U N I V E R S I T Ä T   H A M B U R G

Loss systems in a random environment - embedded Markov chains analysis

Ruslan Krenzler, Hans Daduna

Preprint–No. 2013-02   Mai 2013

DEPARTMENT MATHEMATIK

SCHWERPUNKT MATHEMATISCHE STATISTIK UND STOCHASTISCHE PROZESSE

BUNDESSTR. 55, D – 20146 HAMBURG

Loss systems in a random environment-embedded Markov chains analysis.

Ruslan Krenzler, Hans Daduna

May 16, 2013

We consider state dependent queueing systems which interact with a random environment in both directions: The queue can trigger a change of the environment and the environment can interrupt service of customers and the arrival stream of new customers, such that new arrivals are lost to the system. The systems live in continuous time and we observe them at departure instants only, which results in considering an embedded Markov chain. Our main aim is to identify conditions which enforce the systems to stabilize in a way that the queue and the environment decouple in the sense that the stationary queue length and environment behave independently, i.e., a product form equilibrium exists.

We show that the behaviour of the embedded Markov chain is often considerably different from that of the original continuous time Markov process investigated in [11].

For exponential queueing systems we show that there is a product form equilibrium under rather general conditions. For systems with non-exponential service times more restrictive constraints are needed, which we prove by a counter example where the environment represents an inventory attached to an M/D/1 queue. Such integrated queueing-inventory systems are dealt with in the literature previously. Further applications are, e.g., in modeling unreliable queues.

Keywords: Queuing systems, random environment, product form steady state, loss systems, $M/M/1/\infty$, $M/M/m/\infty$, $M/G/1/\infty$, embedded Markov chains, inventory systems, availability, lost sales, matrix invertibility.

Contents

1 Introduction

Product form networks of queues are common models for easy to perform structural and quantitative first order analysis of complex networks in Operations Research applications. The most prominent representatives of this class of models are the Jackson [7] and Gordon-Newell [5] networks and their generalizations as BCMP [1] and Kelly [9] networks, for a short review see [2].

Over the last years product form models became popular in other areas of operations research as well, especially when constructing integrated models for production (queueing models) and inventory, for a review on integrated queueing-inventory models see the survey paper [12]. A review with emphasis on product form models in this and related areas can be found in the introduction of [11].

We continue in this paper our investigations described in [11]. In that paper we analyzed a *loss system in a random environment*, described by a homogeneous continuous time Markov process $(X, Y) = ((X(t), Y(t)) : t \geq 0)$, where X describes an $M/M/1/\infty$-type queuing system (or a general birth-and-death process) and Y is an environment process (representing, e.g., the stock size of an attached inventory or the availability status of the production system modeled by the queue).

In the present paper for the reader's convenience we take X as a single server queue, and Y as a general environment. X and Y will interact vice versa, in both directions.

The main feature of this class of models is that jumps of the queue may enforce the environment to jump instantaneously, and in the other direction the evolving environment may interrupt the queue in a way, that service is interrupted and no customers are admitted during the interruption interval. For more details see the introduction of [11] and Section 2 below.

The main contribution of [11] is to identify conditions which guarantee that the ergodic continuous time Markov process eventually settles down in a stable state distribution which is of product form, i.e., the limiting and stationary distribution of the vector process (X, Y) is the independent product of the stationary distributions of X and Y, which are both not Markov. In this paper we partly continue that research by observing the systems at departure time instants only.

For investigating $M/G/1/\infty$ queues this is a standard procedure to avoid using supplementary variable technique. Embedded chain analysis was applied by Vineetha [15] who extended the theory of integrated queueing-inventory models with exponential service times to systems with service times which are i.i.d. and follow a general distribution. Our investigations which are reported in this paper were in part motivated by her investigations.

In Section 3 we revisit some of Vineetha's [15] queueing-inventory systems, using similarly embedded Markov chain techniques. In the course of these investigations we found that there arise problems even for purely exponential systems, which we describe in Section 2.2 and 2.3 first, before describing the $M/G/1/\infty$ queue in a random environment and its structural properties.

To emphasize the problems arising from the interaction of the two components of integrated systems, we remind the reader, that for ergodic $M/M/1/\infty$ queues the limiting and stationary distribution of the continuous time queue length process and the Markov chains embedded at either arrival instants or departure instants are the same.

Our first finding is, that even in the case of $M/M/1/\infty$ queues with attached inventory this in general does not hold. This especially implies, that the product form results obtained in [11] do not carry over immediately to the case of loss systems in a random environment observed at departure times from the queue (downward jumps of the generalized birth-and-death process).

A striking observation is that for a system which is ergodic in the continuous time Markovian description the Markov chain embedded at departure instants may be not ergodic. The reason for this is two-fold. Firstly, the embedded Markov chain may have inessential states due to the specified interaction rules. Secondly, even when we delete all inessential states, the resulting single positive recurrent class may be periodic.

We study this problem in depth in Section 2.3 for purely exponential systems, and provide a set of examples which elucidate the problems which one is faced with. Our main result in this section proves the existence of a product form steady state distribution (which is not necessary a limiting distribution) for the Markov chain embedded at departure instants and provides a precise connection between the steady states of the continuous time process and the embedded chain (Theorem 15).

It turns out, that a similar result in the setting with $M/G/1/\infty$ queues is not valid. We are able to give sufficient conditions for the structure of the environment, which guarantee the existence of product form equilibria (Theorem 28).

Unfortunately enough, an analogue to Theorem 15 is not valid for systems with non exponential service times. We prove this for an $M/D/1/\infty$ queue with an attached environment in Section 3.1.1.

Most of our results for systems rely strongly on non-singularity of a certain matrix which reflects important aspects of the system. For systems with a finite environment the regularity of that matrix is proved in an Appendix as a useful lemma which is of interest in its own. This lemma generalizes the well known theorem of invertibility of M-matrices which are irreducible to the case where irreducibility is not required, but only a certain flow condition prevails.

Notations and conventions: $\mathbb{R}_0^+ = [0, \infty)$, $\mathbb{R}^+ = (0, \infty)$,.

All random variables and processes occurring henceforth are defined on a common underlying probability space $(\Omega, \mathcal{F}, P)$.

For all processes considered in this paper we can and will assume that their paths are right continuous with left limits (cadlag).

$1_{[expression]}$ is the indicator function which is 1 if *expression* is true and 0 otherwise.

For any quadratic matrix V we define $diag(V)$ as the matrix with the same diagonal as V, while all other entries are 0.

2 M/M/1/∞ queueing system in a random environment

For the reader's convenience we recall the setting of [11] where we considered a two-dimensional homogeneous strong Markov jump process with cadlag paths $Z := (X, Y) = ((X(t), Y(t)) : t \in [0, \infty))$ with state space $E = \mathbb{N}_0 \times K$. Z describes the development over time of a queue in a random environment, where K is a countable set, the environment space of the process, whereas the queueing state space is $\mathbb{N}_0$. In the typical examples the queueing process X represents some production process, whereas Y describes the status of an attached inventory, or the availability status of the production facility, see [11] for details and examples. We assume throughout that $Z = (X, Y)$ is non-explosive in finite times and irreducible (unless specified otherwise).

A characteristic feature of our systems is that the environment space is partitioned into disjoint components $K := K_W + K_B$. When K represents the inventory size, K_B describes the status "stock out", in the reliability problem K_B describes the status "server broken down". So accordingly K_W indicates for the inventory that there is stock on hand for production, and "server is up" in the other system.

The general interpretation is that whenever the environment process enters K_B the service process is "**B**LOCKED", and the service is resumed immediately whenever the environment process returns to K_W, the server "**W**ORKS" again.

Whenever the environment process stays in K_B new arrivals are lost.

Obviously, it is natural to assume that the set K_W is not empty, while in certain problem settings K_B may be empty, e.g. no break down of the server in the second example.

In the general setting of [11] the arrival stream of customers is Poisson with rate $\lambda^{(n)} > 0$, when there are n customers in the system. The system is a single server under First-Come-First-Served[1] regime (FCFS) with an infinite waiting room and develops over time as follows.

1) If at time t the environment is in state $Y(t) = k \in K_W$ and if there are $X(t) = n$ customers in the queue then service is provided to the customer at the head of the queue with rate $\mu^{(n)} > 0$. As soon as his service is finished he leaves the system and the environment changes with probability R_{km} to state $m \in K$, independent of the history of the system, given k. The stochastic matrix $R \in \mathbb{R}^{K \times K}$

is the jump matrix for the environment, driven by the departure process.

2) If the environment at time t is in state $Y(t) = k \in K_B$ no service is provided to customers in the queue and arriving customers are lost.

3) Whenever the environment at time t is in state $Y(t) = k \in K$ it changes with rate $\nu(k, m)$ to state $m \in K$, independent of the history of the system, given k.
Note, that such changes occur independent of the service and arrival process, while the changes of the environment's status under **1)** are coupled with the service process.

[1] Wer zuerst kommt, mahlt zuerst.

It follows that the non negative transition rates of (X, Y) are for $(n, k) \in E$

$$\begin{aligned} q((n,k) \to (n+1,k)) &= \lambda^{(n)}, \qquad k \in K_W, \\ q((n,k) \to (n-1,m)) &= \mu^{(n)} R_{km}, \qquad k \in K_W, n \geq 1, \\ q((n,k) \to (n,m)) &= \nu(k,m) \in \mathbb{R}_0^+, \quad k \neq m, \\ q((n,k) \to (i,m)) &= 0\,, \qquad \text{otherwise for } \ (n,k) \neq (i,m)\,. \end{aligned}$$

Note, that the diagonal elements of $Q := (q((n,k) \to (i,m)) : (n,k),(i,m) \in E)$ are determined by the requirement that row sums are 0. It is allowed to have positive diagonal entries R_{kk}. R needs not be irreducible, there may exist closed subsets in K. We require $\nu(k,k) = -\sum_{m \in K \setminus \{k\}} \nu(k,m)$ for all $k \in K$, such that

$$\Upsilon = (\nu(k,m) : k, m \in K)$$

is a generator matrix. The Markov process associated with Υ may have absorbing states, i.e., Υ may have zero rows.

Remark 1. It will be convenient to order the state space in the way which is common in matrix analytical investigations, where X is the level process and Y is the phase process. Take on $\mathbb{N}_0$ the natural order and fix a total (linear) order $\preccurlyeq$ on K such that

$$k \in K_W \ \wedge \ l \in K_B \Longrightarrow k \preccurlyeq l, \tag{2.1}$$

holds, and define on $E = \mathbb{N}_0 \times K$ the lexicographic order $\prec$ by

$$(m,k),(n,l) \in E \ \text{ then } \ \big((m,k) \prec (n,l) :\Longleftrightarrow \big[m < n \ \text{ or } \ (m = n \ \text{ and } \ k \preccurlyeq l)\big]\big)\,. \tag{2.2}$$

Recall that the paths of Z are cadlag. With $\tau_0 = \sigma_0 = \zeta_0 = 0$ and

$$\tau_{n+1} := \inf(t > \tau_n : X(t) < X(\tau_n)), \quad n \in \mathbb{N}\,.$$

denote the sequence of departure times of customers by $\tau = (\tau_0, \tau_1, \tau_2, \dots)$, and with

$$\sigma_{n+1} := \inf(t > \sigma_n : X(t) > X(\sigma_n)), \quad n \in \mathbb{N}\,,$$

denote by $\sigma = (\sigma_0, \sigma_1, \sigma_2, \dots)$ the sequence of instants when arrivals are admitted to the system (because the environment is in states of K_W, i.e., not blocking)
and with

$$\zeta_{n+1} := \inf(t > \zeta_n : Z(t) \neq Z(\zeta_n)), \quad n \in \mathbb{N}\,,$$

denote by $\zeta = (\zeta_0, \zeta_1, \zeta_2, \dots)$ the sequence of jump times of Z.

Some notation which will be used henceforth: I_W is a matrix which has ones on its diagonal elements (k,k) with $k \in K_W$ and 0 otherwise. That is

$$(I_W)_{km} = \delta_{km} 1_{[k \in K_W]}\,,$$

and using the ordering (2.1) we have the convenient notation (which is not necessary, but makes reading more comfortable in the proofs below)

$$I_W = \left(\begin{array}{c|cc} & K_W & K_B \\ \hline K_W & \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} & 0 \\ K_B & 0 & 0 \end{array} \right)$$

2.1 Steady state distribution of the continuous time process

We recall from [11] a product form characterization for the steady state distribution of $Z = (X, Y)$.

Theorem 2. **(a)** *Denote for $n \in \mathbb{N}_0$ and $k, m \in K$*

$$\begin{aligned} \tilde{q}_{kk}^{(n)} &= -(1_{[k \in K_W]} \lambda^{(n)} (1 - R_{kk}) + \sum_{m \in K \setminus \{k\}} \nu(k,m)) , \\ \tilde{q}_{km}^{(n)} &= \lambda^{(n)} R_{km} 1_{[k \in K_W]} + \nu(k,m) , \qquad k \neq m \end{aligned} \tag{2.3}$$

and

$$\tilde{Q}^{(n)} = (\tilde{q}_{km}^{(n)} : k, m \in K) .$$

Then the matrices $\tilde{Q}^{(n)}$ are generator matrices for some homogeneous Markov processes.

(b) *If the process $Z = (X, Y)$ is ergodic denote its unique steady state distribution by*

$$\pi = (\pi(n,k) : (n,k) \in E := \mathbb{N}_0 \times K).$$

Then the following properties are equivalent:

(i) *$Z = (X, Y)$ is ergodic with product form steady state*

$$\pi(n,k) = C^{-1} \underbrace{\prod_{i=0}^{n-1} \frac{\lambda^{(i)}}{\mu^{(i+1)}}}_{=:\xi(n)} \theta(k) \tag{2.4}$$

(ii) *The summability condition*

$$C := \sum_{n=0}^{\infty} \prod_{i=0}^{n-1} \frac{\lambda^{(i)}}{\mu^{(i+1)}} < \infty \tag{2.5}$$

holds, and the equation

$$\theta \cdot \tilde{Q}^{(0)} = 0 \tag{2.6}$$

admits a strictly positive stochastic solution $\theta = (\theta(k) : k \in K)$ *which solves also*

$$\forall n \in \mathbb{N}: \quad \theta \cdot \tilde{Q}^{(n)} = 0\,. \tag{2.7}$$

(c) *The (1-dimensional) stationary distribution* π *of* $(X(t), Y(t))$ *(any* t*) fulfills*

$$\begin{aligned} \xi(n) :=& \quad P(X(t) = n), \qquad \theta(k) := P(Y(t) = k), \\ \pi(n,k) & \quad := P(X(t) = n, Y(t) = k) = \xi(n) \cdot \theta(k)\,, \end{aligned}$$

with

$$\xi = (\xi(n) := C^{-1} \prod_{i=0}^{n-1} \frac{\lambda^{(i)}}{\mu^{(i+1)}} : n \in \mathbb{N}_0) \tag{2.8}$$

as the steady state distribution of an ergodic birth-death process with birth rates $\lambda^{(n)}$ *and death rates* $\mu^{(n)}$.

Exploiting the structure of the ordered state space E we can write π as

$$\pi = (\pi^{(0)}, \pi^{(1)}, \pi^{(2)}, \dots) \tag{2.9}$$

with

$$\pi^{(n)} = (\pi(n,k) : k \in K_W, \pi(n,k) : k \in K_B), \quad n \in \mathbb{N}_0\,, \tag{2.10}$$

where we agree that the representation of $(\pi(n,k) : k \in K_W, \pi(n,k) : k \in K_B)$ respects the ordering of K. The most relevant result for the following investigations is the following consequence of the theorem.

Corollary 3. *If in the framework of Theorem 2 the arrival stream is a Poisson-*λ *stream (which is interrupted when the environment process stays in* K_B*) and* $\theta \cdot \tilde{Q}^{(0)} = 0$ *admits a strictly positive stochastic solution then the stationary distribution in case of ergodic* (X, Y) *is of product form*

$$\pi(n,k) = C^{-1} \frac{\lambda^n}{\prod_{i=0}^{n-1} \mu^{(i+1)}} \theta(k) \quad (n,k) \in E, \tag{2.11}$$

with normalization constant C.

Denote by

$$R_W := I_W \cdot R = \left(\begin{array}{c|cc} & K_W & K_B \\ \hline K_W & R|_{K_W \times K_W} & R|_{K_W \times K_B} \\ K_B & 0 & 0 \end{array} \right) \tag{2.12}$$

the matrix with K_W*-rows the rows of* R *and with* K_B*-rows with only zeros.*

Then the equation (2.6) can be written as

$$\theta\left(\lambda\left(R_W - I_W\right) + \Upsilon\right) = 0\,. \tag{2.13}$$

which has the stochastic solution θ.

Proof. (2.11) is part of [11][Corollary 6], and (2.13) is a direct reformulation of (2.6). □

(2.3) in Theorem 2 reveals that the solution of the equation $\theta \cdot \tilde{Q}^{(0)} = 0$ (see (2.6)) does not depend on the values $\mu^{(n)}$. So, changing the service capacity of the queueing system will not change the the steady state of the environment, as long as the system remains stable (ergodic).

The following lemmata will be used in the sequel. It refers to the structure of the continuous time process. We emphasize that the generator Υ is not necessarily irreducible.

Lemma 4. *Let Z be ergodic. Then for any non-empty subset $\tilde{K}_B \subset K_B$ the overall Υ-transition rate from $\tilde{K}_B$ to its complement $\tilde{K}_B^c = K \setminus \tilde{K}_B$ is positive, i.e.,*

$$\forall \quad \tilde{K}_B \subset K_B, \tilde{K}_B \neq \emptyset : \qquad \exists \quad k \in \tilde{K}_B, m \in \tilde{K}_B^c : \nu(k,m) > 0 \tag{2.14}$$

Remark. Consider the directed transition graph of Υ, with vertices K and edges $\mathcal{E}$ defined by $km \in \mathcal{E} \iff \nu(k,m) > 0$. Then the condition (2.14) guarantees the existence of a path from any vertex in K_B to a some vertex in K_W.

Proof. (of Lemma 4) Fix $\tilde{K}_B$ and suppose the system is ergodic and it is started with $Z(0) = (0,k)$, for some $k \in \tilde{K}_B$, i.e., with an empty queue and in an environment state k which blocks the arrival process. From ergodicity it follows that for some $m \in K_W$ must hold

$$P(Z(\sigma_1) = (1,m)|Z(0) = (0,k)) > 0,$$

because there is a positive probability for the first arrival of some customer admitted into the system.

Because no arrival is possible if $m \in K_B$, necessarily $m \in K_W$ holds, and because up to σ_1- no departure or arrival could happen, the only possibility to enter m is by a sequence of transitions triggered by Υ. Because Z is regular this sequence is finite with probability 1. The path from $k \in \tilde{K}_B$ to $m \in K_W$ of the directed transition graph of Υ contains an edge $k_1k_2 \in \mathcal{E}$ with $k_1 \in \tilde{K}_B$ and $k_2 \in \tilde{K}_B^c$. □

Lemma 5. *For any strictly positive $\eta \in \mathbb{R}^+$ the matrix $(-diag(\Upsilon) + \eta I_W)$ is invertible.*

Proof. For any $k \in K_W$ the corresponding diagonal element of the matrix $(-diag(\Upsilon) + \eta I_W)$ is greater than η because $-\nu(k,k) \geq 0$.

If $k \in K_B$, we utilize the ergodicity of Z in continuous time and apply Lemma 4 with $\tilde{K}_B := \{k\}$. The lemma implies that there is some $m \neq k$ with $\nu(k,m) > 0$. It follows $-\nu(k,k) > 0$.

We conclude that the diagonal matrix $(-diag(\Upsilon) + \eta I_W)$ has only strictly positive values on its diagonal and therefore it is invertible. □

2.2 Observing the system at departure instants

Recall that the paths of Z are cadlag and that $\tau = (\tau_0, \tau_1, \tau_2, \dots)$ with $\tau_0 = 0$ denotes the sequence of departure times of customers. Then with $\hat{X}(n) := X(\tau_n)$ and $\hat{Y}(n) := Y(\tau_n)$ for $n \in \mathbb{N}_0$ it is easy to see that

$$\hat{Z} = ((\hat{X}(n), \hat{Y}(n)) : n \in \mathbb{N}_0) \tag{2.15}$$

is a homogeneous Markov chain on state space $E = \mathbb{N}_0 \times K$. If $\hat{Z}$ has a unique stationary distribution, this will be denoted by $\hat{\pi}$.

It will turn out that this Markov chain exhibits interesting structural properties of the loss systems in random environments. E.g., if the vector $\hat{Z}_\infty = (\hat{X}_\infty, \hat{Y}_\infty)$ is distributed according to the (1-dimensional in time) stationary distribution $\hat{\pi}$ of $\hat{Z}(n) = (\hat{X}(n), \hat{Y}(n))$ (any n), then with ξ from (2.8)

$$\xi(n) = P(\hat{X}_\infty = n), \quad \hat{\theta}(k) := P(\hat{Y}_\infty = k), \quad \hat{\pi}(n,k) := P(\hat{X}_\infty = n, \hat{Y}_\infty = k) = \xi(n) \cdot \hat{\theta}(k)\,,$$

but in general we do not have $\hat{\pi}(n,k) > 0$ on the global state space E, because $\hat{\theta}(k) = 0$ may occur. Especially, in general it holds $\theta \neq \hat{\theta}$.

The reason for this seems to be the rather general vice-versa interaction of the queueing system and the environment. Of special importance is the fact that we consider the continuous time systems at departure instants where we have the additional information that right now the influence of the queueing systems on the change of the environment is in force (described by the stochastic matrix R).

The dynamics of $\hat{Z}$ will be described in a way that resembles the $M/G/1$ type matrix analytical models. Recall that the state space E carries an order structure which will govern the description of the transition matrix and, later on, of the steady state vector.

Definition 6. We define the one-step transition matrix $\mathbf{P}$ of $\hat{Z}$ by

$$\begin{aligned} & \left(\mathbf{P}_{(i,k),(j,m)} : (i,k),(j,m) \in E\right) \\ := \quad & \left(P(Z(\tau_1) = (j,m) | Z(0) = (i,k)) : (i,k),(j,m) \in E\right), \end{aligned}$$

and introducing matrices $A^{(i,n)} \in \mathbb{R}^{K \times K}$ and $B^{(n)} \in \mathbb{R}^{K \times K}$ by

$$B^{(n)}_{km} := P(Z(\tau_1) = (n,m) | Z(0) = (0,k)) \tag{2.16}$$

$$A^{(i,n)}_{km} := P(Z(\tau_1) = (i+n-1,m) | Z(0) = (i,k)), \qquad 1 \leq i \tag{2.17}$$

for $k, m \in K$, the matrix $\mathbf{P}$ has the form

$$\mathbf{P} = \begin{pmatrix} B^{(0)} & B^{(1)} & B^{(2)} & B^{(3)} & \dots \\ A^{(1,0)} & A^{(1,1)} & A^{(1,2)} & A^{(1,3)} & \dots \\ 0 & A^{(2,0)} & A^{(2,1)} & A^{(2,2)} & \dots \\ 0 & 0 & A^{(3,0)} & A^{(3,1)} & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}, \tag{2.18}$$

which exploits the structure of the state space as a product of level variables in $\mathbb{N}_0$ and phase variables in K. We emphasize that K is ordered for its own, see (2.1).

For the loss system in a random environment we will solve the equation

$$\hat{\pi}\mathbf{P} = \hat{\pi} \tag{2.19}$$

for a stochastic solution $\hat{\pi}$ which is a steady state distribution of the embedded Markov chain $\hat{Z}$. Because $\hat{Z}$ is in general not irreducible on E there are some subtleties with respect to the uniqueness of a stochastic solution of the equation.

Similarly to structuring π in (2.10) it will be convenient to group $\hat{\pi}$.

Definition 7. We write $\hat{\pi}$ as

$$\hat{\pi} = (\hat{\pi}^{(0)}, \hat{\pi}^{(1)}, \hat{\pi}^{(2)}, \dots) \tag{2.20}$$

with

$$\hat{\pi}^{(n)} = (\hat{\pi}(n,k) : k \in K_W, \hat{\pi}(n,k) : k \in K_B), \quad n \in \mathbb{N}_0\,, \tag{2.21}$$

where we agree that the representation of $(\hat{\pi}(n,k) : k \in K_W, \hat{\pi}(n,k) : k \in K_B)$ respects the ordering of K. Especially we write for $(n,k) \in E$

$$\hat{\pi}^{(n)}(k) := \hat{\pi}(n,k) = P(\hat{X}_\infty = n, \hat{Y}_\infty = k)\,.$$

An immediate consequence of this definition is that the steady state equation (2.19) can be written as

$$\hat{\pi}^{(0)} B^{(n)} + \sum_{i=1}^{n+1} \hat{\pi}^{(i)} A^{(i,n-i+1)} = \hat{\pi}^{(n)}, \qquad n \in \mathbb{N}_0\,. \tag{2.22}$$

2.3 Steady state for the system observed at departure instants

We start our investigation with a detailed analysis of the one-step transition matrix (2.18) and will express the matrices $B^{(n)}$ and $A^{(i,n)}$ from Definition 6 by means of auxiliary matrices W and $U^{(i,n)}$, which reflect the dynamics of the system.

It turns out that the matrix $(\lambda I_W - \Upsilon)$ plays a central role in this analysis and that we shall need especially its inverse. We therefore set in force for the rest of the paper the technical

Overall Assumption (I), that the matrix $(\lambda I_W - \Upsilon)$ is **invertible**.

This assumption is not restrictive for modeling purposes as the following proposition reveals. Further examples and a discussion can be found in the Appendix.

Proposition 8. *Let Z be ergodic with finite environment space K, and Υ be the associated generator driving the continuous changes of the environment. Then for any $\lambda > 0$ the matrix $(\lambda I_W - \Upsilon)$ is invertible.*

Proof. Follows from Lemma 4 and Lemma 33 from the Appendix. □

Proposition 9. *Let Z be ergodic with environment space K partitioned according to $K = K_W + K_B$, with $K_W \neq \emptyset$, and with $|K_B| < \infty$, and $\lambda > 0$ such that $\lambda I_W - \Upsilon$ is surjective on $\ell_\infty(\mathbb{R}^K)$.*

Let the generator matrix $\Upsilon := (\nu(k,m) : k,m \in K) \in \mathbb{R}^{K \times K}$ be uniformizable, i.e. it holds $\inf_{k \in K} \nu(k,k) > -\infty$.

Then the matrix $\lambda I_W - \Upsilon$ is invertible.

Proof. It is immediate, that $\lambda I_W - \Upsilon$ fulfills the assumptions (5.14), (5.15), and (5.16) of Lemma 36 with $\varepsilon(K_W) = \lambda$. The flow condition holds in this setting from the ergodicity of the continuous times process with arguments similar to those in the proof of Lemma 5. We conclude that M is injective.

□

In a first step we analyze the dynamics incorporated in the matrix $A^{(i,n)}$ and $B^{(n)}$.

Lemma 10. *Recall that τ_1 denotes the first departure instant, that σ_1 denotes the first arrival instant of a customer, and that $Y(\sigma_1) \in K_W$ holds.*

For $k \in K, m \in K_W$, we define

$$U_{km}^{(i,n)} := P\left(\left(X(\tau_1), Y(\tau_1^-)\right) = (n+i-1, m) | Z(0) = (i,k)\right), \qquad 1 \leq i, n \in \mathbb{N}_0\,, \quad (2.23)$$

and for $k \in K$ and $m \in K_B$ we prescribe by definition $U_{km}^{(i,n)} = 0$.

Similarly, for $k \in K, m \in K_W$, we define

$$W_{km} := P(Z(\sigma_1) = (1,m) | Z(0) = (1,k))\,, \quad (2.24)$$

and for $k \in K$ and $m \in K_B$ prescribe by definition $W_{km} = 0$.

Then it holds for $A^{(i,n)}$ and $B^{(n)}$ from Definition 6

$$A^{(i,n)} = U^{(i,n)} R\,, \quad (2.25)$$

$$B^{(n)} = W A^{(1,n)} = W U^{(1,n)} R\,. \quad (2.26)$$

Proof. Using the fact, that the paths of the system in continuous time almost sure have left limits, we get for $i \geq 1,\ n \geq 0$ and $k, m \in K$

$$\begin{aligned} A_{km}^{(i,n)} &= P\left((X(\tau_1), Y(\tau_1)) = (i+n-1, m) | Z(0) = (i,k)\right) \\ &= \sum_{h \in K} P\left(\left(X(\tau_1), Y(\tau_1^-)\right) = (i+n-1, h) | Z(0) = (i,k)\right) R_{hm} \\ &= \sum_{h \in K} U_{kh} R_{hm}\,, \end{aligned}$$

which in matrix form is (2.25).

For the property (2.26) we will use the fact, that if the system starts with an empty queue, then the first arrival occurs always before the first departure, $P(\sigma_1 < \tau_1) = 1$.

We obtain for $n \geq 0$ and $k, m \in K$

$$\begin{aligned} B_{km}^{(n)} : &= P((X(\tau_1), Y(\tau_1)) = (n, m) | Z(0) = (0, k)) \\ &= \sum_{h \in K} P\left((X(\tau_1), Y(\tau_1)) = (n, m) \cap Z(\sigma_1) = (1, h) | Z(0) = (0, k)\right) \\ &= \sum_{h \in K} P\left((X(\tau_1), Y(\tau_1)) = (1 + n - 1, m) | Z(\sigma_1) = (1, h) \cap Z(0) = (0, k)\right) \\ &\quad \cdot P\left(Z(\sigma_1) = (1, h) | Z(0) = (0, k)\right) \\ &\overset{SM}{=} \sum_{h \in K} \underbrace{P\left((X(\tau_1), Y(\tau_1)) = (1 + n - 1, m) | Z(0) = (1, h)\right)}_{= U_{hm}^{(1,n)}} \\ &\quad \cdot \underbrace{P\left(Z(\sigma_1) = (1, h) | Z(0) = (0, k)\right)}_{= W_{kh}} \\ &= \sum_{h \in K} W_{kh} A_{hm}^{(1,n)}, \end{aligned}$$

which proves (2.26). □

The proof of Lemma 10 reveals that the stochastic matrix W describes the system's development (queue length $\hat{X}$ and environment $\hat{Y}$ process) if it is started empty, until the next customer enters the system.

The matrix $U^{(i,n)}$ describes the system's development from start of the ongoing service time of the, say, n-th admitted customer, until time τ_n-; to be more precise, we describe an ongoing service and the subsequent departure but without the immediately following jump of the environment triggered by R.

We will use the following properties of the system and its describing process Z:

- the strong Markov (SM) property of Z,
- skip free to the left (SF) property of the system

$$P(Z(\zeta_1) = (n + j, m) | Z(0) = (n, k)) = 0 \qquad \forall j \geq 2. \tag{2.27}$$

- cadlag paths; in particular we are interested in the values of $Y(\tau_1-)$, just before departure instants.

We furthermore have to take into account that matrix multiplication in general is not commutative. We write therefore $\prod_{j=i}^{n+1} B_i$ by definition for $B_i B_{i+1} \ldots B_{n+1}$.

Proposition 11. *Let σ_1 denote the arrival time of the first customer which is admitted to the system, which implies that at time σ_1 the environment is in a non-blocking state.*

For the matrix $W = (W_{km} : k, m \in K)$ from Lemma 10 it holds

$$W = \lambda(\lambda I_W - \Upsilon)^{-1} I_W$$

Proof. Recall that ζ_1 is the first jump time of the system which can be triggered only by Υ or by an arrival conditioned on $\hat{Y}$ being in K_W. It follows for $m \in K_W$

$$
\begin{aligned}
& W_{km} \\
= \; & P(Z(\sigma_1) = (1,m) | Z(0) = (0,k)) \\
= \; & \sum_{h \in K \setminus \{k\}} P(Z(\sigma_1) = (1,m) \cap Z(\zeta_1) = (0,h) | Z(0) = (0,k)) \\
& + \delta_{km} P(Z(\zeta_1) = (1,m) | Z(0) = (0,k)) \\
= \; & \sum_{h \in K \setminus \{k\}} P(Z(\sigma_1) = (1,m) | Z(\zeta_1) = (0,h), Z(0) = (0,k)) P(Z(\zeta_1) = (0,h) | Z(0) = (0,k)) \\
& + \delta_{km} P(Z(\zeta_1) = (1,m) | Z(0) = (0,k)) \\
\overset{SM}{=} \; & \sum_{h \in K \setminus \{k\}} \underbrace{P(Z(\sigma_1) = (1,m) | Z(0) = (0,h))}_{W_{hm}} \underbrace{P(Z(\zeta_1) = (0,h) | Z(0) = (0,k))}_{= \frac{\nu(k,h)}{-\nu(k,k) + \lambda 1_{[k \in K_W]}}} \\
& + \delta_{km} \underbrace{P(Z(\zeta_1) = (1,m) | Z(0) = (0,k))}_{= \frac{\lambda}{-\nu(k,k) + \lambda 1_{[k \in K_W]}} 1_{[k \in K_W]}}
\end{aligned}
$$

The equation above can be written in matrix form

$$
\begin{aligned}
W &= (-diag(\Upsilon) + \lambda I_W)^{-1} ((\Upsilon - diag(\Upsilon))W + \lambda I_W) \\
\Longleftrightarrow (-diag(\Upsilon) + \lambda I_W) W &= (\Upsilon - diag(\Upsilon))W + \lambda I_W \\
\Longleftrightarrow (\lambda I_W - \Upsilon) W &= \lambda I_W \Longrightarrow W = \lambda(\lambda I_W - \Upsilon)^{-1} I_W
\end{aligned}
$$

□

Proposition 12. *Let τ_1 denote the first departure time. For the matrices $U^{(i,n)} = (U^{(i,n)}_{km} : k, m \in K)$ from Lemma 10 it holds*

$$U^{(i,0)} = ((\lambda + \mu^{(i)}) I_W - \Upsilon)^{-1} \mu^{(i)} I_W, \quad 1 \le i, \tag{2.28}$$

and for $1 \le i, n \in \mathbb{N}_0$,

$$U^{(i,n+1)} = U^{(i,n)} \left(\frac{\lambda}{\mu^{(n+i)}} \right) \mu^{(n+1+i)} \left(\lambda I_W + \mu^{(n+1+i)} I_W - \Upsilon \right)^{-1}, \tag{2.29}$$

Proof. Note that ζ_1 is the first jump time of the system, and if this jump is triggered by a departure than $\zeta_1 = \tau_1$.

For $U^{(i,0)}$ with $i \ge 1$ it holds for $k \in K$ and $m \in K_W$:

$$
\begin{aligned}
& U_{km}^{(i,0)} \\
= \; & P\left(\left(X(\tau_1), Y(\tau_1^-)\right) = (i-1,m) | Z(0) = (i,k)\right) \\
= \; & \sum_{h \in K \setminus \{k\}} P\left(\left(X(\tau_1), Y(\tau_1^-)\right) = (i-1,m) \cap Z(\zeta_1) = (i,h) | Z(0) = (i,k)\right) \\
& + \delta_{km} P\left(\left(X(\tau_1), Y(\tau_1^-)\right) = (i-1,k) | Z(0) = (i,k)\right) \\
= \; & \sum_{h \in K \setminus \{k\}} P\left(\left(X(\tau_1), Y(\tau_1^-)\right) = (i-1,m) | Z(\zeta_1) = (i,h), Z(0) = (i,k)\right) \\
& \cdot P\left(Z(\zeta_1) = (i,h) | Z(0) = (i,k)\right) \\
& + \delta_{km} P\left(\left(X(\tau_1), Y(\tau_1^-)\right) = (i-1,k) | Z(0) = (i,k)\right) \\
= \; & \sum_{h \in K \setminus \{k\}} P\left(\left(X(\tau_1), Y(\tau_1^-)\right) = (i-1,m) | Z(0) = (i,h)\right) \frac{\nu(k,h)}{-\nu(k,k) + (\lambda + \mu^{(i)}) 1_{[k \in K_W]}} \\
& + \delta_{km} \frac{\mu^{(i)}}{-\nu(k,k) + (\lambda + \mu^{(i)}) 1_{[k \in K_W]}} \\
= \; & \frac{1}{-\nu(k,k) + (\lambda + \mu^{(i)}) 1_{[k \in K_W]}} \left(\sum_{h \in K \setminus \{k\}} \nu(k,h) U_{hk}^{(i,0)} + \delta_{km} \mu^{(i)} 1_{[k \in K_W]} \right)
\end{aligned}
$$

We write the equation above in a matrix form

$$
\begin{aligned}
U^{(i,0)} &= \left(-diag(\Upsilon) + (\lambda + \mu^{(i)}) I_W\right)^{-1} \cdot \left((\Upsilon - diag(\Upsilon)) U^{(i,0)} + \mu^{(i)} I_W\right) \\
\Longleftrightarrow (-diag(\Upsilon) + (\lambda + \mu^{(i)}) I_W) U^{(i,0)} &= ((\Upsilon - diag(\Upsilon)) U^{(i,0)} + \mu^{(i)} I_W) \\
\Longleftrightarrow ((\lambda + \mu^{(i)}) I_W - \Upsilon) U^{(i,0)} &= \mu^{(i)} I_W \\
\Longrightarrow U^{(i,0)} &= ((\lambda + \mu^{(i)}) I_W - \Upsilon)^{-1} \mu^{(i)} I_W
\end{aligned}
$$

Next we calculate for $n \geq 0$ and $1 \leq i$ the elements of the matrix $U_{km}^{(i,n+1)}$

$$
\begin{aligned}
& U_{km}^{(i,n+1)} \\
= \quad & P\left(\left(X(\tau_1), Y(\tau_1^-)\right) = (n+1+i-1, m) | Z(0) = (i,k)\right) \\
= \quad & \sum_{j=0}^{n+1} \sum_{h\in K} 1_{[(j,h)\neq(i,k)]} P\left(\left(X(\tau_1), Y(\tau_1^-)\right) = (n+i, m) \cap Z(\zeta_1) = (j,h) | Z(0) = (i,k)\right) \\
\overset{SF}{=} \quad & \sum_{h\in K\setminus\{h\}} P\left(\left(X(\tau_1), Y(\tau_1^-)\right) = (n+i, m) \cap Z(\zeta_1) = (i,h) | Z(0) = (i,k)\right) \\
& + P\left(\left(X(\tau_1), Y(\tau_1^-)\right) = (n+i, m) \cap Z(\zeta_1) = (i+1,k) | Z(0) = (i,k)\right) \\
= \quad & \sum_{h\in K\setminus\{h\}} P\left(\left(X(\tau_1), Y(\tau_1^-)\right) = (n+i, m) | Z(\zeta_1) = (i,h), Z(0) = (i,k)\right) \\
& \cdot P\left(Z(\zeta_1) = (i,h) | Z(0) = (i,k)\right) \\
& + P\left(\left(X(\tau_1), Y(\tau_1^-)\right) = (n+i, m) | Z(\zeta_1) = (i+1,k), Z(0) = (i,k)\right) \\
& \cdot P\left(\left(X(\tau_1), Y(\tau_1^-)\right) = (i+1,k) | Z(0) = (i,k)\right) \\
\overset{SM}{=} \quad & \sum_{h\in K\setminus\{h\}} P\left(\left(X(\tau_1), Y(\tau_1^-)\right) = (n+i, m) | Z(0) = (i,h)\right) P((Z(\zeta_1) = (i,h) | Z(0) = (i,k))) \\
& + P\left(\left(X(\tau_1), Y(\tau_1^-)\right) = (n+i, m) | Z(\zeta_1) = (i+1,k)\right) P\left(Z(\zeta_1) = (i+1,k) | Z(0) = (i,k)\right) \\
= \quad & \sum_{h\in K\setminus\{h\}} P(\left(X(\tau_1), Y(\tau_1^-)\right) = (n+i, m) | Z(0) = (i,h)) \\
& \cdot \frac{\nu(k,h)}{-\nu(k,k) + (\mu^{(i)} + \lambda)1_{[k\in K_W]}} \\
& + P(\left(X(\tau_1), Y(\tau_1^-)\right) = (n+i+1-1, m) | Z(0) = (i+1,k)) \\
& \cdot 1_{[k\in K_W]} \frac{\lambda}{-\nu(k,k) + (\mu^{(i)} + \lambda)1_{[k\in K_W]}}
\end{aligned}
$$

The last equation can be written in matrix form as

$$
\begin{aligned}
U^{(i,n+1)} = \left(-diag(\Upsilon) + (\lambda + \mu^{(i)})I_W\right)^{-1} \quad & \cdot \quad \left((\Upsilon - diag(\Upsilon) U^{(i,n+1)} + \lambda I_W U^{(i+1,n)}\right) \\
\Longleftrightarrow \left(-diag(\Upsilon) + (\lambda + \mu^{(i)})I_W\right) U^{(i,n+1)} \quad & = \quad \left(\Upsilon - diag(\Upsilon)\right) U^{(i,n+1)} + \lambda I_W U^{(i+1,n)} \\
\Longleftrightarrow ((\lambda + \mu^{(i)})I_W - \Upsilon) U^{(i,n+1)} \quad & = \quad \lambda I_W U^{(i+1,n)} \\
\Longrightarrow U^{(i,n+1)} \quad & = \quad ((\lambda + \mu^{(i)})I_W - \Upsilon)^{-1} \lambda I_W U^{(i+1,n)}
\end{aligned}
$$

Iterating the last equation n-times and then applying (2.28) leads with $I_W^2 = I_W$ to

$$
\begin{aligned}
U^{(i,n+1)} \quad & = \quad \prod_{j=i}^{n+1+i} \left[\lambda \left((\lambda + \mu^{(j)})I_W - \Upsilon\right)^{-1} I_W\right] \frac{\mu^{(n+1+i)}}{\lambda} \\
\Longrightarrow U^{(i,n+1)} \quad & = \quad U^{(i,n)} \frac{\lambda}{\mu^{(n+i)}} \mu^{(n+1+i)} \left((\lambda + \mu^{(n+1+i)})I_W - \Upsilon\right)^{-1} I_W \qquad (2.30)
\end{aligned}
$$

Finally we verify that the recursion (2.30) is compatible with (2.28):

$$\begin{aligned} U^{(i,1)} &= \underbrace{\lambda\left((\lambda+\mu^{(i)})I_W - \Upsilon\right)^{-1}}_{U^{(i,0)}\frac{\lambda}{\mu^{(i)}}} I_W \lambda \left((\lambda+\mu^{(i+1)})I_W - \Upsilon\right)^{-1} I_W \frac{\mu^{(1+i)}}{\lambda} \\ &= U^{(i,0)}\frac{\lambda}{\mu^{(i)}}\mu^{(1+i)}\left((\lambda+\mu^{(1+i)})I_W - \Upsilon\right)^{-1} I_W \end{aligned}$$

□

We are now ready to evaluate the steady state equations (2.22) of $\hat{Z}$. Because we have a Poisson-λ arrival stream, the marginal steady state (2.8) of the continuous time queue length process X is

$$\xi = (\xi(n) := C^{-1}\prod_{i=1}^{n}\frac{\lambda}{\mu^{(i)}} : n \in \mathbb{N}_0)$$

Recall (2.22)

$$\hat{\pi}^{(0)}B^{(n)} + \sum_{i=1}^{n+1}\hat{\pi}^{(i)}A^{(i,n-i+1)} = \hat{\pi}^{(n)}, \qquad n \in \mathbb{N}_0\,,$$

and the decomposition from Lemma 10:

$$A^{(i,n)} = U^{(i,n)}R, \quad \text{and} \quad B^{(n)} = WU^{(1,n)}R\,.$$

The *conjectured product form steady state* will eventually be realized as

$$\hat{\pi}(n,k) = \xi(n)\cdot\hat{\theta}(k), \text{ for } (n,k)\in E, \text{ and } \hat{\pi}^{n} = \xi(n)\cdot\hat{\theta}, \text{ for } n\in\mathbb{N}_0,$$

with $\hat{\theta}(k) = 0$ for some $k \in K$.

The *idea of the proof* is: The steady state equation is transformed into

$$\xi(n)\cdot\hat{\theta} = \xi(0)\cdot\hat{\theta}\cdot W\cdot U^{(1,n)}\cdot R + \sum_{i=1}^{n+1}\xi(n)\cdot\hat{\theta}\cdot U^{(i,n-i+1)}\cdot R, \quad n\in\mathbb{N}_0. \tag{2.31}$$

We insert $\xi(n)$, cancel C^{-1}, and obtain the "*environment equations*"

$$\hat{\theta} = \hat{\theta}\cdot W\cdot U^{(1,0)}\cdot R + \left(\frac{\lambda}{\mu^{(1)}}\right)\cdot\hat{\theta}\cdot U^{(i,n-i+1)}\cdot R\,, \tag{2.32}$$

and for $n \geq 1$

$$\left(\prod_{i=1}^{n}\frac{\lambda}{\mu^{(i)}}\right)\cdot\hat{\theta} = \hat{\theta}\cdot W\cdot U^{(1,n)}\cdot R + \sum_{i=1}^{n+1}\left(\prod_{j=1}^{i}\frac{\lambda}{\mu^{(i)}}\right)\cdot\hat{\theta}\cdot U^{(i,n-i+1)}\cdot R,\,, \tag{2.33}$$

which we may consider as a sequence of equations with vector of unknowns $\hat{\theta}$. The obvious problem with this system, namely, having an infinite sequence of equations for the same solution, is resolved by the following lemma.

Lemma 13. *For $n \in \mathbb{N}_0$ denote*

$$M^{(n)} := WU^{(1,n)} + \sum_{i=1}^{n+1} \left(\prod_{j=1}^{i} \frac{\lambda}{\mu^{(i)}} \right) \cdot U^{(i,n-i+1)} .$$

Then it holds

$$M^{(0)} = \lambda(\lambda I_W - \Upsilon)^{-1} I_W \tag{2.34}$$

$$M^{(n)} = \left(\prod_{i=1}^{n} \frac{\lambda}{\mu^{(i)}} \right) \lambda(\lambda I_W - \Upsilon)^{-1} I_W, \quad n \geq 1 , \tag{2.35}$$

and consequently

$$M^{(n)} = \left(\prod_{i=1}^{n} \frac{\lambda}{\mu^{(i)}} \right) M^{(0)}, \qquad n \geq 1 . \tag{2.36}$$

Proof. We show that (2.34) holds and compute directly

$$\begin{aligned}
M^{(0)} &= WU^{(1,0)} + \frac{\lambda}{\mu^{(1)}} U^{(1,0)} = \left(W + \frac{\lambda}{\mu^{(1)}} I \right) U^{(1,0)} \\
&= \left(\lambda(\lambda I_W - \Upsilon)^{-1} I_W + \frac{\lambda}{\mu^{(1)}} I \right) \left(-\Upsilon + (\mu^{(1)} + \lambda) I_W \right)^{-1} \mu^{(1)} I_W \\
&= \frac{\lambda}{\mu^{(1)}} (\lambda I_W - \Upsilon)^{-1} \left(\mu^{(1)} I_W - (\lambda I_W - \Upsilon) \right) (\Upsilon - (\mu^{(1)} + \lambda) I_W)^{-1} \mu^{(1)} I_W \\
&= \lambda(\lambda I_W - \Upsilon)^{-1} I_W
\end{aligned}$$

Assume now, that for $n \geq 0$ (2.35) holds for $M^{(n)}$. Then

$$
\begin{aligned}
& M^{(n+1)} \\
= \;& WU^{(n+1,1)} + \sum_{i=1}^{n+2} \left(\prod_{j=1}^{i} \frac{\lambda}{\mu^{(j)}} \right) U^{(i,n-i+2)} \\
= \;& \underbrace{\left(WU^{(1,n)} + \sum_{i=1}^{n+1} \left(\prod_{j=1}^{i} \frac{\lambda}{\mu^{(j)}} \right)^{n} U^{(i,n-i+1)} \right) \frac{\mu^{(n+2)}}{\mu^{(n+1)}} \lambda}_{= M^{(n)} \frac{\mu^{(n+2)}}{\mu^{(n+1)}} \lambda} \left(\lambda I_W + \mu^{(n+2)} I_W - \Upsilon \right)^{-1} I_W \\
& + \left(\prod_{i=1}^{n+2} \frac{\lambda}{\mu^{(i)}} \right) \mu^{(n+2)} (\lambda I_W + \mu^{(n+2)} I_W - \Upsilon)^{-1} I_W \\
= \;& \left(\frac{\lambda}{\mu^{(n+1)}} \mu^{(n+2)} M^{(n)} + \left(\prod_{i=1}^{n+2} \frac{\lambda}{\mu^{(i)}} \right) \mu^{(n+2)} I \right) \left(\lambda I_W + \mu^{(n+2)} I_W - \Upsilon \right)^{-1} I_W \\
= \;& \left(\prod_{i=1}^{n+1} \frac{\lambda}{\mu^{(i)}} \right) \left(\lambda (\lambda I_W - \Upsilon)^{-1} I_W \mu^{(n+2)} + \lambda I \right) \left(\lambda I_W + \mu^{(n+2)} I_W - \Upsilon \right)^{-1} I_W \\
= \;& \lambda \left(\prod_{i=1}^{n+1} \frac{\lambda}{\mu^{(i)}} \right) (\lambda I_W - \Upsilon)^{-1} \left(I_W \mu^{(n+2)} + (\lambda I_W - \Upsilon) \right) \left(\lambda I_W + \mu^{(n+2)} I_W - \Upsilon \right)^{-1} I_W \\
= \;& \lambda \left(\prod_{i=1}^{n+1} \frac{\lambda}{\mu^{(i)}} \right) (\lambda I_W - \Upsilon)^{-1} \left(\mu^{(n+2)} I_W + (\lambda I_W - \Upsilon) \right) \left(\mu^{(n+2)} I_W + (\lambda I_W - \Upsilon) \right)^{-1} I_W \\
= \;& \lambda \left(\prod_{i=1}^{n+1} \frac{\lambda}{\mu^{(i)}} \right) (\lambda I_W - \Upsilon)^{-1} I_W
\end{aligned}
$$

□

Note, that with the definitions in Lemma 13 the sequence of *environment equations* (2.32) and (2.33) reduces to

$$
\hat{\theta} = \hat{\theta} \cdot M^{(0)} \cdot R \,, \quad \text{and for} \quad n \geq 1 : \quad \left(\prod_{i=1}^{n} \frac{\lambda}{\mu^{(i)}} \right) \cdot \hat{\theta} = \hat{\theta} \cdot M^{(n)} \cdot R \,,
$$

and the result in (2.36) says, that all these equations are compatible, in fact, they are the same. Therefore the next lemma will open the path to our main theorem by providing a common solution to all the environment equations.

Lemma 14. (a) *The matrix $M^{(0)}$ is stochastic, i.e., it holds*

$$
M^{(0)} \mathbf{e} = \mathbf{e} \quad \textit{and} \quad M^{(0)}_{km} \geq 0, \qquad \forall k, m \in K \,. \tag{2.37}
$$

(b) *If the continuous time process Z is ergodic with product form steady state π with*

$$
\pi(n,k) = \xi(n) \theta(k), \quad (n,k) \in E \,, \tag{2.38}
$$

then, with the marginal stationary distribution θ *of* Y *in continuous time,*

$$\hat{\theta} = (\theta I_W \mathbf{e})^{-1} \cdot \theta I_W R \tag{2.39}$$

is a stochastic solution of the equation

$$\hat{\theta} \underbrace{\lambda(\lambda I_W - \Upsilon)^{-1} I_W}_{=M^{(0)}} R = \hat{\theta} \,. \tag{2.40}$$

If $\left(I_W - \frac{1}{\lambda}V\right)^{-1}$ *is injective, the stochastic solution* $\hat{\theta}$ *of the equation* (2.40) *is unique.*

(c) *Let* $L := \{k \in K : \exists \; m \in K_W : R_{m,k} > 0\}$ *the set of states of the environment which can be reached from* K_W *by a one-step jump governed by* R*. Then it holds*

$$\hat{\theta}(k) = 0, \quad \forall k \in K \setminus L \,. \tag{2.41}$$

(d) *If* $\hat{\theta}$ *is a stochastic solution of* (2.40) *then* x *defined as*

$$x \;:=\; \hat{\theta} \left(I_W - \frac{1}{\lambda}\Upsilon\right)^{-1} \tag{2.42}$$

is a solution of the steady state equation (2.13) *of the continuous time process* (X, Y)*. Therefore the uniquely determined stationary distribution of* (2.13) *is*

$$\theta \;=\; \left(\hat{\theta}\left(I_W - \frac{1}{\lambda}\Upsilon\right)^{-1} \mathbf{e}\right)^{-1} \hat{\theta}\left(I_W - \frac{1}{\lambda}\Upsilon\right)^{-1} \tag{2.43}$$

Proof. **(a)** Recall, that the matrix Υ is a generator, so is $\Upsilon \mathbf{e} = 0$ and

$$(\lambda I_W - \Upsilon)\mathbf{e} = \lambda I_W \mathbf{e} \tag{2.44}$$

Applying (2.44) to

$$M^{(0)}\mathbf{e} = (\lambda I_W - \Upsilon)^{-1}\lambda I_W \mathbf{e} = (\lambda I_W - \Upsilon)^{-1}(\lambda I_W - \Upsilon)\mathbf{e}$$

proves the first part of the statement (2.37).

In the Lemma 13 we defined the matrix $M^{(0)}$ as $WU^{(0,1)} + \frac{\lambda}{\mu^{(1)}}U^{(0,1)}$, where the entries of W and $U^{(0,1)}$ are probabilities. Because $\frac{\lambda}{\mu^{(1)}}$ is positive, the matrix $M^{(0)}$ is non negative, and $M^{(0)} \cdot R$ as well.

(b) Due to ergodicity of Z with product form steady state, θ is the unique stochastic solution of (see (2.13) in Corollary 3)

$$\theta \left(\lambda \left(R_W - I_W\right) + \Upsilon\right) = 0 \,. \tag{2.45}$$

To prove the existence of a stochastic solution of (2.40) we rewrite (2.45) as

$$\theta(\lambda(I_W R - I_W) + \Upsilon) = 0 \quad \Longleftrightarrow \quad \lambda\theta I_W R = \theta(\lambda I_W - \Upsilon)$$

Multiplying both sides of the last equation with $(\lambda I_W - \Upsilon)^{-1} I_W R$ leads to

$$\begin{aligned}\lambda\theta I_W R(\lambda I_W - \Upsilon)^{-1} I_W R &= \theta(\lambda I_W - \Upsilon)(\lambda I_W - \Upsilon)^{-1} I_W R \\ \Longrightarrow (\theta I_W R)\lambda(\lambda I_W - \Upsilon)^{-1} I_W R &= (\theta I_W R)\end{aligned}$$

One can see that $\theta I_W R$ solves the steady state equation (2.40) and is therefore after normalization a stationary distribution of $M^{(0)} \cdot R$. The normalization constant is

$$\theta I_W \mathbf{e} = \theta I_W \underbrace{R\mathbf{e}}_{=\mathbf{e}} = \theta R_W \mathbf{e}\,.$$

To prove uniqueness of $\hat{\theta}$ we assume that $\hat{\theta}_1$ and $\hat{\theta}_2$ are different non-zero solutions of the equation (2.40) and define

$$x_1 := \hat{\theta}_1 \left(I_W - \frac{1}{\lambda}\Upsilon \right)^{-1} \tag{2.46}$$

$$x_2 := \hat{\theta}_2 \left(I_W - \frac{1}{\lambda}\Upsilon \right)^{-1} \tag{2.47}$$

Both x_1 and x_2 are solutions of the continuous time steady steate equation (2.13). Due to ergodicity of the process (X, Y) and its product form stationary distribution there exits some constant c such that $x_1 = cx_2$ holds.

$$x_1 - cx_2 = 0 \tag{2.48}$$

$$\Longleftrightarrow (\hat{\theta}_1 - c\hat{\theta}_2) \left(I_W - \frac{1}{\lambda}\Upsilon \right)^{-1} = 0 \tag{2.49}$$

Because $\left(I_W - \frac{1}{\lambda}\Upsilon\right)^{-1}$ is injective it follows $\hat{\theta}_1 - c\hat{\theta}_2 = 0$ and thus the uniqueness of the stochastic solution $\hat{\theta}$ of (2.40).

(c) Denote $\hat{\phi} := \hat{\theta}\lambda(\lambda I_W - \Upsilon)^{-1} I_W$. Because I_W has zero K_B-columns, the matrix $\lambda(\lambda I_W - \Upsilon)^{-1} I_W$ has the same property and therefore $\hat{\phi}(k) = 0$ for all $k \in K_B$. It follows for all $k \in K$

$$\hat{\theta}(k) = \sum_{m \in K} \hat{\phi}(m) R_{mk} = \sum_{m \in K_W} \hat{\phi}(m) R_{mk}\,,$$

which is by definition not zero only if $k \in L$.

(d) We show that x defined in (2.42) is a solution of the continuous time steady state equation (2.13).

We write (2.40) in the following form and assume that $\hat{\theta}$ is any stochastic solution

$$\hat{\theta} \left(I_W - \frac{1}{\lambda} V \right)^{-1} I_W R = \hat{\theta} \tag{2.50}$$

$$\tag{2.51}$$

We multiply at the right-hand side of the equation the identity matrix, and obtain

$$\hat{\theta}\left(I_W - \frac{1}{\lambda}\Upsilon\right)^{-1} I_W R = \hat{\theta}\underbrace{\left(I_W - \frac{1}{\lambda}\Upsilon\right)^{-1}\left(I_W - \frac{1}{\lambda}\Upsilon\right)}_{=I}$$

and rewrite it as

$$xI_W R = x\left(I_W - \frac{1}{\lambda}\Upsilon\right) \tag{2.52}$$

with

$$x := \hat{\theta}\left(I_W - \frac{1}{\lambda}\Upsilon\right)^{-1}$$

The equation (2.52) can be transformed directly into the continuos steady state equation (2.13).

$$\Longleftrightarrow x\left(I_W R - I_W + \frac{1}{\lambda}\Upsilon\right) = 0 \Longleftrightarrow x\left(\lambda(R_W - I_W) + \Upsilon\right) = 0$$

□

Theorem 15. *Consider the ergodic Markov process $Z = (Z(t) : t \geq 0)$ which describes the $M/M/1/\infty$ loss system in a random environment.*

(a) *The Markov chain $\hat{Z} = (\hat{Z}(n) : n \in \mathbb{N}_0)$ embedded at departure instants of Z has a stationary distribution $\hat{\pi}$ of product form*

$$\hat{\pi}(n,k) = \xi(n)\hat{\theta}(k), \quad (n,k) \in E\,. \tag{2.53}$$

Here $\xi = (\xi(n) : n \in \mathbb{N}_0)$ is the probability distribution

$$\xi(n) \quad := \quad C^{-1}\left(\prod_{i=1}^{n} \frac{\lambda}{\mu^{(i)}}\right), \quad n \in \mathbb{N}_0, \tag{2.54}$$

with normalization constant C^{-1} and $\hat{\theta}$ is the stochastic solution (2.39) of the equation

$$\hat{\theta}\lambda(\lambda I_W - \Upsilon)^{-1} I_W R = \hat{\theta}\,, \tag{2.55}$$

which is independent of the values of $\mu^{(n)}$.

(b) *Let $L := \{k \in K : \exists \;\; m \in K_W : R_{m,k} > 0\}$ the set of states of the environment which can be reached from K_W by a one-step jump governed by R. Then the states in $\mathbb{N}_0 \times (K \setminus L)$ are inessential for $\hat{Z}$ and consequently for all $n \in \mathbb{N}_0$*

$$\hat{\pi}(n,k) = 0, \quad \forall k \in (K \setminus L) \tag{2.56}$$

Proof. We show that the product form distribution (2.53) with marginal distributions (2.54) and the solution $\hat{\theta}$ of (2.55) solves the steady state equations (3.12) for $n = 0$.

$$\hat{\pi}^{(0)}B^{(0)} + \hat{\pi}^{(1)}A^{(0,1)} = \hat{\pi}^{(0)} \quad \Longleftrightarrow \quad \hat{\theta}\big(\xi(0)B^{(0)} + \xi(1)A^{(0,1)}\big) = \xi(0)\hat{\theta}$$

With matrices W, $U^{(0,1)}$, R and $M^{(0)}$ this equation can be written as

$$\xi(0)\hat{\theta}WU^{(0,1)}R + \xi(1)\hat{\theta}U^{(0,1)}R = \xi(0)\hat{\theta} \quad \Longleftrightarrow \quad \hat{\theta}M^{(0)}R = \hat{\theta}\,, \tag{2.57}$$

which has a stochastic solution $\hat{\theta}$ according to Lemma 14 **(a)**.

We finally show that $\hat{\pi}^{(n)} = \xi(n)\hat{\theta}$ solves all remaining equations for $n \geq 1$:

$$\begin{aligned} \hat{\pi}^{(0)}B^{(1,n)} + \sum_{i=1}^{n+1} \hat{\pi}^{(n)}A^{(i,n-i+1)} &= \hat{\pi}^{(n)} \\ \Longleftrightarrow \xi(0)WU^{(1,n)}R + \sum_{i=1}^{n+1} \xi(n)\hat{\theta}U^{(i,n-i+1)}R &= \xi(n)\hat{\theta} \\ \Longleftrightarrow \xi(0)\hat{\theta}(M^{(n)})R &= \xi(n)\hat{\theta} \end{aligned}$$

Using the property (2.36) of the matrix $M^{(n)}$ the last equation becomes

$$\xi(0)\hat{\theta}\left(\prod_{i=1}^{n} \frac{\lambda}{\mu^{(i)}}\right) M^{(0)}R = \xi(n)\hat{\theta} \quad \Longleftrightarrow \quad \hat{\theta}M^{(0)}R = \hat{\theta}$$

which is again (2.57). Substituting for $M^{(0)}$ the expression (2.34) finally proves the rest of part **(a)**.

For part **(b)** of the theorem we realize from the dynamics of the system that $\mathbb{N}_0 \times L$ are the only states that can be entered just after a departure instant. So, if $\hat{Z}$ is started in some state $\mathbb{N}_0 \times (K \setminus L)$ these states states will never be visited again by $\hat{Z}$ and are therefore inessential, which is in accordance with (2.41). □

Part **(b)** of Theorem 15 shows that in case of $K \setminus L \neq \emptyset$ $\hat{Z}$ is not irreducible on E, hence not ergodic, although Z is ergodic on E. Furthermore, in general $\hat{Z}$ is even on the reduced state space $\hat{E} := \mathbb{N}_0 \times L$ not ergodic. The reason is, that $\hat{Z}$ may have periodic classes as the following example shows.

Example 16. [14](See also Section 5.1 in [11].) We consider an M/M/1/∞-system with attached inventory, i.e. a single server with infinite waiting room under FCFS regime and an attached inventory under (r, S)-policy, which is set in this example to $r = 0$.

There is a Poisson-λ-arrival stream, $\lambda \geq 0$. Customers request for an amount of service time which is exponentially distributed with mean $\mu > 0$.

The server needs for each customer exactly one item from the inventory. The on-hand inventory decreases by one at the moment of service completion. If the inventory is decreased to the reorder point $r = 0$ after the service of a customer is completed, a replenishment order is instantaneously triggered. The replenishment lead times are i.i.d. exponentially distributed with parameter $\nu > 0$. The replenishment fills the inventory up to maximal inventory size $S > 0$.

During the time the inventory is depleted and the server waits for a replenishment order to arrive, no customers are admitted to join the queue ("lost sales").

All service, interarrival and lead times are assumed to be independent.
$X(t)$ is the number of customers present at the server at time $t \geq 0$, and $Y(t)$ is the on-hand inventory at time $t \geq 0$.

The state space of (X,Y) is $E = \{(n,k) : n \in \mathbb{N}_0, k \in K\}$, with $K = \{S, S-1, \ldots, 1, 0\}$. where $S < \infty$ is the maximal size of the inventory at hand.

The inventory management process under $(0,S)$-policy fits into the definition of the environment process by setting

$$K = \{S, S-1, \ldots, 1, 0\}, \qquad K_B = \{0\},$$

$$R_{0,0} = 1, \quad R_{k,k-1} = 1, \quad 1 \leq k \leq S, \qquad \nu(k,m) = \begin{cases} \nu, & \text{if } \ k = 0, m = S \\ 0, & \text{otherwise for } \ k \neq m\,. \end{cases}$$

The queueing-inventory process $Z = (X,Y)$ in continuous time is ergodic iff $\lambda < \mu$. The steady state distribution $\pi = (\pi(n,k) : (n,k) \in E)$ of (X,Y) has product form

$$\pi(n,k) = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\nu}\right)^n \theta(k),$$

where $\theta = (\theta(k) : k \in K)$ with normalization constant C is

$$\theta(k) \quad = \begin{cases} C^{-1}(\frac{\lambda}{\nu}) & k = 0, \\ C^{-1}(\frac{\lambda+\nu}{\lambda})^{k-1} & k = 1, ..., r, \\ C^{-1}(\frac{\lambda+\nu}{\lambda})^{r} & k = r+1, ..., S. \end{cases} \tag{2.58}$$

For the Markov chain $\hat{Z}$ embedded in Z at departure instants we have $L = \{0, 1, \ldots, S-1\}$ and therefore the states $\mathbb{N}_0 \times \{S\}$ are inessential.

From the dynamics of the system determined by the inventory management follows directly that $\hat{Z}$ is periodic with period S and that $\mathbb{N}_0 \times L$ is an irreducible closed set (the single essential class), which is positive recurrent iff $\lambda < \mu$ holds. $\mathbb{N}_0 \times L$ is partitioned into S subclasses $\mathbb{N}_0 \times \{k\}$ which are periodically visited

$$\ldots \to \mathbb{N}_0 \times \{S-1\} \to \mathbb{N}_0 \times \{S-2\} \to \ldots \to \mathbb{N}_0 \times \{0\} \to \mathbb{N}_0 \times \{S-1\} \ldots$$

The following corollary and examples demonstrate the versatility of the class of models under consideration and consequences for the interplay of θ for the continuous time setting and $\hat{\theta}$ for the embedded Markov chain due to special settings of the environment.

Corollary 17. *Consider an ergodic $M/M/1/\infty$ loss system in a random environment with any λ, $\mu^{(n)}$, Υ, and R as defined in Section 2.*

(a) *If $R = I$, then the conditional distribution $\hat{\theta}$ of θ conditioned on L,*

$$\hat{\theta}(k) = \begin{cases} \frac{\theta(k)}{\theta(L)} & \text{if } \ k \in L, \\ 0 & \text{if } \ k \in K \setminus L, \end{cases} \qquad \text{with} \quad \theta(L) := \sum_{m \in L} \theta(m)\,,$$

solves (2.55),

which shows that the embedded chain in this case reveals only the behaviour of the environment on L*, i.e. we loose information incorporated in the continuous time description of the process.*

(b) *If* $\Upsilon = 0$ *then the set* K_B *of blocking states is empty, and therefore* $I_W = I$ *holds. Furthermore,* R *is irreducible and positive recurrent.*

The marginal steady state distribution θ *of* Y *in continuous time is the stationary distribution of* R*, i.e., the solution of* $\theta R = \theta$.

And finally it holds $\theta = \hat{\theta}$*, i.e.,* θ *solves on* K (2.55)*, which shows that the embedded chain exploits in this case the full information about the possible environment of the system.*

Proof. **(a)** is a direct consequence of (2.39) because of $R = I$, and therefore $L = K_W$.

(b) If $\Upsilon = 0$ and $K_B \neq \emptyset$, then from ergodicity the environment process Y must enter K_B in finite time, but once the system entered a blocking state k it can never leave this because of $\nu(k, m) = 0$ for all $m \in K$. Furthermore, from ergodicity of Z with a similar argument, R must be irreducible and positive recurrent.

(2.13) then reduces to $\theta(\lambda(R - I)) = 0$ which is the steady state equation for R.

We substitute $I_W = I$ and $\Upsilon = 0$ into into the left side of equation (2.55) and obtain

$$\hat{\theta}\lambda(\lambda I_W - \Upsilon)^{-1} I_W R = \hat{\theta} R \,,$$

which reduces equation (2.55) to $\hat{\theta} R = \hat{\theta}$, which from irreducibility and positive recurrence has a unique stochastic solution θ. □

Example 18. We consider an $M/M/1/\infty$ system with arrival rate λ, service rate μ, with $\lambda < \mu$, in a random environment. The following examples will address the interrelations between θ and $\hat{\theta}$.

(a) The first example provides a continuum of different environments, which in continuous time have different marginal stationary distributions θ, but all of them have the same $\hat{\theta}$.

The environment is $K = \{1, 2\}$ with $K_B = \{2\}$, and

$$\Upsilon = \left(\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 0 & 0 \\ 2 & \nu & -\nu \end{array}\right) \qquad R = \left(\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 0 & 1 \\ 2 & 0 & 1 \end{array}\right)$$

According to (2.13) the marginal steady state θ of the environment in continuous time is the solution of the equation $\theta(\lambda(R - I) + \Upsilon) = 0$, which is

$$\theta \left(\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & -\lambda & \lambda \\ 2 & \nu & -\nu \end{array}\right) = 0$$

It follows that $\theta = (\frac{\nu}{\lambda+\nu}, \frac{\lambda}{\lambda+\nu})$, which depends on both, λ and ν.

On the other side we have $L = \{2\}$, and therefore the marginal steady state distribution for the environment in the embedded chain is $\hat{\theta} = (0, 1)$
for all λ and ν.

(b) The second example provides two different environments with the same environment space $K = \{1, 2\}$. The point of interest is that in continuous time both have the same marginal stationary distribution (θ) of the environment, but the embedded chains have different marginal stationary distributions ($\hat{\theta}$) of the environment. For both systems holds $\nu = \lambda$.

(b1) The first system is a special case of (a) with $K = \{1, 2\}$, blocking set $K_{1,B} = \{2\}$, and $\nu = \lambda$, i.e.,

$$\Upsilon_1 = \left(\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 0 & 0 \\ 2 & \lambda & -\lambda \end{array}\right) \qquad R_1 = \left(\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 0 & 1 \\ 2 & 0 & 1 \end{array}\right)$$

Using the results from example **(a)**, we immediately get $\theta_1 = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $\hat{\theta}_1 = (0, 1)$.

(b2) The second system has environment space $K = \{1, 2\}$, blocking set $K_{2,B} = \{1\}$, and $\nu = \lambda$, i.e.,

$$\Upsilon_2 = \left(\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & \lambda & -\lambda \\ 2 & 0 & 0 \end{array}\right), \qquad R_2 = \left(\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 1 & 0 \\ 2 & 1 & 0 \end{array}\right).$$

The steady state equation $\theta_2(\lambda(R_2 - I) + \Upsilon_2) = 0$ for θ of the second system is

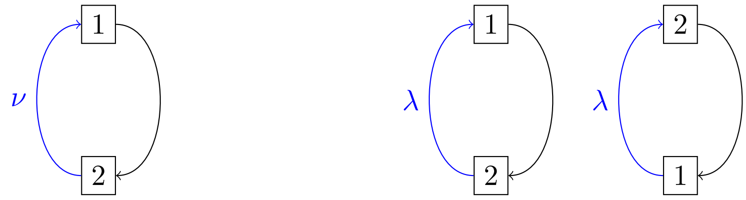
$$\theta_2 \left(\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & -\lambda & \lambda \\ 2 & \lambda & -\lambda \end{array}\right) = 0$$

which is solved by $\theta_2 = \left(\frac{1}{2}, \frac{1}{2}\right)$. Using the same argumentation as in example (a) and the fact that $L_2 = \{1\}$, the marginal steady state distribution of the embedded Markov chain is $\hat{\theta}_2 = (1, 0)$.

Example 19. We consider an $M/M/1/\infty$ system in a random environment with arrival rate λ, service rate μ, with $\lambda < \mu$, in a random environment. The system is ergodic in continuous time and the Markov chain observed at departure instants is ergodic as well. There are no blocking states and therefore no loss of customers occurs, i.e. the stream of admitted customers is Poissonian.

The environment is constructed in a way that the stationary distributions of the of the environment of the continuous time process and of the embedded Markov chains are distinct: $\hat{\theta} \neq \theta$.

We set $K = \{1, 2\}$, $K_W = K$ and $K_B = \emptyset$, and with $\nu_1, \nu_2 > 0$, $\nu_1 \neq \nu_2$ the matrices which govern the environment are $\Upsilon = \begin{pmatrix} -\nu_1 & \nu_1 \\ \nu_2 & -\nu_2 \end{pmatrix}$ and $R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.



(a) Environment states in Example 18 a.

(b) Environment states in Examples 18 b1,b2.

Figure 2.1: The blue lines represent the positive transitions of the Υ matrix and the black lines represent positive transitions of the R matrix.

Then it holds for the generator $\tilde{Q}^n =: \tilde{Q}$, (which is independent of n), see (2.3),

$$\tilde{Q} = (\lambda(R - I) + \Upsilon) = \begin{pmatrix} -\lambda - \nu_1 & \lambda + \nu_1 \\ \lambda + \nu_2 & -\lambda - \nu_2 \end{pmatrix}$$

We calculate θ, which solves $\theta\tilde{Q} = 0$ (see (2.13)) and obtain

$$\theta_1 = \frac{\lambda + \nu_2}{(\lambda + \nu_1) + (\lambda + \nu_2)}, \qquad \theta_2 = \frac{\lambda + \nu_1}{(\lambda + \nu_1) + (\lambda + \nu_2)} \quad \overset{\nu_1 \neq \nu_2}{\Longrightarrow} \theta_1 \neq \theta_2$$

In order to show the uniqueness of $\hat{\theta}$ we calculate $\lambda(\lambda I_W - \Upsilon)^{-1} = \lambda(\lambda I - \Upsilon)^{-1}$

$$\lambda I - \Upsilon = \begin{pmatrix} \lambda + \nu_1 & -\nu_1 \\ -\nu_2 & \lambda + \nu_2 \end{pmatrix}$$

$$\Longrightarrow \lambda(\lambda I - \Upsilon)^{-1} = \frac{\lambda}{(\lambda + \nu_1)(\lambda + \nu_2) - \nu_1\nu_2} \begin{pmatrix} \lambda + \nu_2 & \nu_1 \\ \nu_2 & \lambda + \nu_1 \end{pmatrix}$$

$$\Longrightarrow \lambda(\lambda I - \Upsilon)^{-1} IR \;=\; \frac{\lambda}{(\lambda + \nu_1)(\lambda + \nu_2) - \nu_1\nu_2} \begin{pmatrix} \nu_1 & \lambda + \nu_2 \\ \lambda + \nu_1 & \nu_2 \end{pmatrix}$$

One can see that the stochastic matrix $\lambda(\lambda I_W - \Upsilon)^{-1} I_W R$ is irreducible, therefore there exists a unique stochastic solution of the equation

$$\hat{\theta} = \hat{\theta}\lambda(\lambda I_W - V)^{-1} I_W R\,,$$

which is (with $K_W = K$) by Lemma 14 (b) a multiple of $\theta I_W R = (\theta(2), \theta(1)) \overset{\theta(1) \neq \theta(2)}{\neq} \theta$

3 $M/G/1/\infty$ queueing system in a random environment

Vineetha [15] extended the theory of integrated queueing-inventory models with exponential service times to systems with i.i.d. service times which follow a general distribution.

The lead time is exponential and during stock-out periods lost sales occur. Her approach was classical in that she considered the continuous time Markovian state process at departure instants of customers.

In this section we revisit some of Vineetha's [15] models. We prove some of our results for queues with general environments from the previous sections in the $M/G/1/\infty$ framework, which includes an extension of Vineetha's queueing-inventory systems to queues with state dependent service speeds and with non-exponential service times.

Our main aim is to identify conditions which enforce the systems to stabilize in a way that the queue and the environment decouple in the sense that the stationary queue length and environment behave independently, i.e., a product form equilibrium exists.

It will come out that this is not always possible, but we are able to provide sufficient conditions for the existence of product form equilibria.

Our framework is as in the previous sections: Consider the system at departure instants and utilize Markov chain analysis.

3.1 $M/G/1/\infty$ queueing systems with state dependent service intensities

We first describe a pure queueing model in continuous time which is of $M/G/1/\infty$ type, under FCFS regime, where the single server works with different queue length dependent speeds ("service intensities"), and the customers' service requests are queue length dependent as well.

A review of $M/G/1/\infty$ queueing systems with state dependent arrival and service intensities, which are related to the model described here, and their asymptotic and equilibrium behaviour is provided in the survey of Dshalalow [4].

The arrival stream is Poisson-λ. When a customer enters the single server seeing $n-1 \geq 0$ customers behind him, i.e., the queue length is n, his amount of requested service time is drawn according to a distribution function $B_n : [0,\infty) \to [0,1]$ with $B_n(0) = 0$. The set of all interarrival times and service time requests is an independent collection of variables.

The server works with queue length dependent service speeds $c(n) > 0$, i.e., when at time $t \geq 0$ there are $X(t) = n > 0$ customers in the system (n including the one in service), and if the residual service request of the customer in service at time t is $R(t) = r > 0$, then at time $t+\varepsilon$ his residual service request is

$$R(t+\varepsilon) = r - \varepsilon \cdot c(n), \quad \text{if this is} > 0\,,$$

otherwise at time $t+\varepsilon$ his service expired and he has already departed from the system.

It is a standard observation that the process

$$(X,R) = ((X(t),R(t)) : t \geq 0)$$

is a homogeneous strong Markov process on state space $\mathbb{N}_0 \times \mathbb{R}_0^+$ (with cadlag paths).

With $\tau_0 = 0$ we will denote as in the previous sections by $\tau = (\tau_0, \tau_1, \dots)$ the sequence of departure times of customers. It is a similar standard observation that the process

$$\hat{X} = (\hat{X}(n) := (X(\tau_n), R(\tau_n-)) : n \in \mathbb{N}_0)$$

is a homogeneous Markov chain on state space $\mathbb{N}_0 \times \{0\}$. Because of $R(\tau_n-) = 0\ \forall n$, we prefer to use for this Markov chain on state space $\mathbb{N}_0$ the description

$$\hat{X} = (\hat{X}(n) := X(\tau_n) : n \in \mathbb{N}_0)\,.$$

A little reflection shows that the one-step transition matrix of $\hat{X}$ is a matrix which has the usual skip-fee to the left property, i.e., with $\tilde{p}^{(i,n)}$ defined as

$$\tilde{p}^{(i,n)} := P\left(X(\tau_1) = i + n - 1 | X(0) = i\right)\,, \tag{3.1}$$

it is of the form (empty entries are zero)

$$\tilde{P} \quad := \quad \begin{pmatrix} \tilde{p}^{(1,0)} & \tilde{p}^{(1,1)} & \tilde{p}^{(1,2)} & \tilde{p}^{(1,3)} & \dots \\ \tilde{p}^{(1,0)} & \tilde{p}^{(1,1)} & \tilde{p}^{(1,2)} & \tilde{p}^{(1,3)} & \dots \\ & \tilde{p}^{(2,0)} & \tilde{p}^{(2,1)} & \tilde{p}^{(2,2)} & \dots \\ & & \tilde{p}^{(3,0)} & \tilde{p}^{(3,1)} & \dots \\ & & & & \end{pmatrix}\,, \tag{3.2}$$

which is an upper Hessenberg matrix. A similar one-step transition matrix arises in [4][p.68] where the service requests are state dependent, but no speeds are incorporated.

So for $\tilde{P}$ the row index i indicates the number of customers in system when a service commences (and the service request is drawn according to B_n), and the (varying in row number) column index n indicates the number of customers who arrived during the ongoing service.

Note, that although we have used an intuitive notation for the non zero entries of $\tilde{P}$, the matrix is a fairly general upper Hessenberg matrix: The only restrictions are strict positivity of the $\tilde{p}^{(i,n)}$ and row sum 1.

We will not go into the details of computing $\tilde{P}$, but recall the classical result for state independent service speeds (= 1) in the following subsection.

3.1.1 $M/D/1/\infty$ queueing systems

The classical situation is as follows (See [10, 177+]).

Proposition 20. *For the $M/G/1/\infty$ queuing system with service time distribution $B : [0,\infty) \to [0,1]$, the transition probabilities $\tilde{p}^{(i,n)}$ are independent of i and the transition matrix $\tilde{P}$ has the form*

$$\tilde{P} := \begin{pmatrix} \tilde{p}^{(0)} & \tilde{p}^{(1)} & \tilde{p}^{(2)} & \tilde{p}^{(3)} & \dots \\ \tilde{p}^{(0)} & \tilde{p}^{(1)} & \tilde{p}^{(2)} & \tilde{p}^{(3)} & \dots \\ & \tilde{p}^{(0)} & \tilde{p}^{(1)} & \tilde{p}^{(2)} & \dots \\ & & \tilde{p}^{(0)} & \tilde{p}^{(1)} & \dots \\ & & & & \end{pmatrix} \tag{3.3}$$

with

$$p^{(n)} := \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^n}{n!} dB(t)$$

With $\mu^{-1} < \infty$ we denote the mean service time. Then under $\lambda\mu^{-1} < 1$ the continuous time process and the chain embedded at departure instants are ergodic.

We now recall well known results for standard $M/D/1/\infty$ queues where the service time is deterministic of length $\frac{1}{\mu}$, i.e., the distribution function is $B = \delta_{\frac{1}{\mu}}$ (Dirac measure). We assume $\rho := \lambda/\mu < 1$. Then the queue length process $\hat{X} = (\hat{X}(n)) : n \in \mathbb{N}_0)$ at departure times is an ergodic Markov chain with one-step transition matrix (3.3)

with

$$\tilde{p}^{(n)} := \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^n}{n!} d\delta_{\frac{1}{\mu}}(t) = e^{-\frac{\lambda}{\mu}} \frac{(\frac{\lambda}{\mu})^n}{n!} \tag{3.4}$$

We denote as usual the stationary distribution of of $\hat{X}$ by $\hat{\xi}$, which is the unique stochastic solution of the equation

$$\hat{\xi}\tilde{P} = \hat{\xi}\,. \tag{3.5}$$

We will utilize later on some special values of $\hat{\xi}$ (see [6])

$$\hat{\xi}(0) = (1-\rho) \qquad \hat{\xi}(1) = (1-\rho)\,(e^\rho - 1) \qquad \hat{\xi}(2) = (1-\rho)\,e^\rho(e^\rho - \rho - 1) \tag{3.6}$$

3.2 $M/D/1/\infty$ system with inventory under lost sales

We analyze an $M/D/1/\infty$ queueing system with an attached inventory under (r,S)-policy with lost sales, which is similar to Example 16, but with deterministic service times. We summarize the system's parameters:
Poisson-λ input, deterministic-$\frac{1}{\mu}$ service times, $\rho := \lambda/\mu < 1$. Lead times are exponential-ν. All service, interarrival, and lead times constitute an independent family.
Order policy is (r,S) with $r = 1$ and $S = 2$. When the inventory is depleted no service is provided and new arrivals are rejected (lost sales).

The Markovian state process of the integrated queueing-inventory system relies on the description of the $M/D/1/\infty$ queueing system, given at the beginning of Section 3.1.1.

For the system's description in continuous time we use the supplemented queue length process (X,R), where the R process on $[0,\mu^{-1}]$ denotes the residual service time of the ongoing service as the supplementary variable. We to enlarge this process by adding the inventory size Y.

The joint queueing-inventory process with supplementary variable R will be denoted by $Z = (X,R,Y)$, and lives on state space $\mathbb{N}_0 \times [0,\mu^{-1}] \times \{2,1,0\}$. We consider the system at departure instants, which leads to a one-step transition matrix similar to (3.3).

The dynamics of of the Markov chain $\hat{Z}$ embedded into Z at departure instants will be described in a way that resembles the $M/G/1$ type matrix analytical models.

From the structure of the embedding, we know, that $R(\tau_n-) = 0$ and whenever $X(\tau_n) = 0$ we see $R(\tau_n) = 0$, resp. whenever $X(\tau_n) > 0$ we see $R(\tau_n) = 1/\mu$. We therefore can, without loss of information, delete the R-component of the process, to obtain a Markov chain embedded at departure times

$$\hat{Z} = (\hat{X},\hat{Y}) = ((\hat{X}(n),\hat{Y}(n)) : n \in \mathbb{N}_0)\,, \quad \text{with} \quad \hat{Z}(n) := (\hat{X}(n),\hat{Y}(n)) := (X(\tau_n),Y(\tau_n))\,.$$

The state space of $\hat{Z}$ is $E = \mathbb{N}_0 \times \{2,1,0\}$ where $K = \{2,1,0\}$ is partitioned into $K = K_W + K_B$ with $K_B = \{0\}$ and carries the reversed natural order structure.

We proceed with nomenclature similar to Definition 6 with the obvious modifications, which stem from the observation, that for $i \geq 1$ the probabilities $P(Z(\tau_1) = (i+n-1, m)|Z(0) = (i,k))$ do not depend on i, because service is provided with an intensity which is independent of the queue length. We reuse several of the previous notations but there will be no danger of misinterpretation in this section. Recall, that $(\tau_n : n \in \mathbb{N}_0)$ is the sequence of departure instants

Definition 21. We define the one-step transition matrix $\mathbf{P}$ by

$$\begin{aligned} &\left(\mathbf{P}_{(i,k),(j,m)} : (i,k),(j,m) \in E\right) \\ := \quad &\left(P(Z(\tau_1) = (j,m)|Z(0) = (i,k)) : (i,k),(j,m) \in E\right), \end{aligned}$$

and introducing matrices $A^{(n)} \in \mathbb{R}^{3\times 3}$ and $B^{(n)} \in \mathbb{R}^{3\times 3}$ by

$$B^{(n)}_{km} := P(Z(\tau_1) = (n,m)|Z(0) = (0,k)) \tag{3.7}$$

$$A^{(n)}_{km} := P(Z(\tau_1) = (i+n-1,m)|Z(0) = (i,k)), \qquad 1 \leq i \tag{3.8}$$

for $k, m \in K$, the matrix $\mathbf{P}$ has the form

$$\mathbf{P} = \begin{pmatrix} B^{(0)} & B^{(1)} & B^{(2)} & B^{(3)} & \dots \\ A^{(0)} & A^{(1)} & A^{(2)} & A^{(3)} & \dots \\ 0 & A^{(0)} & A^{(1)} & A^{(2)} & \dots \\ 0 & 0 & A^{(0)} & A^{(1)} & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}. \tag{3.9}$$

We will clarify the structure of the solution of the equation $\hat{\pi}\mathbf{P} = \hat{\pi}$. So, $\hat{\pi}$ is the steady state distribution of the embedded Markov chain $\hat{Z}$. It will become clear that $\hat{Z}$ is in general not irreducible on E.

Similarly to structuring π in (2.9) it will be convenient to group $\hat{\pi}$ as

$$\hat{\pi} = (\hat{\pi}^{(0)}, \hat{\pi}^{(1)}, \hat{\pi}^{(2)}, \dots) \tag{3.10}$$

with

$$\hat{\pi}^{(n)} = (\hat{\pi}(n,2), \hat{\pi}(n,1), \hat{\pi}(n,0)), \qquad n \in \mathbb{N}_0\,. \tag{3.11}$$

An immediate consequence is that the steady state equation can be written as

$$\hat{\pi}^{(0)} B^{(n)} + \sum_{i=1}^{n+1} \hat{\pi}^{(i)} A^{(n-i+1)} = \hat{\pi}^{(n)}, \qquad n \in \mathbb{N}_0\,. \tag{3.12}$$

We determine $A^{(n)}$, $B^{(n)}$ explicitly, distinguishing cases by the initial states $\hat{Z}(0)$.

- $\hat{Z}(0) = (i,0)$, $i \geq 1$: The server waits for replenishment of inventory. The queue length stays at i until the ordered replenishment arrives. Then the inventory is re-stocked to $S = 2$ and the server resumes his work, stochastically identical to a standard

$M/D/1/\infty$-system until the service expires. When the served customer leaves the system, the inventory contains one item.

$$A^{(n)}_{(0,1)} = P(Z(\tau_1) = (i{+}n{-}1,1)|Z(0) = (i,0)) = \int_0^\infty e^{-\lambda t}\frac{(\lambda t)^n}{n!}d\delta_{\frac{1}{\mu}}(t) = e^{-\frac{\nu}{\mu}}\frac{(\frac{\lambda}{\mu})^n}{n!} = \tilde{p}^{(n)}\,.$$

Obviously, from the inventory management regime

$$A^{(n)}_{(0,0)} = A^{(n)}_{(0,2)} = 0\,.$$

• $\hat{Z}(0) = (i,1),\ i \geq 1$: A lead time is ongoing and the server is active serving the first customer in the queue. In this case there are two possible target states for the inventory when the customer currently in service leaves the system.
∘ *Target state* 0: The ongoing service expires before the lead time does. The resulting inventory state after service is finished is 0.

$$\begin{aligned} A^{(n)}_{(1,0)} &= P(Z(\tau_1) = (i+n-1,0)|Z(0) = (i,1)) \\ &= \int_0^\infty e^{-\lambda t}\frac{(\lambda t)^n}{n!}e^{-\nu t}d\delta_{\frac{1}{\mu}}(t) = e^{-\frac{\lambda+\nu}{\mu}}\frac{(\frac{\lambda}{\mu})^n}{n!} = e^{-\frac{\nu}{\mu}}\tilde{p}^{(n)} \end{aligned}$$

∘ *Target state* 1: The ongoing lead expires before the service time does, and the inventory is filled up to $S = 2$ during the ongoing service. The resulting inventory state when service expired is 1. (Additionally, an order is placed, but this does not change the state.)

$$\begin{aligned} A^{(n)}_{(1,1)} &= P(Z(\tau_1) = (i+n-1,1)|Z(0) = (i,1)) \\ &= \int_0^\infty e^{-\lambda t}\frac{(\lambda t)^n}{n!}(1-e^{-\nu t})d\delta_{\frac{1}{\mu}}(t) = (1-e^{-\frac{\nu}{\mu}})e^{-\frac{\lambda}{\mu}}\frac{(\frac{\lambda}{\mu})^n}{n!} = (1-e^{-\frac{\nu}{\mu}})\tilde{p}^{(n)} \end{aligned}$$

Obviously, from the inventory management regime

$$A^{(n)}_{(1,2)} = 0$$

• $\hat{Z}(0) = (i,2),\ i \geq 1$: There are $S = 2$ items on stock, no order is placed and the service is provided just as in a standard $M/D/1/\infty$ system. The resulting inventory state when service expired is 1. (Additionally, an order is placed, but this does not change the state.)

$$A^{(n)}_{(2,1)} = P(Z(\tau_1) = (i+n-1,1)|Z(0) = (i,2)) = \int_0^\infty e^{-\lambda t}\frac{(\lambda t)^n}{n!}d\delta_{\frac{1}{\mu}}(t) = \tilde{p}^{(n)}$$

Obviously, from the inventory management regime

$$A^{(n)}_{(2,0)} = A^{(n)}_{(2,2)} = 0$$

• $\hat{Z}(0) = (0,0)$: The queue is empty, an order is placed. No customers are admitted until replenishment of inventory. When the ongoing lead time expires, inventory is restocked to $S = 2$. Thereafter new customers are admitted, and service starts immediately after the first arrival. When this customer is served, the stock size is 1.

$$B^{(n)}_{(0,1)} = P(Z(\tau_1) = (n,m)|Z(0) = (0,k)) = \int_0^\infty e^{-\lambda t}\frac{(\lambda t)^n}{n!}e^{-\nu t}d\delta_{\frac{1}{\mu}}(t) = \tilde{p}^{(n)}$$

Obviously, from the inventory management regime

$$B^{(n)}_{(0,0)} = B^{(n)}_{(0,2)} = 0$$

• $\hat{Z}(0) = (0,1)$: The queue is empty, there is 1 item on stock, and an order is placed. In this case there are two possible target states for the inventory when the first customer who arrives will be served and leaves the system.
∘ *Target state* 0: The ongoing inter-arrival time expires before the lead time does. The arriving customer's service starts immediately and is finished before the replenishment arrives. The resulting inventory state after service is finished is 0.

$$\begin{aligned} B^{(n)}_{(1,0)} &= P(Z(\tau_1) = (n,0)|Z(0) = (0,1)) \\ &= \frac{\lambda}{\nu+\lambda}\int_0^\infty e^{-\lambda t}\frac{(\lambda t)^n}{n!}e^{-\nu t}d\delta_{\frac{1}{\mu}}(t) = \frac{\lambda}{\nu+\lambda}e^{-\frac{\nu}{\mu}}\tilde{p}^{(n)} \end{aligned}$$

∘ *Target state* 1:
(**1**) The ongoing lead expires before the inter-arrival time does, and the inventory is filled up to $S = 2$ during the ongoing inter-arrival time. Then, until the first departure, the system acts like a standard $M/D/1/\infty$ queue. When the first departure happens, inventory size decreases to 1. (Additionally, an order is placed, but this does not change the state.)
(**2**) The ongoing inter-arrival time expires before the lead time does. The arriving customer's service starts immediately and the replenishment arrives before the service is finished and by the replenishment the stock size increases to 2. The resulting inventory state after service is finished is 1. (Additionally, an order is placed, but this does not change the state.)

$$\begin{aligned} B^{(n)}_{(1,1)} &= P(Z(\tau_1) = (n,1)|Z(0) = (0,1)) \\ &= \frac{\nu}{\nu+\lambda}\int_0^\infty e^{-\lambda t}\frac{(\lambda t)^n}{n!}d\delta_{\frac{1}{\mu}}(t) + \frac{\lambda}{\nu+\lambda}\int_0^\infty e^{-\lambda t}\frac{(\lambda t)^n}{n!}(1-e^{-\nu t})d\delta_{\frac{1}{\mu}}(t) \\ &= \frac{\nu}{\nu+\lambda}\tilde{p}^{(n)} + \frac{\lambda}{\nu+\lambda}(1-e^{-\frac{\nu}{\mu}})\tilde{p}^{(n)} = \left(1 - \frac{\lambda}{\nu+\lambda}e^{-\frac{\nu}{\mu}}\right)\tilde{p}^{(n)} \end{aligned}$$

Obviously, from the inventory management regime

$$B^{(n)}_{(1,2)} = 0$$

• $\hat{Z}(0) = (0,2)$: The queue is empty, there are 2 items on stock, and an inter-arrival time is ongoing. Until the first departure the system develops like a standard $M/D/1/\infty$

queue. After that departure the inventory size is 1. (Additionally, an order is placed, but this does not change the state.)

$$B^{(n)}_{(2,1)} = P(Z(\tau_1) = (n,1)|Z(0) = (0,2)) = \int_0^\infty e^{-\lambda t}\frac{(\lambda t)^n}{n!}d\delta_{\frac{1}{\mu}}(t) = \tilde{p}^{(n)}$$

Obviously, from the inventory management regime

$$B^{(n)}_{(2,0)} = B^{(n)}_{(2,2)} = 0$$

Summarizing the results we have (note, that we ordered the environment in line: $2, 1, 0$)

$$A^{(n)} = \tilde{p}^{(n)} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 - e^{-\frac{\nu}{\mu}} & e^{-\frac{\nu}{\mu}} \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{and} \quad B^{(n)} = \tilde{p}^{(n)} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 - \frac{\lambda}{\nu+\lambda}e^{-\frac{\nu}{\mu}} & \frac{\lambda}{\nu+\lambda}e^{-\frac{\nu}{\mu}} \\ 0 & 1 & 0 \end{pmatrix}.$$

We first prove that the steady state (marginal) queue length distribution of $\hat{X}$ is the steady state distribution $\hat{\xi}$ of the standard $M/D/1/\infty$ queue.

The row sums of $B^{(n)}$ and $A^{(n)}$ are $\tilde{p}^{(n)}$, that is

$$B^{(n)}\mathbf{e} = A^{(n)}\mathbf{e} = \tilde{p}^{(n)}\mathbf{e}, \quad n \in \mathbb{N}_0.$$

Multiplying the steady state equations (3.12) for $\hat{Z}$ with $\mathbf{e}$ leads to

$$\hat{\pi}^{(0)}B^{(n)}\mathbf{e} + \sum_{i=1}^{n+1}\hat{\pi}^{(i)}A^{(i,n-i+1)}\mathbf{e} = \hat{\pi}^{(n)}\mathbf{e} \Longrightarrow \hat{\pi}^{(0)}\mathbf{e}\tilde{p}^{(0)} + \sum_{i=1}^{n+1}\hat{\pi}^{(i)}\mathbf{e}\tilde{p}^{(n+1-i)} = \hat{\pi}^{(n)}\mathbf{e},$$

which is (3.5), which has a unique stochastic solution. Now, $\hat{\pi}^{(i)}\mathbf{e}$ is the steady state (marginal) queue length distribution of $\hat{X}$ and solves (3.5), so we have shown $\hat{\pi}^{(i)}\mathbf{e} = \hat{\xi}(i)$ for all $i \in \mathbb{N}_0$.

Now we are prepared to show that assuming a product form steady state distribution $(\pi(n,k) = \xi(n)\hat{\theta}(k), (n,k) \in E)$ inserted (2.19) leads to a contradiction.

Inserting this product form $\pi(n,k) = \xi(n)\theta(k)$ into the equation for the level $n = 0$ and phase $k = 0$, the steady state equation (2.22) is transformed into

$$\begin{aligned} \hat{\pi}(0,1)B^{(0)}_{(1,0)} + \hat{\pi}(1,1)A^{(0)}_{(1,0)} &= \hat{\pi}(0,0) \\ \Longleftrightarrow \frac{\lambda}{\nu+\lambda}e^{-\frac{\lambda+\nu}{\mu}}\hat{\xi}(0)\hat{\theta}(1) + e^{-\frac{\lambda+\nu}{\mu}}\hat{\xi}(1)\hat{\theta}(1) &= \hat{\xi}(0)\hat{\theta}(0) \\ \Longleftrightarrow e^{-\frac{\lambda+\nu}{\mu}}\left(\frac{\lambda}{\nu+\lambda} + \frac{\hat{\xi}(1)}{\hat{\xi}(0)}\right)\hat{\theta}(1) &= \hat{\theta}(0) \\ \Longleftrightarrow e^{-\frac{\lambda+\nu}{\mu}}\left(\frac{\lambda}{\nu+\lambda} + e^{\rho} - 1\right)\hat{\theta}(1) &= \hat{\theta}(0), \end{aligned} \tag{3.13}$$

and the equation for level $n = 1$ and phase $k = 0$ under this product form assumption is transformed into

$$\begin{aligned} \hat{\pi}(0,1)B^{(1)}_{(1,0)} + \hat{\pi}(1,1)A^{(1)}_{(1,0)} + \hat{\pi}(2,1)A^{(0)}_{(1,0)} &= \hat{\pi}(1,0) \qquad (3.14)\\ \hat{\xi}(0)\hat{\theta}(1)B^{(1)}_{(1,0)} + \hat{\xi}(1)\hat{\theta}(1)A^{(1)}_{(1,0)} + \hat{\xi}(2)\hat{\theta}(1)A^{(0)}_{(1,0)} &= \hat{\xi}(1)\hat{\theta}(0)\\ \Longleftrightarrow \left(\frac{\lambda}{\nu+\lambda}e^{-\frac{\lambda+\nu}{\mu}}\frac{\lambda}{\mu}\hat{\xi}(0) + e^{-\frac{\lambda+\nu}{\mu}}\frac{\lambda}{\mu}\hat{\xi}(1) + e^{-\frac{\lambda+\nu}{\mu}}\hat{\xi}(2)\right)\hat{\theta}(1) &= \hat{\xi}(1)\hat{\theta}(0)\\ \Longleftrightarrow e^{-\frac{\lambda+\nu}{\mu}}\left(\frac{\lambda}{\nu+\lambda}\frac{\lambda}{\mu}\frac{\hat{\xi}(0)}{\hat{\xi}(1)} + \frac{\lambda}{\mu} + \frac{\hat{\xi}(2)}{\hat{\xi}(1)}\right)\hat{\theta}(1) &= \hat{\theta}(0)\\ \Longleftrightarrow e^{-\frac{\lambda+\nu}{\mu}}\left(\frac{\lambda}{\nu+\lambda}\frac{\lambda}{\mu}\frac{1}{e^\rho-1} + \frac{\lambda}{\mu} + \frac{e^\rho(e^\rho-\rho-1)}{(e^\rho-1)}\right)\hat{\theta}(1) &= \hat{\theta}(0) \qquad (3.15) \end{aligned}$$

One can see that the expressions (3.13) and (3.15) are in general not equal. For example, with the parameters $\lambda = 1$, $\mu = 2$ and $\nu = 3$ the $\hat{\theta}(0)$ from (3.13) is approximately equal to $0.122 \cdot \hat{\theta}(1)$ and the $\hat{\theta}(0)$ from the expression (3.15) is approximately equal to $0.145 \cdot \hat{\theta}(1)$.

3.3 $M/G/1/\infty$ queueing systems with state dependent service intensities and product form steady state

In the previous section we have shown by a counterexample, that in general the steady state distribution of an $M/G/1/\infty$ system with (r, S) policy and lost sales does not have a product form. Nevertheless, there are cases where loss systems in a random environment have product form steady states. These systems belong to a class of generalized $M/G/1/\infty$ loss systems, which will be discussed in this subsection. We point out, that the results apply to general birth-and-death processes in a random environment as well.

Definition 22. We consider an $M/G/1/\infty$ queueing system in continuous time with state dependent service intensities (speeds) as described at the beginning of Section 3.1 (p.28) and use the notation introduced there.

The supplemented queue length process (X, R) (queue length, residual service request) is not Markov because we additionally assume that this queueing system is coupled with a **finite** environment $K = K_W + K_B$ with $K_W \neq \emptyset$, driven again by a generator Υ and a stochastic jump matrix R, as described at the beginning of Section 2. The state of the environment process will be denoted by Y again.

We prescribe that the interaction of (X, R) with the environment process Y is via the following principles and restrictions:

(**1**) If the environment process is in a *non-blocking* state k, i.e. $k \in K_W$, the queueing system develops in the same way as an $M/G/1/\infty$ queuing system in isolation, governed by $\tilde{P}$ from (3.2), without any change of the environment until the next departure happens. Holding the environment invariant during this period is guaranteed by $\nu(k, m) = 0$ for all $k \in K_W, m \in K$.

(**2**) If at time t a customer departs from the system, the environment state changes according to the stochastic jump matrix R, independent of the history of the system given $Y(t)$.
(**3**) Whenever the environment process is in a *blocking* state $k \in K_B$, it may change its state with rates governed by the matrix Υ, independent of the queue length and the residual service request.

From these assumptions it is immediate, that $Z = (X, R, Y)$ is a continuous time strong Markov process. We introduce sequences of stopping times for the process $Z = (X, R, Y)$ as before: With $\tau_0 = \sigma_0 = \zeta_0 = 0$ we will denote by
$\tau = (\tau_0, \tau_1, \dots)$ the sequence of departure times of customers,
$\sigma = (\sigma_0, \sigma_1, \dots)$ the sequence of arrival times of customers admitted to the system,
$\zeta = (\zeta_0, \zeta_1, \dots)$ the sequence of jump times of the continuous time process Z.

By standard arguments it is seen that the sequence

$$(X(\tau_n), R(\tau_n-), Y(\tau_n)) : n \in \mathbb{N}_0)$$

is a homogeneous Markov chain on state space $\mathbb{N}_0 \times \{0\} \times K$. Because for all $n \in \mathbb{N}_0$ holds $R(\tau_n-) = 0$ we omit the R-component and consider henceforth the homogeneous Markov chain

$$\hat{Z} = ((\hat{X}(n), \hat{Y}(n)) := (X(\tau_n), Y(\tau_n)) : n \in \mathbb{N}_0)$$

on state space $\mathbb{N}_0 \times K$. The following formulae follow directly from the description.
(**1**) $\Longrightarrow$ for $k \in K_W,\ m \in K$

$$P\left((X(\tau_1), Y(\tau_1-)) = (n+i-1, m) | Z(0) = (i,k)\right) = \delta_{km} \tilde{p}^{(i,n)}\,.$$

(**2**) $\Longrightarrow$ for $k \in K_W,\ m \in K$

$$\begin{aligned} &P\left((X(\tau_1), Y(\tau_1)) = (n+i-1, m) | Z(0) = (i,k)\right) \\ &= \sum_{h \in K} P\left((X(\tau_1), Y(\tau_1-)) = (n+i-1, h) | Z(0) = (i,k)\right) \cdot R_{hm}\,. \end{aligned}$$

(**3**) $\Longrightarrow$ for $k \in K_B,\ m \in K$

$$P\left((X(\zeta_1), Y(\zeta_1)) = (j, m) | Z(0) = (i,k)\right) = \delta_{ij} \frac{\nu(k,m)}{-\nu(k,k)}\,.$$

Note, that in the last expression $k \in K_B$ implies that the queueing system is frozen, and therefore in the denominator of the right side a summand $+1_{[k \in K_W]}(\lambda + \mu 1_{[i>0]})$, which one might have expected, does **not** appear.

Although we have imposed constraints on the behaviour of the environment the model still is a very versatile one. The class of models from Definition 22 encompasses (e.g.) many vacation models. These are models describing a server working on primary and secondary customers, a situation which arises in many computer, communication, and production systems and networks. If one is mainly interested in the service process of primary customers, then working on secondary customers means from the viewpoint of the primary customers, that the server is not available or is interrupted. For more details see e.g. the survey of Doshi [3]. In the classification given there [3][p. 221, 222] the above model is a single server queue with *general nonexhaustive service with nonpreemptive vacations* and *general vacation rule*. Our system fits into these classification because whenever a service expires the server decides (governed by R) whether to perform another service or to wait for the next arriving customer (a state in K_W is selected), or to change its activity for secondary customers (a state in K_B is selected). The sojourn time in this status is completely general distributed by construction, in fact these sojourns in general are neither identical distributed nor independent.

The proposed product form property of $\hat{Z}$ originates from the specific structure of the one-step transition matrix $\mathbf{P}$ of $\hat{Z}$. With some stochastic matrix $H \in \mathbb{R}^{K\times K}$, which we present in all details below,

$$\mathbf{P} = \begin{pmatrix} \tilde{p}^{(1,0)}H & \tilde{p}^{(1,1)}H & \tilde{p}^{(1,2)}H & \tilde{p}^{(1,3)}H & \dots \\ \tilde{p}^{(1,0)}H & \tilde{p}^{(1,1)}H & \tilde{p}^{(1,2)}H & \tilde{p}^{(1,3)}H & \dots \\ 0 & \tilde{p}^{(2,2)}H & \tilde{p}^{(2,1)}H & \tilde{p}^{(2,2)}H & \dots \\ 0 & 0 & \tilde{p}^{(3,0)}H & \tilde{p}^{(3,1)}H & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}. \tag{3.16}$$

We will use an evaluation procedure similar to that used for the $M/M/1/\infty$ in a random environment, by decomposing the matrices $B^{(n)} = WU^{(n,0)}R$ and $A^{(i,n)} = U^{(i,n)}R$.

The next lemma guarantees that the expression $\frac{1}{-\nu(k,k)+1_{[k\in K_W]}}$ in Lemma 24 and Lemma 25 is always well defined.

Lemma 23. *For the system defined in Definition 22 it holds*

$$|\nu(k,k)| > 0, \qquad \forall k \in K_B \tag{3.17}$$

Therefore the expression $\frac{1}{-\nu(k,k)+1_{[k\in K_W]}}$ *is well defined for any* $k \in K$

Proof. The proof uses the same idea as that of Lemma 5. Because Z is ergodic there must be a positive rate $\nu(k,m) > 0$ to leave any blocking state $k \in K_B$. The generator property $|\nu(k,k)| = \sum_{h\neq k} \nu(k,h)$ of the matrix Υ proves the inequality (3.17). □

We now define similar to (2.24) in Lemma 10 a matrix W and determine an explicit representation.

Lemma 24. *For the system from Definition 22 we set for* $k, m \in K$

$$W_{km} := P\left(Z(\sigma_1) = (1,m)|Z(0) = (0,k)\right), \tag{3.18}$$

and remark that $W_{km} = 0$ *for all* $m \in K_B$. *Then it holds*

$$W = (I_W - \Upsilon)^{-1} I_W \tag{3.19}$$

Proof. Basically, the matrix W has the same structure as W in Proposition 11, but we will derive a new representation, which is more suitable in the subsequent proofs. Using the same transformation as in Proposition 11 we get by a first entrance argument

$$\begin{aligned} W_{km} &= \sum_{h\in K\setminus\{k\}} \underbrace{P\left(Z(\sigma_1)=(1,m)|Z(0)=(0,h)\right)}_{=W_{hm}} P\left(Z(\zeta_1)=(0,h)|Z(0)=(0,k)\right) \\ &\quad +\delta_{km} P\left(Z(\zeta_1)=(1,m)|Z(0)=(0,k)\right) \end{aligned}$$

The last term simplifies (with $\nu(k,k)=0$ for $k\in K_W$) to

$$\delta_{km} P\left(Z(\zeta_1)=(1,m)|Z(0)=(0,k)\right) = \delta_{km}\frac{\lambda 1_{[k\in K_W]}}{\lambda 1_{[k\in K_W]}-\nu(k,k)} = \delta_{km} 1_{[k\in K_W]}\,.$$

If $k\neq h$ the expression $P\left(Z(\zeta_1)=(0,h)|Z(0)=(0,k)\right)$ is $\frac{\nu(k,h)}{-\nu(k,k)}$ for $k\in K_B$ and 0 for $k\in K_W$. In both cases we will use the expression $\frac{\nu(k,h)}{-\nu(k,k)+\lambda 1_{[k\in K_W]}}$, which is defined for any $k\in K$ (see Lemma 23); it follows:

$$\begin{aligned} W_{km} &= \sum_{h\in K\setminus\{k\}} W_{hm}\frac{\nu(k,h)}{-\nu(k,k)+\lambda 1_{[k\in K_W]}} + \delta_{km}\frac{\lambda}{-\nu(k,k)+\lambda 1_{[k\in K_W]}} 1_{[k\in K_W]} \\ &= \sum_{h\in K\setminus\{k\}} W_{hm}\frac{\nu(k,h)}{-\nu(k,k)+1_{[k\in K_W]}} + \delta_{km}\underbrace{\frac{1}{-\nu(k,k)+1_{[k\in K_W]}}}_{1 \text{ for } k\in K_W} 1_{[k\in K_W]} \end{aligned}$$

This equation reads in matrix form

$$W = \left(-diag(\Upsilon)+I_W\right)^{-1}\left((\Upsilon-diag(\Upsilon))W+I_W\right)$$

and can finally be transformed into the lemma's statement (3.19):

$$\begin{aligned} (-diag(\Upsilon)+I_W)W &= (\Upsilon-diag(\Upsilon))W+I_W \\ \iff (I_W-\Upsilon)W &= I_W \quad\Longrightarrow W=(I_W-\Upsilon)^{-1}I_W \end{aligned}$$

□

We now determine in a similar way the matrices $U^{(i,n)}$, see the definition (2.23) in Lemma 10 for the exponential case, and determine an explicit representation.

Lemma 25. *In the system from Definition 22 we define for $n\geq 0$ and $i\geq 1$*

$$U^{(i,n)}_{km} := P\left(\left(X(\tau_1),Y(\tau_1^-)\right)=(n+i-1,m)|Z(0)=(i,k)\right)\,.$$

Then for the transition probability matrix U it holds

$$U^{(i,n)} = \tilde{p}^{(i,n)}(I_W-\Upsilon)^{-1}I_W$$

Proof. For $U^{(i,n)}$ with any $n \geq 0$ and $i \geq 1$ it holds:

$$\begin{aligned}
U_{km}^{(i,n)} &= P\left(\left(X(\tau_1), Y(\tau_1^-)\right) = (n+i-1, m) | Z(0) = (i,k)\right) \\
&= \sum_{h \in K \setminus \{k\}} P\left(\left(X(\tau_1), Y(\tau_1^-)\right) = (n+i-1, m) \cap Z(\zeta_1) = (i,h) | Z(0) = (i,k)\right) \\
&\quad + \delta_{km} P\left(\left(X(\tau_1), Y(\tau_1^-)\right) = (n+i-1, k) | Z(0) = (i,k)\right) \\
&= \sum_{h \in K \setminus \{k\}} P\left(\left(X(\tau_1), Y(\tau_1^-)\right) = (n+i-1, m) | Z(\zeta_1) = (i,h), Z(0) = (i,k)\right) \\
&\quad \cdot P\left(Z(\zeta_1) = (i,h) | Z(0) = (i,k)\right) \\
&\quad + \delta_{km} P\left(\left(X(\tau_1), Y(\tau_1^-)\right) = (n+i-1, k) | Z(0) = (i,k)\right) \\
&= \sum_{h \in K \setminus \{k\}} \underbrace{P\left(\left(X(\tau_1), Y(\tau_1^-)\right) = (n+i-1, m) | Z(0) = (i,h)\right)}_{=U_{hm}^{(i,n)}} \\
&\quad \cdot P\left(Z(\zeta_1) = (i,h) | Z(0) = (i,k)\right) \\
&\quad + \delta_{km} P\left(\left(X(\tau_1), Y(\tau_1^-)\right) = (n+i-1, k) | Z(0) = (i,k)\right)
\end{aligned}$$

We analyze the expression $P\left(Z(\zeta_1) = (i,h) | Z(0) = (i,k)\right)$: it is $\frac{\nu(k,h)}{-\nu(k,k)}$ for $k \in K_B$ and 0 for $k \in K_W$. As in the proof of the Lemma 24, we use the combined expression $\frac{\nu(k,h)}{\nu(k,k)+\lambda 1_{[k \in K_W]}} = \frac{\nu(k,h)}{\nu(k,k)+1_{[k \in K_W]}}$ which is valid for any $k \in K$. It follows

$$\begin{aligned}
U_{km}^{(i,n)} &= \sum_{h \in K \setminus \{k\}} U_{hm}^{(i,n)} \frac{\nu(k,h)}{-\nu(k,k) + \lambda 1_{[k \in K_W]}} + \frac{\lambda}{-\nu(k,k) + \lambda 1_{[k \in K_W]}} \delta_{km} \tilde{p}^{(i,n)} 1_{[k \in K_W]} \\
&= \sum_{h \in K \setminus \{k\}} U_{hm}^{(i,n)} \frac{\nu(k,h)}{-\nu(k,k) + 1_{[k \in K_W]}} + \underbrace{\frac{1}{-\nu(k,k) + 1_{[k \in K_W]}}}_{=1 \quad \text{for } k \in K_W} \delta_{km} \tilde{p}^{(i,n)} 1_{[k \in K_W]}
\end{aligned}$$

The equation above, written in matrix form, reads

$$\begin{aligned}
U^{(i,n)} &= (-diag(\Upsilon) + I_W)^{-1}((\Upsilon - diag(\Upsilon))U^{(i,n)} + \tilde{p}^{(i,n)} I_W) \\
\iff (-diag(\Upsilon) + I_W)U^{(i,n)} &= ((\Upsilon - diag(\Upsilon))U^{(i,n)} + \tilde{p}^{(i,n)} I_W) \\
\iff (I_W - \Upsilon)U^{(i,n)} &= \tilde{p}^{(i,n)} I_W \iff U^{(i,n)} = \tilde{p}^{(i,n)} (I_W - \Upsilon)^{-1} I_W.
\end{aligned}$$

□

We are now prepared to evaluate the transition matrix of the $M/G/1/\infty$ system in a random environment from Definition 22. It turns out that it has precisely the form (3.16).

Lemma 26. *Consider the continuous time Markov state process of the system described in Definition 22, and the Markov chain $\hat{Z}$, embedded at departure instants of customers.*

The one-step transition matrix $\mathbf{P}$ *of* $\hat{Z}$

$$\mathbf{P} = \begin{pmatrix} B^{(0)} & B^{(1)} & B^{(2)} & B^{(3)} & \dots \\ A^{(0,1)} & A^{(1,1)} & A^{(2,1)} & A^{(3,1)} & \dots \\ 0 & A^{(0,2)} & A^{(1,2)} & A^{(2,2)} & \dots \\ 0 & 0 & A^{(0,3)} & A^{(1,3)} & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix},$$

is build up by the following block matrices:

$$B^{(n)} = A^{(1,n)} = \tilde{p}^{(1,n)}H \quad \textit{and} \quad A^{(i,n)} = \tilde{p}^{(i,n)}H$$

with

$$H := (I_W - \Upsilon)^{-1} I_W R \tag{3.20}$$

Proof. We analyze the block structure of the matrix $(I_W - \Upsilon)^{-1} I_W$:

$$\begin{aligned}
(I_W - \Upsilon) &= \left(\begin{array}{c|cc} & K_W & K_B \\ \hline K_W & I_W & 0 \\ K_B & -\Upsilon|_{K_B\times K_W} & -\Upsilon|_{K_B\times K_B} \end{array}\right) \\
\Longrightarrow (I_W - \Upsilon)^{-1} &= \left(\begin{array}{c|cc} & K_W & K_B \\ \hline K_W & I_W & 0 \\ K_B & (I_W - \Upsilon)^{-1}|_{K_B\times K_W} & (I_W - \Upsilon)^{-1}|_{K_B\times K_B} \end{array}\right) \\
\Longrightarrow (I_W - \Upsilon)^{-1} I_W &= \left(\begin{array}{c|cc} & K_W & K_B \\ \hline K_W & I_W & 0 \\ K_B & (I_W - \Upsilon)^{-1}|_{K_B\times K_W} & 0 \end{array}\right)
\end{aligned}$$

This leads to the useful property

$$(I_W - \Upsilon)^{-1} I_W (I_W - \Upsilon)^{-1} I_W = (I_W - \Upsilon)^{-1} I_W\,. \tag{3.21}$$

In a completely similar way as in Lemma 10 we can show the following representations

$$A^{(i,n)} = U^{(i,n)}R \qquad \text{and} \qquad B^{(n)} = WU^{(1,n)}R.$$

Inserting the results from Lemma 25 and Lemma 24 we obtain directly

$$A^{(i,n)} = U^{(i,n)}R = \tilde{p}^{(i,n)}(I_W - \Upsilon)^{-1} I_W R\,,$$

$$\begin{aligned}
B^{(n)} &= WU^{(1,n)}R = \tilde{p}^{(1,n)}\left((I_W - \Upsilon)^{-1} I_W\right)^2 R \\
&\overset{(3.21)}{=} \tilde{p}^{(1,n)}(I_W - \Upsilon)^{-1} I_W R = A^{(1,n)}\,,
\end{aligned}$$

which is the proposed result. □

The next step is similar to that in case of the purely exponential system.

Lemma 27. *The matrix $H = (I_W - \Upsilon)^{-1} I_W R$ defined in (3.20) is a stochastic matrix and there exists a stochastic solution $\hat{\theta}$ of the steady state equation*

$$\hat{\theta} H = \hat{\theta} \tag{3.22}$$

Proof. The generator property of Υ leads to

$$(I_W - \Upsilon)\mathbf{e} = I_W \mathbf{e} + \underbrace{\Upsilon \mathbf{e}}_{=0} = I_W \mathbf{e} \tag{3.23}$$

and the stochasticity of R yields $R\mathbf{e} = \mathbf{e}$. Inserting this into the definition of M leads to

$$H\mathbf{e} = (I_W - \Upsilon)^{-1} I_W R \mathbf{e} = (I_W - \Upsilon)^{-1} I_W \mathbf{e} \overset{(3.23)}{=} (I_W - \Upsilon)^{-1}(I_W - \Upsilon)\mathbf{e} = \mathbf{e}$$

Since the matrix $\tilde{p}^{(i,n)}(I_W - \Upsilon)^{-1} I_W$ describes transition probabilities, all its entries are non-negative, therefore the matrix H is stochastic.

Finally, finiteness of K guarantees the existence of a stochastic solution of (3.22). □

Theorem 28. *Consider the $M/G/1/\infty$ in a random environment from Definition 22 with state dependent service speeds and state dependent selection of requested service times. The describing Markov process (X, R, Y) in continuous time is assumed to be ergodic. For the Markov chain $(\hat{X}, \hat{Y})$ embedded at departure points of customers denote the (existing) stationary distribution by $\hat{\pi}$.*

Then $\hat{\pi}$ has product form according to

$$\hat{\pi}(n, k) = \hat{\xi}(n)\hat{\theta}(k), \quad (n, k) \in \mathbb{N}_0 \times K.$$

Here $\hat{\xi}$ is the steady state distribution of the Markov chain with one-step transition matrix (3.2) derived for the queue length process at departure points in a system with the same parameter as under consideration but without environment, that is a solution of

$$\hat{\xi}\tilde{P} = \hat{\xi}, \tag{3.24}$$

and $\hat{\theta}$ is a stochastic solution of the equation

$$\hat{\theta} H = \hat{\theta}(I_W - \Upsilon)^{-1} I_W R = \hat{\theta} \tag{3.25}$$

Proof. According to Lemma 26 the transition matrix $\mathbf{P}$ of the system has block form (3.16), which is the tensor product of $\tilde{P}$ from (3.2) and H:

$$\mathbf{P} = \tilde{P} \otimes H.$$

Let $\hat{\xi}$ be the steady state solution of (3.24), i.e., of the pure queuing system without environment.

Let $\hat{\theta}$ be the stochastic solution of the equation $\hat{\theta}H = \hat{\theta}$, which exists according to Lemma 27. Then from tensor calculus of matrices [13, (2.2.1.9) on p. 53] $\hat{\pi}(n, k) = \hat{\xi}(n)\hat{\theta}(k)$ solves the steady state equation

$$\hat{\pi}\mathbf{P} = (\hat{\xi} \otimes \hat{\theta})\tilde{P} \otimes H = (\hat{\xi}\tilde{P}) \otimes (\hat{\theta}H) = \hat{\xi} \otimes \hat{\theta} = \hat{\pi}.$$

□

4 Applications

We apply the results from Sections 2.2, 2.3, and 3 to queueing-inventory systems which are dealt with in literature recently, see the review in [12].

In any of the following applications the queueing system represents a production facility where raw material arrives and for to assemble a final product from a piece of raw material exactly one item from the stock is needed. This item will formally be taken from the stock when the production of the final product is finished.

4.1 Systems with exponential service requests

Proposition 29. *We consider an exponential single server queue with state dependent service rates, environment dependent replenishment rates, and an attached inventory under (r, S) policy (with $0 \le r < S \in \mathbb{N}$), and lost sales when the inventory is depleted.*

Using the definitions of Section 2.2 we set the environment state space $K := \{0, ..., S\}$ with $K_B = \{0\}$, $X(t)$ the queue length at time t, and $Y(t) = k$ indicates that at time t the stock contains exactly k items. The appropriate transitions intensities are

$$\begin{aligned}
q((n,k) \to (n+1,k)) &= \lambda \qquad k > 0\\
q((n,k) \to (n,S)) &= \nu_k \qquad 0 \le k \le r\\
q((n,k) \to (n-1,k-1)) &= \mu^{(n)} \qquad n > 0, 1 \le k \le S\\
q((n,k) \to (l,m)) &= 0, \qquad \text{otherwise}
\end{aligned}$$

The steady state $\hat{\pi}$ of the Markov chain $(\hat{X}, \hat{Y})$ embedded at departure times has product form

$$\hat{\pi}(n,k) = \xi(n)\hat{\theta}(k), \quad (n,k) \in \mathbb{N}_0 \times K, \tag{4.1}$$

with

$$\xi(n) = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n$$

and

$$\hat{\theta}(k) = \begin{cases} C^{-1} \cdot \prod_{i=1}^{k} \left(\frac{\lambda+\nu_i}{\lambda}\right)^i, & 0 \le k \le r\\ C^{-1} \cdot \prod_{i=1}^{r} \left(\frac{\lambda+\nu_i}{\lambda}\right)^i, & r+1 \le k \le S-1\\ 0 & k = S \end{cases} \tag{4.2}$$

with

$$C = \sum_{k=0}^{r-1} \prod_{i=1}^{k} \left(\frac{\lambda + \nu_i}{\lambda}\right)^i + (S - r) \prod_{i=1}^{r} \left(\frac{\lambda + \nu_i}{\lambda}\right)^i$$

Note that even for the constant values $\nu_k = \nu$ the marginal distribution $\hat{\theta}$ (4.2) differs from the marginal stationary distribution $P(Y(t) = k)$ in continuous time in [14, p. 66].

Proof. According to Theorem 15 the marginal distribution $\hat{\theta}$ is a solution of the equation

$$\hat{\theta}\lambda(\lambda I_W - \Upsilon)^{-1} I_W R = \hat{\theta}$$

We calculate the matrix $\lambda(\lambda I_W - V)^{-1} I_W R$ explicitly. The cautious reader will realize, that we write down the matrices here with indices in an order which inverts the usual one which we prescribed for $\preccurlyeq$ on K in the first part of the paper. This will make reading easier in this special case.

$$(\lambda I_W - \Upsilon) =$$

$$\left(\begin{array}{c|ccccccccccc}
 & 0 & 1 & 2 & \dots & r-1 & r & r+1 & \dots & S-1 & S \\
\hline
0 & \nu_0 & 0 & 0 & & 0 & 0 & 0 & & 0 & -\nu_0 \\
1 & 0 & (\nu_1+\lambda) & 0 & & 0 & 0 & 0 & & 0 & -\nu_1 \\
2 & 0 & 0 & (\nu_2+\lambda) & & 0 & 0 & 0 & & 0 & -\nu_2 \\
\vdots & \vdots & \vdots & \ddots & & & & & & & \vdots \\
r & 0 & 0 & 0 & \dots & 0 & (\nu_r+\lambda) & 0 & \dots & 0 & -\nu_r \\
r+1 & 0 & 0 & 0 & & 0 & 0 & \lambda & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
S-1 & 0 & 0 & 0 & \dots & 0 & 0 & & & \lambda & 0 \\
S & 0 & 0 & 0 & \dots & 0 & 0 & 0 & & 0 & \lambda
\end{array}\right)$$

$$\lambda(\lambda I_W - \Upsilon)^{-1} I_W =$$

$$\left(\begin{array}{c|ccccccccccc}
 & 0 & 1 & 2 & \dots & r-1 & r & r+1 & \dots & S-1 & S \\
\hline
0 & 0 & 0 & 0 & & 0 & 0 & 0 & & 0 & 1 \\
1 & 0 & \frac{\lambda}{\nu_1+\lambda} & 0 & & 0 & 0 & 0 & & 0 & \frac{\nu_1}{\nu_1+\lambda} \\
2 & 0 & 0 & \frac{\lambda}{\nu_2+\lambda} & & 0 & 0 & 0 & & 0 & \frac{\nu_2}{\nu_2+\lambda} \\
\vdots & \vdots & \vdots & \ddots & & & & & & & \vdots \\
r & 0 & 0 & 0 & \dots & 0 & \frac{\lambda}{\nu_r+\lambda} & 0 & \dots & 0 & \frac{\nu_r}{\nu_r+\lambda} \\
r+1 & 0 & 0 & 0 & & 0 & 0 & 1 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
S-1 & 0 & 0 & 0 & \dots & 0 & 0 & & & 1 & 0 \\
S & 0 & 0 & 0 & \dots & 0 & 0 & 0 & & 0 & 1
\end{array}\right)$$

$$R =$$

$$\left(\begin{array}{c|cccccc}
 & 0 & 1 & 2 & \dots & S-1 & S \\
\hline
0 & 1 & 0 & 0 & \dots & 0 & 0 \\
1 & 1 & 0 & 0 & \dots & 0 & 0 \\
2 & 0 & 1 & 0 & & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
S-1 & 0 & 0 & 0 & \dots & 0 & 0 \\
S & 0 & 0 & 0 & \dots & 1 & 0
\end{array}\right)$$

$$\lambda(\lambda I_W - \Upsilon)^{-1} I_W R =$$

$$\left(\begin{array}{c|ccccccccccc}
 & 0 & 1 & 2 & \dots & r-1 & r & r+1 & \dots & S-1 & S \\
\hline
0 & 0 & 0 & 0 & & 0 & 0 & 0 & & 1 & 0 \\
1 & \frac{\lambda}{\nu_1+\lambda} & 0 & 0 & & 0 & 0 & 0 & & \frac{\nu_1}{\nu_1+\lambda} & 0 \\
2 & 0 & \frac{\lambda}{\nu_2+\lambda} 0 & 0 & & 0 & 0 & 0 & & \frac{\nu_2}{\nu_2+\lambda} & 0 \\
\vdots & \vdots & \vdots & \ddots & & & & & & \vdots & 0 \\
r & 0 & 0 & 0 & \dots & \frac{\lambda}{\nu_r+\lambda} & 0 & 0 & & \frac{\nu_r}{\nu_r+\lambda} & 0 \\
r+1 & 0 & 0 & 0 & & 0 & 1 & 0 & & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & 0 \\
 & & & & & & & & \ddots & & \\
S-1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & & 0 & 0 \\
S & 0 & 0 & 0 & \dots & 0 & 0 & 0 & & 1 & 0
\end{array}\right)$$

□

Proposition 30. *We consider an exponential single server queue with state dependent service rates, environment dependent replenishment rates, and an attached inventory under (r, Q) policy (with $0 \le r < Q \in \mathbb{N}$), and lost sales when the inventory is depleted.*

Using the definitions of Section 2.2 we set the environment state space $K := \{0, ..., S\}$ with $K_B = \{0\}$, $X(t)$ the queue length at time t, and $Y(t) = k$ indicates that at time t the stock contains exactly k items. The appropriate transition intensities are

$$\begin{aligned} q((n,k) \to (n+1,k)) &= \lambda \qquad k > 0 \\ q((n,k) \to (n,k+Q)) &= \nu_k \qquad 0 \le k \le r \\ q((n,k) \to (n-1,k-1)) &= \mu^{(n)} \qquad n > 0, 1 \le k \le r+Q \\ q((n,k) \to (l,m)) &= 0, \qquad \textit{otherwise} \end{aligned}$$

The steady state $\hat{\pi}$ has product form

$$\hat{\pi}(n,k) = \xi(n)\hat{\theta}(k), \quad (n,k) \in \mathbb{N}_0 \times K, \tag{4.3}$$

with

$$\xi(n) = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n$$

and

$$\hat{\theta}(k) = \begin{cases} C^{-1} \cdot \prod_{i=1}^{k} \left(\frac{\lambda+\nu_i}{\lambda}\right)^i & , \quad 0 \le k \le r \\ C^{-1} \cdot \prod_{i=1}^{r} \left(\frac{\lambda+\nu_i}{\lambda}\right)^i & , \quad r+1 \le k \le Q-1 \\ C^{-1} \prod_{i=1}^{r} \left(\frac{\lambda+\nu_i}{\lambda}\right)^i - \prod_{i=1}^{k-Q} \left(\frac{\lambda+\nu_i}{\lambda}\right)^i & , \quad Q \le k \le r+Q-1 \\ 0 & , \quad k = r+Q \end{cases} \tag{4.4}$$

with normalization constant

$$C = (Q-r)\prod_{i=1}^{k}\left(\frac{\lambda+\nu_i}{\lambda}\right)^i$$

Proof. According to Theorem 15 the marginal distribution $\hat{\theta}$ is a solution of the equation

$$\hat{\theta}\lambda(\lambda I_W - \Upsilon)^{-1} I_W R = \hat{\theta}$$

We calculate the matrix $\lambda(\lambda I_W - V)^{-1} I_W R$ explicitly (the remark from Proposition 29 on indexing the matrices applies here as well).

$$(\lambda I_W - \Upsilon) =$$

	0	1	2	…	$r-1$	r	$r+1$	…	Q	$1+Q$	$2+Q$	…	$r+Q$
0	ν_0	0	0		0	0	0		$-\nu_0$				0
1	0	$(\nu_1+\lambda)$	0		0	0	0		0	$-\nu_1$			0
2	0	0	$(\nu_2+\lambda)$		0	0	0		0		$-\nu_2$		0
$\vdots$	$\vdots$	$\vdots$	$\ddots$									$\ddots$	$\vdots$
r	0	0	0	…	0	$(\nu_r+\lambda)$	0	…	0				$-\nu_r$
$r+1$	0	0	0		0	0	λ	…	0				0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\vdots$				$\vdots$
Q	0	0	0	…	0	0			λ				0
$1+Q$	0	0	0		0	0							
$2+Q$													
$\vdots$													
$r+Q$	0	0	0	…	0	0	0		0				λ

$$\lambda(\lambda I_W - \Upsilon)^{-1} I_W =$$

	0	1	2	…	$r-1$	r	$r+1$	…	Q	$1+Q$	$2+Q$	…	$r+Q$
0	0	0	0		0	0	0		1	0	0		0
1	0	$\frac{\lambda}{\nu_1+\lambda}$	0		0	0	0		0	$\frac{\nu_1}{\nu_1+\lambda}$			
2	0	0	$\frac{\lambda}{\nu_2+\lambda}$		0	0	0		0		$\frac{\nu_2}{\nu_2+\lambda}$		
$\vdots$	$\vdots$	$\vdots$	$\ddots$									$\ddots$	
r	0	0	0	…	0	$\frac{\lambda}{\nu_r+\lambda}$	0	…	0				$\frac{\nu_r}{\nu_r+\lambda}$
$r+1$	0	0	0		0	0	1	…	0				0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\vdots$				$\vdots$
Q	0	0	0	…	0	0	0		1				0
$1+Q$	0	0	0		0	0	0			1			
$2+Q$	0	0	0		0	0	0				1		
	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$					$\ddots$	
S	0	0	0	…	0	0	0		0	0	0		1

$$R =$$

	0	1	2	…	$r-1+Q$	$r+Q$
0	1	0	0		0	0
1	1	0	0		0	0
2	0	1	0		0	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$		$\vdots$	$\vdots$
$r-1+Q$	0	0	0	…	0	0
$r+Q$	0	0	0	…	1	0

$$\lambda(\lambda I_W - \Upsilon)^{-1} I_W R =$$

	0	1	2	…	$r-1$	r	$r+1$	…	$Q-1$	Q	$1+Q$	$2+Q$	…	$r+Q$
0	0	0	0		0	0	0		1	0	0	0		0
1	$\frac{\lambda}{\nu_1+\lambda}$	0	0		0	0	0			$\frac{\nu_1}{\nu_1+\lambda}$				
2	0	$\frac{\lambda}{\nu_2+\lambda}$			0	0	0				$\frac{\nu_2}{\nu_2+\lambda}$			
$\vdots$	$\vdots$	$\vdots$	$\ddots$									$\ddots$		
r	0	0	0	…	$\frac{\lambda}{\nu_r+\lambda}$	0	0						$\frac{\nu_r}{\nu_r+\lambda}$	
$r+1$	0	0	0		0	1								0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$							$\vdots$
$Q-1$	0	0	0	0	0	0		$\ddots$						
Q	0	0	0	…	0	0	0		1					0
$1+Q$	0	0	0		0	0	0			1				
$2+Q$	0	0	0		0	0	0				1			
	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$					$\ddots$		
$r+Q$	0	0	0	…	0	0	0			0	0	0	1	0

□

Note that even for constant values $\nu_k = \nu$ the marginal distribution $\hat{\theta}$ under (r, Q) policy differs from the marginal steady state distribution $P(Y(t) = k)$ in continuous time from [14, p. 66].

4.2 Systems with non-exponential service requests

Proposition 31. *We consider a single server queue of $M/G/1/\infty$-type, with state dependent service speeds, state dependent selection of requested service times, exponential-ν replenishment times, and an attached inventory under $(r = 0, S)$ policy (with $0 < S \in \mathbb{N}$), and lost sales when the inventory is depleted (see Definition 22).*

We have $K = \{S, S-1, \dots, 1, 0\}$ with $K_B = \{0\}$.

The relevant matrices are the stochastic jump matrix R which represents the downward jumps of the inventory and is

$$R = \left(\begin{array}{c|cc} & 0 \dots S-1 & S \\ \hline 0 & (1, 0, \dots, 0) & 0 \\ 1 & \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} & 0 \\ \vdots & & \\ S & & \end{array}\right)$$

and because the environment moves only for itself if there is stockout, the environment generator Υ has only non zero entries $\nu(0, S) = \nu$, $\nu(0, 0) = -\nu$. So with $K_B = \{0\}$ the requirement of Theorem 28 is fulfilled.

$$\Upsilon = \left(\begin{array}{c|cc} & 0 & 1 \dots S \\ \hline 0 & -\nu & (0, \dots, 0, \nu) \\ 1 & & \\ \vdots & & 0 \\ S & & \end{array}\right)$$

From Theorem 28 we conclude that the Markov chain $(\hat{X}, \hat{Y})$, embedded at departure instants of customers has a stationary distribution $\hat{\pi}$ of product form

$$\hat{\pi}(n, k) = \hat{\xi}(n)\hat{\theta}(k), \quad (n, k) \in \mathbb{N}_0 \times K.$$

Here $\hat{\xi}$ is the steady state distribution of the Markov chain with one-step transition matrix (3.2) derived for the queue length process at departure points in a system with the same parameters as under consideration but without environment, i.e, a solution of $\hat{\xi}\tilde{P} = \hat{\xi}$, and $\hat{\theta}$ is for $k \in \{0, 1, \dots, S\}$

$$\hat{\theta}(k) = \frac{1}{S} \quad k \neq S, \qquad \hat{\theta}(S) = 0 \tag{4.5}$$

According to Theorem 28, $\hat{\theta}$ is a stochastic solution of the equation $\hat{\theta}(I_W - \Upsilon)^{-1} I_W R = \hat{\theta}$ We calculate the matrix $H = (I_W - \Upsilon)^{-1} I_W R$ explicitly.

$$(I_W - \Upsilon) = \left(\begin{array}{c|cc} & 0 & 1 \ldots S \\ \hline 0 & \nu & (0,\ldots,0,-\nu) \\ 1 & 0 & \\ \vdots & \vdots & \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \\ S & 0 & \end{array} \right)$$

$$(I_W - \Upsilon)^{-1} = \left(\begin{array}{c|cc} & 0 & 1 \ldots S \\ \hline 0 & \frac{1}{\nu} & (0,\ldots,0,1) \\ 1 & 0 & \\ \vdots & \vdots & \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \\ S & 0 & \end{array} \right)$$

$$(I_W - \Upsilon)^{-1} I_W = \left(\begin{array}{c|cc} & 0 & 1 \ldots S \\ \hline 0 & 0 & (0,0,\ldots,1) \\ 1 & 0 & \\ \vdots & \vdots & \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \\ S & 0 & \end{array} \right)$$

$$H = (I_W - \Upsilon)^{-1} I_W R = \left(\begin{array}{c|cc} & 0 \ldots S-1 & S \\ \hline 0 & (0,0,\ldots,1) & 0 \\ 1 & & 0 \\ \vdots & \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} & \vdots \\ S & & 0 \end{array} \right)$$

$\hat{\theta}$ defined in (4.5) is the unique solution of the (3.25).

Proposition 32. *We consider a single server queue of $M/G/1/\infty$-type, with state dependent service speeds, state dependent selection of requested service times, inventory management policy (r,Q) or (r,S), and zero lead times (see Definition 22, and note that lost sales do not occur because of zero lead time).*

In the case of (r,S) policy the inventory size after the first delivery will stay on between $r+1$ and S, therefore for long term behaviour of the system we take in account only environment states $K = \{r+1, r+2, ..., S\}$. The zero lead time means $\Upsilon = 0$, $K_B = \emptyset$, and the corresponding R matrix has the form

$$R = \left(\begin{array}{c|cc} & r+1 \ldots S-1 & S \\ \hline r+1 & (0,0,\ldots,0) & 1 \\ r+2 & & \\ \vdots & \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} & 0 \\ S & & \end{array} \right)$$

The steady state distribution has a product form

$$\hat{\pi}(n,k) = \hat{\xi}(n)\hat{\theta}(k)\,, \quad (n,k) \in \mathbb{N}_0 \times K\,,$$

with

$$\hat{\theta}(k) = \frac{1}{S-r}, \quad k \in K. \tag{4.6}$$

Proof. According to Theorem 28 $\hat{\theta}$ is a stochastic solution of the equation $\hat{\theta}(I_W - \Upsilon)^{-1}I_W R = \hat{\theta}$ We calculate the matrix $H = (I_W - \Upsilon)^{-1}I_W R$, which in the case of the model equivalent to

$$\hat{\theta}\underbrace{(I_W - V)^{-1}}_{=I}\underbrace{I_W}_{=I} R = \hat{\theta} \tag{4.7}$$

$$\iff \hat{\theta} R = \hat{\theta} \tag{4.8}$$

with a unique stochastic solution (4.6).

For system under (r, Q) policy with zero lead times (4.6) holds as well , the proof is analogous, we just set $S = r + Q$. □

Remark. Similar results for the steady state of queueing-inventory systems with zero lead times (without speeds) were obtained by Vineetha in [15, Theorem 5.2.1] for the case of i.i.d service times.

5 Appendix

5.1 Useful lemmata

In our proofs we require the matrix $(\lambda I_W - \Upsilon)$ to be invertible, the following lemma is the key to this property in case of finite K.

Lemma 33. *Let $M \in \mathbb{R}^{K\times K}$, where the set of indices is partitioned according to $K = K_W + K_B$, $K_W \neq \emptyset$, and $|K| < \infty$, whose diagonal elements have following properties:*

$$|M_{kk}| = \sum_{k\neq m} |M_{km}|, \qquad \forall k \in K_B \tag{5.1}$$

$$|M_{kk}| > \sum_{k\neq m} |M_{km}|, \qquad \forall k \in K_W \tag{5.2}$$

and it holds the flow condition

$$\forall \tilde{K}_B \subset K_B,\ \tilde{K}_B \neq \emptyset : \quad \exists \quad k \in \tilde{K}_B, \quad m \in \tilde{K}_B^c : \quad M_{km} \neq 0. \tag{5.3}$$

Then M is invertible.

Remark. The Lema 33 does not require the matrix to be irreducible. Since we are interested in systems with reducible matrices Υ which appear in inventory models (see Propositions 29 and 30), we have to modify the proof for irreducible matrices which can be found e.g. in [8, Lemma 4.12].

Proof. We prove the lemma by contradiction, and let $x = (x_k : k \in K)$ be a vector with

$$Mx = 0 \text{ with } x \neq 0. \tag{5.4}$$

The property $Mx = 0$ leads for all $k \in K$ to

$$\begin{aligned} -M_{kk}x_k &= \sum_{k\neq m} M_{km}x_m, \\ \Longrightarrow |M_{kk}||x_k| &\leq \sum_{k\neq m} |M_{km}||x_m|, \\ \Longrightarrow |M_{kk}|\frac{|x_k|}{||x||_\infty} &\leq \sum_{k\neq m} |M_{km}| \underbrace{\frac{|x_m|}{||x||_\infty}}_{\leq 1} \leq \sum_{k\neq m} |M_{km}|, \end{aligned} \tag{5.5}$$

We denote by J the set of indices of elements x_k of x with the largest absolute value

$$J := \{k \in K| \; |x_k| = ||x||_\infty\}.$$

Because of $x \neq 0$ and $|K| < \infty$ the set J is non empty.

First we show that

$$\forall k \in K_W : |x_k| < ||x||_\infty \tag{5.6}$$

holds, which implies

$$K_W \subset J^c. \tag{5.7}$$

For $K_B = \emptyset$ the proof is complete because we have

$$K = K_W \subseteq J^c \subsetneqq K,$$

and so we proceed with the proof for $K_B \neq \emptyset$.

From (5.5) and (5.2) it follows for all $k \in K_W$

$$\begin{aligned} |M_{kk}|\frac{|x_k|}{||x||_\infty} &\leq \sum_{k\neq m} |M_{km}| < |M_{kk}|, \\ \Longrightarrow |M_{kk}|\frac{|x_k|}{||x||_\infty} &< |M_{kk}|, \end{aligned} \tag{5.8}$$

The inequality (5.8) is valid if and only if $\frac{|x_k|}{||x||_\infty}$ is strictly less than 1, which implies $|x_k| < ||x||_\infty$ and therefore (5.7).

Next, we analyze the set $J \subset K_B$. For $k \in J$ we examine the kth row of the equation $Mx = 0$.

For all $k \in J$ it follows from (5.5)

$$\begin{aligned} |M_{kk}| &\leq \sum_{k\neq m} |M_{km}| \underbrace{\frac{|x_m|}{||x||_\infty}}_{\leq 1} \leq \sum_{k\neq m} |M_{km}| \leq |M_{kk}|, \\ \Longrightarrow \sum_{k\neq m} |M_{km}| \underbrace{\frac{|x_m|}{||x||_\infty}}_{\leq 1} &= \sum_{k\neq m} |M_{km}| \end{aligned} \tag{5.9}$$

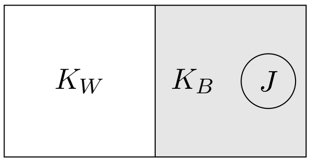


Figure 5.1: Sets in proposition 33. The set K_B is gray.

Because $\frac{|x_m|}{||x||_\infty}$ is strictly less than 1 for all $m \in J^c$, the inequality (5.9) yields

$$M_{km} = 0, \qquad \forall k \in J, m \in J^c$$

Since $K_W \subset J^c$ we have a contradiction to the existence of a path of positive values M_{km} from $k \in J \subset K_B$ to K_W which is guaranteed by (5.3). □

Example 34. This example provides a matrix M which fulfills the requirements of Proposition Lemma 33 and is therefore invertible It is neither irreducible nor strictly diagonal dominant. We set $\lambda, \nu(2,3), \nu(3,2), \nu(4,3), \nu(4,6), \nu(5,4), \nu(6,3) > 0$, all other entries are zero. Figure 5.2 on page 51 shows the resulting flow graph according to the remark below the statement of Corollar 4.

$M =$

$$\left(\begin{array}{c|cccccc}
 & 1 \in K_W & 2 \in K_W & 3 \in K_B & 4 \in K_B & 5 \in K_B & 6 \in K_B \\
\hline
1 \in K_W & \lambda & & & & & \\
2 \in K_W & & (\lambda + \nu(2,3)) & -\nu(2,3) & & & \\
3 \in K_B & & -\nu(3,2) & \nu(3,2) & & & \\
4 \in K_B & & & -\nu(4,3) & (\nu(4,3) + \nu(4,6)) & & -\nu(4,6) \\
5 \in K_B & & & & -\nu(5,4) & \nu(5,4) & \\
6 \in K_B & & & -\nu(6,3) & & & \nu(6,3)
\end{array}\right)$$

Note, that this matrix is of the form $M = \lambda I_W - \Upsilon$ with $\Upsilon =$

$$\left(\begin{array}{c|cccccc}
 & 1 \in K_W & 2 \in K_W & 3 \in K_B & 4 \in K_B & 5 \in K_B & 6 \in K_B \\
\hline
1 \in K_W & 0 & & & & & \\
2 \in K_W & & \nu(2,3) & -\nu(2,3) & & & \\
3 \in K_B & & -\nu(3,2) & \nu(3,2) & & & \\
4 \in K_B & & & -\nu(4,3) & (\nu(4,3) + \nu(4,6)) & & -\nu(4,6) \\
5 \in K_B & & & & -\nu(5,4) & \nu(5,4) & \\
6 \in K_B & & & -\nu(6,3) & & & \nu(6,3)
\end{array}\right),$$

and fits therefore exactly into the realm of our investigations of loss systems in a random environment. **(Hier ist noch ein Bild wieder herzustellen.)**

For infinite K we have the following results.

Proposition 35. *Let $M \in \mathbb{R}^{K \times K}$, be a linear operator on $\ell_\infty(\mathbb{R}^K)$. If for all $k \in K$ holds $|M_{kk}| \geq \sum_{m \in K \setminus \{k\}} |M_{km}| + \varepsilon$ for some $\varepsilon > 0$ and $\sup_{k \in K} |M_{kk}| < \infty$,then M is invertible.*

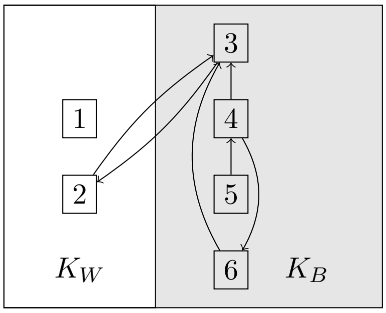


Figure 5.2: Graph from example according to the remark of corollary 4.

Proof. **(1)** Assume $M_{kk} > 0$ for all $k \in K$. Define $\beta := \frac{1}{\sup_{k\in K} M_{kk}}$, then it holds

$$||I - \beta M||_\infty = \sup_{k\in K} \left(|1 - \underbrace{\beta M_{kk}}_{\le 1}| + \beta \underbrace{\sum_{m\in K\setminus\{k\}} |M_{km}|}_{\le M_{kk}-\varepsilon} \right) \tag{5.10}$$

$$\le \sup_{k\in K} (1 - \beta M_{kk} + \beta(M_{kk} - \varepsilon)) < 1 \tag{5.11}$$

Thus M is invertible and it holds

$$M^{-1} = \beta \sum_{n=0}^{\infty} (I - \beta M)^i$$

(2) We define a matrix S with

$$S_{km} = \begin{cases} 1 & k = m,\ M_{kk} > 0 \\ -1 & k = m,\ M_{kk} < 0 \\ 0 & \text{otherwise} \end{cases} \tag{5.12}$$

Then S is a bounded invertible operator with $S^{-1} = S$. According to **(1)** SM is invertible and it holds $M^{-1} = (SSM)^{-1} = (SM)^{-1}S^{-1} = (SM)^{-1}S$ □

Lemma 36. *Let $M \in \mathbb{R}^{K\times K}$, be a linear operator on $\ell_\infty(\mathbb{R}^K)$ where the set of indices is partitioned according to $K = K_W + K_B$, $K_W \neq \emptyset$, and $|K_B| < \infty$, with the following properties:*

Flow condition: *Define a directed graph $(K, \mathcal{E})$ by*

$$(k, m) \in \mathcal{E} :\Leftrightarrow M(k, m) \neq 0\,.$$

Then for any $k \in K_B$ there exists some $m = m(k) \in K_W$ such that there exists a directed path of finite length in $(K, \mathcal{E})$ from k to m.

$$\textit{The sequence} \quad |M_{mm}|, m \in K, \quad \textit{is bounded.} \tag{5.13}$$

$$|M_{kk}| = \sum_{k \neq m} |M_{km}|, \qquad \forall k \in K_B\,. \tag{5.14}$$

$$\sup_{k \in K_W} \sum_{k \neq m} |M_{km}| =: ND(K_W) < \infty\,. \tag{5.15}$$

There exists some $\varepsilon(K_W) > 0$ such that

$$\inf_{m \in K_W} |M_{mm}| = ND(K_W) + \varepsilon(K_W) \tag{5.16}$$

holds.

Then M is injective.

Remark. The sequence $|M_{mm}|, m \in K$, needs not be bounded.

Proof. In the case $K_B = \emptyset$ the matrix M is strictly diagonal dominant and thus invertible according to Proposition 35.

Let $x = (x_k : k \in K) \in \ell_\infty(\mathbb{R}^K)$ be any vector with

$$Mx = 0 \text{ with } x \neq 0 \tag{5.17}$$

(a) To show that

$$\forall k \in K_W : |x_k| < ||x||_\infty \tag{5.18}$$

holds, is a word-by-word analogue of that property in the proof of Proposition 33.

(b) We show: $\{|x_k| : k \in K_W\}$ is uniformly bounded away from $||x||_\infty$ from below.

The property $Mx = 0$ leads for all $k \in K$ to

$$\begin{aligned} -M_{kk}x_k &= \sum_{k \neq m} M_{km}x_m \Longrightarrow \\ |M_{kk}||x_k| &\leq \sum_{k \neq m} |M_{km}||x_m| \leq ||x||_\infty \sum_{k \neq m} |M_{km}| \leq ||x||_\infty ND(K_W)\,, \end{aligned}$$

and therefore

$$\begin{aligned} |x_k| \inf_{m \in K_W} |M_{mm}| &\leq ||x||_\infty ND(K_W) \quad \Longrightarrow \\ |x_k| \leq \frac{ND(K_W)}{\inf_{m \in K_W} |M_{mm}|} ||x||_\infty &= \left(1 - \underbrace{\frac{\varepsilon(K_W)}{ND(K_W) + \varepsilon(K_W)}}_{\in(0,1)} \right) ||x||_\infty \end{aligned}$$

(c) We show: $J := \{k \in K : |x_k| = ||x||_\infty\} \neq \emptyset$ and $K_W \subset J^c$.
The second property follows from **(b)**, while the first property holds, because the set $\{|x_k| : k \in K_W\}$ is uniformly bounded away from $||x||_\infty$ from below and K_B is finite, so there must exist some $k(0) \in K_B$ where $|x_{k(0)}| = ||x||_\infty$ is attained.
(d) To show that

$$M_{km} = 0, \qquad \forall k \in J, m \in J^c$$

holds, is a word-by-word analogue of that property in the proof of Proposition 33. Therefore the flow condition is violated and we have proved the theorem. □

References

[1] F. Baskett, M. Chandy, R. Muntz, and F.G. Palacios. Open, closed and mixed networks of queues with different classes of customers. *Journal of the Association for Computing Machinery*, 22:248–260, 1975.

[2] H. Daduna. Stochastic networks with product form equilibrium. In D.N. Shanbhag and C.R. Rao, editors, *Stochastic Processes: Theory and Methods*, volume 19 of *Handbook of Statistics*, chapter 11, pages 309–364. Elsevier Science, Amsterdam, 2001.

[3] B.T. Doshi. Single server queues with vacations. In H. Takagi, editor, *Stochastic Analysis of Computer and Communication Systems*, pages 217 – 267, Amsterdam, 1990. IFIP, North–Holland.

[4] J. Dshalalow. Queueing systems with state dependent paramenters. In Jewgeni Dshalalow, editor, *Frontiers in Queueing: Models and Applications in Science and Engineering*, chapter 4, pages 61–116. CRC Press, Boca Raton, 1997.

[5] W.J. Gordon and G.F. Newell. Closed queueing networks with exponential servers. *Operations Research*, 15:254–265, 1967.

[6] Donald Gross and Carl M. Harris. *Fundamentals of Queueing Theory*. John Wiley & Sons, Inc., 1974.

[7] J.R. Jackson. Networks of waiting lines. *Operations Research*, 5:518–521, 1957.

[8] Christian Kanzow. *Numerik linearer Gleichungssysteme: Direkte und iterative Verfahren*. Springer, 2005.

[9] F. Kelly. Networks of queues. *Advances in Applied Probability*, 8:416–432, 1976.

[10] L. Kleinrock. *Queueing Theory*, volume I. John Wiley and Sons, New York, 1975.

[11] Ruslan Krenzler and Hans Daduna. Loss systems in a random environment - steady state analysis. `http://preprint.math.uni-hamburg.de/public/papers/prst/prst2012-04.pdf`, November 2012.

[12] A. Krishnamoorthy, B. Lakshmy, and R. Manikandan. A survey on inventory models with positive service time. *OPSEARCH*, 48(2):153–169, 2011.

[13] M.F. Neuts. *Matrix Geometric Solutions in Stochastic Models - An Algorithmic Approach.* John Hopkins University Press, Baltimore, MD, 1981.

[14] M. Schwarz, C. Sauer, H. Daduna, R. Kulik, and R. Szekli. M/M/1 queueing systems with inventory. *Queueing Systems and Their Applications*, 54:55–78, 2006.

[15] K. Vineetha. *Analysis of inventory systems with positive and/negligible service time.* PhD thesis, Department of Statistics, University of Calicut,India, 2008.