(\omega_G,\omega_M) = \int_0^\infty \mathrm{e}^{-\omega_G t}\psi_{k-1,1}(\omega_G,\omega_M)\mathrm{d}B(t) = \beta(\omega_G)\psi_{k-1,1}(\omega_G,\omega_M). \tag{3.3}$$

Finally the case $k > 0, h > 0$:

$$\begin{aligned}\psi_{k,h}(\omega_G,\omega_M) &= \int_0^\infty \mathrm{e}^{-\omega_G t}\left\{\sum_{l=0}^{h-1}\mathrm{e}^{-\mu t}\frac{(\mu t)^l}{l!}\psi_{k-1,h-l+1}(\omega_G,\omega_M)\right.\\ &\quad+\left.\sum_{l=h}^{\infty}\mathrm{e}^{-\mu t}\frac{(\mu t)^l}{l!}\psi_{k-1,1}(\omega_G,\omega_M)\right\}\mathrm{d}B(t). \end{aligned}\tag{3.4}$$

First rewrite (3.4) (for $k > 0, h > 0$) in the following way:

$$\psi_{k,h}(\omega_G,\omega_M) = \sum_{l=0}^{h-1}\psi_{k-1,h-l+1}(\omega_G,\omega_M)a(l,\omega_G) + \psi_{k-1,1}(\omega_G,\omega_M)b(h,\omega_G),$$

where

$$a(l,\omega_G) := \int_0^\infty \mathrm{e}^{-\omega_G t-\mu t}\frac{(\mu t)^l}{l!}\mathrm{d}B(t), \tag{3.5}$$

$$b(h,\omega_G) := \int_0^\infty \mathrm{e}^{-\omega_G t-\mu t}\sum_{l=h}^{\infty}\frac{(\mu t)^l}{l!}\mathrm{d}B(t). \tag{3.6}$$

After a change of variables, we can write:

$$\psi_{k,h}(\omega_G,\omega_M) = \sum_{r=2}^{h+1} \psi_{k-1,r}(\omega_G,\omega_M) a(h-r+1,\omega_G) + \psi_{k-1,1}(\omega_G,\omega_M) b(h,\omega_G).$$

Using a shorthand notation in which we suppress the ω_G and ω_M, the above recursion reads: for $k, h > 0$,

$$\psi_{k,h} = \sum_{r=2}^{h+1} \psi_{k-1,r} a(h-r+1) + \psi_{k-1,1} b(h).$$

Introducing the vectors

$$\bar{\psi}_k := (\psi_{k,N-k-1}, \psi_{k,N-k-2}, \ldots, \psi_{k,1}), \quad k = 0, 1, , \ldots, N-2, \tag{3.7}$$

and the $(N-k) \times (N-k-1)$ matrices $A(N-k), k = 1, \ldots, N-1$, which are given by:

$$A(N-k) = \begin{pmatrix} a(0) & 0 & 0 & ... & ... & ... \\ a(1) & a(0) & 0 & ... & ... & ... \\ a(2) & a(1) & a(0) & ... & ... & ... \\ a(3) & a(2) & a(1) & ... & ... & ... \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & 0 & 0 \\ ... & ... & ... & ... & a(0) & 0 \\ a(N-k-2) & a(N-k-3) & a(N-k-4) & ... & a(1) & a(0) \\ b(N-k-1) & b(N-k-2) & b(N-k-3) & ... & b(2) & b(1) \end{pmatrix}, \tag{3.8}$$

one can write

$$\begin{aligned} \bar{\psi}_k &= \bar{\psi}_{k-1} A(N-k) \\ &= \bar{\psi}_{k-2} A(N-k+1) A(N-k) \\ &= ... \\ &= \bar{\psi}_0 \prod_{j=1}^{k} A(N-j). \end{aligned} \tag{3.9}$$

¿From (3.2) we immediately obtain for $h \geq 1$

$$\begin{aligned} &\psi_{0,h}(\omega_G,\omega_M) \\ = &\frac{\mu}{\mu+\omega_M} \psi_{0,h-1}(\omega_G,\omega_M) + \int_0^\infty \mathrm{e}^{-\omega_G t} \left\{ \sum_{l=h}^{\infty} \mathrm{e}^{-\mu t} \frac{(\mu t)^l}{l!} \frac{\mu}{\mu+\omega_M} \right\} \frac{\omega_M}{\mu+\omega_M} \mathrm{d}B(t), \end{aligned}$$

with

$$\psi_{0,0}(\omega_G,\omega_M) = \beta(\omega_G) \frac{\mu}{\mu+\omega_M},$$

from (3.1). From (3.6) it follows for $h \geq 1$

$$\begin{aligned} &\psi_{0,h}(\omega_G,\omega_M) \\ = &\frac{\mu}{\mu+\omega_M} \psi_{0,h-1}(\omega_G,\omega_M) + b(h,\omega_G) \frac{\mu}{\mu+\omega_M} \frac{\omega_M}{\mu+\omega_M}, \end{aligned} \tag{3.10}$$

and we have

$$\psi_{0,0}(\omega_G,\omega_M) = b(0,\omega_G)\frac{\mu}{\mu+\omega_M}, \tag{3.11}$$

with $b(0,\omega_G) = \beta(\omega_G)$, which is in line with (3.6).

Direct application of the recursion (3.10) - (3.11) now yields for $h \geq 1$

$$\begin{aligned}&\psi_{0,h}(\omega_G,\omega_M)\\ = \;& \beta(\omega_G)\left(\frac{\mu}{\mu+\omega_M}\right)^{h+2} + \frac{\omega_M}{\mu+\omega_M}\frac{\mu}{\mu+\omega_M}\left\{\sum_{l=0}^{h} b(l,\omega_G)\left(\frac{\mu}{\mu+\omega_M}\right)^{h-l}\right\}.\end{aligned} \tag{3.12}$$

Now it is time to return to the joint sojourn time LST of a customer in (first) Q_G and (then) Q_M. As we'll see, the vectors $\bar{\psi}_{k-1}$ play a key role in its determination. Indeed, introducing the vectors

$$\pi(1) = 1, \quad \pi(h) := \mathrm{e}^{-\mu t}(1, \mu t, \frac{(\mu t)^2}{2!}, \ldots, \frac{(\mu t)^{h-2}}{(h-2)!}, \sum_{l=h-1}^{\infty}\frac{(\mu t)^l}{l!})^T, \quad h = 2,3,\ldots, \tag{3.13}$$

we can rewrite (2.4) into:

$$\begin{aligned}&\mathbb{E}[\mathrm{e}^{-\omega_G S_G - \omega_M S_M}] = \mathbb{P}(Z=0)\psi_{0,N-1}(\omega_G,\omega_M)\\ +\;& \sum_{k=1}^{N-1}\int_0^\infty \mathrm{e}^{-\omega_G t}\{\bar{\psi}_{k-1}\pi(N-k)\}\mathrm{d}\mathbb{P}(Z=k, R<t).\end{aligned}$$

The term inside the curly brackets represents a product of two vectors. Finally, using (3.9) and (3.12), we obtain:

$$\begin{aligned}&\mathbb{E}[\mathrm{e}^{-\omega_G S_G - \omega_M S_M}]\\ =\;& \mathbb{P}(Z=0)\left[\beta(\omega_G)\left(\frac{\mu}{\mu+\omega_M}\right)^{N+1} + \frac{\omega_M}{\mu+\omega_M}\frac{\mu}{\mu+\omega_M}\left\{\sum_{l=0}^{N-1} b(l,\omega_G)\left(\frac{\mu}{\mu+\omega_M}\right)^{h-l}\right\}\right]\\ +\;& \sum_{k=1}^{N-1}\int_0^\infty \mathrm{e}^{-\omega_G t}\left\{\bar{\psi}_0\prod_{l=1}^{k-1}A(N-l)\pi(N-k)\right\}\mathrm{d}\mathbb{P}(Z=k, R<t),\end{aligned}$$

an empty product being one. All ingredients for the determination of $\mathbb{E}[\mathrm{e}^{-\omega_G S_G - \omega_M S_M}]$ are now available:

(1) $\mathbb{P}(Z=0)$ and $\mathbb{P}(Z=k, R<t)$ are given by (2.2) and (2.3);

(2) $\psi_{0,N-1}(\omega_G,\omega_M)$ and $\bar{\psi}_0$ from (3.7) is given by (3.1) and (3.2), resp., in (3.12);

(3) the A-matrices are given in (3.8). We only need to determine $A(N-1)$, because $A(k-1)$ is obtained from $A(k)$ by deleting the first column and first row;

(4) the vectors $\pi(k)$ are explicitly given in (3.13).

Example: $N = 2$.

It readily follows from (2.4) that, for $N = 2$,

$$\mathbb{E}[\mathrm{e}^{-\omega_G S_G - \omega_M S_M}] = \psi_{0,1}(\omega_G,\omega_M)[\mathbb{P}(Z=0) + \mathbb{E}[\mathrm{e}^{-\omega_G R}(Z=1)],$$

with (cf. (3.2))

$$\psi_{0,1}(\omega_G,\omega_M) = \beta(\omega_G+\mu)(\frac{\mu}{\mu+\omega_M})^2 + [\beta(\omega_G) - \beta(\omega_G+\mu)]\frac{\mu}{\mu+\omega_M}.$$

First consider the case of deterministic service (service time D) in Q_G. Then $\beta(\omega_G+\mu) = \mathrm{e}^{-\omega_G D}\mathrm{e}^{-\mu D} = \beta(\omega_G)\beta(\mu)$, quickly yielding $\mathbb{E}[\mathrm{e}^{-\omega_G S_G - \omega_M S_M}] = \mathbb{E}[\mathrm{e}^{-\omega_G S_G}]\mathbb{E}[\mathrm{e}^{-\omega_M S_M}]$, confirming that S_G and S_M are in this case independent, as remarked in Section 1.
Next consider the case of exponential service times in Q_G, with mean $1/\alpha$. Then $\mathbb{P}(Z=0) = \frac{\alpha}{\alpha+\mu}$ and $R \sim exp(\alpha)$. A brief calculation confirms that

$$\mathbb{E}[\mathrm{e}^{-\omega_G S_G - \omega_M S_M}] = \frac{\alpha}{\alpha+\mu}\frac{\alpha}{\alpha+\omega_G}(\frac{\mu}{\mu+\omega_M})^2 + \frac{\mu}{\alpha+\mu}(\frac{\alpha}{\alpha+\omega_G})^2\frac{\mu}{\mu+\omega_M},$$

in agreement with a result in [3].

A straightforward calculation shows that

$$\mathrm{cov}(S_G, S_M) = \frac{1}{\mu}[\mathbb{E}(B\mathrm{e}^{-\mu B}) - \mathbb{E}B\mathbb{E}(\mathrm{e}^{-\mu B})] \leq 0, \tag{3.14}$$

the last inequality following from the fact that B and $\mathrm{e}^{-\mu B}$ are negatively correlated. It immediately follows from (3.14) that the covariance is zero in the case of deterministic service at Q_G, and that it equals $-(\frac{\mathbb{E}B}{1+\mu\mathbb{E}B})^2$ in the case of exponential service at Q_G – which is in agreement with Formula (2.18) with $N=2$ in [3].

Now let us turn to the reversed case, as analyzed in [4], viz., the case in which first the M queue is visited and subsequently the G queue. According to Formula (4.4) of [4], the LST of the joint distribution of (first) $\hat{S}_M$ and (then) $\hat{S}_G$ (to prevent confusion, we indicate the successive sojourn times by $\hat{S}_M$ and $\hat{S}_G$) is for $N=2$ customers given by:

$$\mathbb{E}[\mathrm{e}^{-\omega_M \hat{S}_M - \omega_G \hat{S}_G}] =$$
$$\beta(\omega_M)[\frac{\mu\beta(\omega_M)}{\mu+\omega_G-\omega_M} - \frac{\mu\omega_M}{\mu+\omega_G}\frac{\beta(\omega_G+\mu)}{\mu+\omega_G-\omega_M}][\frac{\mu\beta(\mu)}{\mu+\omega_G} + 1 - \beta(\mu)],$$

yielding (with $\beta^{(1)}(\omega)$ the first derivative of $\beta(\omega)$):

$$\mathrm{cov}(\hat{S}_M, \hat{S}_G) = \frac{1}{\mu^2}[\beta(\mu) - 1 - \mu\beta^{(1)}(\mu)]. \tag{3.15}$$

It is readily verified that $\hat{S}_M$ and $\hat{S}_G$ are negatively correlated, just like S_G and S_M. Indeed,

$$\beta(\mu) - \mu\beta^{(1)}(\mu) = \int_{t=0}^{\infty} \mathrm{e}^{-\mu t}(1+\mu t)\mathrm{d}B(t) < \int_{t=0}^{\infty} \mathrm{e}^{-\mu t}\mathrm{e}^{\mu t}\mathrm{d}B(t) = 1.$$

Finally, it follows from (3.14) and (3.15) that

$$\mathrm{cov}(S_G, S_M) - \mathrm{cov}(\hat{S}_M, \hat{S}_G) = \frac{1}{\mu^2}\int_{t=0}^{P\infty}(1 - \mathrm{e}^{-\mu t} - \mu\mathbb{E}B\mathrm{e}^{-\mu t})\mathrm{d}B(t) = \frac{1}{\mu^2}[1 - \frac{\beta(\mu)}{\beta_{exp}(\mu)}],$$

where $\beta_{exp}(\mu) := \frac{1}{1+\mu\mathbb{E}B}$. It is immediately clear that the two covariances are equal when $B(\cdot)$ is exponential (as already remarked in the Introduction).

4 An analytic approach

In this section we determine the generating function $A(x, y, \omega_G, \omega_M)$ of $\psi_{k,h}(\omega_G, \omega_M)$. One might use that generating function to find an expression for $\psi_{k,h}(\omega_G, \omega_M)$. However, a more important goal of determining $A(x,y,\omega_G,\omega_M)$ becomes apparent in the next section: It will turn out that knowledge of $A(x,y,\omega_G,\omega_M)$ is instrumental in solving a long-standing open problem in queueing theory, viz., the determination of the joint sojourn time distribution in the open tandem queue $M/G/1-./M/1$.

We determine

$$A(x,y,\omega_G,\omega_M) := \sum_{k=0}^{\infty}\sum_{h=1}^{\infty} x^k y^h \psi_{k,h}(\omega_G,\omega_M),$$

by using the (recursion) relations (3.1)-(3.4). Notice that we do not restrict ourselves to $k+h \leq N-1$, although that restriction holds in the closed cyclic system with N customers.

Multiplying the lefthand and righthand sides of (3.2) with y^h, and the lefthand and righthand sides of (3.4) with $x^k y^h$, and summing all terms, yields:

$$\begin{aligned}
A(x,y,\omega_G,\omega_M) &= \int_0^{\infty} \mathrm{e}^{-\omega_G t} \sum_{h=1}^{\infty} y^h \sum_{l=0}^{h-1} \mathrm{e}^{-\mu t}\frac{(\mu t)^l}{l!}\left(\frac{\mu}{\mu+\omega_M}\right)^{h-l+1} \mathrm{d}B(t) \\
+ & \int_0^{\infty} \mathrm{e}^{-\omega_G t} \sum_{h=1}^{\infty} y^h \sum_{l=h}^{\infty} \mathrm{e}^{-\mu t}\frac{(\mu t)^l}{l!}\frac{\mu}{\mu+\omega_M} \mathrm{d}B(t) \\
+ & \int_0^{\infty} \mathrm{e}^{-\omega_G t} \sum_{k=1}^{\infty} x^k \sum_{h=1}^{\infty} y^h \sum_{l=0}^{h-1} \mathrm{e}^{-\mu t}\frac{(\mu t)^l}{l!}\psi_{k-1,h-l+1}(\omega_G,\omega_M) \mathrm{d}B(t) \\
+ & \int_0^{\infty} \mathrm{e}^{-\omega_G t} \sum_{k=1}^{\infty} x^k \sum_{h=1}^{\infty} y^h \sum_{l=h}^{\infty} \mathrm{e}^{-\mu t}\frac{(\mu t)^l}{l!}\psi_{k-1,1}(\omega_G,\omega_M) \mathrm{d}B(t) \\
=: & \ I + II + III + IV.
\end{aligned} \tag{4.1}$$

We successively evaluate the four terms I, II, III and IV in the righthand side of (4.1). Interchanging the summations in I leads to a relatively straightforward evaluation:

$$\begin{aligned}
I &= y\left(\frac{\mu}{\mu+\omega_M}\right)^2 \frac{1}{1-\frac{y\mu}{\mu+\omega_M}} \int_0^{\infty} \mathrm{e}^{-\omega_G t-\mu t+\mu y t} \mathrm{d}B(t) \\
&= \frac{\mu}{\mu+\omega_M}\frac{\mu y}{\mu(1-y)+\omega_M}\beta(\omega_G+\mu(1-y)).
\end{aligned} \tag{4.2}$$

In a similar way, after interchanging summations, we have:

$$\begin{aligned}
II &= \frac{\mu}{\mu+\omega_M}\int_0^{\infty} \mathrm{e}^{-\omega_G t}\mathrm{d}B(t) \sum_{l=1}^{\infty} \mathrm{e}^{-\mu t}\frac{(\mu t)^l}{l!}\frac{y-y^{l+1}}{1-y} \\
&= \frac{\mu}{\mu+\omega_M}\frac{y}{1-y}[\beta(\omega_G)-\beta(\omega_G+\mu(1-y))].
\end{aligned} \tag{4.3}$$

To evaluate III, we interchange the summations. We first sum h from $l+1$ to ∞, then k from 1 to ∞ and finally l from 0 to ∞, obtaining

$$III = \frac{x}{y}\beta(\omega_G+\mu(1-y))[A(x,y,\omega_G,\omega_M) - y\sum_{k=0}^{\infty} x^k \psi_{k,1}(\omega_G,\omega_M)]. \tag{4.4}$$

Finally,

$$IV = \frac{xy}{1-y}[\beta(\omega_G)-\beta(\omega_G+\mu(1-y))]\sum_{k=0}^{\infty} x^k \psi_{k,1}(\omega_G,\omega_M). \tag{4.5}$$

It follows from (4.1)-(4.5), bringing the $A(x,y,\omega_G,\omega_M)$ terms to the lefthand side and introducing

$$A_1(x,\omega_G,\omega_M) := \sum_{k=0}^{\infty} x^k \psi_{k,1}(\omega_G,\omega_M),$$

that

$$\begin{aligned} & A(x,y,\omega_G,\omega_M)[1-\frac{x}{y}\beta(\omega_G+\mu(1-y))] \\ = \quad & A_1(x,\omega_G,\omega_M)[\frac{xy}{1-y}\beta(\omega_G)-\frac{x}{1-y}\beta(\omega_G+\mu(1-y))] \\ + \quad & \frac{\mu}{\mu+\omega_M}\frac{y}{1-y}\beta(\omega_G) \\ + \quad & \frac{\mu}{\mu+\omega_M}y[\frac{\mu}{\omega_M+\mu(1-y)}-\frac{1}{1-y}]\beta(\omega_G+\mu(1-y)). \end{aligned} \tag{4.6}$$

The unknown function $A_1(x,\omega_G,\omega_M)$ is determined by the following observation (cf. Cohen [11], p. 250). When $\mu\mathbb{E}B<1$ (which will be the case in the next section, in the steady-state analysis of the open tandem queue consisting of Q_G followed by Q_M), $y-x\beta(\omega_G+\mu(1-y))$ has for $|x|\le 1$, Re $\omega_G \ge 0$ a unique zero $f(x,\omega_G)$ in $|y|\le 1$. That zero is given by

$$f(x,\omega_G) = E[x^N \mathrm{e}^{-\omega_G P}], \tag{4.7}$$

where N and P are the number of customers in a busy period and the length of this busy period, in an $M/G/1$ queue Q_G with arrival rate μ and service time distribution $B(\cdot)$. Obviously, the condition $\mu\mathbb{E}B<1$ is the condition for the steady-state sojourn time distribution in the $M/G/1$ queue Q_G to exist.
$A(x,y,\omega_G,\omega_M)$, being a generating function in x and y, is bounded and analytic in $|x|\le 1$, $|y|\le 1$. Hence the righthand side of (4.6) must be zero for $y=f(x,\omega_G)$, so

$$\begin{aligned} A_1(x,\omega_G,\omega_M) &= \frac{\frac{\mu}{\mu+\omega_M}\frac{f(x,\omega_G)}{1-f(x,\omega_G)}[\beta(\omega_G)-\frac{\omega_M}{\omega_M+\mu(1-f(x,\omega_G))}\beta(\omega_G+\mu(1-f(x,\omega_G)))]}{x\frac{1}{1-f(x,\omega_G)}[\beta(\omega_G+\mu(1-f(x,\omega_G)))-f(x,\omega_G)\beta(\omega_G)]} \\ &= \frac{\mu}{\mu+\omega_M}\frac{\beta(\omega_G)-\frac{\omega_M}{\omega_M+\mu(1-f(x,\omega_G))}\frac{f(x,\omega_G)}{x}}{1-x\beta(\omega_G)}. \end{aligned} \tag{4.8}$$

Remark.
Notice that $A_1(x,\omega_G,0)=\frac{\beta(\omega_G)}{1-x\beta(\omega_G)}$. Indeed, $A_1(x,\omega_G,0)=\sum_{k=0}^{\infty}x^k\psi_{k,1}(\omega_G,0)=\sum_{k=0}^{\infty}x^k\beta(\omega_G)^{k+1}$ as the sojourn time at Q_G consists of $k+1$ service times if the tagged customer finds k other customers ahead of him at the start of a service. This gives the x^k-coefficient of $A_1(x,\omega_G,0)$. Similarly, it is not hard to determine the x^k-coefficient of $A_1(x,\omega_G,\omega_M)$, which is $\psi_{k,1}(\omega_G,\omega_M)$. We leave this to the reader, restricting ourselves to obtaining the x^j-coefficient of one term, viz., of $\frac{\omega_M}{\omega_M+\mu(1-f(x,\omega_G))}$:

$$\frac{\omega_M}{\omega_M+\mu(1-f(x,\omega_G))} = \frac{\omega_M}{\mu+\omega_M}\frac{1}{1-\frac{\mu}{\mu+\omega_M}f(x,\omega_G)} = \sum_{i=0}^{\infty}\frac{\omega_M}{\mu+\omega_M}(\frac{\mu}{\mu+\omega_M})^i f^i(x,\omega_G),$$

yielding the following x^j-coefficient of this term:

$$\sum_{i=0}^{\infty}\frac{\omega_M}{\mu+\omega_M}(\frac{\mu}{\mu+\omega_M})^i\mathbb{E}[\mathrm{e}^{-\omega_G(P_1+\cdots+P_i)}(N_1+\cdots+N_i=j)],$$

where (N_r, P_r) are the number of customers served in an $M/G/1$ busy period and its length (see (4.7)).

Let us now determine $A(x, y, \omega_G, \omega_M)$ by substituting the expression found in (4.8) for $A_1(x, \omega_G, \omega_M)$ into (4.6), using f for $f(x, \omega_G)$ as shorthand notation:

$$
\begin{aligned}
& A(x, y, \omega_G, \omega_M) = \frac{\frac{\mu}{\mu+\omega_M}}{y - x\beta(\omega_G + \mu(1-y))} \\
& \left[y \frac{\beta(\omega_G) - \frac{\omega_M}{\omega_M+\mu(1-f)} \frac{f}{x}}{1 - x\beta(\omega_G)} \frac{x}{1-y} (y\beta(\omega_G) - \beta(\omega_G + \mu(1-y))) \right. \\
+ \quad & \left. \frac{y^2}{1-y} [\beta(\omega_G) - \frac{\omega_M}{\omega_M + \mu(1-y)} \beta(\omega_G + \mu(1-y))] \right] \\
= \quad & \frac{\mu}{\mu+\omega_M} \frac{y}{1-y} \frac{1}{y - x\beta(\omega_G + \mu(1-y))} \left[\frac{x\beta(\omega_G) - \frac{\omega_M}{\omega_M+\mu(1-f)} f}{1 - x\beta(\omega_G)} (y\beta(\omega_G) - \beta(\omega_G + \mu(1-y))) \right. \\
+ \quad & \left. y \left(\beta(\omega_G) - \frac{\omega_M}{\omega_M + \mu(1-y)} \beta(\omega_G + \mu(1-y)) \right) \right]. \qquad (4.9)
\end{aligned}
$$

Note that $y = f$ and also $y = 1$ make the term in large square brackets in the righthand side of (4.9) equal to zero, as should be the case.

5 The tandem queue

In this section we consider the open counterpart of the cyclic queue that was studied in Sections 2 and 3: A tandem network consisting of two FCFS single server queues, fed by an external Poisson arrival stream with rate λ: an $M/G/1$ queue Q_G and an exponential single server queue Q_M. A customer who has been served at Q_G immediately enters Q_M. The service times at Q_G are independent, identically distributed random variables $B_1, B_2, \dots$ with distribution $B(\cdot)$ and LST $\beta(\cdot)$. The service times at Q_M are independent, exponentially distributed with mean $1/\mu$. The arrival process at Q_G and the service times at Q_G and Q_M are all independent.

Blanc, Iasnogorodski and Nain [2] have determined the (transform of the) steady-state joint queue length distribution for this tandem queue, and in particular also the (transform of the) probabilities $\mathbb{P}(X_G = i, R \in (t, t+dt), X_M = j)$, where X_G (X_M) denotes the steady-state queue length in Q_G (Q_M) and R denotes the residual service time of the customer in service at Q_G. By PASTA, this joint distribution is the same at an arrival epoch of a customer in Q_G. We shall denote it by $\mathrm{d}_t G(i, t, j) := \mathbb{P}(X_G^a = i, R \in (t, t+dt), X_M^a = j)$, where X_G^a and X_M^a are steady-state queue lengths at arrival epochs of Q_G. It should be observed that the condition for the existence of the steady-state joint queue length distribution, and the steady-state joint sojourn time distribution, is $\max(\lambda \mathbb{E} B_1, \frac{\lambda}{\mu}) < 1$, which is assumed to hold in the remainder of the paper.

Our goal is to determine the LST of the steady-state joint distribution of the successive sojourn times S_G and S_M of a tagged customer C at (first) Q_G and (then) Q_M. Note that if the two queues were reversed (customers arrive at Q_M and then move to Q_G), then the queue lengths at both queues as well as the sojourn times of a tagged customer are independent. This is well known, and follows from the reversibility of the queue length process in the first queue, which now is an $M/M/1$ queue (and its output process is a Poisson process, turning the second queue into an $M/G/1$ queue). The paper of Blanc et al. [2] filled an important gap in the classical queueing literature by determining the joint queue length distribution in the $M/G/1 - \cdot/M/1$ queue; we aim to fill another gap in that literature by determining the joint sojourn time distribution. Our starting-point is the following expression for the joint sojourn time LST, which is obtained by conditioning on X_G^a and X_M^a:

$$
\begin{aligned}
&\mathbb{E}[\mathrm{e}^{-\omega_G S_G-\omega_M S_M}] = \mathbb{P}(X_G^a=0, X_M^a=0)\psi_{0,0}(\omega_G,\omega_M) \\
+ &\sum_{j=1}^{\infty}\mathbb{P}(X_G^a=0, X_M^a=j)\psi_{0,j}(\omega_G,\omega_M) \\
+ &\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\int_{t=0}^{\infty}\mathrm{e}^{-\omega_G t}\Big\{\sum_{h=0}^{j-1}\mathrm{e}^{-\mu t}\frac{(\mu t)^h}{h!}\psi_{i-1,j-h+1}(\omega_G,\omega_M) \\
+ &\sum_{h=j}^{\infty}\mathrm{e}^{-\mu t}\frac{(\mu t)^h}{h!}\psi_{i-1,1}(\omega_G,\omega_M)\Big\}\mathrm{d}_t G(i,t,j) =: T_1+T_2+T_3. \qquad (5.1)
\end{aligned}
$$

The terms $\psi_{ij}(\omega_G,\omega_M)$, or rather their generating function, have been determined in Section 4. We shall successively determine T_1, T_2 and T_3.

Determination of T_1 and T_2

For

$$T_1 = \mathbb{P}(X_G^a=0, X_M^a=0)\psi_{0,0}(\omega_G,\omega_M),$$

and

$$T_2 = \sum_{j=1}^{\infty}\mathbb{P}(X_G^a=0, X_M^a=j)\psi_{0,j}(\omega_G,\omega_M), \qquad (5.2)$$

we need a result of [2] for their generating function $\Omega(y) := \sum_{j=0}^{\infty} y^j \mathbb{P}(X_G^a=0, X_M^a=j)$. $\Omega(y)$ is shown to satisfy a Fredholm integral equation of the second kind ((4.20) in [2]), and $\Omega(y)$ is determined in Section 5 of [2]. First of all, we conclude that (cf. (3.11)),

$$T_1 = \Omega(0)\beta(\omega_G)\frac{\mu}{\mu+\omega_M}. \qquad (5.3)$$

Next consider T_2. The problem we are facing in (5.2) is that the probabilities in the sum in (5.2) are only known via their generating function $\Omega(y)$, and that the $\psi_{0,j}(\omega_G,\omega_M)$ are only known via their generating function $A(0,y,\omega_G,\omega_M)$. To handle this problem, we resort to the inversion formula for generating functions:

$$\psi_{0,j}(\omega_G,\omega_M) = \frac{1}{2\pi\iota}\int_{D_y}\frac{A(0,y,\omega_G,\omega_M)}{y^{j+1}}\mathrm{d}y,$$

where D_y denotes the unit circle. ¿From (4.9) it follows that

$$A(0,y,\omega_G,\omega_M) = \frac{\mu}{\mu+\omega_M}\frac{y}{1-y}\left[\beta(\omega_G) - \frac{\omega_M}{\omega_M+\mu(1-y)}\beta(\omega_G+\mu(1-y))\right].$$

Hence

$$
\begin{aligned}
T_2 &= \frac{1}{2\pi\iota}\sum_{j=1}^{\infty}\int_{D_y}(\frac{1}{y})^j\mathbb{P}(X_G^a=0, X_M^a=j)\frac{A(0,y,\omega_G,\omega_M)}{y}\mathrm{d}y \\
&= \frac{1}{2\pi\iota}\int_{D_y}\frac{\mu}{\mu+\omega_M}\frac{1}{1-y}\left[\beta(\omega_G) - \frac{\omega_M}{\omega_M+\mu(1-y)}\beta(\omega_G+\mu(1-y))\right] \\
&\quad \cdot\mathbb{E}\left[(\frac{1}{y})^{X_M^a}(X_G^a=0, X_M^a>0)\right]\mathrm{d}y. \qquad (5.4)
\end{aligned}
$$

For $|y| > 1$, $\mathbb{E}[(\frac{1}{y})^{X_M^a}(X_G^a = 0, X_M^a > 0)] = \Omega(\frac{1}{y}) - \Omega(0)$ is analytic. Furthermore, $y = 1$ is a removable singularity of the integrand of (5.4). However, the term within square brackets in (5.4) has a pole $y = \frac{\mu+\omega_M}{\mu}$ with absolute value larger than 1, and also $\beta(\omega_G + \mu(1-y))$ may have poles for $|y| > 1$. We can evaluate the contour integral in (5.4) in the following way: Take a large positive L. Consider the closed contour consisting of the unit circle, the straight lines from ι to ιL and from ιL to ι, and the large circle with radius L. In the end we are letting $L \to \infty$. The contributions of the integrals along the two straight lines will cancel, and the contribution of the integral along the large circle will vanish when $L \to \infty$. Using Cauchy's residue theorem, the integral over the closed contour equals, on the one hand, the integral over D_y in (5.4); on the other hand, it equals minus the sum of the residues of the integrand of (5.4) for its poles outside the unit circle. As observed above, one pole is $y = \frac{\mu+\omega_M}{\mu}$; it has residue

$$Residue = -\frac{\mu}{\mu+\omega_M}\beta(\omega_G - \omega_M)[\Omega(\frac{\mu}{\mu+\omega_M}) - \Omega(0)]. \tag{5.5}$$

The only other possible poles are the poles of $\beta(\omega_G + \mu(1-y))$. We can only determine those when we have specified $\beta(\cdot)$. Below we consider a specific example.

Example: exp(α) distributed service times in Q_G
If the service time distribution $B(\cdot)$ is exp(α), then $\beta(\omega_G + \mu(1-y)) = \frac{\alpha}{\alpha+\omega_G+\mu(1-y)}$ has one pole $y = 1 + \frac{\alpha+\omega_G}{\mu}$ outside the unit circle. The sum of minus the residues at this pole and at $y = \frac{\mu+\omega_M}{\mu}$ gives

$$T_2 = (1-\frac{\lambda}{\alpha})(1-\frac{\lambda}{\mu})\frac{\alpha}{\alpha+\omega_G}\frac{\mu}{\mu+\omega_M}\frac{\lambda}{\omega_M+\mu-\lambda}\frac{\omega_G+\mu+\alpha+\omega_M-\lambda}{\omega_G+\mu+\alpha-\lambda}. \tag{5.6}$$

One could also evaluate the contour integral in (5.4) by summing the residues of the poles *inside* the unit circle. In this particular case, one can do that by observing that, in this case of two $M/M/1$ queues in series, there is the well-known product-form result (going back to R.R.P. Jackson [18]):

$$\Omega(y) = \sum_{j=0}^{\infty} y^j (1-\frac{\lambda}{\alpha})(1-\frac{\lambda}{\mu})(\frac{\lambda}{\mu})^j = (1-\frac{\lambda}{\alpha})\frac{1-\frac{\lambda}{\mu}}{1-\frac{\lambda}{\mu}y}.$$

Hence $\Omega(\frac{1}{y}) - \Omega(0)$ in (5.4) has one pole $y = \frac{\lambda}{\mu}$ inside the unit circle D_y. Its residue equals the expression for T_2 in (5.6). To give additional insight into this kind of calculation, let us mention a third way to evaluate T_2. Starting-point now is (5.2), where we substitute $\mathbb{P}(X_G^a = 0, X_M^a = j) = (1-\frac{\lambda}{\alpha})(1-\frac{\lambda}{\mu})(\frac{\lambda}{\mu})^j$ and use (3.12) for $\psi_{0,j}(\omega_G, \omega_M)$ with

$$b(l, \omega_G) = \frac{\alpha}{\alpha+\omega_G}\frac{(\frac{\mu}{\mu+\omega_M})^{l+1} - (\frac{\mu}{\mu+\alpha+\omega_G})^{l+1}}{(\frac{\mu}{\mu+\omega_M}) - (\frac{\mu}{\mu+\alpha+\omega_G})},$$

to get (5.6) once more.

Remark.
We have determined T_2 in three different ways for the case of exponential service times in Q_G, to give more insight in term T_2. On the one hand we want to convince the reader that T_2 can be evaluated without an exceptional effort. On the other hand we want to point out that there are quite a few technicalities which have to be handled. They mainly concern a careful determination of the poles of the integrand of (5.4), but we also would like to mention the following three technicalities. (i) We have changed summation and integration to get (5.4). (ii) $\Omega(y)$ is not explicitly given in [2];

the authors of [2] only need the real part of $\Omega(y)$ on a circle, but there is analytic continuation. (iii) In (5.5) we need the real part of $\omega_G - \omega_M$ to be non-negative. In the above $\exp(\alpha)$ example, the only pole of $\beta(\omega_G - \omega_M)$ occurs at $\alpha + \omega_G = \omega_M$, and the integrand in (5.4) now appears to have a *double* pole at $y = \frac{\mu+\omega_M}{\mu}$.

Determination of T_3

Let us finally consider T_3, which we split in an obvious way into T_{31} and T_{32}. We first determine T_{32}:

$$\begin{aligned}
T_{32} &= \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\int_{t=0}^{\infty} \mathrm{e}^{-\omega_G t}\sum_{h=j}^{\infty}\mathrm{e}^{-\mu t}\frac{(\mu t)^h}{h!}\psi_{i-1,1}(\omega_G,\omega_M)\mathrm{d}_t G(i,t,j)\\
&= \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\int_{t=0}^{\infty} \mathrm{e}^{-\omega_G t}\frac{1}{2\pi\iota}\int_{D_z}\frac{1-z\mathrm{e}^{-\mu(1-z)t}}{(1-z)z^{j+1}}\mathrm{d}z\,\frac{1}{2\pi\iota}\int_{D_x}\frac{A_1(x,\omega_G,\omega_M)}{x^i}\mathrm{d}x\;\mathrm{d}_t G(i,t,j)\\
&= \left(\frac{1}{2\pi\iota}\right)^2\int_{D_z}\frac{1}{1-z}\int_{D_z} A_1(x,\omega_G,\omega_M)\left(\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\int_{t=0}^{\infty}\mathrm{d}_t G(i,t,j)\frac{1}{z^j}\frac{1}{x^i}\mathrm{e}^{-\omega_G t}\frac{1}{z}-\right.\\
&\qquad\left.-\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\int_{t=0}^{\infty}\mathrm{d}_t G(i,t,j)\frac{1}{z^j}\frac{1}{x^i}\mathrm{e}^{-\omega_G t}\mathrm{e}^{-\mu(1-z)t}\right)\mathrm{d}x\;\mathrm{d}z\;.
\end{aligned}$$

Here we have twice used the inversion formula for a generating function:

$$\psi_{i-1,1}(\omega_G,\omega_M) = \frac{1}{2\pi\iota}\int_{D_x}\frac{A_1(x,\omega_G,\omega_M)}{x^i}\mathrm{d}x,$$

and

$$\sum_{h=j}^{\infty}\mathrm{e}^{-\mu t}\frac{(\mu t)^h}{h!} = \frac{1}{2\pi\iota}\int_{D_z}\frac{1-z\mathrm{e}^{-\mu(1-z)t}}{(1-z)z^{j+1}}\mathrm{d}z, \tag{5.7}$$

the integration being over the unit circles D_x and D_z, respectively. The integrand in (5.7) is obtained by direct evaluation for $|z| < 1$ of

$$\begin{aligned}
&\sum_{j=0}^{\infty} z^j\sum_{h=j}^{\infty}\mathrm{e}^{-\mu t}\frac{(\mu t)^h}{h!} = \sum_{h=0}^{\infty}\mathrm{e}^{-\mu t}\frac{(\mu t)^h}{h!}\sum_{j=0}^{h} z^j\\
=\;&\sum_{h=0}^{\infty}\mathrm{e}^{-\mu t}\frac{(\mu t)^h}{h!}\frac{1-z^{h+1}}{1-z} = \frac{1-z\mathrm{e}^{-\mu t(1-z)}}{1-z}.
\end{aligned}$$

Introducing for $|r_1| \le 1$, $|r_2| \le 1$, $\mathrm{Re}\,\omega \ge 0$:

$$\Phi(r_1,\omega,r_2) := \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\int_{t=0}^{\infty} r_1^i r_2^j \mathrm{e}^{-\omega t}\mathrm{d}_t G(i,t,j), \tag{5.8}$$

$\Phi(r_1,\omega,r_2)$ being a function which follows from the analysis in [2], we find: $T_{32} =$

$$\left(\frac{1}{2\pi\iota}\right)^2\int_{D_z}\frac{1}{1-z}\int_{D_z} A_1(x,\omega_G,\omega_M)\left(\frac{1}{z}\Phi(\frac{1}{x},\omega_G,\frac{1}{z}) - \Phi(\frac{1}{x},\omega_G+\mu(1-z),\frac{1}{z})\right)\mathrm{d}x\;\mathrm{d}z\;. \tag{5.9}$$

We evaluate T_{31} in a similar way:

$$\begin{aligned}
T_{31} &= \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\int_{t=0}^{\infty} \mathrm{e}^{-\omega_G t}\sum_{h=0}^{j-1}\mathrm{e}^{-\mu t}\frac{(\mu t)^h}{h!}\psi_{i-1,j-h+1}(\omega_G,\omega_M)\mathrm{d}_t G(i,t,j) \\
&= \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\int_{t=0}^{\infty} \mathrm{e}^{-\omega_G t}\sum_{h=0}^{j-1}\mathrm{e}^{-\mu t}\frac{(\mu t)^h}{h!}(\frac{1}{2\pi\iota})^2\int_{D_x}\int_{D_y}\frac{A(x,y,\omega_G,\omega_M)}{x^i y^{j-h+2}}\mathrm{d}x\ \mathrm{d}y\ \mathrm{d}_t G(i,t,j) \\
&= \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\int_{t=0}^{\infty} \mathrm{e}^{-\omega_G t}(\frac{1}{2\pi\iota})^2\int_{D_x}\int_{D_y}\frac{A(x,y,\omega_G,\omega_M)}{x^i y^{j+2}}\mathrm{d}x\ \mathrm{d}y\ \mathrm{d}_t G(i,t,j)\sum_{h=0}^{j-1}\frac{1}{2\pi\iota}\int_{D_z}\frac{\mathrm{e}^{-\mu(1-yz)t}}{z^{h+1}}\mathrm{d}z \\
&= \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\int_{t=0}^{\infty} \mathrm{e}^{-\omega_G t}(\frac{1}{2\pi\iota})^2\int_{D_x}\int_{D_y}\frac{A(x,y,\omega_G,\omega_M)}{x^i y^{j+2}}\mathrm{d}x\mathrm{d}y \\
&\quad \frac{1}{2\pi\iota}\int_{D_z}\mathrm{e}^{-\mu t(1-yz)}[\frac{1-(\frac{1}{z})^j}{z-1}]\mathrm{d}z\ \mathrm{d}_t G(i,t,j).
\end{aligned}$$

Finally, using the definition in (5.8), we can write:

$$\begin{aligned}
T_{31} &= (\frac{1}{2\pi\iota})^3\int_{D_x}\int_{D_y}\int_{D_z}\frac{A(x,y,\omega_G,\omega_M)}{y^2(z-1)} \\
&\quad [\Phi(\frac{1}{x},\omega_G+\mu(1-yz),\frac{1}{y})-\Phi(\frac{1}{x},\omega_G+\mu(1-yz),\frac{1}{yz})]\mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z. \qquad (5.10)
\end{aligned}$$

Combining (5.1), (5.3), (5.4), (5.9) and (5.10), we have obtained an expression for the joint LST of S_G and S_M in the open tandem queue $M/G/1-\cdot/M/1$. This LST is expressed in contour integrals of terms which are known: $A_1(x,\omega_G,\omega_M)$ and $A(x,y,\omega_G,\omega_M)$ were derived in the previous section, while $\Omega(y)$ and $\Phi(r_1,\omega,r_2)$ are in principle known from [2]. Evaluation of the contour integrals that appear in the joint LST expression can be done explicitly once the service time LST is specified.

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