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for Markovian network processes
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Correlation formulas for Markovian network processes in a random environment

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Abstract

We consider stochastic Markovian processes, which describe e.g. queueing network processes, in a random environment. The environment influences the network by determining random breakdown of nodes, and the necessity of repair thereafter. Starting from an explicit steady state distribution of product form available in the literature, we notice that this steady state distribution does not provide information about the correlation structure in time and space (over nodes). We study this correlation structure via one step correlations for the queueing-environment process. Although formulas for absolute values of these correlations are rather complicated, it turns out that differences of correlations of related networks are surprisingly simple and have a nice structure. We therefore compare two networks in a random environment having the same invariant distribution, and focus on questions such as: What happens to the time behaviour of the processes when in such a network the environment changes or the

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rules for travelling are perturbed? We show that evaluating these comparison formulas we can compare spectral gaps and asymptotic variances of related processes.

1 Introduction

We consider classical stochastic networks of the Jacksonian type in a random environment. For a general introduction into the problem of Markov processes in a random environment with applications to stochastic networks, see [Zhu94], [Eco05], [BM13]. These stochastic network systems have recently found interest as a general model for queueing networks in connection with other areas of Operations Research, e.g. inventory theory and reliability theory. The interaction of network and environment in these models is in the first system that the service processes of the queues decreases the inventories and the inventory restricts the possibility of serving customers due to limited stock at hand or that in the second system external forces let servers break down which requires repair.

We concentrate in the present paper on the second framework: There are external forces which generate random breakdowns of servers in the network and the subsequent repair is also performed under random influences. We allow the environment to be of a rather general structure, which implies that nodes may break down in isolation or in groups, and that batch repair is possible as well.

For this framework there is a product form extension of Jackson's steady state result at hand, which provides in case of ergodicity the joint steady state distribution of the environment (represented by the set of broken down nodes) and the joint queue length vector in a product form: The environment status and the queue lengths seem to decouple asymptotically and in steady state (which is the essence of Jackson's theorem in case of pure queueing systems).

Clearly, this does not mean that the environment and the queue lengths are independent: The environment is assumed here to be a Markov process for its own, but it strongly influences the service provided by the nodes and even the arrival streams there and, furthermore, the nodes interact as well - the interactions are carried by the traveling customers. These dependencies are not expressed by the one-dimensional (in time) marginal process distribution, which is a multidimensional (in space) product form distribution. In fact, very little is known about the dependence structure of the interacting processes here. Therefore we study in this paper the correlation structure in time of the environment-queue length process via the one-step correlations, which in time as well as in space exhibit complex dependence behaviour.

To be more precise, our main interest is focused on the following scenarios: Compare two networks in a random environment which have the same invariant (product form) distribution, and therefore are in some sense variants of one another. Typical questions are: What happens to the time behaviour of a network when in such a network the rules for traveling (routing chains) are perturbed, or, when the environment changes ?

Our main results are comparison theorems and formulas which provide differences of one step correlations in related, resp. perturbed networks. Although the formulas for absolute values of the one step correlations are rather complicated, it turns out that differences of correlations of related networks are surprisingly simple and have a nice structure. As a consequence, whenever we have obtained quantities connected to one step correlations for some reference network as an anchor (possibly from simulations or numerical evaluations), we can perform easily explicit performance analysis, especially sensitivity analysis by varying, e.g., breakdown and repair probabilities or routing probabilities.

The structure of the paper is as follows. In Section 2 we describe the stochastic networks and the influence of the environment via Markovian breakdown and repair processes, which results in a non-Markovian structure of the queue size processes alone. We end this section with citing the steady state distribution for these networks.

In Section 3 we derive the explicit formulas for the one step correlations in time for the joint environment-network process and show that for the interesting comparison problems these formulas simplify considerably.

In Section 4 we show that our results allow to compare the spectral gaps and asymptotic variances of different systems by evaluating our previous formulas suitably. Comparison results for spectral gaps allow to compare speed of convergence to stationarity for networks in L^2 norm. An Appendix in Section 5 comprises the main technical proofs.

Notation and conventions:
For a set M we denote by $2^M = \mathcal{P}(M)$ the set of all subsets of M.
For sets A, B we write $A \subseteq B$ for A which is a subset of B or equals B, and we write $A \subset B$ for A which is a subset of B but does not equal B.
Throughout, the node set of our graphs (networks) are denoted by $\tilde{J} := \{1, \ldots, J\}$, and the "extended node set" is $\tilde{J}_0 := \{0, 1, \ldots, J\}$, where "0" refers to the external source and sink of the network.
We denote the diagonal matrix with a vector ξ on the diagonal and zero otherwise by $diag(\xi)$.
e_j is the standard j-th base vector in $\mathbb{N}^J$ if $1 \leq j \leq J$ and e_0 is the J-dimensional zero vector.
We will use the following abbreviations:
For $D \subseteq \{1, \ldots, J\}$ and $\underline{n} = (n_j : j \in \tilde{J}) \in \mathbb{N}^J$ we write $\underline{n}_D := (n_j : j \in D) \in \mathbb{N}^{|D|}$ and

$\underline{n}_{\tilde{J}\setminus D} := (n_j : j \in \tilde{J} \setminus D) \in \mathbb{N}^{|\tilde{J}\setminus D|}$, and will, as usual, identify
$\underline{n} = ((n_j : j \in D), (n_j : j \in \tilde{J} \setminus D)) = (n_j : j \in D, n_j : j \in \tilde{J} \setminus D)$.
Similarly we use for $\mathbb{N}^J$-valued random variables with $X_t = X(t) = \underline{n} = (n_j : j \in \tilde{J})$ the self explaining abbreviations $X_D(t) = (n_j : j \in D) \in \mathbb{N}^{|D|}$ and $X(t) = (X_D(t), X_{\tilde{J}\setminus D}(t))$.

We use the following notation. For $(\tilde{\mathbb{E}}, \mathcal{E}, \tilde{\pi})$ and functions $f, g : (\tilde{\mathbb{E}}, \mathcal{E}) \to (\mathbb{R}, \mathbb{B})$ we define the inner product of f, g with respect to $\tilde{\pi}$, whenever the following integral exists:

$$\langle f, g \rangle_{\tilde{\pi}} = \int_{\tilde{\mathbb{E}}} f(x) \cdot g(x) \; \tilde{\pi}(dx).$$

We denote by $L^2 := L^2(\tilde{\mathbb{E}}, \tilde{\pi})$ the space of square integrable functions with respect to $\tilde{\pi}$, and $\|f\|_{\tilde{\pi}} = (\langle f, f \rangle_{\tilde{\pi}})^{(1/2)}$.

All random variables occurring in the sequel are defined on a common underlying probability space $(\Omega, \mathcal{F}, P)$.

2 Stochastic networks in a random environment

2.1 Stochastic networks

A Jackson network [Jac57]) consists of J nodes numbered $1, \ldots, J$, where indistinguishable customers arrive, are served, possibly at several stations, and eventually depart from the network. The nodes are exponential single servers with state dependent service rates and with an infinite waiting room under first–come–first–served (FCFS) regime. If at node j there are $n_j > 0$ customers present, either in service or waiting, then service is provided there at rate $\mu_j(n_j) > 0$; we set $\mu_j(0) := 0$. All customers follow the same rules.

We shall need later on a slight extension of the standard Jackson network models. This is described in terms of an irreducible stochastic routing matrix

$$R = [r_{ij}]_{i,j=0,\ldots,J}, \tag{2.1}$$

where the artificial "node 0" represents the source and the sink of all customers. Strict inequality may hold for $r_{00} \geq 0$, which means that some arriving customers may be rejected.

Customer arrive in a Poisson stream of intensity $\lambda > 0$ which is split (independently) according to the first row $r_0 := (r_{0i} : i = 0, 1, \ldots, J)$ of R. Then at nodes $j = 1, 2, \ldots, J$ we observe independent Poisson-λ_j arrival streams with $\lambda_j = \lambda r_{0j}$, while a portion $\lambda_0 = \lambda r_{00}$ of the arriving customers is rejected (lost).

Routing is Markovian, a customer departing from node i immediately proceeds to node j with probability $r_{ij} \geq 0$, and departs from the network with probability r_{j0}.

Then the **traffic equations for the admitted customers**

$$\eta_j = \lambda_j + \sum_{i=1}^{J} \eta_i r_{ij}, \qquad j = 1, \ldots, J, \tag{2.2}$$

have a unique solution which we denote by $\eta = (\eta_j : j = 1, \ldots, J)$. Note, that (2.2) only counts for the admitted customers because of $\lambda_j = \lambda \cdot r_{0j}, j = 1, \ldots, J$, and $\lambda_1 + \ldots + \lambda_J = \lambda(1 - r_{00})$. For $r_{00} = 0$, R is the so-called extended routing matrix of standard Jackson networks, see [DS08][(3.2)].

Let $\mathbf{X} = (X_t : t \geq 0)$ denote the vector process recording the joint queue lengths in the network at time t. $X_t = (X_1(t), \ldots, X_J(t)) \in \mathbb{N}^J$ reads: at time t there are $X_j(t)$ customers present at node j, either in service or waiting. The assumptions put on the system imply that $\mathbf{X}$ is a strong Markov process on state space $\mathbb{N}^J$ with generator $Q^{\mathbf{X}} = (Q^{\mathbf{X}}(\underline{n}, \underline{m}) : \underline{m}, \underline{n} \in \mathbb{N}^J)$ which is given for $g : \mathbb{N}^J \to \mathbb{R}$ by

$$\begin{aligned}(Q^{\mathbf{X}} g)(\underline{n}) \;=\; & \sum_{j=1}^{J} \lambda_j (g(\underline{n} + e_j) - g(\underline{n})) + \sum_{j=1}^{J} (1 - \delta_{0 n_j}) \mu_j(n_j) r_{j0} (g(\underline{n} - e_j) - g(\underline{n})) \\ & + \sum_{j=1}^{J} (1 - \delta_{0 n_j}) \mu_j(n_j) \sum_{i=1}^{J} r_{ji} (g(\underline{n} - e_j + e_i) - g(\underline{n}))\end{aligned} \tag{2.3}$$

We assume throughout that $\mathbf{X}$ is ergodic and that $\sup\{\mu_j(k) : j \in \{1, \ldots, J\}, k \in \mathbb{N}\} < \infty$ holds, so that $Q^{\mathbf{X}}$ is a bounded operator, i.e., $\inf_{\underline{n} \in \mathbb{N}^J} Q^{\mathbf{X}}(\underline{n}, \underline{n}) > -\infty$.

For an ergodic network process $\mathbf{X}$ Jackson's theorem [Jac57] states that the unique steady–state and limiting distribution π on $\mathbb{N}^J$ is

$$\pi(\underline{n}) = \pi(n_1, \ldots, n_J) = \prod_{j=1}^{J} \left(C(j)^{-1} \prod_{k=1}^{n_j} \frac{\eta_j}{\mu_j(k)} \right), \quad \underline{n} = (n_1, \ldots, n_J) \in \mathbb{N}^J, \tag{2.4}$$

with normalizing constants $C(j)$ for marginal distributions of $\mathbf{X}$.

2.2 Breakdown-repair processes

We are interested in stochastic networks, where the nodes due to external environment influences can breakdown and are repaired periodically. A simple but common situation is that the breakdown-repair process is Markov of its own, and the network reacts on this random environment driven perturbations. To describe these Markovian processes we consider a set of J stations or devices (nodes) numbered $1, \dots, J$. The stations are unreliable, break down randomly and are repaired thereafter. The repair time is random as well.

We assume that the availability status of the system can be described by a homogeneous Markov process

$$\mathbf{Y} = (Y(t) : t \geq 0), \quad Y(t) : (\Omega,\mathcal{F},P) \to (2^{\tilde{J}}, \mathcal{P}(2^{\tilde{J}})).$$

The state $Y(t) = D$ indicates that at time $t \geq 0$ the stations included in $D \subseteq \tilde{J}$ are broken down and under repair, while stations in $\tilde{J} \setminus D \subseteq \tilde{J}$ are functioning ("are up").

The transition rates (breakdown and repair intensities) of the failure process $\mathbf{Y}$ are

Definition 2.1. *Take any pair of functions* $A : 2^{\tilde{J}} \to [0, \infty)$ *and* $B : 2^{\tilde{J}} \to [0, \infty)$, *subject to* $A(\emptyset) = 1$ *and* $B(\emptyset) = 1$ *and for* $D, I, H \subseteq \tilde{J}$ *(we set* $0/0 = 0$ *and* $1/0 = \infty$*)*

$$\frac{A(I)}{A(D)} < \infty \quad \forall\, D \subset I \subseteq \tilde{J} \quad \textit{and} \quad \frac{B(D)}{B(H)} < \infty \quad \forall\, H \subset D \subseteq \tilde{J}\,.$$

With these functions define breakdown and repair rates as follows:

$$q^{\mathbf{Y}}(D, I) = \frac{A(I)}{A(D)}, \quad D \subset I \subseteq \tilde{J}\,,$$

for breakdowns of nodes in non-empty set $I \setminus D$ *if nodes in* D *are already down, and*

$$q^{\mathbf{Y}}(D, H) = \frac{B(D)}{B(H)}, \quad H \subset D \subseteq \tilde{J}, \,,$$

for finishing repair of nodes in non-empty set $D \setminus H$ *if nodes in* D *are under repair.*

For all other pairs $G, H \subseteq \tilde{J}, G \neq H$, *we set* $q^{\mathbf{Y}}(G, H) = 0$, *and for all* $D \subseteq \tilde{J}$ *we set* $q^{\mathbf{Y}}(D, D) = -\sum_{H \subseteq \tilde{J}, H \neq D} q^{\mathbf{Y}}(D, H)$.

Example 2.2. If nodes break down independently of one another with rate $a(i), i \in \tilde{J}$, and are individually repaired with rate $b(i), i \in \tilde{J}$, independent of other nodes, we obtain a network structure which is a node set in a typical dynamical random graph on a prescribed network, where nodes disappear and return later on. We have

$$D \subset I \subseteq \tilde{J} : \;\; q^{\mathbf{Y}}(D, I) = \prod_{i \in I \setminus D} a(i)\,, \;\; \text{and} \;\; H \subset D \subseteq \tilde{J} : \;\; q^{\mathbf{Y}}(D, H) = \prod_{i \in D \setminus H} b(i)\,.$$

The construction of the respective arcs is described in Section 2.3.

The generator $Q^{\mathbf{Y}} = (q^{\mathbf{Y}}(K,L) : K, L \subseteq \tilde{J})$ of $\mathbf{Y}$ is defined for real functions $f : 2^{\tilde{J}} \to \mathbb{R}$, by

$$(Q^{\mathbf{Y}} f)(D) = \sum_{H \subset D} \frac{B(D)}{B(H)}(f(H) - f(D)) + \sum_{I \supset D} \frac{A(I)}{A(D)}(f(I) - f(D)) \tag{2.5}$$

By inspection we see that the probability measure

$$\hat{\pi} := \left(\hat{\pi}(D) := \hat{C}^{-1}\frac{A(D)}{B(D)}, \quad D \in 2^{\tilde{J}}\right) \tag{2.6}$$

with normalization constant $\hat{C}^{-1}$ fulfills

$$\hat{\pi}(D) \cdot q^{\mathbf{Y}}(D,G) = \hat{\pi}(G) \cdot q^{\mathbf{Y}}(G,D), \quad \forall D, G \in 2^{\tilde{J}} \text{ with } D \subseteq G \text{ or } D \supseteq G\,,$$

which implies that the breakdown and repair process $\mathbf{Y}$ is reversible with respect to $\hat{\pi}$.

Example 2.3. In the Example 2.2 we obtain with normalization constant $\hat{C}^{-1}$

$$\hat{\pi} := \left(\hat{\pi}(D) := \hat{C}^{-1}\prod_{i \in D} \frac{a(i)}{b(i)}, \quad D \in 2^{\tilde{J}}\right)\,. \tag{2.7}$$

From these explicit formulas (and similar ones) we can directly perform a parametric analysis of the impact of breakdown and repair rate functions.

2.3 Rerouting

The network process and the breakdown-repair process (availability process) interact and we have to fix rules for the interaction regime. The general rule is:

(1) Whenever a station is broken down and under repair, service is interrupted and the customers present there are frozen, while new customers are not admitted to this station.

(2) Therefore we have to define a new routing mechanism. Examples of how to do this to obtain explicit steady states can be found in [SD03][Sections 5, 6]. We describe an abstract "rerouting scheme", which encompasses the three schemes described there.

Assumption 2.4 (Rerouting schemes in open networks)**.** *Consider a Jackson network (with possible customer rejection ($r_{00} \geq 0$)) with routing matrix* (2.1) *and traffic equations for the admitted customers* (2.2) *where $\lambda_j = \lambda r_{0j}$.*

Let $D \subseteq \tilde{J}$ be the set of nodes of the network which are down. Then the routing probabilities are restricted to nodes in $\tilde{J}_0 \setminus D$ and are defined by some routing matrix

$$R^D = [r^D_{ij}]_{i,j \in \tilde{J}_0 \setminus D}\,. \tag{2.8}$$

The associated traffic equations for the admitted customers similar to (2.2) *are*

$$\eta_j^D = \lambda_j^D + \sum_{i\in\tilde{J}\setminus D} \eta_i^D r_{ij}^D, \qquad j \in \tilde{J}\setminus D, \qquad \text{with } \lambda_j^D := \lambda \cdot r_{0,j}^D, \quad j \in \tilde{J}\setminus D\,, \tag{2.9}$$

and **are assumed** *to be solved by*

$$\eta_j^D = \eta_j, \quad j \in \tilde{J}\setminus D, \tag{2.10}$$

where the η_j *are from the solution of* (2.2).
For convenience we set $R^\emptyset := R$*, and similarly other expressions, if necessary.*

Note that under the new rerouting scheme (with nodes in set D broken down) $\lambda_0^D := \lambda \cdot r_{0,0}^D$ is the new rejection rate. The following lemma will be useful.

Lemma 2.5. *If* $(\eta_j^D, j \in \tilde{J}\setminus D)$ *solves the traffic equations* (2.9) *for the admitted customers, when nodes in* D *are broken down and rerouting is according to Assumption 2.4, then with* $\eta_0^D := \lambda$ *the vector* $\hat{\eta}^D := (\eta_j^D, j \in \tilde{J}_0 \setminus D)$ *solves the equation* $x = x \cdot R^D$.

Proof. The traffic equations (2.9) can be written as

$$\eta_j^D = \lambda \cdot r_{0j}^D + \sum_{i\in\tilde{J}\setminus D} \eta_i^D r_{ij}^D, \quad j \in \tilde{J}\setminus D\,. \tag{2.11}$$

Summing (2.11) over $j \in \tilde{J}\setminus D$ yields

$$\lambda(1 - r_{00}^D) = \sum_{i\in\tilde{J}\setminus D} \eta_i^D r_{i0}^D\,,$$

which is the missing equation of $x = x \cdot R^D$, with the required solution inserted. □

Remark 2.6. Prescribing rerouting by (2.9) is not constructive, but a constructive approach will be not necessary for our main applications. A detailed description of rerouting schemes which fulfill the requirements of Assumption 2.4 is given in [Sau06][Section 2].

When considering rerouting schemes which are used in the literature it may happen that the rerouting chain on certain subsets $\tilde{J}\setminus D$ is not irreducible, for details see [Sau06][Proof of Theorem 1.2.29]. This makes the computations more involved, but leads to the same results as those we shall present below.

The following example describes a way that is common to resolve blocking situations in networks with blocking of stations due to full buffers or blocking due to resource sharing, and is called blocking principle Repetitive Service - Random Destination (rs–rd). For applications in modeling of communication protocols in systems with finite buffers or for ALOHA-type protocols see [Kle76], Section 5.11. Within the abstract framework of reversible processes it occurs in [Lig85], Proposition II.5.10.

Example 2.7 (RS–RD with reversible routing in open exponential networks)**.** When nodes in $D \subseteq \tilde{J}$ of a Jackson network (with possible customer rejection) are down and under repair, rerouting is restricted to $\tilde{J}_0 \setminus D$ by

$$r_{ij}^{D} = \begin{cases} r_{ij}, & i,j \in \tilde{J}_0 \setminus D,\ i \neq j, \\ r_{ii} + \sum_{k \in D} r_{ik}, & i \in \tilde{J}_0 \setminus D,\ i = j\,. \end{cases} \tag{2.12}$$

The external arrival rates at the Jackson network are in this situation: $\lambda_j^D = \lambda r_{0j}^D = \lambda r_{0j} = \lambda_j$ for nodes $j \in \tilde{J} \setminus D$ and $\lambda_j^D = \lambda r_{0j}^D = 0$ for nodes $j \in D$ $(j \neq 0)$.

Note that even if $r_{00} = 0$, external arrivals may be rejected with positive probability because arrivals to nodes under repair are "rerouted" ($\equiv$ rejected): With $\lambda_0 = \lambda \cdot r_{00}$

$$r_{00}^{D} = r_{00} + \sum_{k \in D} r_{0k} = \sum_{k \in D \cup \{0\}} \frac{\lambda_k}{\lambda} \geq 0\,.$$

A standard assumption in the literature for this rerouting scheme to apply is that the original routing Markov chain, resp., its transition matrix is reversible:

$$\eta_j \cdot r_{ji} = \eta_i \cdot r_{ij} \text{ for all } i,j \in \tilde{J}_0 \text{ with } \eta_j \text{ from (2.2) and } \eta_0 := \lambda. \tag{2.13}$$

We shall set this assumption always in force when investigating this protocol.

The next rerouting regime seems to be nearly trivial. Nevertheless, it is often implemented as reaction to detected failures in complex production systems to maintain high production quality, e.g. in automotive industry.

Example 2.8 (Stalling)**.** Whenever a node failure occurs and a node (or more) break down, all arrival processes are shut down and all ongoing services are interrupted. So no customers move in the network until all nodes are repaired. For nodes in D being down we have

$$r_{ij}^{D} = \begin{cases} 0, & i,j \in \tilde{J}_0 \setminus D,\ i \neq j, \\ 1, & i \in \tilde{J}_0 \setminus D,\ i = j\,. \end{cases} \tag{2.14}$$

We assume that the stopped nodes which are up wait in warm standby. They can therefore break down without serving and have to be repaired then also.

The next rerouting regime is a typical reaction in case of random walks in a random network with disappearing nodes. The random walker jumps over the gap, possibly iterated, until a target node of the random walker allows him to settle down. - In Markov chain theory it occurs when taboo sets are investigated.

Example 2.9 (Skipping)**.** When the nodes in D are down, moving customers skip these nodes. If the destination node i of a customer lies in D the customer performs this jump but does not stay there. Instead he selects according to R another node, say k with probability $r(i,k)$. If $k \notin D$ he settles down, otherwise he immediately selects another node according to R, and so on. This results in a new routing matrix R^D with

$$r_{jk}^{\bar{D}} = r_{jk} + \sum_{i \in \bar{D}} r_{ji} r_{ik}^{D} \quad \textit{for } k, j \in \tilde{J}_0 \setminus D \tag{2.15}$$

with

$$r_{ik}^{D} = r_{ik} + \sum_{l \in D} r_{il} r_{lk}^{D} \quad \textit{for } i \in D, k \in \tilde{J}_0 \setminus D. \tag{2.16}$$

The external arrival rates during a breakdown of $\bar{I}$ are $\lambda_k^D = 0$ for $k \in D$ and

$$\lambda_j^D = \lambda_j + \sum_{i \in D} \lambda_i r_{ij}^D \quad \textit{for } j \in \tilde{J} \setminus D\,. \tag{2.17}$$

2.4 Networks with breakdown and repair: Product formula

We shall exploit the product form of the invariant distribution of the Markov process describing an unreliable Jackson network in order to find its correlations. In this section we formally define this process by giving the corresponding generator and prove the product formula. A Markovian state process for this system requires that the state space $\mathbb{N}^J$ of the Jackson network process $\mathbf{X}$ is supplemented by a coordinate $\mathbf{Y}$ which indicates the set of broken down stations. Operating on these states we define a Markov process $\mathbf{Z} = (\mathbf{Y}, \mathbf{X})$ describing the degradable network with the state space $\tilde{\mathbb{E}} = 2^{\tilde{J}} \times \mathbb{N}^J$. Elements of $\tilde{\mathbb{E}}$ are $\mathbf{n} = (D, \underline{n}) = (D, n_1, n_2, \ldots, n_J) \in \tilde{\mathbb{E}}$, where the first coordinate in $\mathbf{n}$ we call the availability coordinate. The interpretation is: The set D is the set of servers in *down status.* At node $i \in D$ there are n_i customers waiting for server being repaired. We denote by $\mathcal{E}$ the set of all subsets of $\tilde{\mathbb{E}}$.

Definition 2.10. *The Markov process* $\mathbf{Z} = (Z(t), t \geq 0)$ *defined by the infinitesimal generator (transition intensity matrix)* $Q^{\mathbf{Z}} = (q^{\mathbf{Z}}(\boldsymbol{n}, \boldsymbol{n}') : \boldsymbol{n}, \boldsymbol{n}' \in \tilde{\mathbb{E}}\ )$ *via*

$$\begin{aligned}
&(Q^{\mathbf{Z}} f)(D, n_1, n_2, \ldots, n_J) = \\
= &\sum_{j \in \tilde{J} \setminus D} \lambda r_{0j}^D (f(D, \underline{n} + e_j) - f(\boldsymbol{n})) + \sum_{j \in \tilde{J} \setminus D} (1 - \delta_{0 n_j}) \mu_j(n_j) r_{j0}^D (f(D, \underline{n} - e_j) - f(\boldsymbol{n})) \\
&+ \sum_{j \in \tilde{J} \setminus D} (1 - \delta_{0 n_j}) \mu_j(n_j) \sum_{i \in \tilde{J} \setminus D} r_{ji}^D (f(D, \underline{n} - e_j + e_i) - f(\boldsymbol{n})) \qquad (2.18) \\
&+ \sum_{H \subset D} \frac{B(D)}{B(H)} (f(H, \underline{n}) - f(\boldsymbol{n})) + \sum_{I \supset D} \frac{A(I)}{A(D)} (f(I, \underline{n}) - f(\boldsymbol{n}))
\end{aligned}$$

is called **unreliable Jackson network** *process.*

Theorem 2.11. (Product form for Jackson networks with breakdown and repair) *[Sau06][Theorem 2.4.1] Under the Assumption 2.4, if* $\mathbf{Z}$ *is ergodic then the steady state is with* π *from* (2.4) *and* $\hat{\pi}$ (2.6) *of the product form:*

$$\begin{aligned}\tilde{\pi}(D,\, n_1, n_2, \dots, n_J) &= \hat{\pi}(D) \cdot \pi(n_1, n_2, \dots, n_J) \\ &= \hat{\pi}(D) \cdot \prod_{j=1}^{J} \left(C_j^{-1} \prod_{i=1}^{n_j} \frac{\eta_j}{\mu_j(i)} \right), \quad (D,\, n_1, \dots, n_J) \in \tilde{\mathbb{E}}\,. \end{aligned} \tag{2.19}$$

$\eta = (\eta_1, \dots, \eta_J)$ *is the solution of the traffic equation* (2.2) *for admitted customers when all nodes are up,* C_j *is the normalization constant for the local queue length process at node* j *which is finite if and only if the unreliable network is ergodic. We denote*

$$C = C(J) = \sum_{(n_1,\dots,n_J)\in\mathbb{N}^J} \left[\prod_{j=1}^{J} \prod_{i=1}^{n_j} \left(\frac{\eta_j}{\mu_j(i)} \right) \right] = \prod_{j=1}^{J} C_j\,.$$

Proof. Because the proof gives some structural insight, we give a shortened version of [Sau06][Theorem 2.4.1] for the sake of completeness. For all $(D;\, n_1, n_2, \dots, n_J) \in \tilde{\mathbb{E}}$ we have to check the global balance equations

$$\begin{aligned} &\tilde{\pi}(D;\, n_1, n_2, \dots, n_J) \Bigg(\sum_{j\in\tilde{J}\setminus D} (1-\delta_{0n_j}) \mu_j(n_j)(1 - r_{jj}^D) + \sum_{j\in\bar{J}\setminus D} \lambda_j^D \\ &\qquad + \sum_{I\supset D} q^{\mathbf{Y}}(D, I) + \sum_{H\subset D} q^{\mathbf{Y}}(D, H) \Bigg) \\ &= \sum_{j\in\tilde{J}\setminus D} (1-\delta_{0n_j}) \tilde{\pi}(D;\, n_1, n_2, \dots, n_j - 1, \dots, n_J) \lambda_j^D \\ &+ \sum_{j\in\bar{J}\setminus D} (1-\delta_{0n_j}) \sum_{\substack{k\in\tilde{J}\setminus D \\ k\neq j}} \tilde{\pi}(D;\, n_1, \dots, n_k + 1, \dots, n_j - 1, \dots, n_J) \mu_k(n_k+1) r_{kj}^D \\ &+ \sum_{j\in\tilde{J}\setminus D} \tilde{\pi}(D;\, n_1, n_2, \dots, n_j + 1, \dots, n_J) \mu_j(n_j+1) r_{j0}^D \\ &+ \sum_{I\supset D} \tilde{\pi}(D;\, n_1, n_2, \dots, n_J) q^{\mathbf{Y}}(D, I) + \sum_{H\subset D} \tilde{\pi}(D;\, n_1, n_2, \dots, n_J) q^{\mathbf{Y}}(D, H) \end{aligned} \tag{2.20}$$

We first equate the second line of the left hand side with the last line of the right hand side and see with inserting the product form

$$\begin{aligned} &\pi(n_1, n_2, \dots, n_J) \cdot \hat{\pi}(D) \Bigg(\sum_{I\supset D} q^{\mathbf{Y}}(D, I) + \sum_{H\subset D} q^{\mathbf{Y}}(D, H) \Bigg) \\ &= \sum_{I\supset D} \pi(n_1, n_2, \dots, n_J) \cdot \hat{\pi}(I)\, q^{\mathbf{Y}}(I, D) + \sum_{H\subset D} \pi(n_1, n_2, \dots, n_J) \cdot \hat{\pi}(H)\, q^{\mathbf{Y}}(H, D). \end{aligned}$$

which is obviously the global balance equation associated with the generator $Q^{\mathbf{Y}}$, and thus solved by $\hat{\pi}$.

The reminder terms of (2.20) constitute balance equations corresponding to a Jackson network on the node set $\tilde{J} \setminus D$ with routing scheme r^D, the traffic equations of which (using Assumption 2.4) are solved by $(\eta_j : j \in \tilde{J} \setminus D)$. Renormalizing, the problem is reduced to balance equations for a standard Jackson network when $\hat{\pi}(D)$ is canceled. □

In [SD03] the result of Theorem 2.11 is proved for more general breakdown and repair schemes: These allow the breakdown and repair rates to depend on the load (queue lengths) of nodes. The question whether in this framework results similar to those in the following sections can be derived is still open and part of our ongoing research.

Remark 2.12. Theorem 2.11 is not covered by the product form results for networks in a random environment of Zhu [Zhu94] and Economou [Eco05]. In both papers the central assumption is that under different environment states the ratio ”local arrival rate/local service rate” is independent of the environment status. This is obviously not the case in our systems.

3 One step correlation

Recall $\lambda_j = \lambda \cdot r_{0j}, j = 1, \ldots, J$, and $\lambda_1 + \ldots + \lambda_J = \lambda(1 - r_{00})$ and that we therefore consider only admitted customers even if all nodes are up. We will not mention this further in this section. For the network process $\mathbf{Z}$ with generator $Q^{\mathbf{Z}}$ and stationary distribution $\tilde{\pi}$ consider *one step* correlation expressions

$$\langle f, Q^{\mathbf{Z}} g \rangle_{\tilde{\pi}}. \tag{3.1}$$

If $f = g$, then (3.1) is (the negative of) a quadratic form, because $-Q^{\mathbf{Z}}$ is positive definite. (3.1) occurs in the definition of Cheeger's constant because division of (3.1) with $f = g$ by $\langle f, f \rangle_{\tilde{\pi}}$ yields Rayleigh quotients. It also occurs in the definition of the corresponding Dirichlet form. This is helpful to bound the second largest eigenvalue of $Q^{\mathbf{Z}}$ and to prove the corresponding Poincare inequality for the corresponding Markovian process, see e.g., [Ch05]. Furthermore, (3.1) can be utilized to determine the asymptotic variance of costs or performance measures associated with Markovian network processes and to compare the asymptotic variances of two such related processes. It is possible to compare the correlations for $\mathbf{Z}$ with that of the related process $\mathbf{Z}'$ with the same stationary distribution $\tilde{\pi}$, using

$$\langle f, Q^{\mathbf{Z}} g \rangle_{\tilde{\pi}} - \langle f, Q^{\mathbf{Z}'} g \rangle_{\tilde{\pi}},$$

which will be given explicitly in Section 3.2. Because we are dealing with processes having bounded generators, properties connected with (3.1) can be turned into properties of

$$\langle f, I + \varepsilon Q^{\mathbf{Z}} g\rangle_{\tilde{\pi}} = E_{\tilde{\pi}}(f(Z_0)g(Z_\tau)) \tag{3.2}$$

where I is the identity operator, and $\varepsilon > 0$ is sufficiently small such that $I + \varepsilon Q^{\mathbf{Z}}$ is a stochastic matrix (i.e. the transition matrix of the uniformized chain), and $\tau \sim \exp(\varepsilon)$ (exponentially distributed). This enables one to directly apply discrete time methods to characterize properties of continuous time processes.

3.1 Correlation formulas

Due to the product form steady state distribution of $\mathbf{Z}$ the one step correlation $\langle f, Q^{\mathbf{Z}} g\rangle_{\tilde{\pi}}$ splits immediately into two terms having an intuitive interpretation. Namely, the one step correlation is the sum of weighted one step conditional correlations (i) of the environment process Y, and (ii) of network processes, which for a fixed time point seem to behave conditionally independent of the environment.

As will be seen immediately, it is illuminating, to define for all $D \subseteq \{1, \ldots, J\}$ the generators $Q^{\mathbf{X}_{\tilde{J}\setminus D}}$ of certain "synthetic subnetworks" on node set $\tilde{J}\setminus D$ with overall arrival rate λ, service rates as prescribed in the Definition 2.10, and routing matrix R^D. We have a splitting formula

Proposition 3.1. *For unreliable Jackson network processes* $\mathbf{Z} = (\mathbf{Y}, \mathbf{X})$ *as in Theorem 2.11 the one step correlations splits as follows*

$$\begin{aligned}
\langle f, Q^{\mathbf{Z}} g\rangle_{\tilde{\pi}} = \sum_{\underline{n}\in\mathbb{N}^J} \pi(\underline{n}) \left\{ \sum_{D\subseteq\{1,\ldots,J\}} \hat{\pi}(D) f(D,\underline{n}) \left(Q^{\mathbf{Y}} g(\cdot,\underline{n})\right)(D) \right\} \\
+ \sum_{D\subseteq\{1,\ldots,J\}} \hat{\pi}(D) \sum_{\underline{n}_D\in\mathbb{N}^{|D|}} \pi_D(\underline{n}_D) \\
\left\{ \sum_{\underline{n}_{\tilde{J}\setminus D}\in\mathbb{N}^{J-|D|}} \pi_{\tilde{J}\setminus D}(\underline{n}_{\tilde{J}\setminus D}) f(D,(\underline{n}_D,\underline{n}_{\tilde{J}\setminus D})) \left(Q^{\mathbf{X}_{\tilde{J}\setminus D}} g(D,(\underline{n}_D,(\cdot)_{\tilde{J}\setminus D}))\right)(\underline{n}_{\tilde{J}\setminus D}) \right\}
\end{aligned}$$

The proof of Proposition 3.1 and the next correlation formula are postponed to Section 5.

This more explicit correlation formula seems to be of limited use directly, but will yield remarkable simplifications when used for differences.

Proposition 3.2. *For unreliable Jackson network processes* $\mathbf{Z} = (\mathbf{Y}, \mathbf{X})$ *as in Theorem 2.11, with* ξ^D *the probability solution of the equation* $x^D = x^D \cdot R^D$ *(which applies, when nodes in* D *are down), the* **one-step correlation formula** *is*

$$\langle f, Q^{\mathbf{Z}} g\rangle_{\tilde{\pi}} =$$

$$= \sum_{D\subseteq\{1,\dots,J\}} \sum_{\underline{n}\in\mathbb{N}^J} \tilde{\pi}(D,\underline{n}) \left\{ \sum_{H\subset D} q^{\mathbf{Y}}(D,H) f(D,\underline{n}) g(H,\underline{n}) + \sum_{I\supset D} q^{\mathbf{Y}}(D,I) f(D,\underline{n}) g(I,\underline{n}) \right\}$$

$$+ \sum_{D\subseteq\{1,\dots,J\}} \sum_{\underline{n}\in\mathbb{N}^J} \tilde{\pi}(D,\underline{n}) \frac{\lambda}{\xi_0^D} \left\{ \sum_{j\in\tilde{J}_0\setminus D} \sum_{i\in\tilde{J}_0\setminus D} \xi_j^D r_{ji}^D f(D,\underline{n}+e_j) g(D,\underline{n}+e_i) \right\}$$

$$- \sum_{D\subseteq\{1,\dots,J\}} \sum_{\underline{n}\in\mathbb{N}^J} \tilde{\pi}(D,\underline{n}) f(D,\underline{n}) g(D,\underline{n}) \left\{ \sum_{H\subset D} q^{\mathbf{Y}}(D,H) + \sum_{I\supset D} q^{\mathbf{Y}}(D,I) + \lambda + \sum_{j\in\tilde{J}\setminus D} \mu_j(n_j) \right\}.$$

3.2 Comparison of one step correlations

The following formulas for differences of one step correlations will give additional insight into various properties of networks, for example to speed of convergence or asymptotic variance. They display how, e.g., the routing and the breakdown and repair affects correlations in networks. The proofs of these applications follow those ideas which are used to prove the following theorems. We therefore present some details of this proof here and give only hints for proving the applications.

Our first result describes the reaction of an unreliable network to changes of the routing behaviour of the customers.

Theorem 3.3. *Suppose* $\mathbf{Z} = (\mathbf{Y}, \mathbf{X})$ *is an ergodic unreliable Jackson network process with a routing matrix* R *and* $\mathbf{Z}' = (\mathbf{Y}', \mathbf{X}')$ *is another Jackson network process having the same arrival and service intensities and failure-repair rates but with routing matrix* $R' = [r'_{ij}]_{i,j=0,1,\dots,J}$*, such that the solutions of the traffic equation derived from* R *and for* R' *coincide (denoted by* η*). Assume that both networks follow some rerouting mechanism for which the Assumption 2.4 holds. Then for arbitrary real functions* $f, g \in L^2$

$$\langle f, Q^{\mathbf{Z}} g\rangle_{\tilde{\pi}} - \langle f, Q^{\mathbf{Z}'} g\rangle_{\tilde{\pi}} = E_{\tilde{\pi}}\left[\frac{\lambda}{\xi_0^{Y_t}} \Big(tr\big(W^{g,f}(Y_t, X_t) \cdot diag(\xi^{Y_t}) \cdot (R^{Y_t} - R'^{Y_t}) \big) \Big) \right],$$

where ξ^D *is the probability solution of* $x^D = x^D \cdot R^D$ *(see Lemma 2.5),* $tr(A)$ *denotes trace of* A*, and with* $e_0 = (0, \dots, 0)$

$$W^{g,f}(D,\underline{n}) = [g(D,\underline{n}+e_i) f(D,\underline{n}+e_j)]_{i,j\in\tilde{J}_0\setminus D}.$$

Proof. Because the external arrival streams are the same, and the traffic equations have the same solution η, and the rerouting mechanisms have property (2.10), for any availability status D the rerouting schemes on $\tilde{J} \setminus D$ have the same solution of the traffic equation. It follows from Lemma 2.5 that for all D the probability solution of the equations $x^D = x^D \cdot R^D$ and $x^D = x^D \cdot R'^D$ are in both systems the same. Because of $q^{\mathbf{Y}} = q^{\mathbf{Y}'}$ we immediately have from Proposition 3.2 the reduction

$$\langle f, Q^{\mathbf{Z}} g\rangle_{\tilde{\pi}} - \langle f, Q^{\mathbf{Z}'} g\rangle_{\tilde{\pi}} =$$

$$= \sum_{D\subseteq\{1,\dots,J\}} \sum_{\underline{n}\in\mathbb{N}^J} \tilde{\pi}(D,\underline{n}) \frac{\lambda}{\xi_0^D} \left\{ \sum_{j\in\tilde{J}_0\setminus D} \sum_{i\in\tilde{J}_0\setminus D} \xi_j^D r_{ji}^D f(D,\underline{n}+e_j) g(D,\underline{n}+e_i) \right\}$$

$$- \sum_{D\subseteq\{1,\dots,J\}} \sum_{\underline{n}\in\mathbb{N}^J} \tilde{\pi}(D,\underline{n}) \frac{\lambda}{\xi_0^D} \left\{ \sum_{j\in\tilde{J}_0\setminus D} \sum_{i\in\tilde{J}_0\setminus D} \xi_j^D {r'}_{ji}^D f(D,\underline{n}+e_j) g(D,\underline{n}+e_i) \right\}$$

$$= \sum_{D\subseteq\{1,\dots,J\}} C^{-1}\hat{\pi}(D) \sum_{\underline{n}_D\in\mathbb{N}^{|D|}} \prod_{\ell\in D} \prod_{i=1}^{n_\ell} \left(\frac{\eta_\ell}{\mu_\ell(i)}\right)$$

$$\sum_{\underline{n}_{\tilde{J}\setminus D}\in\mathbb{N}^{J-|D|}} \prod_{\ell\in\tilde{J}\setminus D} \prod_{i=1}^{n_\ell} \left(\frac{\eta_\ell}{\mu_\ell(i)}\right) \left[\frac{\lambda}{\xi_0^D} \left\{ \sum_{j\in\tilde{J}_0\setminus D} \sum_{i\in\tilde{J}_0\setminus D} \xi_j^D r_{ji}^D f(D,\underline{n}+e_j) g(D,\underline{n}+e_i) \right\} \right.$$

$$\left. - \frac{\lambda}{\xi_0^D} \left\{ \sum_{j\in\tilde{J}_0\setminus D} \sum_{i\in\tilde{J}_0\setminus D} \xi_j^D {r'}_{ji}^D f(D,\underline{n}+e_j) g(D,\underline{n}+e_i) \right\} \right]$$

We interpret in the last two lines for fixed D and $\underline{n}_D$ and $i,j \in \tilde{J}_0 \setminus D$ the expressions

$$f(D,\underline{n}+e_j) =: f(D,(\underline{n}_D, \underline{n}_{\tilde{J}\setminus D}+e_j)) \quad \text{and} \quad g(D,\underline{n}+e_i) =: g(D,(\underline{n}_D, \underline{n}_{\tilde{J}\setminus D}+e_i))$$

as functions of $\underline{n}_{\tilde{J}\setminus D}$ only, and see that the resulting expressions have exactly the structure of the functions dealt with in Proposition 4.1 of [DS08].

After renormalization of the densities $\prod_{\ell\in\tilde{J}\setminus D} \prod_{i=1}^{n_\ell} \left(\frac{\eta_\ell}{\mu_\ell(i)}\right)$, which is in fact result of conditioning on $\{Y(t) = D, X_D(t) = \underline{n}_D\}$, we obtain $\langle f, Q^{\mathbf{Z}} g\rangle_{\tilde{\pi}} - \langle f, Q^{\mathbf{Z}'} g\rangle_{\tilde{\pi}}$

$$= \sum_{D\subseteq\{1,\dots,J\}} \sum_{\underline{n}_D\in\mathbb{N}^{|D|}} P(\{Y(t)=D, X_D(t)=\underline{n}_D\})$$

$$E_{\tilde{\pi}} \left[\frac{\lambda}{\xi_0^D} \left(tr(W^{g,f}(D,(\underline{n}_D, X_{\tilde{J}_0\setminus D})) \cdot diag\xi^D \cdot (R^D - {R'}^D)) \right) |\{Y(t)=D, X_D(t)=\underline{n}_D\} \right],$$

and deconditioning eventually finishes the proof. □

Our next result describes the reaction of a network to changes of the breakdown and repair mechanisms. It turns out that it is possible to write the formula for the difference of one step correlations for the corresponding generators of networks in terms of scalar products with respect to invariant measure for the failure processes using the difference of the corresponding intensity matrices of the failure processes.

Theorem 3.4. *Suppose* $\mathbf{Z} = (\mathbf{Y}, \mathbf{X})$ *is an ergodic unreliable Jackson network process with a routing matrix* $R = [r_{ij}]_{i,j=0,1,\dots,J}$ *and* $\mathbf{Z}' = (\mathbf{Y}', \mathbf{X}')$ *is another Jackson network process having the same arrival and service intensities, and with the same routing regimes, described by* R *and rerouting fulfilling Assumption 2.4.*

The breakdown-repair process for $\mathbf{Z}$ *is given in Definition 2.1 and for* $\mathbf{Z}'$ *is defined similarly via functions* $A',\ B' : 2^{\tilde{J}} \to [0, \infty)$, *subject to the indicated restrictions there.*

Then the breakdown and repair rates for $\mathbf{Y}'$ *are:*

$$q^{\mathbf{Y}'}(D, I) = \frac{A'(I)}{A'(D)}, \quad D \subset I \subseteq \tilde{J}, \qquad \textit{and } \ q^{\mathbf{Y}'}(D, H) = \frac{B'(D)}{B'(H)}, \quad H \subset D \subseteq \tilde{J}, .$$

The processes $\mathbf{Y}$ *and* $\mathbf{Y}'$ *are Markov with generators* $Q^{\mathbf{Y}} = (q^{\mathbf{Y}}(K, L) : K, L \subseteq \tilde{J})$ *of* $\mathbf{Y}$ *and* $Q^{\mathbf{Y}'} = (q^{\mathbf{Y}'}(K, L) : K, L \subseteq \tilde{J})$ *of* $\mathbf{Y}'$ *as defined in* (2.5) *for* $\mathbf{Y}$ *and similar for* $\mathbf{Y}'$.

Assume that the stationary distributions of $\mathbf{Y}$ *and* $\mathbf{Y}'$ *are identical, denoted by*

$$\hat{\pi} := \left(\hat{\pi}(D) := \hat{C}^{-1} \frac{A(D)}{B(D)} = \hat{C}'^{-1} \frac{A'(D)}{B'(D)}, \quad D \in 2^{\tilde{J}} \right). \tag{3.3}$$

Then for arbitrary real functions $f, g : \tilde{\mathbb{E}} \to \mathbb{R}$ *holds*

$$\langle f, Q^{\mathbf{Z}} g \rangle_{\tilde{\pi}} - \langle f, Q^{\mathbf{Z}'} g \rangle_{\tilde{\pi}} = E_\pi \left[\langle f(\circ, X_t), Q^{\mathbf{Y}} g(\cdot, X_t)(\circ) \rangle_{\hat{\pi}} - \langle f(\circ, X_t), Q^{\mathbf{Y}'} g(\cdot, X_t)(\circ) \rangle_{\hat{\pi}} \right] =$$

$$= E_\pi \left[\langle f(\circ, X_t), (Q^{\mathbf{Y}} - Q^{\mathbf{Y}'}) g(\cdot, X_t)(\circ) \rangle_{\hat{\pi}} \right].$$

Proof. Interchanging summations, regrouping terms, and exploiting the product form structure of the state distributions (which are identical for $\mathbf{Z} = (\mathbf{Y}, \mathbf{X})$ and $\mathbf{Z}' = (\mathbf{Y}', \mathbf{X}')$) in the correlation formula of Proposition 3.2, we obtain

$$\begin{aligned}
&\langle f, Q^{\mathbf{Z}} g \rangle_{\tilde{\pi}} = \\
= &\sum_{\underline{n} \in \mathbb{N}^J} \pi(\underline{n}) \sum_{D \subseteq \{1,\dots,J\}} \hat{\pi}(D) \ \left\{ \sum_{H \subset D} q^{\mathbf{Y}}(D, H) f(D, \underline{n}) g(H, \underline{n}) + \sum_{I \supset D} q^{\mathbf{Y}}(D, I) f(D, \underline{n}) g(I, \underline{n}) \right\} \\
&- \sum_{\underline{n} \in \mathbb{N}^J} \pi(\underline{n}) \sum_{D \subseteq \{1,\dots,J\}} \hat{\pi}(D) f(D, \underline{n}) g(D, \underline{n}) \left\{ \sum_{H \subset D} q^{\mathbf{Y}}(D, H) + \sum_{I \supset D} q^{\mathbf{Y}}(D, I) \right\}
\end{aligned}$$

$$+ \sum_{D\subseteq\{1,\ldots,J\}} \sum_{\underline{n}\in\mathbb{N}^J} \tilde{\pi}(D,\underline{n}) \frac{\lambda}{\xi_0^D} \left\{ \sum_{j\in\tilde{J}_0\setminus D} \sum_{i\in\tilde{J}_0\setminus D} \xi_j^D r_{ji}^D f(D,\underline{n}+e_j) g(D,\underline{n}+e_i) \right\}$$
$$- \sum_{D\subseteq\{1,\ldots,J\}} \sum_{\underline{n}\in\mathbb{N}^J} \tilde{\pi}(D,\underline{n}) f(D,\underline{n}) g(D,\underline{n}) \left\{ \lambda + \sum_{j\in\tilde{J}\setminus D} \mu_j(n_j) \right\}.$$

Because the arrival streams, the service rates, the routing, and the steady states of the processes $\mathbf{Z} = (\mathbf{Y}, \mathbf{X})$ and $\mathbf{Z}' = (\mathbf{Y}', \mathbf{X}')$ are the same, for $\langle f, Q^{\mathbf{Z}'} g\rangle_{\tilde{\pi}}$ the last two lines in the respective formula are identical to those in the displayed formula. The difference becomes therefore

$$\begin{aligned}
&\langle f, Q^{\mathbf{Z}} g\rangle_{\tilde{\pi}} - \langle f, Q^{\mathbf{Z}'} g\rangle_{\tilde{\pi}} = \\
= &\sum_{\underline{n}\in\mathbb{N}^J} \pi(\underline{n}) \sum_{D\subseteq\{1,\ldots,J\}} \hat{\pi}(D) \left\{ \sum_{H\subset D} q^{\mathbf{Y}}(D,H) f(D,\underline{n}) g(H,\underline{n}) + \sum_{I\supset D} q^{\mathbf{Y}}(D,I) f(D,\underline{n}) g(I,\underline{n}) \right. \\
&- \sum_{\underline{n}\in\mathbb{N}^J} \pi(\underline{n}) \sum_{D\subseteq\{1,\ldots,J\}} \hat{\pi}(D) f(D,\underline{n}) g(D,\underline{n}) \left\{ \sum_{H\subset D} q^{\mathbf{Y}}(D,H) + \sum_{I\supset D} q^{\mathbf{Y}}(D,I) \right\} \\
&- \sum_{\underline{n}\in\mathbb{N}^J} \pi(\underline{n}) \sum_{D\subseteq\{1,\ldots,J\}} \hat{\pi}(D) \left\{ \sum_{H\subset D} q^{\mathbf{Y}'}(D,H) f(D,\underline{n}) g(H,\underline{n}) + \sum_{I\supset D} q^{\mathbf{Y}'}(D,I) f(D,\underline{n}) g(I,\underline{n}) \right\} \\
&+ \sum_{\underline{n}\in\mathbb{N}^J} \pi(\underline{n}) \sum_{D\subseteq\{1,\ldots,J\}} \hat{\pi}(D) f(D,\underline{n}) g(D,\underline{n}) \left\{ \sum_{H\subset D} q^{\mathbf{Y}'}(D,H) + \sum_{I\supset D} q^{\mathbf{Y}'}(D,I) \right\}
\end{aligned}$$

For fixed $\underline{n}$ we interpret in the last two lines $f(D,\underline{n})$ and $g(D,\underline{n})$ as functions of D parametrized by $X_t(\omega) = \underline{n}$. This leads to

$$\begin{aligned}
\langle f, Q^{\mathbf{Z}} g\rangle_{\tilde{\pi}} - \langle f, Q^{\mathbf{Z}'} g\rangle_{\tilde{\pi}} &= \sum_{\underline{n}\in\mathbb{N}^J} \pi(\underline{n}) \left[\langle f(\circ,\underline{n}), Q^{\mathbf{Y}} g(\cdot,\underline{n})(\circ)\rangle_{\hat{\pi}} - \langle f(\circ,\underline{n}), Q^{\mathbf{Y}'} g(\cdot,\underline{n})(\circ)\rangle_{\hat{\pi}} \right] \\
&= E_\pi \left[\langle f(\circ,X_t), Q^{\mathbf{Y}} g(\cdot,X_t)(\circ)\rangle_{\hat{\pi}} - \langle f(\circ,X_t), Q^{\mathbf{Y}'} g(\cdot,X_t)(\circ)\rangle_{\hat{\pi}} \right]. \qquad (3.4)
\end{aligned}$$

□

In both Theorems 3.3 and 3.4 we prove a reduction of complexity: We show that one can reduce comparing operators and resulting one step correlations via functions on infinite state space to comparing matrix operators via functions on finite state space.

The Theorems 3.3 and 3.4 are valid for all square integrable functions f, g on $\tilde{\mathbb{E}}$. This opens the way to compare more intricate correlations for multidimensional marginals in time of the network processes with unreliable nodes according to concordance ordering utilizing the abstract setting for Markov processes of [DS08][Theorem 5.2].

For a concise notation we introduce the standard difference operators for functions on $\mathbb{N}^J$. For all $f \in L^2$ and all $j = 0, 1, \ldots, J$, we define (recall e_0 is the zero vector)

$$\mathcal{D}_j f : \tilde{\mathbb{E}} \to \mathbb{R}, \quad (D, \underline{n}) \to \mathcal{D}_j f(D, \underline{n}) := f(D, \underline{n} + e_j) - f(D, \underline{n}),$$

and

$$\mathcal{D} f : \tilde{\mathbb{E}} \to \mathbb{R}^{J+1-|D|}, \quad (D, \underline{n}) \to (\mathcal{D}_j f(D, \underline{n}), j \in \tilde{J}_0 \setminus D).$$

That way we can treat $\mathcal{D}f(D, \underline{n})$ as a vector of the dimension corresponding to the size of D, and the corresponding routing matrices R^D as operators on it. Moreover, it is possible to consider the corresponding scalar products generated by invariant vectors ξ^D, and write the formula for the difference of one step correlations (which are scalar products with respect to invariant measures for the network process $\tilde{\pi}$) in terms of scalar products with respect to invariant measures ξ^D for the routing processes.

Corollary 3.5. *For unreliable Jackson network processes $\mathbf{Z} = (\mathbf{Y}, \mathbf{X})$ as in Theorem 2.11, with ξ^D the probability solution of the equation $x^D = x^D \cdot R^D$ we have*

$$\begin{aligned} \langle f, Q^{\mathbf{Z}} g \rangle_{\tilde{\pi}} &= E_{\tilde{\pi}} \left[\frac{\lambda}{\xi_0^{Y_t}} \left\langle (\mathcal{D} + Id) f(Y_t, X_t), R^{Y_t} (\mathcal{D} + Id) g(Y_t, X_t) \right\rangle_{\xi^{Y_t}} \right] \\ &+ E_{\pi} \left[\langle f(\circ, X_t), Q^{\mathbf{Y}} g(\cdot, X_t)(\circ) \rangle_{\hat{\pi}} \right] \\ &- E_{\tilde{\pi}} \left[f(Y_t, X_t) g(Y_t, X_t) (\lambda + \underline{\mu}_{Y_t}) \right] \end{aligned}$$

where ξ^D is the probability solution of $x^D = x^D \cdot R^D$ (see Lemma 2.5), and $\underline{\mu}_D = \sum_{j \in \tilde{J} \setminus D} \mu_j(n_j)$ is the total service rate for nodes which are up.

Proof. Take the correlation formula from Proposition 3.2, use conditioning as in Theorem 3.3, and insert the suitable difference operators. □

We can now reformulate the result of Theorem 3.3 in a more compact form which immediately relates our results to methods dealt with in optimizing MCMC simulation.

Corollary 3.6. *For unreliable Jackson network processes $\mathbf{Z}, \mathbf{Z}'$ as in Theorem 3.3 we have*

$$\langle f, Q^{\mathbf{Z}} g \rangle_{\tilde{\pi}} - \langle f, Q^{\mathbf{Z}'} g \rangle_{\tilde{\pi}} = E_{\tilde{\pi}} \left[\frac{\lambda}{\xi_0^{Y_t}} \left\langle (\mathcal{D} + Id) f(Y_t, X_t), \left(R^{Y_t} - R'^{Y_t} \right) (\mathcal{D} + Id) g(Y_t, X_t) \right\rangle_{\xi^{Y_t}} \right],$$

4 Applications

4.1 Comparison of spectral gaps

Let $\mathbf{Z}$ be a continuous time homogeneous ergodic Markov process with stationary probability $\tilde{\pi}$ and generator $Q^{\mathbf{Z}}$. Let $\tilde{\pi}(f) = \int_{\tilde{\mathbb{E}}} f(x)\tilde{\pi}(dx)$ The spectral gap of $\mathbf{Z}$, resp. $Q^{\mathbf{Z}}$ is

$$Gap(Q^{\mathbf{Z}}) = \inf\{\langle f, -Q^{\mathbf{Z}} f\rangle_{\tilde{\pi}} : f \in L^2(\tilde{\mathbb{E}}, \tilde{\pi}), \tilde{\pi}(f) = 0,\ \langle f, f\rangle_{\tilde{\pi}} = 1\}.$$

The spectral gap determines for $\mathbf{Z}$ the speed of convergence to equilibrium $\tilde{\pi}$ in $L^2(\tilde{\mathbb{E}}, \tilde{\pi})$ with norm $\|\cdot\|_{\tilde{\pi}}$: $Gap(Q^{\mathbf{Z}})$ is the largest number Δ such that for the transition semigroup $P = (P_t : t \geq 0)$ of $\mathbf{Z}$ holds

$$\|P_t f - \tilde{\pi}(f)\|_{\tilde{\pi}} \leq e^{-\Delta t}\|f - \tilde{\pi}(f)\|_{\tilde{\pi}} \quad \forall f \in L^2(\tilde{\mathbb{E}}, \tilde{\pi}).$$

It should be noted that one has to be careful which class of functions is used for the definition of spectral gap. For a discussion and more references see the introduction of [LS13].

We shall utilize the following orderings to compare routings, failure processes and then correlations, see [Pes73].

Definition 4.1. *Let $R = [r_{ij}]$ and $R' = [r'_{ij}]$ be transition matrices on a finite set $\mathbb{E}$ such that $\xi R = \xi R' = \xi$ for a probability vector ξ.*

We say that R' is smaller than R in the positive semidefinite order, $R' \prec_{pd} R$, if the matrix $R - R'$ is positive semidefinite.

We say that R' is smaller than R in the Peskun order, $R' \prec_P R$, if for all $j, i \in \mathbb{E}$ with $i \neq j$ holds $r'_{ji} \leq r_{ji}$.

Peskun used the latter order to compare reversible transition matrices with the same stationary distribution and their asymptotic variance, and Tierney [Tie98] has shown (in a more general setting, i.e. using operators rather than matrices) that the main property used in the proof of Peskun, namely that "$R \prec_P R'$ implies $R' \prec_{pd} R$", holds without reversibility assumptions.

Example 4.2. For any transition matrix $R = [r_{ij}]$ holds for $R' = Id$, where Id is the identity matrix of the same dimension as R,

$$R' \prec_P R\,, \tag{4.1}$$

which says that the family of transition matrices R of a fixed dimension has a (unique) minimal element. If we consider different routing schemes, resp. rerouting schemes, therefore STALLING from Example 2.8 is an extremal (re-)routing scheme.

Proposition 4.3. *Consider two ergodic unreliable Jackson networks with state processes* $\mathbf{Z}$ *and* $\mathbf{Z}'$ *and with the same arrival and service intensities, and the same failure-repair rates. Assume that the equations* $x = x \cdot R$ *and* $x = x \cdot R'$ *have the same stochastic solution* ξ*, and the Assumption 2.4 holds, i.e. both networks follow some rerouting mechanism according to* (2.9) *with the property* (2.10).

If $R^D \prec_{pd} R'^D$ *for all* D *(also for* $D = \emptyset$*), then*

$$Gap(Q^{\mathbf{Z}'}) \leq Gap(Q^{\mathbf{Z}}).$$

Proof. From Corollary 3.6 we have for all $f \in L^2$

$$\langle f, -Q^{\mathbf{Z}} f\rangle_{\tilde{\pi}} - \langle f, -Q^{\mathbf{Z}'} f\rangle_{\tilde{\pi}} = E_{\tilde{\pi}} \left[\frac{\lambda}{\xi_0^{Y_t}} \left\langle (\mathcal{D} + Id) f(Y_t, X_t), \left(R'^{Y_t} - R^{Y_t}\right)(\mathcal{D} + Id) f(Y_t, X_t) \right\rangle_{\xi^{Y_t}} \right],$$

and from the product formula we rewrite this formula as

$$\langle f, -Q^{\mathbf{Z}} f\rangle_{\tilde{\pi}} - \langle f, -Q^{\mathbf{Z}'} f\rangle_{\tilde{\pi}} = E_{\pi} E_{\hat{\pi}} \left[\frac{\lambda}{\xi_0^{Y_t}} \left\langle (\mathcal{D} + Id) f(Y_t, X_t), \left(R'^{Y_t} - R^{Y_t}\right)(\mathcal{D} + Id) f(Y_t, X_t) \right\rangle_{\xi^{Y_t}} \right] =$$

$$= E_{\pi} \sum_{D} \hat{\pi}(D) \left[\frac{\lambda}{\xi_0^{D}} \left\langle (\mathcal{D} + Id) f(D, X_t), \left(R'^{D} - R^{D}\right)(\mathcal{D} + Id) f(D, X_t) \right\rangle_{\xi^{D}} \right].$$

From our Assumption on $\prec_{pd}$ ordering of routings we therefore have for all $f \in L^2$

$$\langle f, -Q^{\mathbf{Z}} f\rangle_{\tilde{\pi}} \geq \langle f, -Q^{\mathbf{Z}'} f\rangle_{\tilde{\pi}}$$

Taking infima on both sides over the set $\{f \in L^2 : \tilde{\pi}(f) = 0, \ \langle f, f\rangle_{\tilde{\pi}} = 1\}$ we get the result. □

As discussed above, a sufficient condition for Proposition 4.3 is that $R'^D \prec_P R^D$ holds for any D, because we then have $R^D \prec_{pd} R'^D$.

Computing spectral gaps for Markov processes with multidimensional state space is challenging, in many cases nearly impossible. Exceptions are multidimensional independent birth-death processes, because for birth-death processes explicit results are known, see e.g. [Doorn02], and Liggett has proved that the gap of independent processes is the minimum of the spectral gap of the marginal processes [Lig89][Theorem 6.2].

We will show that the gap of the joint queue length network process $\mathbf{Z}$ (with unreliable nodes) can be bounded from below by the gap of a related process consisting of identical breakdown-repair process and related multidimensional birth-death process with **conditionaly independent** components.

The first result is in the realm of the abstract rerouting framework of Assumption 2.4, while the second result is for rerouting schemes which are defined by concrete recipes described in the Examples 2.7, 2.8, and 2.9.

Proposition 4.4. *Consider an ergodic Jackson network process* $\mathbf{Z}$ *with unreliable servers as in Theorem 2.11. Assume that for all* $D \subseteq \{1, 2, \ldots, J\}$ *with* $\tilde{\pi}(D) > 0$ *the routing matrix* $R^D = [r^D_{ij}]_{i,j \in \tilde{J}_0 \setminus D}$ *has strict positive entrance and departure probabilities* ($r^D_{0i} > 0, r^D_{i0} > 0$) *for every node* $i \in \tilde{J} \setminus D$.

Assume further that for all $D \subseteq \{1, 2, \ldots, J\}$ *the rerouting* R^D *fulfills overall balance for all network nodes which are up*

$$\eta^D_j \sum_{i \in \tilde{J} \setminus D} r^D_{ji} = \sum_{i \in \tilde{J} \setminus D} \eta^D_i r^D_{ij}, \quad \forall j \in \tilde{J} \setminus D. \tag{4.2}$$

Then there exists an ergodic Jackson network process process $\mathbf{Z}'$ *with unreliable servers as in Theorem 2.11 with the same stationary distribution* $\tilde{\pi}$ *as* $\mathbf{Z}$*, such that*

$$Gap(Q^{\mathbf{Z}'}) \leq Gap(Q^{\mathbf{Z}}) \,.$$

The nodes of $\mathbf{Z}'$ *are perturbed by a common breakdown-repair regime identical to that of* $\mathbf{Z}$*, and for any given set* D *of broken down nodes the joint network process on* $\tilde{J} \setminus D$ *consists of conditionally independent birth-death processes, and the coordinate birth and death processes on the* i*-th coordinate have birth rate* λr^D_{0i} *and state dependent death rate* $\mu_i(n_i) r^D_{i0}$.

Proof. In order to obtain a lower bound for the spectral gap using birth and death processes the idea is to allow in the comparison network for each node i, which is up, that any customer who enters node i from the external source after being served only to feed back (possibly iteratively) to node i or to depart from the network. This results in updating the service rates suitably. Recall that the original network processes and the constructed comparison processes are additionally perturbed by the same failure mechanism.

Consider the situation when nodes in D are down. Directly from the formula (2.18) for the generator of the network process $\mathbf{Z}$, it is clear that after reducing movements inside the network, and allowing only for movements into the network from outside, or from the network into outside, or feedback, we get as long as the reliability level D does not change transitions for changing the queue lengths which look identical as those of the generator of independent birth and death processes such that on the i-th coordinate the birth rate equals λr^D_{0i} and the state dependent death rate equals $\mu_i(n_i) r^D_{i0}, i \in \tilde{J} \setminus D$.

Now, in order to be able to apply a formula for differences of one step correlations we have to show on every reliability level D that such a modification is possible within a class of networks with extended routings having the same stationary solution. For this reason we need the assumption on overall balance (4.2).

More precisely, we define R'^D by $r'^D_{i0} = r^D_{i0}$, $r'^D_{0i} = r^D_{0i}$, for all i, $r'^D_{ij} = 0$ for $j \neq i,\ i, j \in \tilde{J} \setminus D$, and $r'^D_{ii} = 1 - r^D_{i0}$ for $i \in \tilde{J} \setminus D$. With the routing R'^D the network process $\mathbf{Z}'$ (when nodes in D are down) develops as a vector of independent birth and death processes for the up nodes.

For $j \in \{1, \ldots, J\}$ let η'^D_j be the solution of the traffic equations for R'^D. We have directly

$$\eta'^D_j = \lambda r^D_{0j} + \eta'^D_j r'^D_{jj}, \qquad j \in \tilde{J} \setminus D\,, \tag{4.3}$$

and the solution of this system is uniquely defined. We show that these equations are solved (for each D) by $\eta^D = (\eta^D_j, j \in \tilde{J} \setminus D)$ as well. Inserting η^D into (4.3) we obtain with $r'^D_{jj} = 1 - r^D_{jo}$ for $j \in \tilde{J} \setminus D$

$$\eta^D_j = \lambda r^D_{0j} + \eta^D_j (1 - r^D_{jo}) = \lambda r^D_{0j} + \eta^D_j \sum_{i \in \tilde{J} \setminus D} r^D_{ji} = \lambda r^D_{0j} + \eta^D_j \sum_{i \in \tilde{J} \setminus D} \eta^D_i r^D_{ij}\,,$$

which is the traffic equation when nodes in D are down in $\mathbf{Z}$ and has the unique solution η^D.

The last step is done by first observing that η^D is from Assumption 2.4 the restriction of η, the solution of the traffic equation when all nodes in $\mathbf{Z}$ are up, to $\tilde{J} \setminus D$, and, secondly, by considering the above constructed system of independent birth-death processes for the reliability level $\emptyset$ as the comparison system when all nodes are up with routing R', and the R'^D as rerouting scheme for this network on reliability level D.

Because $\eta = \eta'$ and $\eta^D = \eta'^D$ for all D, the η'^D are the restriction of η', the solution of the traffic equation when all nodes in $\mathbf{Z}'$ are up, to $\tilde{J} \setminus D$.

Note that for all D

$$R'^D \prec_P R^D,$$

therefore $R^D \prec_{pd} R'^D$, and the result follows from Proposition 4.3. □

Corollary 4.5. *Consider an ergodic Jackson network process $\mathbf{Z}$ with unreliable servers as in Theorem 2.11 with the RS-RD procedures of rerouting. Assume that the routing matrix $R = [r_{ij}]_{i,j \in \tilde{J}_0}$ is reversible and has strict positive entrance and departure probabilities ($r_{0i} > 0, r_{i0} > 0$) for every node $i \in \tilde{J}$.*

Then there exists an ergodic Jackson network process process $\mathbf{Z}'$ with unreliable servers as in Theorem 2.11 with the same stationary distribution $\tilde{\pi}$ as $\mathbf{Z}$, such that

$$Gap(Q^{\mathbf{Z}'}) \leq Gap(Q^{\mathbf{Z}})\,.$$

The nodes of $\mathbf{Z}'$ *are perturbed by a common breakdown-repair regime identical to that of* $\mathbf{Z}$*, and for any given set* D *of broken down nodes the joint network process on* $\tilde{J} \setminus D$ *consists of conditionally independent birth-death processes, and the coordinate birth and death processes on the i-th coordinate have birth rate* λr_{0i} *and state dependent death rate* $\mu_i(n_i) r_{i0}$*.*

Proof. From reversibility of R follows that for any D the overall balance (4.2) holds. So Proposition 4.4 applies. □

Corollary 4.6. *Consider two ergodic unreliable Jackson networks with state processes* $\mathbf{Z}$ *and* $\mathbf{Z}'$ *and with the same arrival and service intensities, and the same failure-repair rates. Assume that the equations* $x = x \cdot R$ *and* $x = x \cdot R'$ *have the same stochastic solution* ξ*, and the Assumption 2.4 holds. Moreover assume that* $\mathbf{Z}'$ *follows the stalling rerouting scheme. If* $R \prec_{pd} R'$ *then*

$$Gap(Q^{\mathbf{Z}'}) \leq Gap(Q^{\mathbf{Z}}).$$

Proof. For the stalling scheme $R'^D = Id$ for all $D \neq \emptyset$, and $R'^\emptyset = R'$, hence the assumptions of Proposition 4.3 are fulfilled. □

Taking for R' the routing corresponding to parallel birth and death processes in the above corollary we obtain

Corollary 4.7. *Consider an ergodic Jackson network process* $\mathbf{Z}$ *with unreliable servers as in Theorem 2.11. Assume that* $r_{0i} > 0, r_{i0} > 0$ *for every node* $i \in \tilde{J}$*.*

Assume further that the routing R *fulfills overall balance for all network nodes which are up*

$$\eta_j \sum_{i \in \tilde{J}} r_{ji} = \sum_{i \in \tilde{J}} \eta_i r_{ij}, \quad \forall j \in \tilde{J}. \tag{4.4}$$

Then there exists an ergodic process $\mathbf{Z}'$ *with the same stationary distribution* $\tilde{\pi}$ *as* $\mathbf{Z}$*, such that*

$$Gap(Q^{\mathbf{Z}'}) \leq Gap(Q^{\mathbf{Z}}),$$

where $\mathbf{Z}'$ *consists of independent birth-death processes, which are perturbed by a common breakdown-repair regime identical to that of* $\mathbf{Z}$*, moreover* $\mathbf{Z}'$ *obeys the stalling rerouting scheme and the coordinate birth and death processes has on the i-th coordinate the birth rate* λr_{0i} *and the state dependent death rate* $\mu_i(n_i) r_{i0}$*.*

Remark: From irreducibility of $\mathbf{Z}$, for $i, j \in \tilde{J}, i \neq j$ we obtain from (4.2) $\eta_j r_{ji}^{\tilde{J}\setminus\{i,j\}} = \eta_i r_{ij}^{\tilde{J}\setminus\{i,j\}}$, but this does not mean that the matrix R is reversible, because $r_{ij}^{\tilde{J}\setminus\{i,j\}} \neq r_{ij}$ may hold.

The lower bound $Gap(\mathbf{Z}')$ in the previous statements is of interest, because it has constitutive processes with conditionally independent coordinates. From [Lig89][Theorem 2.6] it is known, that the gap of a process with independent coordinates is the minimum of the gaps of the coordinate processes. Unfortunately enough, this theorem does not apply here directly, because the coordinate birth-death processes are controlled by the common breakdown-repair process. However, the comparison result of Proposition 4.3 can be used to obtain upper bounds for spectral gaps. This topic will be considered in a separate paper.

Nevertheless the bound is of practical value, because the bounding process $\mathbf{Z}'$ is reversible with respect to $\tilde{\pi}$, which can be seen by checking the local balance equations. As a consequence, the bounding techniques for reversible processes, e.g., using Cheeger constants, found in the literature can be applied directly.

In [LS13][Example 6.2] it is shown, that the bounds obtained via Proposition 4.4 can be very good for networks with reliable nodes. They compare the bound for an example provided by Ignatiouk-Robert and Tibi [IRT12]:

This is the network described in Section 2.1 with state independent service rates $\mu_j = \mu$, and routing matrix which fulfills $r_{ii} = 0, \forall i = 0, 1, \dots, J$, and $r_{0i} > 0, \forall i = 1, \dots, J$. Furthermore, for all $i, j = 1, \dots, J, i \neq j$, holds complete symmetry by $r_{ij} = p \in (0, 1/(j-1)$, which results in $r_{i0} = 1 - p(J-1) > 0, \forall i = 1, \dots, J$.

It is assumed that no breakdowns (and repair) occur.

In this symmetric network the partial balance (for $D = \emptyset$) holds if and only if $r_{0i} = 1/J, \forall i = 1, \dots, J$, which implies $\lambda_j = \lambda/J, \forall j = 1, \dots, J$ and $\eta_j = \lambda/(J(1-p(J-1))), \forall j = 1, \dots, J$.

We denote by

$$\mu_{i_0} := \min_{1 \le j \le J} \mu_j,$$

and, recalling the bound of the spectral gap obtained for birth-death processes by van Doorn [Doorn02], we obtain from the companion result of Proposition 4.4 (see [DS08][Proposition 4.4])

$$Gap(\mathbf{Z}') \geq \left(\sqrt{\mu_{i_0}(1 - p(J-1))} - \sqrt{\frac{\lambda}{J}} \right)^2 .$$

It is easy to check, that for this setting the Assumptions (3.10) and (3.11) of Corollary 3.4 in [IRT12] are fulfilled, which results in an upper bound for L^2 spectral gap

$$Gap(\mathbf{Z}) \leq \frac{1+p}{1-p(J-2)} \left(\sqrt{\mu_{i_0}(1 - p(J-1))} - \sqrt{\frac{\lambda}{J}} \right)^2 . \tag{4.5}$$

For $p \to 1/(J-1)$ the factor $\frac{1+p}{1-p(J-2)}$ tends monotonously to J, while for $p \to 0$ it tends monotonously decreasing to 1.

A consequence which elaborates on the implication *Peskun yields positive definiteness* is, that if we perturb routing of customers in the networks by shifting transition probability mass from non diagonal entries into the diagonal (leaving the routing equilibrium fixed) then the speed of convergence of the perturbed process is smaller.

The existence of L^2 spectral gap (that is the question when $Gap(Q^{\mathbf{Z}}) > 0$) for unreliable networks is a related topic. It is a common knowledge that for networks with constant service rates (not depending on the number of customers at node) the spectral gap for classical Jackson network exists. For service rates that can depend on the number of customers the problem is more delicate. An *iff* characterization in terms of properties of service rates is given in Lorek and Szekli [LS13]. A special feature of such processes is that the existence of L^2 spectral gap is directly related to the tail properties of the stationary distribution. For references and details see [LS13].

An analogue to Peskun ordering and positive semidefinite order of transition matrices to generator matrices is as follows.

Definition 4.8. *Let $Q = (q(x, y) : x, y \in \mathbb{E})$ and $Q' = (q'(x, y) : x, y \in \mathbb{E})$ be generator matrices on a finite set $\mathbb{E}$ such that $\hat{\pi}Q = \hat{\pi}Q' = 0$ holds for a probability vector $\hat{\pi}$.*

We say that Q' is smaller than Q in the positive semidefinite order (for generators), $Q' \prec_{pd} Q$, if the matrix $Q - Q'$ is positive semidefinite.

We say that Q' is smaller than Q in the Peskun order (for generators), $Q' \prec_P Q$, if for all $x, y \in \mathbb{E}$ with $x \neq y$ holds $q'(x, y) \leq q(x, y)$.

Lemma 4.9. *Let $Q = (q(x, y) : x, y \in \mathbb{E})$ and $Q' = (q'(x, y) : x, y \in \mathbb{E})$ be generator matrices on a finite set $\mathbb{E}$ such that $\hat{\pi}Q = \hat{\pi}Q' = 0$ holds for a probability vector $\hat{\pi}$. Then*

$$Q \prec_P Q' \Longrightarrow Q' \prec_{pd} Q\,. \tag{4.6}$$

Proof. From $q(x, y) \leq q'(x, y)$ for all $x, y \in \mathbb{E}$ with $x \neq y$ follows for all $x \in \mathbb{E}$ that $q'(x, x) \leq q(x, x)$ holds. So

$$Q' - Q := (q'(x, y) - q(x, y) : x, y \in \mathbb{E})$$

is a generator matrix as well. This implies that $-(Q' - Q)$ is positive semidefinite. □

A direct consequence of Definition 4.8, this lemma, and of Theorem 3.4 is (in the spirit of the previous statements of this section)

Corollary 4.10. *Suppose* $\mathbf{Z} = (\mathbf{Y}, \mathbf{X})$ *and* $\mathbf{Z}' = (\mathbf{Y}', \mathbf{X}')$ *are ergodic unreliable Jackson network processes, having the same arrival and service intensities, and with the same routing regimes, described by* $R = [r_{ij}]_{i,j=0,1,\ldots,J}$ *and rerouting fulfilling Assumption 2.4.*

The breakdown-repair process for $\mathbf{Z}$ *is given in Definition 2.1 and for* $\mathbf{Z}'$ *is defined similarly via functions* $A', B' : 2^{\tilde{J}} \to [0, \infty)$*, as given in Theorem 3.4.*

The processes $\mathbf{Y}$ *and* $\mathbf{Y}'$ *are Markov with generators* $Q^{\mathbf{Y}} = (q^{\mathbf{Y}}(K, L) : K, L \subseteq \tilde{J})$ *of* $\mathbf{Y}$ *and* $Q^{\mathbf{Y}'} = (q^{\mathbf{Y}'}(K, L) : K, L \subseteq \tilde{J})$ *of* $\mathbf{Y}'$ *as defined in* (2.5) *for* $\mathbf{Y}$ *and similar for* $\mathbf{Y}'$*.*

Assume that the stationary distributions of $\mathbf{Y}$ *and* $\mathbf{Y}'$ *are identical, denoted by* $\hat{\pi}$*.*
If $Q^{\mathbf{Y}} \prec_{pd} Q^{\mathbf{Y}'}$ *holds, then*

$$Gap(Q^{\mathbf{Y}'}) \leq Gap(Q^{\mathbf{Y}})$$

Proof. Follows from the relation (3.4). □

An easy to understand property is that whenever the breakdown-repair process $\mathbf{Y}$ of $\mathbf{Z}$ is uniformly faster than the breakdown-repair process $\mathbf{Y}'$ of $\mathbf{Z}'$, i.e., for all $x \neq y$ holds $q'(x, y) \leq q(x, y)$, we have $Gap(Q^{\mathbf{Y}'}) \leq Gap(Q^{\mathbf{Y}})$. This follows directly from Lemma 4.9.

So, for example, if we have $A(D) = \kappa^{|D|} \cdot A'(D)$, and $B(D) = \kappa^{|D|} \cdot B'(D), A \in 2^{\tilde{J}}$, for some $\kappa > 1$, then $Q^{\mathbf{Y}'} \prec_P Q^{\mathbf{Y}}$, and these breakdown-repair processes fulfill the requirement of Corollary 4.10.

4.2 Asymptotic variance

Peskun [Pes73] and Tierney [Tie98] derived comparison theorems for the asymptotic variance of Markov chains for application to optimal selection of MCMC transition kernels in discrete time. These asymptotic variances occur as variance in the limiting distribution of central limit theorems (CLTs) for the MCMC estimators. For our network processes $\mathbf{Z}$ we consider Markov chain $(X_k, k \geq 1)$, say with transition matrix

$$K = I + \varepsilon Q^{\mathbf{Z}}$$

(with $\varepsilon > 0$ sufficiently small). Under some regularity conditions on a homogeneous Markov chain with one step transition kernel K we can obtain CLT of the form

$$\sqrt{n}\Big(\frac{1}{n}\sum_{k=1}^{n} f(X_k) - E_{\tilde{\pi}}(f(X_t))\Big) \xrightarrow{D} N(0, v(f, K)),$$

where the asymptotic variance is

$$v(f,K) = \langle f, f\rangle_{\tilde\pi} - \tilde\pi(f) + 2\sum_{k=1}^{\infty}\langle f, K^k f\rangle_{\tilde\pi}.$$

Regularity conditions under which CLT holds for such Markov chains is a related topic which we shall study in a separate paper. For reversible chains with positive spectral gap it is possible to give conditions in terms of the service rates of our network but a general *iff* characterization in terms of the service rates seem to be an open problem. The next proposition gives a possibility to compare asymptotic variances provided they are finite.

Proposition 4.11. *Consider two ergodic unreliable Jackson networks with the same arrival and service intensities, and state processes* $\mathbf{Z}$ *and* $\mathbf{Z}'$*. Assume that the routing matrices* R *and* R' *are reversible with respect to* ξ*. Both networks follow a rerouting mechanism according to* (2.9) *with the property* (2.10)*, such that* R^D *and* R'^D *are reversible with respect to* ξ^D*.*

If R^D *and* R'^D *are ordered for all* D *in positive definite order,* $R'^D \prec_{pd} R^D$*, then for any function* $f \in L_0^2(\tilde{\mathbb{E}}, \tilde\pi) := \{g \in L^2(\tilde{\mathbb{E}}, \tilde\pi) : \pi(g) = 0\}$ *holds*

$$v(f, I + \varepsilon Q^{\mathbf{Z}}) \geq v(f, I + \varepsilon Q^{\mathbf{Z}'}). \tag{4.7}$$

Proof. For standard Jackson networks without breakdown and repair it is well known that reversibility of the routing matrix R implies reversibility of the joint queue length process. A direct way to prove this is to check the local balance equations with respect to the stationary distribution π. It is easy to see that this way of proof verifies reversibility of the processes $\mathbf{Z}$ and $\mathbf{Z}'$ here as well. The reason is that the breakdown and repair process is reversible with respect to $\hat\pi$, and that for fixed D and $\mathbb{N}_D$ intensities of possible transitions on $\mathbb{N}^{|\tilde J\setminus D|}$ balance locally with respect to the densities $\prod_{\ell\in\tilde J\setminus D}\prod_{i=1}^{n_\ell}\left(\frac{\eta_\ell}{\mu_\ell(i)}\right)$.

Because $\mathbf{Z}$ and $\mathbf{Z}'$ are irreducible we can apply a result of Mira and Geyer [MG99][Theorem 4.2], which states that under this condition (4.7) is equivalent to ordering of the one step correlations for $f \in L_0^2(\mathbb{E}, \tilde\pi)$. The letter statement can be shown exactly as in the proof of Proposition 4.3. □

As in our previous statements, Peskun ordering of the kernels is a sufficient condition for $\prec_{pd}$ ordering, which recovers Tierney's Theorem 4 in [Tie98].

5 Proof of the correlation formulas

For $f, g : \mathbb{E} \to \mathbb{R}$ and the steady state probability $\tilde{\pi}$ of $\mathbf{Z}$ we are interested in the one-step correlation expressions

$$\langle f, Q^{\mathbf{Z}} g\rangle_{\tilde{\pi}} = \sum_{D\subseteq\{1,\dots,J\}} \sum_{\underline{n}\in\mathbb{N}^J} \tilde{\pi}(D,\underline{n}) f(D,\underline{n}) \cdot (Q^{\mathbf{Z}} g)(D,\underline{n}) \tag{5.1}$$

$$= \sum_{D\subseteq\{1,\dots,J\}} \sum_{\underline{n}\in\mathbb{N}^J} \tilde{\pi}(D,\underline{n}) f(D,\underline{n}) \sum_{L\subseteq\{1,\dots,J\}} \sum_{\underline{m}\in\mathbb{N}^J} q^{\mathbf{Z}}(D,\underline{n}; L,\underline{m}) g(L,\underline{m}).$$

Recall that the steady state of $\mathbf{Z}$ is of product form: For $(D,\ n_1,\dots,n_J) \in \mathbb{E}$

$$\tilde{\pi}(D,\ n_1,n_2,\dots,n_J) = \hat{\pi}(D)\cdot\pi(n_1,n_2,\dots,n_J) = \hat{\pi}(D)\cdot C^{-1}\prod_{j=1}^{J}\prod_{i=1}^{n_j}\left(\frac{\eta_j}{\mu_j(i)}\right)$$

Here $\eta = (\eta_1,\dots,\eta_J)$ is the solution of the traffic equation for the admitted customers of the corresponding standard Jackson network (2.2) with $D = \emptyset$ and $\lambda_j = \lambda\cdot r_{0j}$, and C is the normalization constant for the (marginal) joint queue length process

$$C = \sum_{(n_1,\dots,n_J)\in\mathbb{N}^J}\left[\prod_{j=1}^{J}\prod_{i=1}^{n_j}\left(\frac{\eta_j}{\mu_j(i)}\right)\right] = C(J)\,.$$

Proof. (of Proposition 3.1) We evaluate directly

$$\langle f, Q^{\mathbf{Z}} g\rangle_{\tilde{\pi}} = C^{-1} \sum_{D\subseteq\{1,\dots,J\}} \sum_{\underline{n}\in\mathbb{N}^J} \hat{\pi}(D) \prod_{\ell=1}^{J}\prod_{i=1}^{n_\ell}\left(\frac{\eta_\ell}{\mu_\ell(i)}\right) f(D,\underline{n}) \Bigg\{ \sum_{j\in\tilde{J}\setminus D} \lambda_j^D g(D,\underline{n}+e_j) \tag{5.2}$$
$$+ \sum_{j\in\tilde{J}\setminus D}(1-\delta_{0n_j})\mu_j(n_j) r_{j0}^D g(D,\underline{n}-e_j) + \sum_{j\in\tilde{J}\setminus D}(1-\delta_{0n_j}) \sum_{i\in\tilde{J}\setminus D, i\neq j} \mu_j(n_j) r_{ji}^D g(D,\underline{n}-e_j+e_i)$$
$$-\left(\sum_{j\in\tilde{J}\setminus D}\lambda_j^D + \sum_{j\in\tilde{J}\setminus D}(1-\delta_{0n_j})\mu_j(n_j)(1-r_{jj}^D)\right) g(D,\underline{n})$$
$$+\sum_{H\subset D} q^{\mathbf{Y}}(D,H) g(H,\underline{n}) + \sum_{I\supset D} q^{\mathbf{Y}}(D,I) g(I,\underline{n}) - \left(\sum_{H\subset D} q^{\mathbf{Y}}(D,H) + \sum_{I\supset D} q^{\mathbf{Y}}(D,I)\right) g(D,\underline{n}) \Bigg\}.$$

Interchanging summations, regrouping terms, and exploiting the product form structure of the state state distributions, we obtain

$$\langle f, Q^{\mathbf{Z}} g\rangle_{\tilde{\pi}} =$$

$$\sum_{\underline{n}\in\mathbb{N}^J} \pi(\underline{n}) \Bigg[\sum_{D\subseteq\{1,\dots,J\}} \hat{\pi}(D) \quad \Bigg\{ \sum_{H\subset D} q^{\mathbf{Y}}(D,H) f(D,\underline{n}) g(H,\underline{n}) + \sum_{I\supset D} q^{\mathbf{Y}}(D,I) f(D,\underline{n}) g(I,\underline{n}) \Bigg\}$$

$$-f(D,\underline{n}) g(D,\underline{n}) \Bigg\{ \sum_{H\subset D} q^{\mathbf{Y}}(D,H) + \sum_{I\supset D} q^{\mathbf{Y}}(D,I) \Bigg\} \Bigg]$$

$$+ \sum_{D\subseteq\{1,\dots,J\}} \hat{\pi}(D) \sum_{\underline{n}_D\in\mathbb{N}^{|D|}} \prod_{\ell\in D} \left(C_\ell^{-1} \prod_{i=1}^{n_\ell} \frac{\eta_\ell}{\mu_\ell(i)} \right)$$

$$\Bigg[\sum_{\underline{n}_{\tilde{J}\setminus D}\in\mathbb{N}^{J-|D|}} \prod_{\ell\in\tilde{J}\setminus D} \left(C_\ell^{-1} \prod_{i=1}^{n_\ell} \frac{\eta_\ell}{\mu_\ell(i)} \right) f(D,(\underline{n}_D,\underline{n}_{\tilde{J}\setminus D}))$$

$$\Bigg\{ \sum_{j\in\tilde{J}\setminus D} \lambda_j^D g(D,\underline{n}+e_j) + \sum_{j\in\tilde{J}\setminus D} (1-\delta_{0n_j}) \mu_j(n_j) r_{j0}^D g(D,(\underline{n}_D,\underline{n}_{\tilde{J}\setminus D}-e_j))$$

$$+ \sum_{j\in\tilde{J}\setminus D} (1-\delta_{0n_j}) \sum_{i\in\tilde{J}\setminus D, i\neq j} \mu_j(n_j) r_{ji}^D g(D,(\underline{n}_D,\underline{n}_{\tilde{J}\setminus D}-e_j+e_i))$$

$$- \left(\sum_{j\in\tilde{J}\setminus D} \lambda_j^D + \sum_{j\in\tilde{J}\setminus D} (1-\delta_{0n_j}) \mu_j(n_j)(1-r_{jj}^D) \right) g(D,(\underline{n}_D,\underline{n}_{\tilde{J}\setminus D})) \Bigg\} \Bigg]$$

For each fixed $D, \underline{n}_D$ the terms in the last squared brackets are identical to the one step correlation of a Jackson network in equilibrium on node set $\tilde{J}\setminus D$ (with the respective transition rates) with respect to the functions $f(D,(\underline{n}_D,(\cdot)_{\tilde{J}\setminus D}))$ and $g(D,(\underline{n}_D,(\cdot)_{\tilde{J}\setminus D}))$.

We have agreed to denote the generator of such network by $Q^{\mathbf{X}_{\tilde{J}\setminus D}}$, and its steady state by $\pi_{\tilde{J}\setminus D}$, which leads to the proposed formula with the aid of the synthetic networks. □

Proof. (of Proposition 3.2) We restart with the expression (5.2) and observe that for fixed $D\subseteq\{1,\dots,J\}$ the contribution of $-r_{jj}^D$ in the negative terms would be exactly the contribution in the double sum of $i\in\tilde{J}\setminus D, i=j$ in the positive terms, where for $i=j$ would occur $g(D,\underline{n}-e_j+e_j)=g(D,\underline{n})$ otherwise. Together with $\mu_j(0)=0\ \forall j$, incorporating these contributions simplifies our expression to

$$-C^{-1}\sum_{D\subseteq\{1,\dots,J\}}\sum_{\underline{n}\in\mathbb{N}^J}\hat{\pi}(D)\prod_{\ell=1}^{J}\prod_{i=1}^{n_\ell}\left(\frac{\eta_\ell}{\mu_\ell(i)}\right)f(D,\underline{n})g(D,\underline{n}) \tag{5.3}$$

$$\left[\left(\sum_{H\subset D}q^{\mathbf{Y}}(D,H)+\sum_{I\supset D}q^{\mathbf{Y}}(D,I)\right)+\left(\sum_{j\in\tilde{J}\setminus D}\lambda_j^D+\sum_{j\in\tilde{J}\setminus D}\mu_j(n_j)\right)\right]$$

$$+C^{-1}\sum_{D\subseteq\{1,\dots,J\}}\sum_{\underline{n}\in\mathbb{N}^J}\hat{\pi}(D)\prod_{\ell=1}^{J}\prod_{i=1}^{n_\ell}\left(\frac{\eta_\ell}{\mu_\ell(i)}\right)f(D,\underline{n})$$

$$\left\{\sum_{H\subset D}q^{\mathbf{Y}}(D,H)g(H,\underline{n})+\sum_{I\supset D}q^{\mathbf{Y}}(D,I)g(I,\underline{n})\right\}$$

$$+C^{-1}\sum_{D\subseteq\{1,\dots,J\}}\sum_{\underline{n}\in\mathbb{N}^J}\hat{\pi}(D)\prod_{\ell=1}^{J}\prod_{i=1}^{n_\ell}\left(\frac{\eta_\ell}{\mu_\ell(i)}\right)f(D,\underline{n})\left\{\sum_{j\in\tilde{J}\setminus D}\lambda_j^D g(D,\underline{n}+e_j)\right.$$

$$\left.+\sum_{j\in\tilde{J}\setminus D}\mu_j(n_j)r_{j0}^D g(D,\underline{n}-e_j)+\sum_{j\in\tilde{J}\setminus D}\sum_{i\in\tilde{J}\setminus D}\mu_j(n_j)r_{ji}^D g(D,\underline{n}-e_j+e_i)\right\}$$

Our main work will be concerned with the simplified expression represented by the last two lines of the formula above. Isolating this, we have

$$C^{-1}\sum_{D\subseteq\{1,\dots,J\}}\sum_{\underline{n}\in\mathbb{N}^J}\hat{\pi}(D)\prod_{\ell=1}^{J}\prod_{i=1}^{n_\ell}\left(\frac{\eta_\ell}{\mu_\ell(i)}\right)f(D,\underline{n})\left\{\sum_{j\in\tilde{J}\setminus D}\lambda_j^D g(D,\underline{n}+e_j)\right.$$

$$\left.+\sum_{j\in\tilde{J}\setminus D}\mu_j(n_j)r_{j0}^D g(D,\underline{n}-e_j)+\sum_{j\in\tilde{J}\setminus D}\sum_{i\in\tilde{J}\setminus D}\mu_j(n_j)r_{ji}^D g(D,\underline{n}-e_j+e_i)\right\}$$

$$=C^{-1}\sum_{D\subseteq\{1,\dots,J\}}\hat{\pi}(D)\sum_{\underline{n}_D\in\mathbb{N}^D}\prod_{\ell\in D}\prod_{i=1}^{n_\ell}\left(\frac{\eta_\ell}{\mu_\ell(i)}\right)$$

$$\left[\sum_{\underline{n}_{\tilde{J}\setminus D}\in\mathbb{N}^{\tilde{J}\setminus D}}\prod_{\ell\in\tilde{J}\setminus D}\prod_{i=1}^{n_\ell}\left(\frac{\eta_\ell}{\mu_\ell(i)}\right)f(D,\underline{n})\left\{\sum_{j\in\tilde{J}\setminus D}\lambda_j^D g(D,\underline{n}+e_j)\right.\right.$$

$$\left.\left.+\sum_{j\in\tilde{J}\setminus D}\mu_j(n_j)r_{j0}^D g(D,\underline{n}-e_j)+\sum_{j\in\tilde{J}\setminus D}\sum_{i\in\tilde{J}\setminus D}\mu_j(n_j)r_{ji}^D g(D,\underline{n}-e_j+e_i)\right\}\right]$$

$$= C^{-1} \sum_{D\subseteq\{1,\dots,J\}} \hat{\pi}(D) \sum_{\underline{n}_D\in\mathbb{N}^D} \prod_{\ell\in D} \prod_{i=1}^{n_\ell} \left(\frac{\eta_\ell}{\mu_\ell(i)}\right)$$

$$\Bigg[\sum_{\underline{n}_{\tilde{J}\setminus D}\in\mathbb{N}^{\tilde{J}\setminus D}} \prod_{\ell\in\tilde{J}\setminus D} \prod_{i=1}^{n_\ell} \left(\frac{\eta_\ell}{\mu_\ell(i)}\right) \Bigg\{ \sum_{j\in\tilde{J}\setminus D} f(D,\underline{n})\lambda g(D,\underline{n}+e_j) r_{0j}^D$$

$$+ \sum_{j\in\tilde{J}\setminus D} \mu_j(n_j) f(D,\underline{n}) g(D,\underline{n}-e_j) r_{j0}^D + \sum_{j\in\tilde{J}\setminus D} \sum_{i\in\tilde{J}\setminus D} \mu_j(n_j) f(D,\underline{n}) g(D,\underline{n}-e_j+e_i) r_{ji}^D \Bigg\}\Bigg]$$

In the last line, if $n_j > 0$, the expression $\mu_j(n_j)$ cancels against a factor in the steady state probability, otherwise the respective summands vanish.

Canceling in the relevant expressions the factors $\mu_j(n_j) > 0$, shifting the summation indices thereafter, reduces the last expression to

$$\sum_{D\subseteq\{1,\dots,J\}} \sum_{\underline{n}\in\mathbb{N}^J} \tilde{\pi}(D,n) \Bigg\{ \sum_{j\in\tilde{J}\setminus D} \lambda f(D,\underline{n}) g(D,\underline{n}+e_j) r_{0j}^D + \sum_{j\in\tilde{J}\setminus D} \eta_j f(D,\underline{n}+e_j) g(D,\underline{n}) r_{j0}^D$$

$$+ \sum_{j\in\tilde{J}\setminus D} \sum_{i\in\tilde{J}\setminus D, i\neq j} \eta_j f(D,\underline{n}+e_j) g(D,\underline{n}+e_i) r_{ji}^D + \sum_{j\in\tilde{J}\setminus D} \eta_j f(D,\underline{n}+e_j) g(D,\underline{n}+e_j) r_{jj}^D \Bigg\}.$$

This simple expression seems to be noteworthy for its own, and could be substituted into the correlation expression. But we can do even better. Write the last expression as (we underbrace some intuition and use $\eta_j = \eta_j^D$ for $j \in \tilde{J} \setminus D$)

$$\sum_{D\subseteq\{1,\dots,J\}} \sum_{\underline{n}\in\mathbb{N}^J} \tilde{\pi}(D,n) \Bigg\{ \sum_{j\in\tilde{J}\setminus D} \underbrace{\lambda f(D,\underline{n}) g(D,\underline{n}+e_j) r_{0j}^D}_{0\to j} + \sum_{j\in\tilde{J}\setminus D} \underbrace{\eta_j^D f(D,\underline{n}+e_j) g(D,\underline{n}) r_{j0}^D}_{j\to 0}$$

$$+ \sum_{j\in\tilde{J}\setminus D} \sum_{i\in\tilde{J}\setminus D} \underbrace{\eta_j^D f(D,\underline{n}+e_j) g(D,\underline{n}+e_i) r_{ji}^D}_{j\to i} + \underbrace{\lambda f(D,\underline{n}) g(D,\underline{n}) r_{00}^D}_{0\to 0} - \lambda f(D,\underline{n}) g(D,\underline{n}) r_{00}^D \Bigg\}$$

From Lemma 2.5 we know that with $\eta_0^D := \lambda$ the vector $\hat{\eta}^D := (\eta_j^D, j \in \tilde{J}_0 \setminus D)$ solves the equation $x = x \cdot R^D$. Denote by $\xi^D := (\xi_i^D : i \in \tilde{J}_0 \setminus D)$ the stochastic solution of $x = x \cdot R^D$. Inserting this into the last expression yields

$$\sum_{D\subseteq\{1,\ldots,J\}} \frac{\lambda}{\xi_0^D} \sum_{\underline{n}\in\mathbb{N}^J} \tilde{\pi}(D,n) \left\{ \underbrace{\xi_0^D r_{00}^D f(D,\underline{n}) g(D,\underline{n})}_{0\to 0} + \sum_{j\in\tilde{J}\setminus D} \underbrace{\xi_0^D r_{0j}^D f(D,\underline{n}) g(D,\underline{n}+e_j)}_{0\to j} \right.$$
$$\left. + \sum_{j\in\tilde{J}\setminus D} \underbrace{\xi_j^D r_{j0}^D f(D,\underline{n}+e_j) g(D,\underline{n})}_{j\to 0} + \sum_{j\in\tilde{J}\setminus D} \sum_{i\in\tilde{J}\setminus D} \underbrace{\xi_j^D r_{ji}^D f(D,\underline{n}+e_j) g(D,\underline{n}+e_i)}_{j\to i} \right\} \quad (5.4)$$
$$+ \sum_{D\subseteq\{1,\ldots,J\}} \sum_{\underline{n}\in\mathbb{N}^J} \tilde{\pi}(D,n)\lambda f(D,\underline{n}) g(D,\underline{n}) r_{00}^D$$

We are now ready to insert (5.4) into the complete correlation expressions (5.1), resp. (5.3). Let $e_0 = (0,0,\ldots,0)$.

$$\langle f, Q^{\mathbf{Z}} g\rangle_{\tilde{\pi}} = \sum_{D\subseteq\{1,\ldots,J\}} \sum_{\underline{n}\in\mathbb{N}^J} \tilde{\pi}(D,\underline{n}) f(D,\underline{n}) \sum_{L\subseteq\{1,\ldots,J\}} \sum_{\underline{m}\in\mathbb{N}^J} q^{\mathbf{Z}}(D,\underline{n};L,\underline{m}) g(L,\underline{m})$$
$$= C^{-1} \sum_{D\subseteq\{1,\ldots,J\}} \sum_{\underline{n}\in\mathbb{N}^J} \hat{\pi}(D) \prod_{\ell=1}^{J} \prod_{i=1}^{n_\ell} \left(\frac{\eta_\ell}{\mu_\ell(i)}\right) f(D,\underline{n})$$
$$\left\{ \sum_{H\subset D} q^{\mathbf{Y}}(D,H) g(H,\underline{n}) + \sum_{I\supset D} q^{\mathbf{Y}}(D,I) g(I,\underline{n}) \right\}$$
$$+ C^{-1} \sum_{D\subseteq\{1,\ldots,J\}} \sum_{\underline{n}\in\mathbb{N}^J} \hat{\pi}(D) \prod_{\ell=1}^{J} \prod_{i=1}^{n_\ell} \left(\frac{\eta_\ell}{\mu_\ell(i)}\right) \frac{\lambda}{\xi_0^D}$$
$$\left\{ \underbrace{\xi_0^D r_{00}^D f(D,\underline{n}) g(D,\underline{n})}_{0\to 0} + \sum_{j\in\tilde{J}\setminus D} \underbrace{\xi_0^D r_{0j}^D f(D,\underline{n}) g(D,\underline{n}+e_j)}_{0\to j} \right.$$
$$\left. + \sum_{j\in\tilde{J}\setminus D} \underbrace{\xi_j^D r_{j0}^D f(D,\underline{n}+e_j) g(D,\underline{n})}_{j\to 0} + \sum_{j\in\tilde{J}\setminus D} \sum_{i\in\tilde{J}\setminus D} \underbrace{\xi_j^D r_{ji}^D f(D,\underline{n}+e_j) g(D,\underline{n}+e_i)}_{j\to i} \right\}$$
$$- C^{-1} \sum_{D\subseteq\{1,\ldots,J\}} \sum_{\underline{n}\in\mathbb{N}^J} \hat{\pi}(D) \prod_{\ell=1}^{J} \prod_{i=1}^{n_\ell} \left(\frac{\eta_\ell}{\mu_\ell(i)}\right) \lambda f(D,\underline{n}) g(D,\underline{n}) r_{00}^D$$
$$- C^{-1} \sum_{D\subseteq\{1,\ldots,J\}} \sum_{\underline{n}\in\mathbb{N}^J} \hat{\pi}(D) \prod_{\ell=1}^{J} \prod_{i=1}^{n_\ell} \left(\frac{\eta_\ell}{\mu_\ell(i)}\right) f(D,\underline{n}) g(D,\underline{n})$$
$$\left[\left(\sum_{H\subset D} q^{\mathbf{Y}}(D,H) + \sum_{I\supset D} q^{\mathbf{Y}}(D,I) \right) + \left(\sum_{j\in\tilde{J}\setminus D} \lambda_j^D + \sum_{j\in\tilde{J}\setminus D} \mu_j(n_j) \right) \right]$$

$$
\begin{aligned}
&= \sum_{D\subseteq\{1,\dots,J\}} \sum_{\underline{n}\in\mathbb{N}^J} \tilde{\pi}(D,\underline{n}) \left\{ \sum_{H\subset D} q^{\mathbf{Y}}(D,H) f(D,\underline{n}) g(H,\underline{n}) + \sum_{I\supset D} q^{\mathbf{Y}}(D,I) f(D,\underline{n}) g(I,\underline{n}) \right\} \\
&+ \sum_{D\subseteq\{1,\dots,J\}} \sum_{\underline{n}\in\mathbb{N}^J} \tilde{\pi}(D,\underline{n}) \frac{\lambda}{\xi_0^D} \left\{ \sum_{j\in(\tilde{J}\cup\{0\})\setminus D} \sum_{i\in(\tilde{J}\cup\{0\})\setminus D} \xi_j^D r_{ji}^D f(D,\underline{n}+e_j) g(D,\underline{n}+e_i) \right\} \\
&- \sum_{D\subseteq\{1,\dots,J\}} \sum_{\underline{n}\in\mathbb{N}^J} \tilde{\pi}(D,\underline{n}) \lambda f(D,\underline{n}) g(D,\underline{n}) r_{00}^D \\
&- \sum_{D\subseteq\{1,\dots,J\}} \sum_{\underline{n}\in\mathbb{N}^J} \tilde{\pi}(D,\underline{n}) f(D,\underline{n}) g(D,\underline{n}) \left\{ \left[\sum_{H\subset D} q^{\mathbf{Y}}(D,H) + \sum_{I\supset D} q^{\mathbf{Y}}(D,I) \right] \right. \\
&\qquad\qquad \left. + \left[\sum_{j\in\tilde{J}\setminus D} \lambda_j^D + \sum_{j\in\tilde{J}\setminus D} \mu_j(n_j) \right] \right\}
\end{aligned}
$$

This yields finally the desired correlation formula from Proposition 3.2:

$$
\begin{aligned}
&\langle f, Q^{\mathbf{Z}} g\rangle_{\tilde{\pi}} \\
= &\sum_{D\subseteq\{1,\dots,J\}} \sum_{\underline{n}\in\mathbb{N}^J} \tilde{\pi}(D,\underline{n}) \left\{ \sum_{H\subset D} q^{\mathbf{Y}}(D,H) f(D,\underline{n}) g(H,\underline{n}) + \sum_{I\supset D} q^{\mathbf{Y}}(D,I) f(D,\underline{n}) g(I,\underline{n}) \right\} \\
&+ \sum_{D\subseteq\{1,\dots,J\}} \sum_{\underline{n}\in\mathbb{N}^J} \tilde{\pi}(D,\underline{n}) \frac{\lambda}{\xi_0^D} \left\{ \sum_{j\in(\tilde{J}\cup\{0\})\setminus D} \sum_{i\in(\tilde{J}\cup\{0\})\setminus D} \xi_j^D r_{ji}^D f(D,\underline{n}+e_j) g(D,\underline{n}+e_i) \right\} \\
&- \sum_{D\subseteq\{1,\dots,J\}} \sum_{\underline{n}\in\mathbb{N}^J} \tilde{\pi}(D,\underline{n}) f(D,\underline{n}) g(D,\underline{n}) \left\{ \left[\sum_{H\subset D} q^{\mathbf{Y}}(D,H) + \sum_{I\supset D} q^{\mathbf{Y}}(D,I) \right] \right. \\
&\qquad\qquad \left. + \left[\underbrace{\sum_{j\in\tilde{J}\setminus D} \lambda_j^D + \lambda r_{00}^D}_{=\lambda} + \sum_{j\in\tilde{J}\setminus D} \mu_j(n_j) \right] \right\}
\end{aligned}
$$

□

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