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Jennifer Sommer, Hans Daduna, Bernd Heidergott

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UND STOCHASTISCHE PROZESSE

BUNDESSTR. 55, D – 20146 HAMBURG

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Jennifer Sommer* Hans Daduna† Bernd Heidergott‡

Abstract

We study generalized Jackson networks with single server stations, where some nodes have an infinite supply of work. We allow in our model simultaneous breakdown of groups of servers, and group repair strategies, thereby capturing the feature that repairing several servers simultaneously may lead to more efficient repair actions and thus may reduce the repair time. In addition, we allow for servers to have infinite supply. We will establish the existence of a steady-state distribution of the queue-length vector at stable nodes for different type of failure-regimes. As it turns out, the distribution of the failure/repair regime and the steady-state distribution of the queue-length vector at stable nodes decouple in a product-form way, and we will provide closed-form solutions for the long run throughput of the network. It is a well known problem in reliability, that the distribution of the time between breakdowns is hard to estimate. We illustrate the impact of our results with an analysis of parameter insecurity of the throughput of the systems for different breakdown regimes.

Keywords: Jackson networks; group breakdowns, group repairs; infinite supply queues;

1 Introduction

Open Jackson networks are a well established class of models in, e.g., production, telecommunication, computer systems; for surveys see [Kel79] and [CY01]. Although real life systems typically don't meet the rather restrictive conditions for Jackson networks, such as exponential service times, independence of service times amongst servers, and infinite waiting capacity at service stations, Jackson networks are typically used in performance analysis of complex networks for coming up with an approximate indication of the long-run system performance. Indeed, simulation of complex networks is (i) time consuming, and (ii) the true specification of service time distributions and possible stochastic dependencies between the behavior of nodes is often not known, which makes Jackson networks a valuable tool in performance analysis.

In this paper, we study Jackson networks with the following two additional features:

- We allow for breakdown and repair of individual servers and for *simultaneous breakdown and repair of groups of servers* (i.e., repair can be grouped). This allows (i) to model simultaneous breakdown of groups of servers, and (ii) model group repair strategies. For example, repairing several servers simultaneously may lead to more efficient repair actions and thus may reduce the repair time.
- We allow for servers to have *infinite supply*. Infinite supply has the aim to utilize the capacity of a server to the fullest. For example, in service center models it is typically assumed that an agent, when not answering a call, switches to low priority works such as answering email and

*Hamburg Port Consulting GmbH, Hamburg, email: j_mylosz@hotmail.com

†Department of Mathematics, University of Hamburg, email: daduna@uni-hamburg.de

‡Department of Econometrics and Operations Research and Tinbergen Institute, Vrije Universiteit Amsterdam, email: bheidergott@feweb.vu.nl

administrative duties. In a traffic model of a highway network, an infinite supply node represents a highway segment with an on-ramp during a rush-hour period where a constant flow of vehicles requiring access to the highway is present. In production processes, an additional inventory with raw material guarantees that the machine will not idle even when there is sometimes no external demand.

This paper provides an analysis of networks enjoying both of the above features within one framework. The key contribution of the paper is that we obtain a product-form solution for the steady-state distribution of the queue-length vector at stable nodes for different type of failure-regimes (a precise definition of "stable node" will be provided later in the text). In addition, our analysis shows that the distribution of the failure/repair process and the steady-state distribution of the queue-length vector at stable nodes decouple in a product-form way. Elaborating on these results we establish closed-form solutions for the long-run throughput of subnetworks and of the network, respectively.

With our product-form result at hand, we further investigate the impact of the distribution of the time between breakdowns of the individual servers on the throughput of the network. This is motivated by the fact that during usual operation breakdowns are to be avoided and typically only censored observations are available, which is in contrast to repairs (indeed, repair times are observable and can often be influenced by a decision maker). We will provide a robustness analysis of the system throughput with respect to this parameter insecurity and will show how a similar analysis for sojourn times is derived. Our analysis shows how for different breakdown and repair regimes, the corresponding risk profiles for system-oriented and customer-oriented performance metrics can be evaluated. It is worth noting that this efficient risk analysis step is only possible due to the simple closed-form solutions obtained for the performance measures. We believe that a robustness analysis as performed in this paper should become a part of performance modeling.

The paper is organized as follows. In Section 2 we present a motivating example, and we will introduce the basic notation and concepts with the help of this example. A literature review is provided in Section 3. The main technical analysis is carried out in Section 4. In Section 5, we provide explicit solutions for availability and performance measures. Eventually, Section 6 is devoted to robustness analysis of the performance measures. Most of the technical proofs are postponed to the Appendix.

2 Motivating Example

We explain the basic setup for our analysis with the following example. Consider the Jackson network with $J = 7$ nodes, depicted in Figure 1, where we assume throughout that servers are FCFS single-server stations with infinite waiting capacity. The set of all nodes is denoted by $\tilde{J} = \{1, \ldots, J\}$. In addition we will denote the outside world, that is, the source as well as the sink, as node 0. We identify the labeling of the service rate μ_i with the server labels, so that μ_i refers to server i. There are two arrival streams with arrival rate λ_1 and λ_2, respectively, and throughout the paper it is assumed that arrival processes are of Poisson type. Routing is Markovian and possible routes are indicated by arrows. Specifically, the probability that a customer after finishing service at server, say, i, moves to server j, is given by $r(i, j)$. Apart from these classical features of a Jackson network, we assume that there is a set of nodes V such each $j \in V$ has *infinite supply*. This is indicated in the figure by dotted arrows. We complement this definition by introducing $W = \tilde{J} \setminus V$ as the set of all nodes without infinite supply. In this example, $V = \{1, 5, 7\}$ and $W = \{2, 3, 4, 6\}$. For $j \in V$, customers in the infinite supply chain have low priority, where customers arriving either from the outside or from another server have high priority with preemptive-resume regime: Service of a low priority customer is interrupted as soon as a high priority customer arrives. When a low priority customer is served and fed into the network, he becomes a high priority customer. Routing decisions, service times, and inter-arrival times are assumed to be mutually independent. The network displayed in Figure 1 is inspired by a model of a highway network. Nodes represent road segments and nodes with infinite supply model road segments with an on-ramp, where it is assumed that there is constant flow of incoming traffic to the on-ramp. Note that

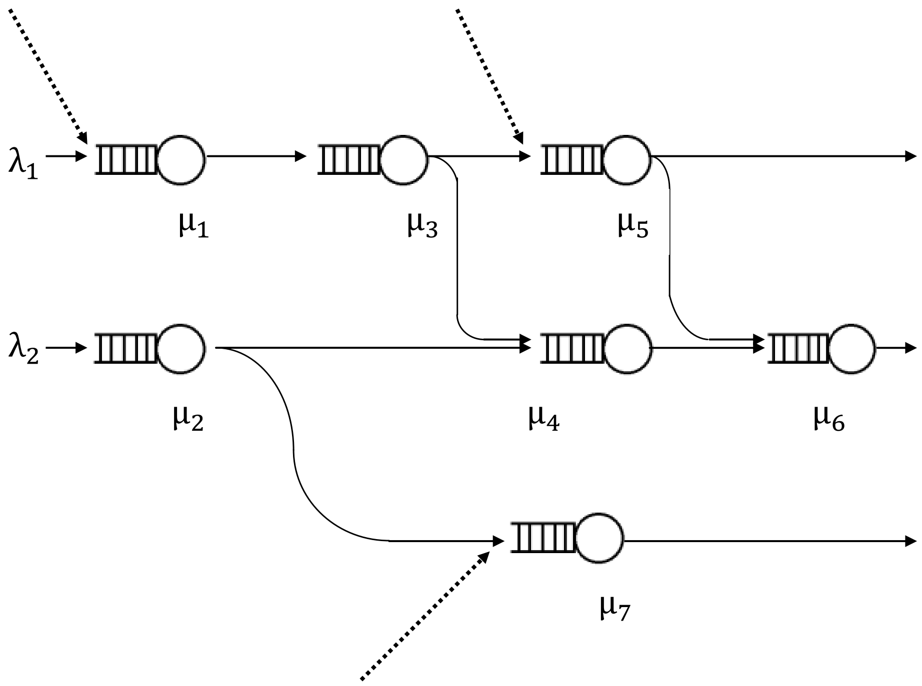

Figure 1: Example of queueing network with infinite supply

in queueing models for traffic systems one typically only models the flow in one direction and fits the service rate of the queues to the traffic characteristics, see, for example, [VWV00, WV06].

Suppose that $\lambda_1 < \mu_1, \mu_3$ and $\mu_1 > \mu_3$. Then, without infinite supply at node 1, node 3 is stable in the classical definition as the rate with which customers arrive to node 3 is smaller than the service rate. In case of infinite supply at node 1, however, node 1 acts as Poisson source and the incoming traffic rate at node 3, which is then μ_1, is larger than the service rate, which causes node 3 to become instable. Whether a node is stable or not can be decided from the traffic equations of the network and details are provided in Section 4.1. We let $S \subseteq \tilde{J}$ denote the set of stable nodes and by U the set of unstable nodes.

To complete this introductory example, we denote the set of nodes that are unreliable by D. For example, take $D = \{3, 5\}$. Breakdown and repair can follow complicated schemes: Nodes may break down isolated or in groups, and repair may happen similarly. It is not required that nodes which are broken down simultaneously are repaired at the same time. We make the simplifying assumption that breakdown and repair intensities do not depend on queue lengths. As will become clear from our analysis, in this model stability of nodes is independent of the breakdown/repair regime. In other words, it is not possible to "create" instability in a network by choosing a misguided repair action.

Definition 2.1 *If the nodes in $\emptyset \subseteq I \subseteq D$ are broken down, then*

1. *if $I \subset H \subseteq D$, the nodes in $H \setminus I$ break down with rate $\alpha(I, H) \geq 0$,*
2. *if $\emptyset \subseteq K \subset I$, the nodes in $I \setminus K$ are repaired with rate $\beta(I, K) \geq 0$.*

$\alpha(\cdot,\cdot), \beta(\cdot,\cdot)$ may be constructed from any pair of non-negative functions $A, B : \mathcal{P}(D) \to [0, \infty)$, subject to $A(\emptyset) = B(\emptyset) = 1$ and for all $I \subset H \subseteq D$ with $A(H)/A(I) < \infty$ and for all $\emptyset \subseteq K \subset I$ $B(I)/B(K) < \infty$ (where we set $0/0 = 0$).

With these functions we set for all subsets of down nodes $I \subseteq D$

$$\alpha(I,H) = \frac{A(H)}{A(I)}, \quad I \subset H \subseteq D, \quad \text{and} \quad \beta(I,K) = \frac{B(I)}{B(K)}, \quad \emptyset \subseteq K \subset I. \tag{1}$$

The parametric form (1) of the breakdown and repair rates is selected because it stems from a versatile recipe for to construct correlated multidimensional birth-death processes. A statistical procedure to check whether this form is justified is to determine in a first step all possible values $A(I) = \alpha(\emptyset, I\})$ and $B(I) = \beta(I, \emptyset), I \subseteq D$, and then to check stepwise (1).

In our example, there are possible breakdown scenarios $\{3\}$, $\{5\}$, $\{3,5\}$. We let $A(\emptyset) = B(\emptyset) = 1$ and assume that the rate with which a breakdown of server $i = 3,5$ occurs is given by $A(\{i\}) = \tau_i$, and the corresponding repair rate by $B(\{i\}) = \rho_i$. In the same vein, let $A(\{3,5\}) = \tau_3 + \tau_5$ be the rate with which the operating system enters breakdown state $\{3,5\}$, and let $B(\{3,5\}) = 2\min(\rho_3, \rho_5)$ be the rate with which the systems jumps from breakdown state $\{3,5\}$ to the state $\emptyset$ with all server operating. The rates $A(I)$ and $B(K)$ are the basic input data for our model. Note that these rates do, for example, not cover transition rate from $\{3\}$ to $\{3,5\}$. Following (1) we now construct transition rate mappings α and β covering all possible intermediate state transitions. More specifically, for $i \in D$ let $\alpha(\emptyset, \{i\}) = \tau_i$ and $\beta(\{i\}, \emptyset) = \rho_i$, $\alpha(\emptyset, \{3,5\}) = \tau_3 + \tau_5$ and $\beta(\{3,5\}, \emptyset) = 2\min(\rho_3, \rho_5)$. Eventually, given that node $i \in D$ is broken down, the breakdown rate of the other node, say, j be given by

$$\alpha(\{i\}, \{3,5\}) = \frac{\tau_3 + \tau_5}{\tau_i}$$

whereas from breakdown scenario $\{3,5\}$ node $i \in D$ alone has repair rate

$$\beta(\{3,5\}, \{j\}) = \frac{2\min(\rho_3, \rho_5)}{\rho_i}.$$

All other values for breakdown and repair rates are zero.

Note that these breakdown and repair rates from (1) define a generator for a Markov process $Y = \{Y_t : t \geq 0\}$ on state space $(\mathcal{P}(D), 2^{\mathcal{P}(D)})$. By inspection we see that

$$\pi := (\pi(K) := A(K)/B(K), \; K \in \mathcal{P}(D)) \tag{2}$$

fulfills

$$\pi(K) \cdot \alpha(K, K \cup G) = \pi(K \cup G) \cdot \beta(K \cup G, K)$$

for all $K, G \in \mathcal{P}(D)$ which implies that, after normalization, π is the steady state of the breakdown and repair process. Even more, we have proved that Y is reversible.

In our example, we have only three possible breakdown scenarios and the repair process has states $\emptyset, \{3\}, \{5\}, \{3,5\}$. The stationary distribution π of Y reads

$$\pi(\emptyset) = c^{-1}, \; \pi(\{3\}) = c^{-1}\frac{\tau_3}{\rho_3}, \; \pi(\{5\}) = c^{-1}\frac{\tau_5}{\rho_5}, \; \pi(\{3,5\}) = c^{-1}\frac{\tau_3 + \tau_5}{2\min(\rho_3, \rho_5)},$$

with normalizing constant

$$c = 1 + \frac{\tau_3}{\rho_3} + \frac{\tau_5}{\rho_5} + \frac{\tau_3 + \tau_5}{2\min(\rho_3, \rho_5)}.$$

3 Literature Review and Related Work

Investigation of generalized Jackson networks with infinite supply has recently found much interest in the literature and it turned out that the feature of infinite supply makes analysis of the network considerably harder than that of classical product form networks of the BCMP and Kelly type.

Infinite supply of lower-priority work ($\equiv$ infinite virtual queues $\equiv$ IVQ) is used frequently, e.g., in [LY75] in an $M/G/1$ queueing system to utilize idle times. Recent works using this concept of infinite supply are, e.g., [Guo08] where generalized Jackson networks are considered or [KNW09] where a push-pull network with infinite supply is investigated.

A special class of multi-class queueing networks with virtual infinite buffers has been introduced in [KW02] and [AW05]. For single-class ergodic networks of Jackson type with infinite supply of work at some nodes G. Weiss [Wei05] has obtained a product-form solution of the steady state queue length distribution at nodes without infinite supply. He discussed as an example a particular computer communication system that works according to MAN (metropolitan area network) Ethernet RPR (resilient packet ring), where ring traffic has priority over the traffic generated at nodes.

Another application from a different field where such model fits is in wireless sensor networks. The nodes (sensors) continuously sense their environment and have to forward the data to a central station (sink). This is usually not possible by direct communication, so the nodes act additionally as transmission stations for data from other sensor nodes. If forwarding transmissions from other nodes has priority, the own data constitute the infinite buffer which generates the infinite supply for the node.

The work of Guo [Guo08] and of Guo, Lefeber, Nazarathy, Weiss, Zhang [GLN13] is on general multi-class queueing networks with IVQs under different scheduling policies for the servers. These policies guide the nodes' decisions how to dedicate their activities to either the regular standard queues or the infinite virtual queues. The key research question is the interplay of the production of jobs from the IVQs and stability of the standard queues.

Another class of models where additional work is added whenever a server becomes idle are *queues with vacations*. If a server observes an empty queue "he goes away to serve at some other place a customer", and returns thereafter. If he finds customers waiting there, he immediately starts servicing them, but when on his return his queue is empty again, he takes "another vacation" from his main queue to serve somewhere else, and so on. For a survey, see [Dos90].

4 Jackson Networks with infinite Supply: Breakdown of Nodes

In this section, we consider Jackson networks which at some nodes have infinite supply and where some nodes break down randomly and are repaired thereafter. Breakdowns of nodes in standard Jackson networks were investigated in [SD03] and [MD09]. It turns out that breakdown of nodes with infinite supply require a more specific regime to control breakdown and repair. The consequences of breakdown of a node are as follows. Whenever a node breaks down,

- service at this node is interrupted, customers (of high as well as of low priority) are frozen there to wait for restart of the service, which is resumed at the point where it was preempted,
- no new customers are admitted to enter that node,
- customers who select a broken down node to visit are rerouted according one of the classical rules: stalling, skipping or blocking rs-rd , which will be defined next,
- all these rules, if applicable, are valid for both classes, high and low priority, of customers.

Note, that rerouting does not apply directly to low priority customers, because on departure from a node with infinite supply they are transformed immediately to high priority, and only thereafter necessary rerouting is in force. We follow [SD03] and distinguish the following three rerouting rules:

Stalling: Whenever a node breaks down the service system is frozen, i.e., all arrival processes are interrupted and the service anywhere in the network is stopped. Thus every movement of customers inside the network and arrivals to the network from the outside are stopped until all broken down nodes are repaired again, i.e., if nodes in $I \neq \emptyset$ are broken down then for all $i \in \tilde{J}$ the I-dependent rates are set to $\lambda_i^I = \mu_i^I = 0$. We assume that the stopped nodes which are in up status are waiting in warm standby, i.e., they can break down although they are stalled. Stalling is applied, for example, in the automotive industry for decreasing variability of the flow of materials. Indeed, stalling prevents that servers continue sending parts to a server that is broken down and thereby prevents piling up inventory.

Skipping: Customers are not allowed to enter down nodes and have to skip these nodes. I.e., if the next destination of a customer is a down node, the customer jumps to this node spending no time there and immediately performs the next jump according to his routing regime until he arrives at a node in up status or leaves the network. Whenever a breakdown of a subset $I \subseteq D$ occurs, customers are rerouted according to the following routing matrix $R^I = (r^I(i,j) : i,j \in \{0\} \cup \tilde{J} \setminus I)$:

$$r^I(j,k) = r(j,k) + \sum_{i \in I} r(j,i) r^I(i,k) \quad \textit{for } k,j \in \{0\} \cup \tilde{J} \setminus I \tag{3}$$

with

$$r^I(i,k) = r(i,k) + \sum_{l \in I} r(i,l) r^I(l,k) \quad \textit{for } i \in I, k \in \{0\} \cup \tilde{J} \setminus I. \tag{4}$$

The external arrival rates during a breakdown of I are

$$\lambda_j^I = \lambda_j + \sum_{i \in I} \lambda_i r^I(i,j) \quad \textit{for } j \in \tilde{J} \setminus I \tag{5}$$

and $\lambda_k^I = 0$ for $k \in I$. The service intensities are

$$\mu_i^I = \begin{cases} \mu_i, & i \in \tilde{J} \setminus I, \\ 0, & \text{otherwise.} \end{cases}$$

Blocking rs-rd: Broken down stations are blocked. A customer whose next destination is a down node stays at his present node to obtain another service there. After the repeated service (rs) the customer chooses his next destination anew according to his routing matrix (random destination (rd)). Whenever a breakdown of a subset $I \subseteq D$ occurs, customers are rerouted according to the following routing matrix $R^I = (r^I(i,j) : i,j \in \{0\} \cup \tilde{J} \setminus I)$ with

$$r^I(i,j) = \begin{cases} r(i,j), & i,j \in \{0\} \cup \tilde{J} \setminus I, \ i \neq j, \\ r(i,j) + \sum_{k \in I} r(i,k), & i \in \{0\} \cup \tilde{J} \setminus I, \quad i = j. \end{cases} \tag{6}$$

The external arrival rates during a breakdown of I are $\lambda_j^I = \lambda_j$ for $j \in \tilde{J} \setminus I$ and $\lambda_j^I = 0$ otherwise as well as the service intensities are

$$\mu_i^I = \begin{cases} \mu_i, & i \in \tilde{J} \setminus I, \\ 0, & \text{otherwise.} \end{cases}$$

With these preparing definitions we summarize our construction. Consider a Jackson network with infinite supply and unreliable nodes and rerouting regime is either stalling, skipping, or blocking rs-rd with the respective rerouting matrices R^I. Denote $R^\emptyset := R$. Then the joint availability-queue length process is described by the Markov process $(Y, X) = ((Y(t); X_1(t), ..., X_J(t)) : t \in \mathbb{R}_+)$ on the state space $\mathcal{P}(D) \times \mathbb{N}^J$ with transition rates matrix $Q = (q(z,z') : z, z' \in \mathcal{P}(D) \times \mathbb{N}^J)$ defined for all $z = (n_1, \ldots, n_J)$

and all $I \subseteq D, i, j \in \tilde{J} \setminus I, i \neq j$:

$$q(I, n_1, ..., n_i, ..., n_J; I, n_1, ..., n_i + 1, ..., n_J) = \lambda_i^I + \sum_{k \in V \setminus I} \mu_k^I r^I(k, i) 1_{\{0\}}(n_k),$$

$$q(I, n_1, ..., n_i, ..., n_J; I, n_1, ..., n_i - 1, ..., n_J) = \mu_i^I r^I(i, 0) 1_{\mathbb{N}_+}(n_i),$$

$$q(I, n_1, ..., n_i, ..., n_j, ..., n_J; I, n_1, ..., n_i - 1, ..., n_j + 1, ..., n_J) = \mu_i^I r^I(i, j) 1_{\mathbb{N}_+}(n_i),$$

$$q(I, n_1, ..., n_J; H, n_1, ..., n_J) = \alpha(I, H) = \frac{A(H)}{A(I)}, \quad I \subset H \subseteq D,$$

$$q(I, n_1, ..., n_J; K, n_1, ..., n_J) = \beta(I, K) = \frac{B(I)}{B(K)}, \quad K \subset I \subseteq D,$$

$$q(I, n_1, ..., n_J; I, n_1, ..., n_J) = -\sum_{i \in \tilde{J} \setminus I} \lambda_i^I - \sum_{i \in \tilde{J} \setminus I} \sum_{k \in V \setminus I} \mu_k^I r^I(k, i) 1_{\{0\}}(n_k)$$
$$- \sum_{i \in \tilde{J} \setminus I} \mu_i^I (1 - r^I(i, i)) 1_{\mathbb{N}_+}(n_i) - \sum_{I \subset H \subseteq D} \alpha(I, H) - \sum_{K \subset I \subseteq D} \beta(I, K),$$

and $q(z, z') = 0$ otherwise for $z \neq z'$.

Theorem 4.1 *With the above definitions:*

(i) If at time t all nodes are up, i.e., $Y(t) = \emptyset$, then the departure streams from node $j \in V$ is a Poisson process with rate μ_j. Thus the departure stream from $j \in V$ to $i \in \tilde{J}$ is Poisson with rate $\mu_j r(j, i)$.

(ii) Whenever nodes in $I \neq \emptyset$ are broken down and either skipping or blocking rs-rd is in force, the departure stream of node $j \in V \setminus I$ with infinite supply in up-status is Poisson with rate μ_j. The departure stream from $j \in V \setminus I$ to $i \in \tilde{J} \setminus I$ is Poisson with rate $\mu_j r^I(j, i)$, where $r^I(j, i)$ is determined by the rerouting regime in force. If a node $k \in V$ with infinite supply is broken down, i.e., $k \in V \cap I$, the Poisson departure stream of this node is interrupted until its server is repaired. In case of stalling, all Poisson arrival streams stop whenever a breakdown occurs ($I \neq \emptyset$), and the Poisson streams are reactivated when all nodes recur to the up-status.

Proof: Consider the network in its different availability states of nodes:
Case I: All nodes are in up-status ($I = \emptyset$), all departure times from node $j \in V$

- are independent of the state of the node due to the infinite supply,
- and therefore have independent and identically exponentially distributed interdeparture times with rate μ_j, the service rate of the node.

Thus, the departure stream of node $j \in V$ is a Poisson process with rate μ_j. Hence for node $i \in \tilde{J}$, the arrival stream from node $j \in V$ is a Poisson process with rate $\mu_j r(j, i)$, because a portion of $r(j, i)$ of the departure stream is directed to node $i \in \tilde{J}$.

Case II: Nodes in $I \neq \emptyset$ are broken down and rerouting is according to blocking rs-rd or skipping, all departure times from node $j \in V \setminus I$

- are independent of the state of the node due to the infinite supply,
- and therefore have independent and identically exponentially distributed inter departure times with rate μ_j, the service rate of the node.

Therefore the departure stream of node $j \in V$ is a Poisson process with rate μ_j. Thus, for node $i \in \tilde{J} \setminus I$, the arrival stream from node $j \in V \setminus I$ is a Poisson process with rate $\mu_j r^I(j, i)$, because a portion of $r^I(j, i)$ of the departure stream is directed to node $i \in \tilde{J} \setminus I$ (decomposition of a Poisson process by some probability independent of all the process and of its times of events). this holds for all states $I \subseteq D$ of the availability process Y which means that whenever a node with infinite supply breaks down, its Poisson departure stream is interrupted until the node is repaired.

Case III: Nodes in $I \neq \emptyset$ are broken down, under stalling all network processes - except breakdown and repair processes - are frozen, i.e., no service is provided in the network and there are no arrivals. Thus, as long as not all nodes are in up status, there are no customer flows in the network. □

4.1 The Traffic Equations under Breakdowns

Different traffic equations required for analysis of the long-time behavior are provided in the subsequent definition. We start with providing the traffic equations for networks without breakdowns and repairs.

Definition 4.2 *The* general traffic equations *for Jackson networks with infinite supply but no breakdowns and repairs are*

$$\eta_i = \lambda_i + \sum_{j \in W} \min(\eta_j, \mu_j) r(j,i) + \sum_{j \in V} \mu_j r(j,i), \quad i \in \tilde{J}\,. \tag{7}$$

A node i is stable if η_i determined by (7) *is strictly less than μ_i, otherwise the node is unstable.*

It is worth noting that for networks without unstable nodes the traffic equations in Definition 7 reduce to the (standard) traffic equations of a Jackson network with infinite supply *[Wei05]*

$$\eta_i = \lambda_i + \sum_{j \in W} \eta_j r(j,i) + \sum_{j \in V} \mu_j r(j,i), \quad i \in \tilde{J}. \tag{8}$$

In the following we present the version of the traffic equations which are appropriate for the specific breakdown regime.

Definition 4.3 *The (standard) traffic equations for unreliable Jackson networks with infinite supply are as follows:*

(i) In case of stalling,

$$\eta_i = \lambda_i + \sum_{j \in W} \eta_j r(j,i) + \sum_{j \in V} \mu_j r(j,i), \; i \in \tilde{J}, \tag{9}$$

as long as all nodes are in up status ($I = \emptyset$). Otherwise $\eta_i^I = 0$ for all $i \in \tilde{J}$.

(ii) In case of rerouting according to blocking rs-rd or skipping,

$$\eta_i^I = \lambda_i^I + \sum_{j \in W \setminus I} \eta_j^I r^I(j,i) + \sum_{j \in V \setminus I} \mu_j^I r^I(j,i), \; i \in \tilde{J} \setminus I, \tag{10}$$

for all $I \subseteq D$.

The traffic equations for some $I \subseteq D$ remain valid only as long as the availability status is unchanged. Whenever the availability status of the system changes, the traffic equations are adapted according to the new set of broken down nodes. Thus each traffic equation (10) may have different solutions for different I. The following two lemma show under which constraints the solution of the traffic equation (10) remains the same on $\tilde{J} \setminus I$ for all $\emptyset \subseteq I \subseteq D$.

Lemma 4.4 *Consider a Jackson network where nodes in $D \subseteq \tilde{J}$ are unreliable and nodes in $V \subseteq \tilde{J}$ have an infinite supply of work. $W := \tilde{J} \setminus V$ is the set of nodes without infinite supply. Let for all nodes $i \in W$ without infinite supply hold $\eta_i < \mu_i$ where $(\eta_i : i \in \tilde{J})$ is the unique solution of* (8)*, resp.* (9)*. In case of breakdowns of nodes we assume that customers are rerouted according to the* **blocking rs-rd** *regime.*

(i) If the following reversibility constraints hold:

$$\eta_i r(i,j) = \eta_j r(j,i) \quad \forall i,j \in W, \tag{11}$$
$$\eta_i r(i,j) = \mu_j r(j,i) \quad \forall i \in W, j \in V, \tag{12}$$

Then for all nodes $i \in W \setminus I$ the solution η_i^I of the traffic equation (10) *equals η_i for all $I \subseteq D$.*

(ii) Let (11) *and* (12) *hold. If we additionally require the reversibility constraint*

$$\mu_i r(i,j) = \mu_j r(j,i) \quad \forall i,j \in V, \tag{13}$$

Then for the solution of (10) *holds $\eta_i^I = \eta_i$ for all $i \in V \setminus I$ and all $I \subseteq D$, as well.*

Proof: (i): We make the ansatz $\eta_i = \eta_i^I$ for all $i \in W \setminus I$ and all $I \subseteq D$. We then obtain with the solution η_i of the traffic equations (8) for any $I \subseteq D$: $\forall i \in W \setminus I$

$$\begin{aligned}
\eta_i &= \lambda_i^I + \sum_{j \in W \setminus I} \eta_j r^I(j,i) + \sum_{j \in V \setminus I} \mu_j r^I(j,i) \\
&= \lambda_i + \eta_i \Big(r(i,i) + \sum_{k \in I} r(i,k) \Big) + \sum_{j \in W \setminus I, j \neq i} \eta_j r(j,i) + \sum_{j \in V \setminus I} \mu_j r(j,i) \\
&= \lambda_i + \sum_{k \in I \cap W} \underbrace{\eta_i r(i,k)}_{\overset{(11)}{=} \eta_k r(k,i)} + \sum_{k \in I \cap V} \underbrace{\eta_i r(i,k)}_{\overset{(12)}{=} \mu_k r(k,i)} + \sum_{j \in W \setminus I} \eta_j r(j,i) + \sum_{j \in V \setminus I} \mu_j r(j,i) \\
&= \lambda_i + \sum_{j \in W} \eta_j r(j,i) + \sum_{j \in V} \mu_j r(j,i) = (8).
\end{aligned}$$

(ii): For any $I \subseteq D$ holds $\forall i \in V \setminus I$:

$$\begin{aligned}
\eta_i^I &\overset{(10)}{=} \lambda_i^I + \sum_{j \in W \setminus I} \underbrace{\eta_j^I}_{\overset{\text{(i)}}{=} \eta_j} r^I(j,i) + \sum_{j \in V \setminus I} \mu_j r^I(j,i) \\
&\overset{(6)}{=} \lambda_i + \sum_{j \in W \setminus I} \eta_j r(j,i) + \mu_i \Big(r(i,i) + \sum_{k \in I} r(i,k) \Big) + \sum_{j \in V \setminus I, j \neq i} \mu_j r(j,i) \\
&= \lambda_i + \sum_{j \in W \setminus I} \eta_j r(j,i) + \sum_{k \in I \cap W} \underbrace{\mu_i r(i,k)}_{\overset{(12)}{=} \eta_k r(k,i)} + \sum_{k \in I \cap V} \underbrace{\mu_i r(i,k)}_{\overset{(13)}{=} \mu_k r(k,i)} + \sum_{j \in V \setminus I} \mu_j r(j,i) \\
&= \lambda_i + \sum_{j \in W} \eta_j r(j,i) + \sum_{j \in V} \mu_j r(j,i) \overset{(8)}{=} \eta_i.
\end{aligned}$$

□

Remark 4.5 *The reversibility constraints* (11), (12) *and* (13) *are different from the classical reversibility constraints which are the local balance equations of the routing process. But the interpretation of* (11), (12) *and* (13) *is the same: the departure rate from i in the direction j equals the departure rate from j in the direction i.*

For rerouting in order to skip broken down nodes, we assume that the unreliable nodes in V are *rate stable* according to [KNW09][p. 76], i.e., these nodes have equal input and output rates. Because such node $i \in V \cap D$ has precisely an output rate (of priority jobs) μ_i due to the infinite supply, it is on average, fully loaded already by customers of preemptive priority. So, intuitively, there are only a few low priority customers (from the infinite supply) served.

Lemma 4.6 *Let for all nodes $i \in W$ without infinite supply hold $\eta_i < \mu_i$ where $(\eta_i : i \in \tilde{J})$ is the unique solution of* (8)*. In case of breakdowns of nodes we assume that customers are rerouted according to the* **skipping** *regime. Let the following constraint* **(rate stability)** *hold:*

$$\eta_i = \mu_i \quad \forall i \in V \cap D. \tag{14}$$

Then for all nodes $i \in \tilde{J} \setminus I$ the solution η_i^I of the traffic equation (10) *for all $I \subseteq D$ equals η_i.*

Proof: We make the ansatz $\eta_i = \eta_i^I$ for all $i \in W \setminus I$ and all $I \subseteq D$. We then obtain with the solution η_i of the traffic equations (8) for any $I \subseteq D$: $\forall i \in W \setminus I$

$$\begin{aligned}
&\lambda_i^I + \sum_{j \in W \setminus I} \eta_j r^I(j,i) + \sum_{j \in V \setminus I} \mu_j r^I(j,i) \\
&= \lambda_i + \sum_{k \in I} \lambda_k r^I(k,i) + \sum_{j \in W \setminus I} \eta_j \Big(r(j,i) + \sum_{k \in I} r(j,k) r^I(k,i) \Big) + \sum_{j \in V \setminus I} \mu_j \Big(r(j,i) + \sum_{k \in I} r(j,k) r^I(k,i) \Big) \\
&= \underbrace{\lambda_i + \sum_{j \in W \setminus I} \eta_j r(j,i) + \sum_{j \in V \setminus I} \mu_j (r(j,i)}_{\overset{(8)}{=} \eta_i - \sum_{j \in I \cap W} \eta_j r(j,i) - \sum_{j \in I \cap V} \mu_j r(j,i)} + \sum_{k \in I} r^I(k,i) \underbrace{\Big(\lambda_k + \sum_{j \in W \setminus I} \eta_j r(j,k) + \sum_{j \in V \setminus I} \mu_j r(j,k) \Big)}_{\overset{(8)}{=} \eta_k - \sum_{j \in I \cap W} \eta_j r(j,k) - \sum_{j \in I \cap V} \mu_j r(j,k)} \\
&= \eta_i - \sum_{j \in I \cap W} \eta_j r(j,i) - \sum_{j \in I \cap V} \mu_j r(j,i) + \\
&\quad + \sum_{k \in I} \eta_k r^I(k,i) - \sum_{k \in I} r^I(k,i) \sum_{j \in I \cap W} \eta_j r(j,k) - \sum_{k \in I} r^I(k,i) \sum_{j \in I \cap V} \mu_j r(j,k) \\
&= \eta_i - \sum_{j \in I \cap W} \eta_j \underbrace{\Big(r(j,i) + \sum_{k \in I} r(j,k) r^I(k,i) \Big)}_{\overset{(4)}{=} r^I(j,i)} + \sum_{k \in I} \eta_k r^I(k,i) - \sum_{j \in I \cap V} \mu_j \underbrace{\Big(r(j,i) + \sum_{k \in I} r(j,k) r^I(k,i) \Big)}_{\overset{(4)}{=} r^I(j,i)} \\
&= \eta_i + \sum_{k \in I \cap W} \eta_k r^I(k,i) - \sum_{j \in I \cap W} \eta_j r^I(j,i) + \sum_{k \in I \cap V} \eta_k r^I(k,i) - \sum_{j \in I \cap V} \mu_j r^I(j,i) \\
&= \eta_i + \sum_{k \in I \cap V} \underbrace{(\eta_k - \mu_k)}_{\overset{(14)}{=} 0} r^I(k,i) = \eta_i.
\end{aligned} \tag{15}$$

Since $\eta_j^I = \eta_i$ holds for all $j \in W \setminus I$ and all $I \subseteq D$, it follows for all $i \in V \setminus I$ and $I \subseteq D$:

$$\eta_i^I = \lambda_i^I + \sum_{j \in W \setminus I} \eta_j^I r^I(j,i) + \sum_{j \in V \setminus I} \mu_j r^I(j,i) = \lambda_i^I + \sum_{j \in W \setminus I} \eta_j r^I(j,i) + \sum_{j \in V \setminus I} \mu_j r^I(j,i)$$

which is the very left side of (15) with $i \in V \setminus I$. The above computations in (15) are valid for all $i \in \tilde{J} \setminus I$, hence it follows $\eta_i^I = \eta_i$ for all $i \in V \setminus I$ and $I \subseteq D$, too. □

We illustrate the adjusted traffic equations with the example presented in Section 2, where nodes in $D = \{3,5\}$ are unreliable. Then the condition (14) requires $\eta_5 = \mu_5$ because node $5 \in D \cap V$. Therefore node 5 is not stable and (7) is the relevant traffic equation. Due to the feed forward structure of the network we can evaluate the arrival rates directly.

Example 4.7 *For $I = \emptyset$, η is just the solution of the standard traffic equation* (7) *given by:*

$$\begin{aligned}
\eta_1 &= \lambda_1,\ \eta_2 = \lambda_2,\ \eta_3 = \mu_1,\ \eta_4 = r(3,4)\mu_1 + r(2,4)\lambda_2, \\
\eta_5 &= r(3,5)\mu_1,\ \eta_6 = r(3,4)\mu_1 + r(2,4)\lambda_2 + r(5,6)\mu_5,\ \eta_7 = r(2,7)\lambda_2.
\end{aligned}$$

Now suppose nodes in $D = \{3,5\}$ are down and consider stalling, then $\eta_i = 0$ for all i.
In case of skipping, we obtain $\eta_i^I = \eta_i$, for $i \in \{1,2,4,6,7\}$, and $\eta_i^I = 0$, for $i = 3,5$. Eventually, blocking rs-rd can not be implemented in this network, if $\eta_i^I = \eta_i$, for $i \in \{1,2,4,6,7\}$ is required because the routing chain is not reversible.

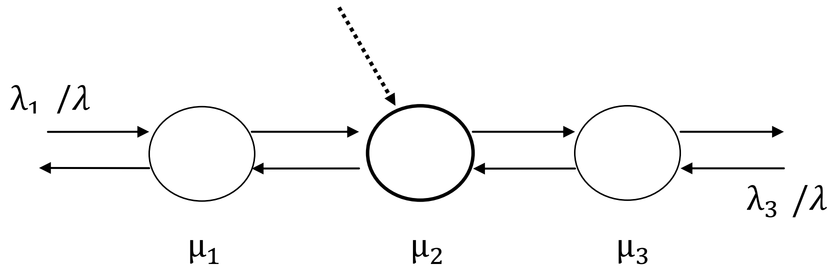

Figure 2: The Two-Way-Tandem Network

Blocking rs-rd does not apply to the system put forward in Example 4.7. In the following we will present two examples of networks that do meet the requirements. One will be a network having a linear topology and the other one having a star-shaped topology.

Example 4.8 (Two-way tandem network) *Consider a network with* $J = \{1, 2, 3\}$*,* $V = \{2\}$*,* $W = \{1, 3\}$*, and* $D = V$*, i.e., all infinite supply nodes are prone to failure. Routing is given by*

$$r(1,2) = a, r(1,0) = 1-a, r(2,3) = b, r(2,1) = 1-b, r(3,0) = c, r(3,2) = 1-c,$$

and

$$r(0,1) = \frac{\lambda_1}{\lambda_1+\lambda_2}, r(0,3) = \frac{\lambda_2}{\lambda_1+\lambda_2},$$

for $0 < a, b < 1$ *and* $\lambda_i > 0$*,* $i = 1, 2$*. The network can be visualized as a two-way tandem of three nodes, see Figure 2. The infinite supply is depicted by a dashed arrow pointing to server 2, and the nodes that are prone to failure are depicted by bold circles. Note that by incorporating node* 0 *into the network, the linear topology is transformed into a ring.*

For ease of analysis we parameterize the model in the following way. We let $\lambda_1 = (1-a)t$*, for* $t > 0$*, and* $\lambda_2 = at$*, and* $b = 1 - c$*. The service rates are set to*

$$\mu_1 > t,\ \mu_2 = t\frac{a}{c} \quad \text{and} \quad \mu_3 > t\frac{a}{c}.$$

The standard traffic equations (7) then have the solution

$$\eta_1 = t, \eta_2 = \eta_3 = t\frac{a}{c}.$$

Note that this implies that $U = \{2\}$*.*

With this choice of parameters it can be seen after some tedious algebra that the network satisfies the reversibility conditions put forward in Lemma 4.4 and rate stability in Lemma 4.6 thus all three blocking disciplines apply.

Recall that $D = \{2\}$*. Hence,* $\emptyset, \{2\}$ *are the only two possible breakdown scenarios and with the breakdown and repair rates from the motivating example in Section 2 we have*

$$\pi(\emptyset) = \frac{\rho_2}{\rho_2+\tau_2} \quad \text{and} \quad \pi(\{2\}) = \frac{\tau_2}{\rho_2+\tau_2}.$$

Example 4.9 (Star-shaped network) *Consider a network with* $J = \{1, 2, \ldots, 6\}$*,* $V = \{2, 3, 4\}$*,* $W = \{1, 5, 6\}$*, and* $D = V$*, i.e., all infinite supply nodes are prone to failure. Jobs arrive from the outside with*

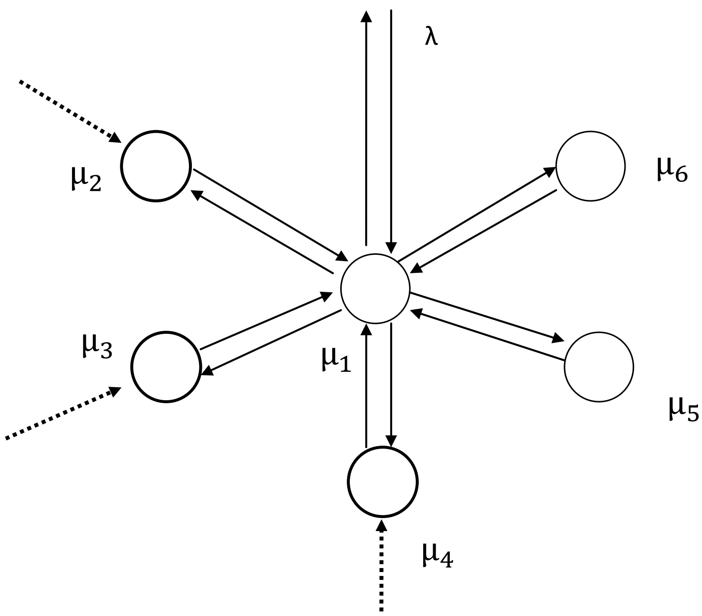

Figure 3: The Star-Shaped Network

rate λ to node 1. From node 1 they go with probability $r/5$ to any of the nodes 2 to 6, for $r \in (0,1)$. After finishing service at node $i = 2, \ldots, 6$, jobs are send back to the central node 1. From there they leave the system with probability $1-r$, or are sent back to service according to one of the servers in the set $\{2, \ldots, 6\}$ according to the routing scheme described above. The network can be visualized as a star-shaped network with one central node and five nodes in the periphery, see Figure 3. The infinite supply is depicted by a dashed arrow pointing to server $2, 3, 4$, and the nodes that are prone to failure are depicted by bold circles. The traffic equations are given by

$$\eta_1 = \lambda + \mu_2 + \mu_3 + \mu_4 + \eta_5 + \eta_6,$$

where $\eta_5 = \eta_6 = r\eta_1/5$, provided that nodes $5, 6$ are stable. For **blocking rs-rd** *and* **skipping** *to be applicable, we let*

$$\eta_i = \frac{r}{5}\eta_1 = \mu_i, \quad 2 \le i \le 4,$$

which implies $\eta_1 = \lambda/(1-r)$, and thus

$$\frac{r}{5}\frac{1}{1-r}\lambda = \mu_i = \eta_i, \quad 2 \le i \le 4.$$

Eventually, we let

$$\frac{r}{5}\frac{1}{1-r}\lambda < \mu_5, \mu_6,$$

in order to let $5, 6$ be stable nodes. Indeed, this choice implies $\eta_i < \mu_i$, $i = 5, 6$. The above conditions imply that the reversibility conditions put forward in Lemma 4.4 and the rate stability condition put forward in Lemma 4.6 are satisfied for the star-shaped network.

Recall that $D = \{2,3,4\}$. *Hence,* $\emptyset, \{2\}, \{3\}, \{4\}, \{2,3\}, \{3,4\}, \{2,4\}$ *and* D *are the possible breakdown scenarios and with the breakdown and repair rates from the motivating example in Section 2 we have*

$$\pi(\emptyset) = \left(1 + \sum_{i=2}^{4} \frac{\tau_i}{\rho_i} + \frac{\tau_2+\tau_3}{2\min(\rho_2,\rho_3)} + \frac{\tau_2+\tau_4}{2\min(\rho_2,\rho_4)} + \frac{\tau_3+\tau_4}{2\min(\rho_3,\rho_4)} + \frac{\tau_2+\tau_3+\tau_4}{3\min(\rho_3,\rho_3,\rho_4)}\right)^{-1}$$

and

$$\pi(\{i\}) = \pi(\emptyset)\frac{\tau_i}{\rho_i},$$

for $i = 2,3,4$,

$$\pi(\{i,j\}) = \pi(\emptyset)\frac{\tau_i+\tau_j}{2\min(\rho_i,\rho_j)},$$

for $2 \le i,j \le 4$, *with* $i \neq j$, *and*

$$\pi(D) = \pi(\emptyset)\frac{\tau_2+\tau_3+\tau_4}{3\min(\rho_3,\rho_3,\rho_4)}.$$

4.2 Long-Time Behavior

Theorem 4.10 *Let* $W \cap U = \emptyset$, *so all nodes without infinite supply are stable. Denote by* $\eta = (\eta_1, ..., \eta_J)$ *the unique solution of the traffic equations* (8).
Under **stalling** *regime, it holds that:*

(i) For nodes without infinite supply, the joint marginal limiting distribution is:

$$\lim_{t\to\infty} P(Y(t) = I; X_i(t) = n_i : i \in W) = \left(\sum_{K\subseteq D} \frac{A(K)}{B(K)}\right)^{-1} \frac{A(I)}{B(I)} \prod_{i\in W}\left(1-\frac{\eta_i}{\mu_i}\right)\left(\frac{\eta_i}{\mu_i}\right)^{n_i}, \qquad (16)$$

for all $I \subseteq D$ *and all* $(n_i : i \in W) \in \mathbb{N}^{|W|}$, *and this is a stationary distribution on* W *as well.*

(ii) If the global network process is started with an initial distribution which has the marginal (16) *on* W, *the arrival stream from* $i \in W$ *to* $j \in V$ *is Poisson with rate* $\eta_i r(i,j)$ *whenever all nodes are in up status. And these streams are independent given the nodes are up.*

(iii) If the global network process is started with an initial distribution which has the marginal (16) *on* W, *then the marginal limiting distribution for a stable node* $i \in V$ *with* $r(i,i) = 0$ *is:*

$$\lim_{t\to\infty} P(Y(t) = I; X_i(t) = n_i) = \left(\sum_{K\subseteq D} \frac{A(K)}{B(K)}\right)^{-1} \frac{A(I)}{B(I)} \left(1-\frac{\eta_i}{\mu_i}\right)\left(\frac{\eta_i}{\mu_i}\right)^{n_i}, \qquad (17)$$

for all $I \subseteq D$ *and all* $n_i \in \mathbb{N}$, *if and only if* $\eta_i < \mu_i$, *and this is a one-dimensional stationary distribution as well.*
If $\eta_i \ge \mu_i$ *for node* $i \in V$, *then for its limiting probability holds for all* $I \subseteq D$ *and all* $n_i \in \mathbb{N}$:

$$\lim_{t\to\infty} P(Y(t) = I; X_i(t) = n_i) = 0, \qquad (18)$$

if the global network process is started with initial distribution which has marginal (16) *on* W.

The proof is postponed to the appendix.

Theorem 4.11 *Assume that $W \cap U = \emptyset$, so all nodes without infinite supply are stable. Denote by $\eta = (\eta_1, ..., \eta_J)$ the unique solution of the traffic equations* (8)*. In case of breakdowns customers are rerouted according to the* **blocking rs-rd** *regime or the* **skipping** *regime. If blocking rs-rd is in force we require the reversibility-constraints* (11) *and* (12)*. If skipping is in force, let* (14) *hold. Then for nodes without infinite supply the joint marginal limiting distribution is given by:*

$$\lim_{t\to\infty} P(Y(t) = I; X_i(t) = n_i : i \in W) = \left(\sum_{K\subseteq D} \frac{A(K)}{B(K)}\right)^{-1} \frac{A(I)}{B(I)} \prod_{i\in W} \left(1 - \frac{\eta_i}{\mu_i}\right) \left(\frac{\eta_i}{\mu_i}\right)^{n_i}, \qquad (19)$$

for all $I \subseteq D$ and all $(n_i : i \in W) \in \mathbb{N}^{|W|}$, and this is a stationary distribution on W as well.

The results of Goodmann and Massey [GM84] are on classical Jackson networks where some nodes are not stable. They prove a product-form *limiting distribution* for the stable subnetwork, but there is no such result for the *stationary distribution*, i.e. for the finite time horizon. The reason is that the exploding unstable nodes will influence the stable part of the network and over any finite (transient) time horizon $[0, t]$ the departure streams from the unstable nodes fail to be of Poisson type. Put differently, only the limiting distribution is known. Fortunately, the proofs of Theorem 4.10(i) and (iii) and of Theorem 4.11 allow for establishing the following result on the *transient phase* of the network process.

Corollary 4.12 *Under the conditions put forward in Theorem 4.10 and Theorem 4.11, the process*

$$(Y, X_W) := (Y, X_i : i \in W)$$

is an ergodic homogeneous Markov process of its own. If, in addition, for $i \in V$ it holds that $\eta_i < \mu_i$, then the process

$$(Y, X_i)$$

is an ergodic homogeneous Markov process of its own for $i \in V$.

Remark 4.13 *In the setting of Theorem 4.11 a statement as in Theorem 4.10(iii) cannot be proved with the methods used here. This is due to the properties of the rerouting regimes skipping and blocking rs-rd. Whenever nodes in $I \neq \emptyset$ are down, immediate feedback may emerge even at nodes in V with $r(i, i) = 0$. If $i \in V \setminus I$ and $r(i, j) > 0$ for at least one $j \in I$ holds, then $r^I(i, i) > 0$ may occur.*

On the other hand, if the network's topology prevents occurrence of feedback by **skipping** *or* **rs-rd regime** *in case of breakdown it is possible to prove a counterpart to Theorem 4.10(iii) in the setting of Theorem 4.11.*

Our motivating example from Section 2 in Figure 1 is a feed forward network according to the following definition. Feed forward networks are an important subclass of Jackson networks.

Definition 4.14 *A network with node set $\tilde{J}$ with $|\tilde{J}| = J$ is a feed forward network if there exist an enumeration $\tilde{J} = \{1, 2, \ldots, J\}$ of the nodes such that*

$$r(i, j) > 0 \Longrightarrow i < j \qquad (20)$$

holds.

Note that a feed forward network can not be reversible and therefore in case of breakdowns we must recur to stalling or skipping as rerouting scheme. The following property of feed forward networks is intuitive.

Lemma 4.15 *If in a feed forward network with node set $\tilde{J} = \{1, 2, \ldots, J\}$ a subset $\emptyset \subseteq I \subseteq \tilde{J}$ of nodes is down and either skipping or stalling is applied as rerouting scheme, then it holds*

$$r^I(i, j) > 0 \Longrightarrow i < j\,, \qquad (21)$$

and therefore there is no immediate feedback at all nodes.

Theorem 4.16 *Consider a feed forward network with node set $\tilde{J} = \{1, 2, \ldots, J\}$. Assume that $W \cap U = \emptyset$, so all nodes without infinite supply are stable. Denote by $\eta = (\eta_1, ..., \eta_J)$ the unique solution of the traffic equations* (8). *In case of breakdowns customers are rerouted according to the* **skipping** *regime. Assume that* (14) *holds.*

If the global network process is started with an initial distribution which has the marginal (16) *on W, then the marginal limiting distribution for a stable node $i \in V$ is:*

$$\lim_{t\to\infty} P(Y(t) = I; X_i(t) = n_i) = \left(\sum_{K\subseteq D} \frac{A(K)}{B(K)}\right)^{-1} \frac{A(I)}{B(I)} \left(1 - \frac{\eta_i}{\mu_i}\right)\left(\frac{\eta_i}{\mu_i}\right)^{n_i}, \tag{22}$$

for all $I \subseteq D$ and all $n_i \in \mathbb{N}$, if and only if $\eta_i < \mu_i$, and this is a one-dimensional stationary distribution as well.
If $\eta_i \geq \mu_i$ for node $i \in V$, then for its limiting probability holds for all $I \subseteq D$ and all $n_i \in \mathbb{N}$:

$$\lim_{t\to\infty} P(Y(t) = I; X_i(t) = n_i) = 0, \tag{23}$$

if the global network process is started with initial distribution which has marginal (16) *on W.*

The proof of this theorem is analogously to that of Theorem 4.10, part (iii) with using Lemma 4.15.

5 Computation of Availability and Performance Measures

We discuss in this section some implications of the results found it the previous section. Evaluating these performance metrics is possible due to explicit access to the joint distribution of availability and (some of the) queue lengths. Standard performance evaluation requires ergodicity of the underlying Markov processes which allows to approximate long-time average cost functions by integrals of the cost function under the stationary distribution. Unfortunately, for the networks with unstable nodes that are non-ergodic, the stationary distribution fails to exist. Our framework overcomes this restriction and allows to investigate even non-ergodic networks across subnetworks where stabilization in the long run occurs. As stated in Corollary 4.12 some important subnetworks of stable nodes can be considered as networks of their own. Therefore for these parts we can extend the traditional analysis directly, as will be seen below. But we emphasize that even if there exists no equilibrium on the stable subnetworks, performance analysis for long-time averages of cost functions is possible via integrals of the cost function under the limiting distribution on stable nodes, respective subnets, for details we refer to Section 4.2 in [MD09] and Section 4.6.4 in [Myl13]. In the following, we will state our results for the setting put forward in Corollary 4.12.

Due to the load-independent breakdown and repair rates, the availability process Y in Theorem 4.10 and Theorem 4.11 is an ergodic Markov process of its own with unique limiting and stationary distribution

$$\pi(I) = \left(\sum_{K\subseteq D} \frac{A(K)}{B(K)}\right)^{-1} \cdot \frac{A(I)}{B(I)} \quad \forall I \subseteq D. \tag{24}$$

From this the stationary (time) point availability of a Jackson network with infinite supply and unreliable nodes (or subnetworks thereof) may be computed similar to [SD03, p.185] as

$$\text{PA}(H)(t) := \sum_{K\subseteq D\setminus H} \pi(K), \quad \text{for } H \subseteq D, t \geq 0$$

where $\pi(I)$ is the probability that exactly the nodes in $I \subseteq D$ are under repair, given by (24).

An important performance metric is in any system the throughput which in classical Jackson networks is easily accessible. Computation of throughput in our systems is different from classical Jackson networks due to the infinite supply and random availability of service. We first define the stationary throughput of a subnetwork and use as abbreviation for the marginal queue length distributions for some node set $M \subseteq \tilde{J}$ and $(n_j : j \in M)$

$$\pi_M(I, n_j : j \in M) := \sum_{(n_\ell : \ell \in \tilde{J} \setminus M) \in \mathbb{N}^{J-|M|}} \pi(I, (n_j : j \in M), (n_\ell : \ell \in \tilde{J} \setminus M)) \tag{25}$$

If $M = \{j\}$ we abbreviate $\pi_{\{j\}} = \pi_j$.

Definition 5.1 *Consider a Jackson network with infinite supply where nodes without infinite supply are stable and nodes in $D \subseteq \tilde{J}$ are unreliable. The stationary throughput $\mathrm{TH_i}$ of a node $i \in W$ without infinite supply is*

$$\mathrm{TH_i} = \sum_{(I, n_j : j \in W) \in \mathcal{P}(D) \times \mathbb{N}^{|W|}} \pi_W(I, n_j : j \in W) \mu_i^I 1_{\mathbb{N}_+}(n_i),$$

The stationary throughput $\mathrm{TH_j}$ of a stable node $j \in V$ with infinite supply is

$$\mathrm{TH_j} = \sum_{(I, n_j) \in \mathcal{P}(D) \times \mathbb{N}} \pi_j(I, n_j) \mu_j^I.$$

The stationary throughput $\mathrm{TH_W}$ of the subnetwork of nodes without infinite supply is

$$\mathrm{TH_W} = \sum_{\mathrm{I} \subseteq \mathrm{D}} \sum_{\mathrm{i} \in \mathrm{W} \setminus \mathrm{I}} \mathrm{TH_i} \mathrm{r^I(i, 0)}\,,$$

and the stationary throughput $\mathrm{TH_S}$ of the subnetwork of stable nodes is

$$\mathrm{TH_S} = \mathrm{TH_W} + \sum_{\mathrm{I} \subseteq \mathrm{D}} \sum_{\mathrm{j} \in (\mathrm{V} \cap \mathrm{S}) \setminus \mathrm{I}} \mathrm{TH_j} \mathrm{r^I(j, 0)}.$$

Proposition 5.2 *Denote by $\eta = (\eta_1, ..., \eta_J)$ the unique solution of the traffic equations* (7)*. We assume that all nodes without infinite supply of work are stable, i.e., $W \cap U = \emptyset$. So in this situation the traffic equations* (7) *reduce to* (8)*.*
In case of breakdowns **stalling** *regime is in force. Then the stationary throughput at nodes $i \in W$ (no infinite supply) is $\mathrm{TH_i} = \eta_\mathrm{i} \cdot \pi(\emptyset)$.*

Let $r(i, i) = 0$ hold for all $i \in V \cap S$ then the stationary throughput at stable nodes $i \in V \cap S$ with infinite supply is $\mathrm{TH_i} = \mu_\mathrm{i} \cdot \pi(\emptyset)$ and the throughput of the stable subnetwork is

$$\mathrm{TH_S} = \pi(\emptyset) \cdot \Big(\lambda - \sum_{\mathrm{i} \in \mathrm{V}} (\eta_\mathrm{i} - \mu_\mathrm{i}) - \sum_{\mathrm{j} \in \mathrm{V} \cap \mathrm{U}} \mu_\mathrm{j} \mathrm{r(j, 0)}\Big).$$

Proof: The proof uses the results of Theorem 4.10. For $i \in W$ we get

$$\begin{aligned}
\mathrm{TH_i} &= \sum_{(I, n_j : j \in W) \in \mathcal{P}(D) \times \mathbb{N}^{|W|}} \pi_W(I, n_j : j \in W) \underbrace{\mu_i^I}_{=0 \text{ if } I \neq \emptyset} 1_{\mathbb{N}_+}(n_i) = \sum_{n_i \in \mathbb{N}} \pi_i(\emptyset, n_i) \mu_i 1_{\mathbb{N}_+}(n_i) \\
&= \sum_{n_i \in \mathbb{N}} \pi_i(\emptyset, n_i + 1) \mu_i = \sum_{n_i \in \mathbb{N}} \pi(\emptyset) \left(1 - \frac{\eta_i}{\mu_i}\right) \left(\frac{\eta_i}{\mu_i}\right)^{n_i + 1} \mu_i = \eta_i \cdot \pi(\emptyset),
\end{aligned}$$

and for $j \in V \cap S$:

$$\mathrm{TH_j} = \sum_{(I, n_j) \in \mathcal{P}(D) \times \mathbb{N}} \pi_j(I, n_j) \underbrace{\mu_i^I}_{=0 \text{ if } I \neq \emptyset} = \underbrace{\sum_{n_j \in \mathbb{N}} \pi_j(\emptyset, n_j)}_{=\pi(\emptyset)} \mu_i = \mu_i \cdot \pi(\emptyset).$$

The throughput of the stable subnetwork is obtained after some direct manipulations

$$\begin{aligned}\mathrm{TH_S} &= \sum_{i\in S}\mathrm{TH_i r(i,0)} = \sum_{\mathrm{i\in W}}\eta_\mathrm{i}\mathrm{r(i,0)}\cdot\pi(\emptyset) + \sum_{\mathrm{j\in V\cap S}}\mu_\mathrm{j}\mathrm{r(j,0)}\cdot\pi(\emptyset)\\ &=\Big(\lambda - \sum_{i\in V}(\eta_i-\mu_i) - \sum_{j\in V\cap U}\mu_j r(j,0)\Big)\cdot\pi(\emptyset).\end{aligned}$$

□

Proposition 5.3 *Denote by $\eta = (\eta_1, ..., \eta_J)$ the unique solution of the traffic equations (7). We assume that all nodes without infinite supply are stable, i.e., $W \cap U = \emptyset$. So in this situation (7) reduces to (8).*

1. *If the rerouting is according to* **blocking rs-rd** *and if (11), (12) and (13) hold, the stationary throughput at a node $j \in W$ is $\mathrm{TH_j} = \eta_\mathrm{j}\cdot\sum_{\mathrm{I\subseteq D, j\notin I}}\pi(\mathrm{I})$ and the stationary throughput of the subnetwork W is*

$$\mathrm{TH_W} = \sum_{\mathrm{I\subseteq D}}\pi(\mathrm{I})\cdot\Big(\sum_{\mathrm{i\in\tilde{J}\setminus I}}\lambda_\mathrm{i} - \sum_{\mathrm{i\in V\setminus I}}\mu_\mathrm{i}\mathrm{r(i,0)} - \sum_{\mathrm{i\in V\setminus I}}(\eta_\mathrm{i}-\mu_\mathrm{i})\Big).$$

2. *If the rerouting is according to* **skipping** *and if (14) holds, the stationary throughput at a node $j \in W$ is $\mathrm{TH_j} = \eta_\mathrm{j}\cdot\sum_{\mathrm{I\subseteq D, j\notin I}}\pi(\mathrm{I})$ and the stationary throughput of the subnetwork W is*

$$\mathrm{TH_W} = \sum_{\mathrm{I\subseteq D}}\pi(\mathrm{I})\cdot\Big(\lambda(1-\mathrm{r^I(0,0)}) - \sum_{\mathrm{i\in V\setminus I}}\mu_\mathrm{i}\mathrm{r^I(i,0)} - \sum_{\mathrm{i\in V\setminus I}}(\eta_\mathrm{i}-\mu_\mathrm{i})\Big).$$

Proof: The proof uses the results of Theorem 4.11. For nodes $i \in W$ we get

$$\begin{aligned}\mathrm{TH_i} &= \sum_{(I,n_j:j\in W)\in\mathcal{P}(D)\times\mathbb{N}^{|W|}}\pi_W(I,n_j:j\in W)\mu_i^I 1_{\mathbb{N}_+}(n_i)\\ &= \sum_{(I,n_i)\in\mathcal{P}(D)\times\mathbb{N}}\pi_i(I,n_i)\underbrace{\mu_i^I}_{=\mu_i\text{ if }i\notin I}1_{\mathbb{N}_+}(n_i) = \sum_{I\subseteq D,i\notin I}\sum_{n_i\in\mathbb{N}}\pi_i(I,n_i+1)\mu_i\\ &= \sum_{I\subseteq D,i\notin I}\pi(I)\sum_{n_i\in\mathbb{N}}\Big(1-\frac{\eta_i}{\mu_i}\Big)\Big(\frac{\eta_i}{\mu_i}\Big)^{n_i+1}\mu_i = \eta_i\cdot\sum_{I\subseteq D,i\notin I}\pi(I),\end{aligned}$$

The throughput of the subnetwork W is

$$\mathrm{TH_W} = \sum_{I\subseteq D}\sum_{i\in W\setminus I}\mathrm{TH_i r^I(i,0)} = \sum_{\mathrm{I\subseteq D}}\pi(\mathrm{I})\sum_{\mathrm{i\in W\setminus I}}\eta_\mathrm{i}\mathrm{r^I(i,0)}.$$

Since $\eta_i = \eta_i^I$ holds for all $i \in \tilde{J}\setminus I$ and $I \subseteq D$ (see Lemma 4.4 or Lemma 4.6 resp.), η_i solves

$$\eta_i = \lambda_i^I + \sum_{j\in W\setminus I}\eta_j r^I(j,i) + \sum_{j\in V\setminus I}\mu_j r^I(j,i).$$

Summing over all $i \in \tilde{J}\setminus I$ on both sides yields

$$\begin{aligned}&\sum_{i\in\tilde{J}\setminus I}\eta_i = \sum_{i\in\tilde{J}\setminus I}\lambda_i^I + \sum_{i\in\tilde{J}\setminus I}\sum_{j\in W\setminus I}\eta_j r^I(j,i) + \sum_{i\in\tilde{J}\setminus I}\sum_{j\in V\setminus I}\mu_j r^I(j,i)\\ \Leftrightarrow\ &\sum_{i\in\tilde{J}\setminus I}\lambda_i^I = \sum_{i\in W\setminus I}\eta_i r^I(i,0) + \sum_{i\in V\setminus I}\mu_i r^I(i,0) + \sum_{i\in V\setminus I}(\eta_i-\mu_i)\\ \Leftrightarrow\ &\sum_{i\in W\setminus I}\eta_i r^I(i,0) = \sum_{i\in\tilde{J}\setminus I}\lambda_i^I - \sum_{i\in V\setminus I}\mu_i r^I(i,0) - \sum_{i\in V\setminus I}(\eta_i-\mu_i). \qquad (26)\end{aligned}$$

In case of (i) equation (26) is equivalent to

$$\sum_{i\in W\setminus I} \eta_i r^I(i,0) = \sum_{i\in \tilde{J}\setminus I} \lambda_i - \sum_{i\in V\setminus I} \mu_i r(i,0) - \sum_{i\in V\setminus I} (\eta_i - \mu_i).$$

In case of (ii) equation (26) is equivalent to

$$\sum_{i\in W\setminus I} \eta_i r^I(i,0) = \lambda \underbrace{\sum_{i\in \tilde{J}\setminus I} r^I(0,i)}_{=(1-r^I(0,0))} - \sum_{i\in V\setminus I} \mu_i r^I(i,0) - \sum_{i\in V\setminus I} (\eta_i - \mu_i).$$

□

We conclude this section by noting that that the asymptotic mean queue-length at a stable node is easily computable due the product form of the asymptotic distribution. Indeed, following Theorem 4.10, Theorem 4.11 and Theorem 4.16, the asymptotic mean queue-length at a stable node can be computed like in the standard Jackson network case. Evoking Little's law, see [Stid72, Stid74], mean waiting times at stable nodes follow just as easy. For example, for a stable node which is unreliable, i.e., $j \in W \cap D$, we obtain under skipping as rerouting scheme the mean stationary sojourn time W_j as

$$W_j = \frac{1}{\mu_j - \eta_j} \left(\sum_{I\subseteq D, j\notin I} \pi(I) \right)^{-1}.$$

6 Robustness Analysis of the Throughput

As pointed out in the introduction, breakdown rates are hard to estimate via historical data. Therefore, one is typically confronted with uncertainty about the true value of the parameters defining the distributions of the time between breakdowns, in our case the failure rate. This is known as *parameter uncertainty* in the literature, see, for example, [HaMe92], for a discussion on integration of parameter uncertainty into queueing models and [Hend03] for a discussion on parameter insecurity from a broader perspective. In the following, we will apply our results to a robustness analysis of our model against the statistical insecurity in the breakdown rates.

In modeling parameter insecurity the choice of the distribution is of importance and one typically chooses a particular distribution based on - possible incomplete - knowledge that is available. For example, if the mean and the variance is known, and, if in addition, we know that the parameter may take values in $\mathbb{R}$, the most general distribution is the normal distribution, where "most general" refers to the fact that this distribution maximizes the entropy. On the other hand, when, due to expert knowledge, it is known that the parameter falls into interval, say, $[a, b]$, then the uniform distribution on $[a, b]$ is the entropy maximizing distribution; see, for example, [Kull59].

In the following we make the reasonable assumption that the "true" breakdown rates τ_i are not revealed to us and we therefore assume that τ_i follows a given distribution F_i. By virtue of Proposition 5.2 and Proposition 5.3, assuming the τ_i to be random only affects the stationary distribution π of the availability process Y, see (2) for a definition of π. Hence, instances of the throughput can be easily obtained by sampling the τ_i's according to their assumed distribution and evaluating the realization of π. Creating a sufficient number of samples, the density and the cumulative distribution function of the throughput can be estimated and evaluated for further robustness analysis.

A similar robustness analysis can be performed for mean sojourn times via application of Little's law.

6.1 Example 1

We revisit the example put forward in Section 2, see Figure 1. Nodes in $D = \{3, 5\}$ are unreliable, see Example 4.7. For illustrating purposes, we analyze the throughput at node 5, denoted by TH_5. Sampling

$(\tau_3(\omega), \tau_5(\omega))$ and computing TH_5 via Proposition 5.2 and Proposition 5.3 yields a realization $\mathrm{TH}_5(\omega)$ of the throughput for the respective breakdown regime.

We denote by

$$\mathrm{TH}_{\mathbf{5}}^{\text{est.}} = \mathbb{E}[\mathrm{TH}_5(\omega)]$$

the expected throughput, where the expectation is taken with respect to the distribution of $(\tau_3(\omega), \tau_5(\omega))$. Alternatively, let $\hat{\tau}_i$ denote the value of the point estimator for τ_i, for $i = 3, 5$. Computing the throughput with rates $\hat{\tau}_3$ and $\hat{\tau}_5$, then yields $\mathrm{TH}_{\mathbf{5}}^{\text{analytical}}$. Note that in standard queuing applications one typically computes $\mathrm{TH}_{\mathbf{5}}^{\text{analytical}}$. In the following numerical examples we will illustrate to what extend this may lead to misguided decisions. To this end, we will also estimate the relative value at risk

$$\mathrm{RVaR} = \frac{\mathrm{TH}_{\mathbf{5}}^{\text{analytical}} - q_{0.05}(\mathrm{TH}_5)}{\mathrm{TH}_{\mathbf{5}}^{\text{analytical}}},$$

which yields the relative maximal discrepancy between the actual throughput TH_5 and the analytical throughput that can occur with probability of at least 0.05. The RVaR is helpful in assessing the risk involved in considering the sample-based rates $\hat{\tau}_i$ as true values. We will compare the value at risk for the different rerouting regimes under breakdowns.

For the numerical example, we let $\mu_5 = 3$ and set the breakdown and repair rates to

$$\tau_3 = 1.0, \tau_5 = 5.0, \rho_3 = 1.1, \rho_5 = 1.01.$$

It is worth noting that the throughput at note 5 has under stalling the explicit solution

$$\frac{\mu_5}{1 + \frac{\tau_3}{\rho_3} + \frac{\tau_5}{\rho_5} + \frac{\tau_3 + \tau_5}{2 \min(\rho_3, \rho_5)}}$$

and under skipping

$$\frac{\mu_5 \left(1 + \frac{\tau_3}{\rho_3}\right)}{1 + \frac{\tau_3}{\rho_3} + \frac{\tau_5}{\rho_5} + \frac{\tau_3 + \tau_5}{2 \min(\rho_3, \rho_5)}},$$

and is independent of the arrival rates and the service rates at nodes other than 5. Put differently, the throughput of node 5 to the outside is independent of the arrival rates and the service rates at the other nodes. It is worth noting that **blocking rs-rd** cannot be applied to this system as the reversibility conditions put forward in Lemma 4.4 do not hold.

For $i = 3, 5$, let $\hat{\tau}_i$ denote the failure rate in the analytical model, which is assumed to be fixed, and let τ_i with

$$\tau_i = \hat{\tau}_i + \rho \cdot (\hat{\tau}_i(U - (1/2))),$$

denote the actual failure rate, where U is uniformly distributed on $[0, 1]$ modeling the parameter insecurity, and ρ denotes the level of insecurity. For the experiments, we let $\rho = 0.5$ (small insecurity), which means that τ_i may differ up to 25 % from $\hat{\tau}_i$, $\rho = 1$ (medium insecurity, which means that τ_i may differ up to 50 % from $\hat{\tau}_i$, $\rho = 2$ (high insecurity), which means that τ_i may differ up to 100 % from $\hat{\tau}_i$. For the statistical analysis we sample $N = 10^6$ realizations of the long-run throughput. Table 1 gives the results under stalling, and Table 2 gives the results under skipping.

The tables illustrate that the risk profile of the two disciplines is significantly different. Not surprisingly, the throughput under skipping is larger than under stalling, which stems from the additional factor τ_5/ρ_5 in the expression for the throughput under skipping. The RVaR, however, of stalling is smaller than that of skipping, which means that stalling is more robust with respect to parameter insecurity in the breakdown rate. We illustrate the different risk profiles of the two breakdown schemes by plotting a histogram of the throughput values. Figure 4 shows the resulting histogram for stalling and Figure 5 shows the resulting histogram for skipping.

The figures illustrate that the risk profile of the two disciplines is significantly different, and thereby support our findings in Tables 1 and Table 2.

	$\rho = 0.5$	$\rho = 1$	$\rho = 2$
$\mathrm{TH}_{\mathbf{5}}^{\mathrm{est.}}$	0.2472	0.25747	0.3282
$\mathrm{TH}_{\mathbf{5}}^{\mathrm{analytical}}$	0.2441	0.2441	0.2441
VaR	0.03594	0.06268	0.0997
RVaR	0.1472	0.2567	0.4084

Table 1: Output analysis for Stalling varying ρ.

	$\rho = 0.5$	$\rho = 1$	$\rho = 2$
$\mathrm{TH}_{\mathbf{5}}^{\mathrm{est.}}$	0.4712	0.4883	0.5998
$\mathrm{TH}_{\mathbf{5}}^{\mathrm{analytical}}$	0.4661	0.4661	0.4661
VaR	0.0798	0.1440	0.2447
RVaR	0.1712	0.3089	0.5251

Table 2: Output analysis for Skipping varying ρ.

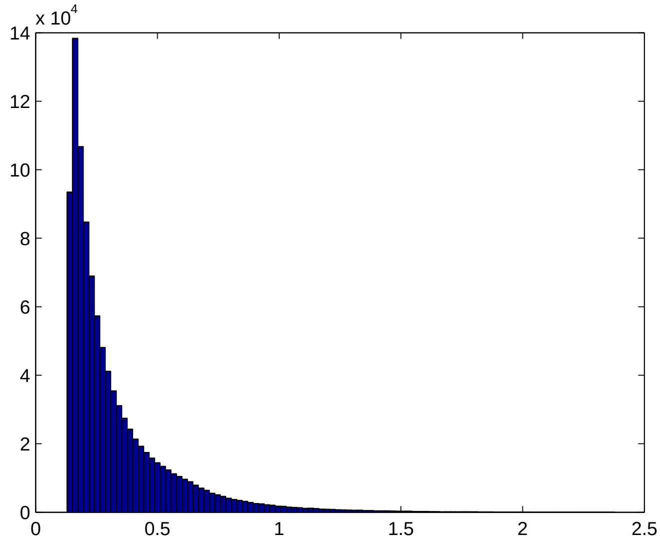

Figure 4: Histogram for Throughput at node 5 under Stalling

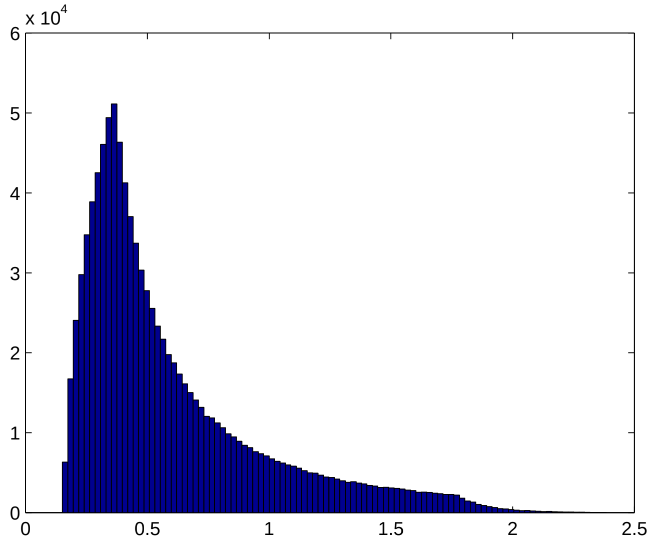

Figure 5: Histogram for Throughput at node 5 under Skipping

	$\rho = 0.5$	ρ= 1	$\rho = 2$
$\mathbf{TH_3}^{\text{est.}}$	0.3365	0.3466	0.4025
$\mathbf{TH_3}^{\text{analytical}}$	0.3333	0.3333	0.3333
VaR	0.0434	0.0769	0.1250
RVaR	0.1304	0.2307	0.3750

Table 3: Output analysis for blocking rs-rd in the Two-Way Tandem for varying ρ.

The reported fact that stalling is more robust with respect to parameter insecurity than skipping is not a general property of the breakdown disciplines. For example, letting $\tau_3 = 5$ and $\tau_5 = 1$ we obtain for $\rho = 0.5$ a RVaR of stalling of 0.1448 and a RVaR of skipping of 0.04530, which shows that in this case skipping is more robust than stalling. It is also worth noting that according to our experience other distributional models for the parameter insecurity may lead to other results.

6.2 Example 2: Two-Way Tandem Network

We revisit the linear network in Example 4.8. For the numerical experiment we choose $a = b = c = 1/2$, and $\lambda_1 = \lambda_2 = 1$. The throughput at node 3 under **blocking rs-rd** is then given by

$$\mathrm{TH}_3 = \eta_3 \frac{\rho_2}{\rho_2 + \tau_2},$$

where $\eta_3 = 2$ for the numerical setting. We model the parameter insecurity as described in Example 6.1.

For the statistical analysis we sample $N = 10^6$ realizations of the long-run throughput. Table 3 gives the results under **blocking rs-rd**. Figure 6 shows the resulting histogram for the throughput at node 5 under **blocking rd-rs** together with a non-parametric density fit.

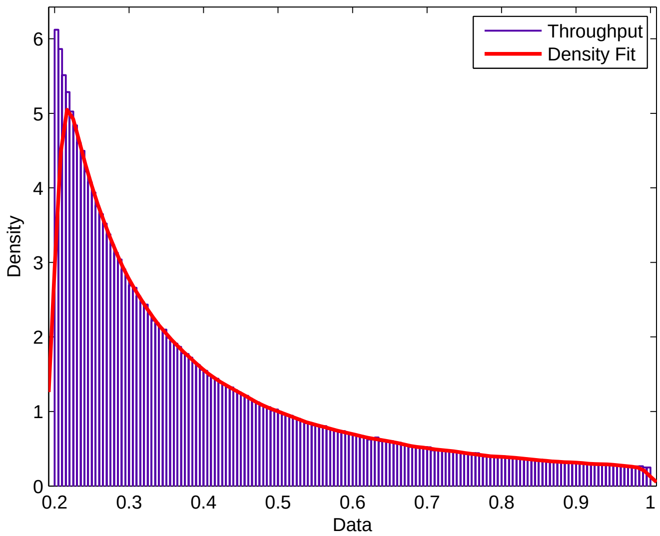

Figure 6: Statistical Analysis of Throughput at node 3 in the Two-Way-Tandem under **blocking rd-rs** at $\rho = 2$

	$\rho = 0.5$	$\rho = 1$	$\rho = 2$
$\mathbf{TH_4}^{\text{est.}}$	0.4668	0.4676	0.4720
$\mathbf{TH_4}^{\text{analytical}}$	0.4666	0.4666	0.4666
VaR	0.0385	0.07859	0.1711
RVaR	0.0825	0.1683	0.3654

Table 4: Output analysis for skipping in the Star-Network for varying ρ.

6.3 Example 3: Open Star-Network

We revisit Example 4.9. Then, the throughput at station 4 under skipping is given by

$$\mathrm{TH}_4 = \eta_4\Big(\pi(\emptyset) + \pi(\{2\}) + \pi(\{3\}) + \pi(\{2,3\})\Big),$$

where the explicit equations for the probabilities are provided in Example 4.9. For the numerical experiment we let $r = 1/2$ and $\lambda = 5$, so that $\eta_4 = 1$. Furthermore, we choose $\tau_i = 1$ and $\rho_i = 2$, for $i = 2, 3, 4$. We model the parameter insecurity as described in Example 6.1. For the statistical analysis we sample $N = 10^6$ realizations of the long-run throughput. Table 4 gives the results under skipping.

Figure 7 shows the resulting histogram for the throughput at node 4 under skipping with a non-parametric density fit.

Conclusion

In this article we have integrated simultaneous breakdown and repair of servers together with infinite supply servers in one frame-work. We obtained closed-form analytical solutions of the steady-state queue length distribution at stable nodes. Numerical examples have illustrated that robustness analysis of such networks with respect to parameter insensitivity is an insightful tool that becomes feasible due to the

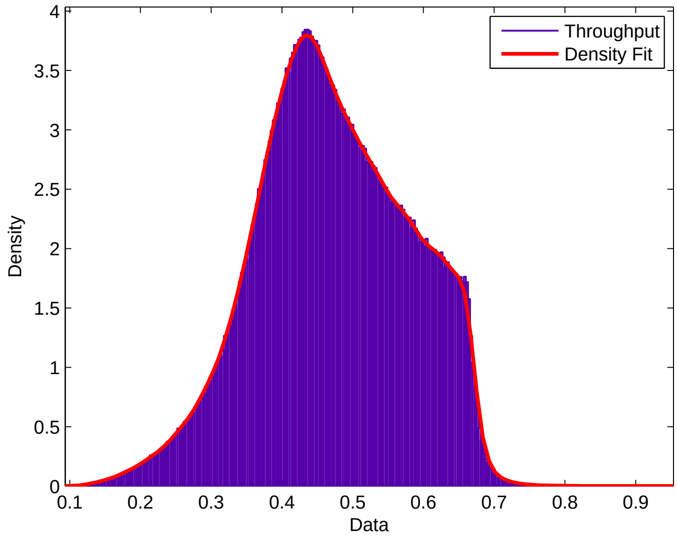

Figure 7: Statistical Analysis of Throughput at node 4 in the Star-Network under Skipping for $\rho = 2$

product-form type solution. Future research will be on extending our results to state-dependent and, more generally to path history dependent failure rates.

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Appendix

7 Proof of Theorem 4.10

(i): Assume first that all nodes are in up status ($I = \emptyset$). We start the proof with evocation of the Subnetwork Argument from the proof of theorem 13 in [SDH14]. It guarantees that the subnetwork W constitutes as a Jackson network where the source and sink represent $\{0\} \cup V$. The corresponding queueing process $\tilde{X} := ((\tilde{X}_i(t) : i \in W) : t \in \mathbb{R}_+)$ is a Markov process of its own. The traffic equations of the described subnetwork W are given by

$$\tilde{\eta}_i = \tilde{\lambda}_i + \sum_{j \in W} \tilde{\eta}_j r(j,i), \quad i \in W, \quad \text{where } \tilde{\lambda}_i := \lambda_i + \sum_{j \in V} \mu_j r(j,i),$$

so $\eta_i = \tilde{\eta}_i$ holds for all $i \in W$. According to Jackson's theorem (see [Jac57]), $\tilde{X}$ has the unique stationary and limiting distribution

$$\lim_{t\to\infty} P(X_i(t) = n_i : i \in W) = \prod_{i\in W} \left(1 - \frac{\eta_i}{\mu_i}\right)\left(\frac{\eta_i}{\mu_i}\right)^{n_i}, \quad \forall (n_i : i \in W) \in \mathbb{N}^{|W|}, \tag{27}$$

because $\eta_i < \mu_i$ for all $i \in W$ holds. Thus, even if the subnetwork V of nodes with infinite supply is not in equilibrium, the equilibrium on the subnetwork W of nodes without infinite supply is preserved, if the initial distribution has the joint marginal (27).

This joint queue length process $\tilde{X}$ is coupled with an availability process Y which only depends on the interaction of the nodes in $D \subseteq \tilde{J}$ but not on their load. Whenever a node in D breaks down, stalling occurs, so all nodes go into a warm standby and all arrivals and services are interrupted until all nodes recur to the up status. The network process $(Y, \tilde{X})$ is a Markov process on the state space $\mathcal{P}(D) \times \mathbb{N}^{|W|}$. the balance equations for the subnetwork W are for all $(\emptyset, n_k : k \in W) \in \{\emptyset\} \times \mathbb{N}^{|W|}$ given by

$$\begin{aligned}
&\pi(\emptyset, n_k : k \in W)\left(\sum_{i\in W}\Big(\lambda_i + \sum_{j\in V}\mu_j r(j,i)\Big) + \sum_{i\in W}\mu_i(1 - r(i,i))1_{\mathbb{N}_+}(n_i) + \sum_{\emptyset\neq I\subseteq D}\alpha(\emptyset, I)\right)\\
&= \sum_{i\in W}\pi(\emptyset, n_k : k \in W\setminus\{i\}, n_i - 1)\cdot\Big(\lambda_i + \sum_{j\in V}\mu_j r(j,i)\Big)\cdot 1_{\mathbb{N}_+}(n_i) +\\
&+ \sum_{i\in W}\pi(\emptyset, n_k : k \in W\setminus\{i\}, n_i + 1)\cdot\mu_i\Big(1 - \sum_{j\in W} r(i,j)\Big) +\\
&+ \sum_{i\in W}\sum_{j\in W\setminus\{i\}}\pi(\emptyset, n_k \in W\setminus\{i,j\}, n_i + 1, n_j - 1)\cdot\mu_i r(i,j)\cdot 1_{\mathbb{N}_+}(n_j)\\
&+ \sum_{\emptyset\neq I\subseteq D}\pi(I, n_k : k \in W)\cdot\beta(I, \emptyset),
\end{aligned} \tag{28}$$

and for all $(I, n_k : k \in W) \in \mathcal{P}(D) \times \mathbb{N}^{|W|}$ with $I \neq \emptyset$

$$\begin{aligned}
&\pi(I, n_k : k \in W)\left(\sum_{I\subset H\subseteq D}\alpha(I,H) + \sum_{\emptyset\neq K\subset I}\beta(I,K)\right)\\
&= \sum_{\emptyset\neq K\subset I}\pi(K, n_k : k \in W)\cdot\alpha(K,I) + \sum_{I\subset H\subseteq D}\pi(H, n_k : k \in W)\cdot\beta(H,I).
\end{aligned} \tag{29}$$

We have to show, that (16) solves these equations. In the following we denote

$$\hat{\pi}(I, n_k : k \in W) := \frac{A(I)}{B(I)}\prod_{i\in W}\left(\frac{\eta_i}{\mu_i}\right)^{n_i}$$

for all $(I, n_k : k \in W) \in \mathcal{P}(D) \times \mathbb{N}^{|W|}$, which is (16) before normalization, and plug it into the above balance equations instead of $\pi(I, n_k : k \in W)$.

In the first equation (28) the term

$$\hat{\pi}(\emptyset, n_k : k \in W)\alpha(\emptyset, I) = \hat{\pi}(\emptyset, n_k : k \in W)A(I) = \hat{\pi}(I, n_k : k \in W)B(I)$$

on the left-hand side is equal to the term $\hat{\pi}(I, n_k : k \in W)\beta(I, \emptyset) = \hat{\pi}(I, n_k : k \in W)B(I)$ on the right-hand side for each $\emptyset \neq I \subseteq D$. the remainder of (28) is the global balance equation of a classical Jackson network which has the solution (see [Jac57])

$$\hat{\pi}(\emptyset, n_k : k \in W) := \hat{\pi}(n_k : k \in W) = \prod_{i\in W}\left(\frac{\eta_i}{\mu_i}\right)^{n_i}.$$

Consider the second equation (29) for some fixed $I \neq \emptyset$. For any $K \subset I$, $K \neq \emptyset$, the term

$$\hat{\pi}(I, n_k : k \in W)\beta(I, K) = \hat{\pi}(I, n_k : k \in W)\frac{B(I)}{B(K)} = \hat{\pi}(\emptyset, n_k : k \in W)\frac{A(I)}{B(K)}$$

on the left-hand side is equal to the term on the right-hand side

$$\hat{\pi}(K, n_k : k \in W)\alpha(K, I) = \hat{\pi}(K, n_k : k \in W)\frac{A(I)}{A(K)} = \hat{\pi}(\emptyset, n_k : k \in W)\frac{A(I)}{B(K)}.$$

Moreover, for any $I \subset H \subseteq D$ the term

$$\hat{\pi}(I, n_k : k \in W)\alpha(I, H) = \hat{\pi}(I, n_k : k \in W)\frac{A(H)}{A(I)} = \hat{\pi}(\emptyset, n_k : k \in W)\frac{A(H)}{B(I)}$$

on the left-hand side is equal to the term

$$\hat{\pi}(H, n_k : k \in W)\beta(H, I) = \hat{\pi}(H, n_k : k \in W)\frac{B(H)}{B(I)} = \hat{\pi}(\emptyset, n_k : k \in W)\frac{A(H)}{B(I)}$$

on the right-hand side.
The proof of (i) is finished by normalization, which is possible because $\eta_i < \mu_i$ holds for all $i \in W$.

(ii): It is well known that ergodic Jackson networks have, in equilibrium, Poisson departure streams from node i to the sink with rate $\tilde{\eta}_i \tilde{r}(i, 0)$, see [Mel79, Example 7.1]. From the proof of (i), we know that the subset W behaves like an ergodic Jackson network with unreliable nodes of its own with $\tilde{\lambda}_i := \lambda_i + \sum_{j \in V} \mu_j r(j, i)$ and

$$\tilde{\eta}_i \tilde{r}(i, 0) = \eta_i\Big(1 - \sum_{j \in W} r(i, j)\Big) = \eta_i\Big(r(i, 0) + \sum_{j \in V} r(i, j)\Big).$$

Hence, if the subnetwork W is in equilibrium, as long as all nodes are in up status, departures to the sink from nodes $i \in W$ are Poisson streams with rate $\eta_i r(i, 0)$ and departures from $i \in W$ to any node $j \in V$ are also Poisson streams with rate $\eta_i r(i, j)$, because a portion of $r(i, j)/(r(i, 0) + \sum_{j \in V} r(i, j))$ of the departure stream from node $i \in W$ is directed to $j \in V$.

(iii): Under the condition that all nodes $j \in \tilde{J}$ are in up status, we start the proof with evocation of the **M/M/1 Argument** from the proof of theorem 13 in [SDH14].

This argument leads to the conclusion, that if the subnetwork W is in equilibrium and if $r(i, i) = 0$ holds, node $i \in V$ behaves as an $M/M/1-$ system of its own. The corresponding queue length process $\hat{X}$ is a birth-death process on state space $\mathbb{N}$ with birth rates $\hat{\lambda}_i = \eta_i$ and death rates μ_i.

This queue length process $\hat{X}$ is here coupled with an availability process Y on $\mathcal{P}(D)$, $D \subseteq \tilde{J}$, where breakdown and repair of nodes only depend on the interaction of the nodes but not on their queue length. Whenever a node in D breaks down, stalling occurs, so all nodes go into a warm standby and all arrivals and services are interrupted until all nodes recur to the up status.
The network process $(Y, \hat{X})$ is a Markov process on the state space $\mathcal{P}(D) \times \mathbb{N}$. The balance equations are

$$\begin{aligned}&\pi_i(\emptyset, n_i)\Big(\hat{\lambda}_i + \mu_i 1_{\mathbb{N}_+}(n_i) + \sum_{\emptyset \neq I \subseteq D} \alpha(\emptyset, I)\Big) \\ &= \pi_i(\emptyset, n_i - 1) \cdot \hat{\lambda}_i \cdot 1_{\mathbb{N}_+}(n_i) + \pi_i(\emptyset, n_i + 1) \cdot \mu_i + \sum_{\emptyset \neq I \subseteq D} \pi_i(I, n_i) \cdot \beta(I, \emptyset)\end{aligned} \tag{30}$$

for all $(\emptyset, n_i) \in \{\emptyset\} \times \mathbb{N}$ and

$$\begin{aligned}&\pi_i(I, n_i)\Big(\sum_{I \subset H \subseteq D} \alpha(I, H) + \sum_{\emptyset \neq K \subset I} \beta(I, K)\Big) \\ &= \sum_{\emptyset \neq K \subset I} \pi_i(K, n_i) \cdot \alpha(K, I) + \sum_{I \subset H \subseteq D} \pi_i(H, n_i) \cdot \beta(H, I)\end{aligned} \tag{31}$$

for all $(I, n_i) \in \mathcal{P}(D) \times \mathbb{N}$ with $I \neq \emptyset$.

We have to show, that (17) solves these equations. In the following we set

$$\hat{\pi}_i(I, n_i) := \frac{A(I)}{B(I)} \left(\frac{\eta_i}{\mu_i}\right)^{n_i}$$

for all $(I, n_i) \in \mathcal{P}(D) \times \mathbb{N}$ as the non normalized proposed solution density.

In the first equation (30) the term

$$\hat{\pi}_i(\emptyset, n_i)\alpha(\emptyset, I) = \hat{\pi}_i(\emptyset, n_i)A(I) = \hat{\pi}_i(I, n_i)B(I)$$

on the left-hand side is equal to the term $\hat{\pi}_i(I, n_i)\beta(I, \emptyset) = \hat{\pi}_i(I, n_i)B(I)$ on the right-hand side for each $\emptyset \neq I \subseteq D$. the remainder of (30) is the global balance equation of an $M/M/1-$system which has the solution

$$\hat{\pi}_i(\emptyset, n_i) := \hat{\pi}_i(n_i) = \left(\frac{\eta_i}{\mu_i}\right)^{n_i},$$

since $\hat{\lambda}_i = \eta_i$ holds.

Consider the second equation (31) for some fixed $I \neq \emptyset$. For any $K \subset I$, $K \neq \emptyset$, the term

$$\hat{\pi}_i(I, n_i)\beta(I, K) = \hat{\pi}_i(I, n_i)\frac{B(I)}{B(K)} = \hat{\pi}_i(\emptyset, n_i)\frac{A(I)}{B(K)}$$

on the left-hand side is equal to the term on the right-hand side

$$\hat{\pi}_i(K, n_i)\alpha(K, I) = \hat{\pi}_i(K, n_i)\frac{A(I)}{A(K)} = \hat{\pi}_i(\emptyset, n_i)\frac{A(I)}{B(K)}.$$

Moreover, for any $I \subset H \subseteq D$ the term

$$\hat{\pi}_i(I, n_i)\alpha(I, H) = \hat{\pi}_i(I, n_i)\frac{A(H)}{A(I)} = \hat{\pi}_i(\emptyset, n_i)\frac{A(H)}{B(I)}$$

on the left-hand side is equal to the term

$$\hat{\pi}_i(H, n_i)\beta(H, I) = \hat{\pi}_i(H, n_i)\frac{B(H)}{B(I)} = \hat{\pi}_i(\emptyset, n_i)\frac{A(H)}{B(I)}$$

on the right-hand side. The proof of (iii) is finished by normalization, which is possible from $\eta_i < \mu_i$.

The limiting probability (18) for unstable nodes with infinite supply follows from the same arguments as in the proof of theorem 15 in [MD09].

8 Proof of Theorem 4.11

Consider the subset W of nodes without infinite supply. For any subset $I \subseteq D$ of broken down nodes, we have the following facts for the subset $W \setminus I$ which remain in force as long as I is unchanged:

- All service times of all up-nodes are exponentially distributed and the service discipline at all nodes is FCFS.
- Routing of customers is Markovian: A customer completing service at node $i \in W \setminus I$ will either move to some node $j \in W \setminus I$ with probability $r^I(i, j)$ or leave the subnetwork with probability $1 - \sum_{j \in W \setminus I} r^I(i, j)$.

- At each node $i \in W \setminus I$, we have external arrivals from the source which are independent Poisson streams with rate $\lambda_i^I \geq 0$. Furthermore all arrivals from nodes $j \in V \setminus I$ with infinite supply into nodes $i \in W \setminus I$ are independent Poisson streams at rate $\mu_j r^I(j,i)$, see Theorem 4.1. The sum of independent Poisson streams is a Poisson stream, hence the arrival stream from the outside of the subset $W \setminus I$ into each node $i \in W \setminus I$ is a Poisson process with rate $\lambda_i^I + \sum_{j\in V\setminus I} \mu_j r^I(j,i)$.
- All service times and all interarrival times are independent of each other.

Let $\tilde{X} := ((\tilde{X}_i(t) : i \in W \setminus I) : t \in \mathbb{R}_+)$ be the queueing process of this subnetwork. The process is supplemented with a Markov process $Y = (Y(t) : t \in \mathbb{R}_+)$ which describes the availability status of the nodes and therefore gives information on how long the network process on the subnet $W \setminus I$ lives until it jumps to the next Markov process on some randomly chosen subnet $W\setminus K$, $K \subseteq D$. Rerouting is according to the blocking rs-rd regime (skipping, resp.). The balance equations of the joint availability-queue length process $(Y, \tilde{X}_i : i \in W)$ are $\forall (I, n_i : i \in W) \in \mathcal{P}(D) \times \mathbb{N}^{|W|}$

$$\begin{aligned}
&\pi(I, n_k : k \in W) \cdot \Big(\sum_{i\in W\setminus I} \Big(\lambda_i^I + \sum_{j\in V\setminus I} \mu_j r^I(j,i)\Big) + \\
&+ \sum_{i\in W\setminus I} \mu_i (1 - r^I(i,i)) \cdot 1_{\mathbb{N}_+}(n_i) + \sum_{I\subset H\subseteq D} \alpha(I,H) + \sum_{K\subset I\subseteq D} \beta(I,K)\Big) \\
&= \sum_{i\in W\setminus I} \pi(I, n_k : k \in W\setminus\{i\}, n_i - 1) \cdot \Big(\lambda_i^I + \sum_{j\in V\setminus I} \mu_j r^I(j,i)\Big) \cdot 1_{\mathbb{N}_+}(n_i) \\
&+ \sum_{i\in W\setminus I} \pi(I, n_k : k \in W\setminus\{i\}, n_i + 1) \cdot \mu_i \Big(1 - \sum_{j\in W\setminus I} r^I(i,j)\Big) + \\
&+ \sum_{i\in W\setminus I} \sum_{j\in W\setminus I, j\neq i} \pi(I, n_k : k \in W\setminus\{i,j\}, n_i + 1, n_j - 1) \cdot \mu_i r^I(i,j) \cdot 1_{\mathbb{N}_+}(n_j) \\
&+ \sum_{K\subset I\subseteq D} \pi(K, n_k : k \in W) \cdot \alpha(K,I) + \sum_{I\subset H\subseteq D} \pi(H, n_k : k \in W) \cdot \beta(H,I)\,.
\end{aligned} \tag{32}$$

We have to show that the distribution given by (19) solves equation (32) for all $(n_i : i \in W) \in \mathbb{N}^{|W|}$ and all $I \subseteq D$. In the following we set

$$\hat{\pi}(I, n_k : k \in W) := \frac{A(I)}{B(I)} \prod_{i\in W} \left(\frac{\eta_i}{\mu_i}\right)^{n_i}$$

for all $(n_i : i \in W) \in \mathbb{N}^{|W|}$ and all $I \subseteq D$, and consider equation (32) for some fixed $I \subseteq D$.

For any $K \subset I$, $K \neq \emptyset$, the term

$$\hat{\pi}(I, n_k : k \in W)\beta(I,K) = \hat{\pi}(I, n_k : k \in W)\frac{B(I)}{B(K)} = \hat{\pi}(\emptyset, n_k : k \in W)\frac{A(I)}{B(K)}$$

on the left-hand side is equal to the term on the right-hand side

$$\hat{\pi}(K, n_k : k \in W)\alpha(K,I) = \hat{\pi}(K, n_k : k \in W)\frac{A(I)}{A(K)} = \hat{\pi}(\emptyset, n_k : k \in W)\frac{A(I)}{B(K)}.$$

Moreover, for any $I \subset H \subseteq D$ the term

$$\hat{\pi}(I, n_k : k \in W)\alpha(I,H) = \hat{\pi}(I, n_k : k \in W)\frac{A(H)}{A(I)} = \hat{\pi}(\emptyset, n_k : k \in W)\frac{A(H)}{B(I)}$$

on the left-hand side is equal to the term

$$\hat{\pi}(H, n_k : k \in W)\beta(H, I) = \hat{\pi}(H, n_k : k \in W)\frac{B(H)}{B(I)} = \hat{\pi}(\emptyset, n_k : k \in W)\frac{A(H)}{B(I)}$$

on the right-hand side. The remainder of (32) is

$$\begin{aligned}
&\hat{\pi}(I, n_k : k \in W) \cdot \Big(\sum_{i \in W \setminus I} \Big(\lambda_i^I + \sum_{j \in V \setminus I} \mu_j r^I(j,i) \Big) + \sum_{i \in W \setminus I} \mu_i (1 - r^I(i,i)) \cdot 1_{\mathbb{N}_+}(n_i) \Big) \\
&= \sum_{i \in W \setminus I} \hat{\pi}(I, n_k : k \in W \setminus \{i\}, n_i - 1) \cdot \Big(\lambda_i^I + \sum_{j \in V \setminus I} \mu_j r^I(j,i) \Big) \cdot 1_{\mathbb{N}_+}(n_i) + \\
&\quad + \sum_{i \in W \setminus I} \hat{\pi}(I, n_k : k \in W \setminus \{i\}, n_i + 1) \cdot \mu_i \Big(1 - \sum_{j \in W \setminus I} r^I(i,j) \Big) + \\
&\quad + \sum_{i \in W \setminus I} \sum_{j \in W \setminus I, j \neq i} \hat{\pi}(I, n_k : k \in W \setminus \{i,j\}, n_i + 1, n_j - 1) \cdot \mu_i r^I(i,j) \cdot 1_{\mathbb{N}_+}(n_j).
\end{aligned} \tag{33}$$

With $\eta_i^I = \lambda_i^I + \sum_{j \in W \setminus I} \eta_j^I r^I(j,i) + \sum_{j \in V \setminus I} \mu_j r^I(j,i)$ (see (10)) this is equivalent to

$$\begin{aligned}
&\hat{\pi}(I, n_k : k \in W) \cdot \Big(\sum_{i \in W \setminus I} \Big(\eta_i^I - \sum_{j \in W \setminus I} \eta_j^I r^I(j,i) \Big) + \sum_{i \in W \setminus I} \mu_i (1 - r^I(i,i)) \cdot 1_{\mathbb{N}_+}(n_i) \Big) \\
&= \sum_{i \in W \setminus I} \hat{\pi}(I, n_k : k \in W \setminus \{i\}, n_i - 1) \cdot \Big(\eta_i^I - \sum_{j \in W \setminus I} \eta_j^I r^I(j,i) \Big) \cdot 1_{\mathbb{N}_+}(n_i) + \\
&\quad + \sum_{i \in W \setminus I} \hat{\pi}(I, n_k : k \in W \setminus \{i\}, n_i + 1) \cdot \mu_i \Big(1 - \sum_{j \in W \setminus I} r^I(i,j) \Big) + \\
&\quad + \sum_{i \in W \setminus I} \sum_{j \in W \setminus I, j \neq i} \hat{\pi}(I, n_k : k \in W \setminus \{i,j\}, n_i + 1, n_j - 1) \cdot \mu_i r^I(i,j) \cdot 1_{\mathbb{N}_+}(n_j).
\end{aligned}$$

Under the required condition of either (11) and (12) in case of blocking rs-rd or (14) in case of skipping holds $\eta_i = \eta_i^I$ for all $i \in W \setminus I$ and all $I \subseteq D$ for the respective reduced traffic equations. therefore from Lemma 4.4 or Lemma 4.6 respectively this is equivalent to

$$\begin{aligned}
&\hat{\pi}(I, n_k : k \in W) \cdot \Big(\sum_{i \in W \setminus I} \Big(\eta_i - \sum_{j \in W \setminus I} \eta_j r^I(j,i) \Big) + \sum_{i \in W \setminus I} \mu_i (1 - r^I(i,i)) \cdot 1_{\mathbb{N}_+}(n_i) \Big) \\
&= \sum_{i \in W \setminus I} \hat{\pi}(I, n_k : k \in W \setminus \{i\}, n_i - 1) \cdot \Big(\eta_i - \sum_{j \in W \setminus I} \eta_j r^I(j,i) \Big) \cdot 1_{\mathbb{N}_+}(n_i) + \\
&\quad + \sum_{i \in W \setminus I} \hat{\pi}(I, n_k : k \in W \setminus \{i\}, n_i + 1) \cdot \mu_i \Big(1 - \sum_{j \in W \setminus I} r^I(i,j) \Big) + \\
&\quad + \sum_{i \in W \setminus I} \sum_{j \in W \setminus I, j \neq i} \hat{\pi}(I, n_k : k \in W \setminus \{i,j\}, n_i + 1, n_j - 1) \cdot \mu_i r^I(i,j) \cdot 1_{\mathbb{N}_+}(n_j).
\end{aligned}$$

Plugging in $\hat{\pi}(I, n_k : k \in W) = \frac{A(I)}{B(I)} \prod_{i \in W} \left(\frac{\eta_i}{\mu_i} \right)^{n_i}$ yields

$$\begin{aligned}
&\sum_{i \in W \setminus I} \Big(\eta_i - \sum_{j \in W \setminus I} \eta_j r^I(j,i) \Big) + \sum_{i \in W \setminus I} \mu_i (1 - r^I(i,i)) \cdot 1_{\mathbb{N}_+}(n_i) \\
&= \sum_{i \in W \setminus I} \frac{\mu_i}{\eta_i} \cdot \Big(\eta_i - \sum_{j \in W \setminus I} \eta_j r^I(j,i) \Big) \cdot 1_{\mathbb{N}_+}(n_i) + \sum_{i \in W \setminus I} \frac{\eta_i}{\mu_i} \cdot \mu_i \Big(1 - \sum_{j \in W \setminus I} r^I(i,j) \Big) + \\
&\quad + \sum_{i \in W \setminus I} \sum_{j \in W \setminus I, j \neq i} \frac{\eta_i}{\mu_i} \frac{\mu_j}{\eta_j} \mu_i r^I(i,j) \cdot 1_{\mathbb{N}_+}(n_j)
\end{aligned}$$

$$\Leftrightarrow 0 = -\sum_{i\in W\setminus I} \frac{\mu_i}{\eta_i} \cdot \sum_{j\in W\setminus I, j\neq i} \eta_j r^I(j,i)\cdot 1_{\mathbb{N}_+}(n_i) + \sum_{i\in W\setminus I}\sum_{j\in W\setminus I, j\neq i} \frac{\mu_j}{\eta_j}\eta_i r^I(i,j)\cdot 1_{\mathbb{N}_+}(n_j).$$

Thus $\hat{\pi}(I, n_k : k\in W) = \frac{A(I)}{B(I)}\prod_{i\in W}\left(\frac{\eta_i}{\mu_i}\right)^{n_i}$ solves the balance equations (32). The last step of proving (i) is by normalizing $\hat{\pi}$, which is possible because $\eta_i < \mu_i$ holds for all $i \in W$.