U N I V E R S I T Ä T   H A M B U R G

Non-ergodic Jackson Networks with Infinite Supply – Local Stabilization and Local Equilibrium Analysis

Jennifer Sommer, Hans Daduna, Bernd Heidergott

2014 - 04

DEPARTMENT MATHEMATIK

SCHWERPUNKT MATHEMATISCHE STATISTIK
UND STOCHASTISCHE PROZESSE

BUNDESSTR. 55, D – 20146 HAMBURG

Non-ergodic Jackson Networks with infinite Supply – local Stabilization and local Equilibrium Analysis

Jennifer Sommer
Hamburg Port Consulting GmbH, P.O. Box 11 14 06, 20414 Hamburg
Email: j.sommer@hpc-hamburg.de

Hans Daduna
Center of Mathematical Statistics and Stochastic Processes Department of Mathematics
University of Hamburg, Bundesstrasse 55, 20146 Hamburg
Email: Daduna@math.uni-hamburg.de

Bernd Heidergott
Tinbergen Institute and Department of Econometrics and Operations Research
Vrije Universiteit Amsterdam, De Boelelaan 1105, 1081 HV Amsterdam
Email: bheidergott@feweb.vu.nl

Abstract

Classical Jackson networks are a well established tool for the analysis of complex systems. In this paper we analyze Jackson networks with the additional features that (i) nodes may have an infinite supply of low priority work and (ii) nodes may be unstable in the sense that the queue length at these nodes grows beyond any bound. We provide the limiting distribution of the queue length distribution at stable nodes, which turns out to be of product-form. A key step in establishing this result is the development of a new algorithm based on adjusted traffic equations for detecting instable nodes. Our results complement the results known in the literature for the sub-cases of Jackson networks with either infinite supply nodes or unstable nodes by providing an analysis of the significantly more challenging case of networks with both types of nonstandard nodes present. Building on our product-form results, we provide closed-form solutions for common customer and system oriented performance measures.

Keywords: Jackson Network, stability, instability, product-form solution, bottleneck analysis, shortest paths

1 Introduction

Open Jackson networks and their generalized successors (BCMP and Kelly networks) are by now a well established class of models in, e.g., production, telecommunication, computer systems. Their constituents are arrival processes to stations (nodes) with servers, servicing of customers, routing among the stations and departures of customers. For surveys see [Kel79] and [CY01].

Todays networks are typically very complex and meet the conditions of classical Jackson networks only locally. As Goodman and Massey [GM84] show in their pioneering paper, if a set of nodes in a complex network fails to be stable, i.e., the queue length at these nodes builds up over time unboundedly, other parts of the networks can operate in a stable manner and the asymptotic queue length distribution at stable nodes has a closed form solution.

The analysis of locally instable Jackson networks becomes even more challenging when some nodes in the network, although stable, i.e., building no infinite queues over time, are required to be fully utilized for working without intermediate idling; see Weiss [Wei05]. This often occurs in production control where a machine is monitored over the time and by some external control additional raw material is supplied whenever there occurs the possibility that the queue empties. Such a system can be modeled by adding to the node which represents the fully utilized machine an infinite buffer (infinite supply or infinite virtual

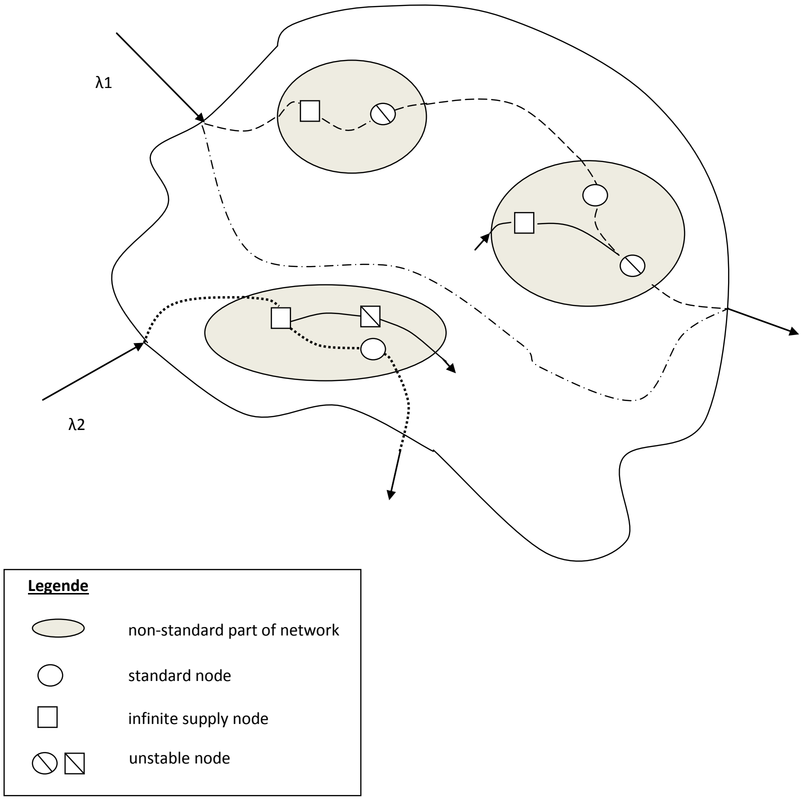

Figure 1: Example of queueing network with non-standard sub-networks

queue (IVQ)) of raw material from which the server takes material whenever the regular queue empties. Whenever a piece of such additional material is completely processed, it is send out as standard item (customer) into the residual network's production process. Implementing an infinite supply at some nodes has the aim to utilize capacities to the fullest, avoid idle times completely and therefore enhance productivity. However, infinite supply nodes act as additional sources to the network and thereby may lead to instability at downstream nodes.
For ease of exposition, we denote the node set of our network by $\tilde{J} = \{1, ..., J\}$. We will distinguish specific subsets of $\tilde{J}$:

- nodes in $V \subseteq \tilde{J}$ have an infinite supply of work; and nodes in $W := \tilde{J} \setminus V$ operate without infinite supply;
- nodes in S are stable; and nodes in $U = \tilde{J} \setminus S$ are unstable.

In this paper we provide an analysis of locally stable Jackson networks such that $V, U \neq \emptyset$ and possibly $V \cap U \neq \emptyset$. To illustrate the type of networks we address in this paper, consider the large scale network depicted in Figure 1. The network has two arrival streams with arrival intensity λ_1 and λ_2, respectively. The network contains three nonstandard sub-networks indicated by the gray-shaded areas, where the term "non-standard" refers to the fact that the subnetworks contain nodes with infinite supply, unstable nodes or both. The figure illustrates the possible cases. For example, in the subnet on top, a node is instable due to the fact that it is preceded by an infinite supply node the service rate of which is lager than the service rate of this note (and this node might become stable if the infinite supply is removed). In the subnetwork on the RHS, a node is unstable due to the load arriving from a stable and an infinite supply node. Finally, in the subnetwork on the bottom, there are two infinite supply nodes in a row the second of which is unstable. Note that customers arriving via arrival stream λ_2 and traversing the net on the dotted path only visit stable nodes, some of which are fully utilized. Customers arriving via arrival stream λ_1 and traversing the net on the dashed path will pass through unstable nodes and will therefore possibly not be able to leave the network in finite time. Customers from arrival stream λ_1 traversing the network on the intermediate path only visit stable nodes without infinite supply.

The difficulty to deal with analysing networks like the one depicted in Figure 1 stems from the simple observation that we have two binary classifications of the nodes which can interact in any way: Stable versus unstable and nodes with infinite supply versus standard nodes.
Our main results can be classified as "product form result", which in the classical setting says that for a vector valued Markov process the stationary distribution (at a fixed time point t) is the product of the stationary marginal distributions at t, i.e., the coordinates at t seem to decouple. More specifically, we find either stationary and limiting distributions for subsets of nodes in the well known form of Jackson's Theorem [Jac57], respectively limiting distributions in the sense of Goodman and Massey [GM84]. In the infinite supply literature, see, e.g., Weiss [Wei05], the setting $V \neq \emptyset$ and $S = \tilde{J}$ is studied, i.e., no unstable nodes; whereas Goodman and Massey [GM84] analyze the case $U \neq \emptyset$ and $V = \emptyset$. In this paper, we develop the missing theory for the case of networks with infinite supply and unstable nodes, thereby combining the problem settings of Weiss [Wei05] and of Goodman and Massey [GM84]. For an overview of existing literature and our contribution, see Table 1, where the theorems refer to the main theorems proved in this paper.

	$U = \emptyset, S = \tilde{J}$ all nodes stable (ergodic)	$U \neq \emptyset, S \subset \tilde{J}$ some nodes unstable (non − ergodic)
$V = \emptyset, W = \tilde{J}$ no infinite supply	classical theory	Goodman and Massey [GM84]
$V \neq \emptyset, W \subset \tilde{J}$ infinite supply	Weiss[Wei05] Theorem 7	Theorem 13, Theorem 14

Table 1: Overview of results from paper.

A key step in establishing the result on the limiting distribution is a new algorithm for detecting instable nodes in the combined framework. We believe that this algorithm is of great practical value as it allows for a stability analysis of complex networks.
The paper is organized as follows. The technical analysis of Jackson Networks with infinite supply and unstable nodes is provided in Section 2. Explicit closed-form solutions to common performance measures are provided in Section 3. Moreover, we show how our results can be applied to identify bottlenecks in stable networks, and we discus customer-related performance characteristics such as mean shortest travel times along stable paths.

2 Jackson Networks with infinite Supply and Unstable Nodes

For our analysis we introduce the following conventions:

- $A \subset B$ means that A is a strict subset of B, $A \subseteq B$ means $(A \subset B \vee A = B)$.
- $\mathbb{N}_+ = \{1, 2, 3, \dots\}$ and $\mathbb{N} = \mathbb{N}_+ \cup \{0\}$. $\mathbb{R}_+ = [0, \infty)$.
- All random variables occurring are defined on a common probability space $(\Omega, \mathcal{F}, P)$.
- For a set D we denote by $\mathcal{P}(D) = 2^D$ the set of all subsets of D.

Definition 1 (Jackson network). *[Jac57] We consider a standard Jackson network with node set $\tilde{J} = \{1, ..., J\}$. At node j an external Poisson(λ_j)-arrival stream ($\lambda_j \geq 0$) generates jobs. We set $\lambda := \lambda_1 + \cdots + \lambda_J \geq 0$ for the total arrival rate of such customers. The stations (nodes) are single servers with exponential(μ_j) distributed service times for all the jobs to be served, have infinite waiting room and operate under first-come-first-served (FCFS) regime. Customers (jobs) are indistinguishable. All interarrival and service times constitute a set of independent random variables.*
Routing is Markovian: Given the departure node i the selection of the next node is independent of the previous history. A customer departing from node i immediately proceeds to node j with probability $r(i, j) \geq 0$ and departs from the network with probability $r(i, 0)$ (the artificial node 0 represents the outside, source and sink, of the network, $r(0, 0) := 0$, $r(0, i) := \lambda_i/\lambda$). The routing matrix $R = (r(i, j) : i, j \in \{0, 1, ..., J\})$ is stochastic and irreducible.

Denote by $X_j(t)$ the local queue length at node j at time $t \geq 0$. Then from the independence assumptions and the memoryless property of the underlying distributions it follows that the joint queue length process $X = ((X_1(t), ..., X_J(t)) : t \in \mathbb{R}_+)$ describing the network's evolution is a Markov process on $\mathbb{N}^J$. The principles for adding an infinite supply of work (IVQ) at a node, say i_0, are as follows:

- Whenever all jobs queued at i_0 (which have high priority) have departed and the node is idle, a job (which has low priority) from the infinite supply depot is taken to be served there. When service is completed, that job is converted into a high priority job, departs, and is routed according to the routing matrix R.
- If during the service of a job from the infinite supply a regular job (from the outside or from a different node) arrives at that node i_0, this new job has preemptive priority and the job from the infinite supply depot is sent back to the depot immediately.
- Thus jobs from the infinite supply have lower priority. But after its initial service is completed, a low priority job turns into a high priority job.
- Nodes with infinite supply are busy all the time, hence their service capacity is fully utilized.
- As long as a job has low priority, it is not counted in the state space as a queued job, so the state description of the node does not change with its arrival at i_0.

It is easy to see that the queue length process $X = ((X_1(t), ..., X_J(t)) : t \in \mathbb{R}_+)$ of a Jackson network with infinite supply at nodes in V is a Markov process on $\mathbb{N}^J$ with transition rates matrix $Q = (q(z, z') : z, z' \in \mathbb{N}^J)$, which is derived in [Wei05], and is independent of whether $U = \emptyset$ or $U \neq \emptyset$. For all $z = (n_1, \ldots, n_J)$ and all $i, j \in \tilde{J}, i \neq j$ we have

$$
\begin{aligned}
&q(n_1, ..., n_i, ..., n_J; n_1, ..., n_i + 1, ..., n_J) = \lambda_i + \sum_{j \in V} \mu_j r(j, i) 1_{\{0\}}(n_j), \\
&q(n_1, ..., n_i, ..., n_J; n_1, ..., n_i - 1, ..., n_J) = \mu_i r(i, 0) 1_{\mathbb{N}_+}(n_i), \\
&q(n_1, ..., n_i, ..., n_j, ..., n_J; n_1, ..., n_i - 1, ..., n_j + 1, ..., n_J) = \mu_i r(i, j) 1_{\mathbb{N}_+}(n_i), \\
&q(n_1, ..., n_J; n_1, ..., n_J) = -\sum_{i \in \tilde{J}} \lambda_i - \sum_{i \in \tilde{J}} \sum_{j \in V} \mu_j r(j, i) 1_{\{0\}}(n_j) - \sum_{i \in \tilde{J}} \mu_i (1 - r(i, i)) 1_{\mathbb{N}_+}(n_i),
\end{aligned}
$$

and $q(z, z') = 0$ otherwise for $z \neq z'$.
Note, that an ongoing service at time $t \geq 0$ of a low priority job at node $i_0 \in V$ is detected by $X_{i_0}(t) = 0$. The principle of not counting the extra arrivals (from the infinite supply) at node i_0 in its state, and counting after first departure from i_0 these arrivals at (other) nodes is also used in [CHT01]. In [Wei05] it is assumed that jobs departing from some node j will be transferred only to nodes $i \neq j$, i.e., no feedback is allowed, $r(j, j) = 0$. We do not impose this condition in general and point out, that we can not remedy this problem of immediate feedback by reducing the service rate and setting the feedback probability to 0. The reason is: A low priority job being fed back, reenters its departure node according to the above rules as a high priority job.
The following property of the networks will be fundamental.

Theorem 2. [Wei05, Proposition 1(iii)] *Consider a Jackson network where nodes in $V \subseteq \tilde{J}$ have an infinite supply of work. Then the departure streams from nodes $j \in V$ are independent Poisson streams with rates μ_j and therefore the departure stream from $j \in V$ to $i \in \tilde{J}$ is Poisson with rate $\mu_j r(j, i)$.*

Example 1. *While for standard open Jackson networks the total arrival intensity necessarily fulfills $\lambda > 0$ to have a Markov process which is irreducible on $\mathbb{N}^J$, in case of infinite supply $\lambda = 0$ is allowed. A typical example is investigated by Adan and Weiss [AW05, Wei05].*
There are two nodes with infinite supply and service rates $\mu_i, i = 1, 2$, and routing matrix which, for $i = 1, 2$ fulfills $r(i, i) = 0, r(i, 0) > 0, i = 1, 2$ and $r(i, j) > 0, i, j = 1, 2, i \neq j$. We therefore have $W = \emptyset$ and $V = \tilde{J}$. Using the compensation method, Adan and Weiss [AW05] computed the steady state distribution, whenever it exists. This steady state distribution is not of product form.

2.1 Literature Review

Investigation of generalized Jackson networks with infinite supply has recently found much interest in the literature and it turned out that the feature of infinite supply makes analysis of the network considerably harder than that of classical product form networks of the BCMP and Kelly type. [Wei05] considers the case $V \neq \emptyset$ and $S = \tilde{J}$ with the additional condition that for $j \in V$ a customer finishing service at j is *not* directly re-routed to j. For this class of networks, product form steady-state results are provided in [Wei05] and compared with results on a special class of multi-class queueing networks with virtual infinite buffers, introduced in [KW02] and [AW05]. Specifically, [Wei05] considered Jackson networks with infinite supply with jobs of two priority classes, and with only one server at each station which can serve both classes of jobs. Jobs moving between the stations are of high priority, the infinite buffer at some stations is filled only with jobs of lower priority. Once completely served at their first station the lower priority jobs turn into higher priority jobs on their subsequent path through the network.

Infinite supply of lower-priority work (infinite virtual queues $\equiv$ IVQ) is used frequently. Early work using this concept of infinite supply are, [LY75] using IVQ attached to an $M/G/1$ queueing system to utilize idle times. More recent are [Guo08] where generalized Jackson networks are considered and [KNW09] where a push-pull network with infinite supply is investigated.

The work of Guo [Guo08] and of Guo, Lefeber, Nazarathy, Weiss, Zhang [GLN13] is on general multi-class queueing networks with IVQs under different scheduling policies for the servers. These policies guide the nodes' decisions how to dedicate their activities to either the regular standard queues or the infinite virtual queues. The key research question is the interplay of the production of jobs from the IVQs and stability of the standard queues.

Another class of models where additional work is added whenever a server becomes idle are *queues with vacations*. If a server observes an empty queue "he goes away to serve at some other place a customer", and returns thereafter. If he finds customers waiting there, he immediately starts servicing them, but when on his return his queue is empty again, he takes "another vacation" from his main queue to serve somewhere else, and so on. For a survey, see [Dos90].

Another application from a different field where such model fits in are wireless sensor networks. The nodes (sensors) continuously sense their environment and have to forward the data to a central station (sink). This is usually not possible by direct communication, so the nodes act additionally as transmission stations for data from other sensor nodes. If forwarding transmissions from other nodes has priority, the own data constitute the infinite buffer which generates the infinite supply for the node. A particular computer communication system that works in a similar way is according to [Wei05] an MAN (metropolitan area network) Ethernet RPR (resilient packet ring), in which ring traffic has priority over the traffic generated at nodes.

2.2 The Traffic Equations

In the product form theorem for ergodic Jackson networks the overall arrival rate at a node is a main ingredient in the steady state distribution. These mean values (expected number of arrivals per time unit in steady state) are obtained as solutions of the (standard) traffic equations for ergodic networks. Goodman and Massey [GM84] observed that in case of non ergodic networks with overloaded nodes a modified set of traffic equations provides valuable information about the individual nodes' asymptotic behaviour. In [Wei05] traffic equations for a Jackson network with infinite supply are derived under the condition $r(j, j) = 0, \forall j$ (which is skipped here). The general principle of *flux in = flux out* is modified as follows:

A node $i \in V$ has an infinite supply of work which is activated whenever this node is empty. The additional customers from the infinite supply depot are not counted in the state space as regular customers to the queue length until they leave the generator node after completed service. Assuming that, on average, all nodes are neither fully loaded nor overloaded, the input rate η_i of high priority customers at node i with infinite supply is less than its output rate of high priority customers. From Theorem 2 we know that node $i \in V$ generates a Poisson departure stream with rate μ_i. Therefore the output rate in the traffic equation (1) below is μ_i for all nodes with infinite supply instead of η_i, the input rate.

Definition 3. *The (standard) traffic equations of a Jackson network with infinite supply are*

$$\eta_i = \lambda_i + \sum_{j \in W} \eta_j r(j, i) + \sum_{j \in V} \mu_j r(j, i), \quad i \in \tilde{J}. \tag{1}$$

Example 2. *In the Example 4 of Adan and Weiss [AW05, Wei05] we find*

$$\eta_i = \mu_{3-i} r(3-i, i), \quad i = 1, 2\,.$$

Lemma 4. *The traffic equations* (1) *have a unique solution* $\eta = (\eta_1, ..., \eta_J)$.

Proof. In order to solve (1), consider the traffic equations in matrix notation partitioned according to the sets V and W:

$$\eta_W = \lambda_W + \eta_W R_{WW} + \mu_V R_{VW}, \tag{2}$$

$$\eta_V = \lambda_V + \eta_W R_{WV} + \mu_V R_{VV}. \tag{3}$$

From irreducibility of R, $(\mathbf{I} - R_{WW})^{-1}$ exists and is positive. Therefore (2) may be transformed into

$$\eta_W = (\lambda_W + \mu_V R_{VW})(\mathbf{I} - R_{WW})^{-1}, \tag{4}$$

which is the unique solution of (2). Inserting this into (3) yields the unique solution of (3), too. □

In our investigation of non ergodic networks where nodes may have an additional infinite supply we need more general traffic equations when there are nodes which, on average, are fully loaded or overloaded by high priority customers. This combines the traffic equations from [Wei05] and [GM84] into a unique setting.

Definition 5. *The* general traffic equations *for Jackson networks with infinite supply are*

$$\eta_i = \lambda_i + \sum_{j \in W} \min(\eta_j, \mu_j) r(j, i) + \sum_{j \in V} \mu_j r(j, i), \quad i \in \tilde{J}\,. \tag{5}$$

A node i is stable *if η_i determined by* (5) *is strictly less than μ_i, otherwise the node is* unstable.

The above traffic equations are motivated by the following considerations:

(i) For node $j \in V$ with infinite supply the output rate of high priority jobs is μ_j, which usually is not the overall arrival rate η_j.

(ii) For a stable node $j \in W$ (no infinite supply) the overall arrival rate η_j is the maximal departure rate as well, which can be met by the node because of $\mu_j > \eta_j$.

(iii) For an unstable node $j \in W$ the overall arrival rate η_j in general cannot be met by the node's capacity, because it can maximally process at rate μ_j.

(iv) The arguments in (ii) and (iii) lead to the departure rates $\min(\eta_j, \mu_j)$ from nodes $j \in W$.

Lemma 6. *The general traffic equations* (5) *have a unique solution which we denote by* $\eta = (\eta_1, ..., \eta_J)$.

In the proof of Lemma 6, the main argument is the existence of an algorithm by which the unique solution of (5) is determined in at most J steps. The structure of the algorithm is similar to that of Goodman and Massey for networks without IVQs, but as will be seen the proof is much more elaborated.

Algorithm 1. *Consider a Jackson network with J nodes where nodes in V have an infinite supply of work. Nodes in $W := \tilde{J} \setminus V$ work without infinite supply. Initially it is not known which nodes are stable and which are unstable.*

1. *Assume that all nodes are unstable. Based on this assumption, let $(\eta_i(1) : i \in \tilde{J})$ be the first estimate for the solution $(\eta_i : i \in \tilde{J})$ of the traffic equations* (5)*, i.e., $(\eta_i(1) : i \in \tilde{J})$ is the solution of the traffic equations:*

$$\eta_i(1) = \lambda_i + \sum_{j=1}^{J} \mu_j r(j, i) \quad \forall\ i \in \tilde{J},$$

which trivially exists and is unique, because all parameters at the right-hand side of the equations are given. Since the departure rate at each node $i \in W$ without infinite supply is $\min(\eta_i, \mu_i)$*, the estimate $\eta_i(1)$ overestimates the true value, so*

$$\eta_i(1) \geq \eta_i \quad \textit{holds for all } i \in \tilde{J}\,. \tag{6}$$

- *If $\eta_i(1) \geq \mu_i$ holds for all $i \in \tilde{J}$, all nodes are unstable and for the first estimate holds $\eta_i = \eta_i(1)$ $\forall\ i \in \tilde{J}$. Stop here.*
- *If $\eta_i(1) \geq \mu_i$ holds for all $i \in W$, then all nodes in W are unstable. If $\eta_{i_*}(1) < \mu_{i_*}$ holds for some nodes $i_* \in V$, then $\mu_{i_*} > \eta_{i_*}$ holds due to (6), so these nodes i_* are stable. But due to the infinite supply at these nodes, the traffic equations do not change with this information, so $\eta_i = \eta_i(1)$ holds $\forall\ i \in \tilde{J}$ and the set of stable nodes is identified as $S(1) := \{i : \eta_i(1) < \mu_i\} \subseteq V$. Stop here.*
- *If for at least one node $i_* \in W$ holds $\eta_{i_*}(1) < \mu_{i_*}$, then $\mu_{i_*} > \eta_{i_*}$ holds due to (6), so this node i_* is stable. But since $\eta_{i_*}(1)$ is obtained under the assumption that all nodes are unstable, we only know $\eta_{i_*}(1) \geq \eta_{i_*}$ as in (6). Set $S(1) := \{i : \eta_i(1) < \mu_i\}$ and proceed to the next step.*

2. *All nodes $i \in S(1)$ will eventually be stable. Assume that all other nodes $i \in \tilde{J} \setminus S(1)$ are unstable. Based on this assumption, let $(\eta_i(2) : i \in \tilde{J})$ be the second estimate for $(\eta_i : i \in \tilde{J})$, i.e., $(\eta_i(2) : i \in \tilde{J})$ is the solution of the traffic equations (with $U(1) = \tilde{J} \setminus S(1)$):*

$$\eta_i(2) = \lambda_i + \sum_{j \in S(1)\cap W} \eta_j(2) r(j,i) + \sum_{j \in U(1)\cup V} \mu_j r(j,i) \quad \forall\ i \in \tilde{J},$$

which exists and is unique, see Proof of Lemma 6. Again, $(\eta_1(2), ..., \eta_J(2))$ is at most an over-estimation, but the assumptions are more conservative than those for $(\eta_i(1) : i \in \tilde{J})$. It holds: $\eta_i \leq \eta_i(2) \leq \eta_i(1)$ $\forall\ i \in \tilde{J}$ and $\eta_i(2) < \mu_i$ $\forall\ i \in S(1)$.

- *If $S(1) = S(2) := \{i : \eta_i(2) < \mu_i\}$, then $\eta_i(2) = \eta_i$ holds $\forall\ i \in \tilde{J}$. Stop here.*
- *If $S(1) \neq S(2)$ (so $S(1) \subset S(2)$) and $(S(2) \setminus S(1)) \cap W = \emptyset$, then $\eta_i(2) = \eta_i$ holds $\forall\ i \in \tilde{J}$, and $S(2)$ is the true set of stable nodes. Stop here.*
- *If $S(1) \neq S(2)$ (so $S(1) \subset S(2)$) and $(S(2) \setminus S(1)) \cap W \neq \emptyset$, then $\eta_{i_*}(2) > \eta_{i_*}$ holds for at least one node $i_* \in \tilde{J}$. Iterate 2. with $S(2)$ as new set of stable nodes.*

Result of the Algorithm: *The algorithm stops after at most J iterations and provides* OUTPUT $\eta = (\eta_i : i \in \tilde{J})$, THE OVERALL ARRIVAL RATES AT THE NODES AND $S \subseteq \tilde{J}$, THE SET OF STABLE NODES.

Proof of Lemma 6. The traffic equations (5) are solved by an algorithm which recursively builds a sequence of vectors $\eta(n) = (\eta_1(n), ..., \eta_J(n))$, $n \in \mathbb{N}_+$, together with a sequence of sets $S(n) := \{i : \eta_i(n) < \mu_i\}$ of nodes, which are detected within the first n steps as being stable, for which holds:

(i) $S(0) := \emptyset$,

(ii) $S(n-1) \subseteq S(n)$ $\forall n \geq 1$,

(iii) $\exists! 0 < n^* \leq J : \quad S(n^*-1) \subset S(n^*) = S(n^*+1)$,

(iv) $\eta(n+1)$ solves the following $S(n) \cap W$-partition of traffic equations with $U(n) := \tilde{J} \setminus S(n)$)

$$\eta(n+1)_{S(n)\cap W} = \lambda_{S(n)\cap W} + \eta(n+1)_{S(n)\cap W} R_{S(n)\cap W\ \ S(n)\cap W} + \mu_{U(n)\cup V} R_{U(n)\cup V\ \ S(n)\cap W}, \quad (7)$$

$$\eta(n+1)_{U(n)\cup V} = \lambda_{U(n)\cup V} + \eta(n+1)_{S(n)\cap W} R_{S(n)\cap W\ \ U(n)\cup V} + \mu_{U(n)\cup V} R_{U(n)\cup V\ \ U(n)\cup V}. \quad (8)$$

We show that the sequence $\eta(n)$ delivered by that algorithm converges to the unique solution η of the traffic equations (5) in at most J iterations, if a unique solution exists (which will be shown in this proof later on):
If $S(n) \subseteq S(n+1)$ holds for all $n \in \tilde{J}$, then there exists $n^* \leq J$ with $S(n^*) = S(n^*+1)$, so the set of stable nodes will be found in at most J iterations and $\eta(n^*) = \eta$ will be the solution of the traffic equations.
We therefore show:

a) $\forall n \in \mathbb{N}$: $\eta_i(n) \geq \eta_i(n+1) \quad \forall\ i \in \tilde{J} \quad \Rightarrow \quad S(n) \subseteq S(n+1)$,

b) $\eta_i(n) \geq \eta_i(n+1)$ holds for all $i \in \tilde{J}, n \in \mathbb{N}_+$.

Proof of a): For all $i \in S(n)$ holds by definition $\eta_i(n) < \mu_i$. From $\mu_i > \eta_i(n) \geq \eta_i(n+1)$ follows $i \in S(n+1)$ and therefore $S(n) \subseteq S(n+1)$ holds for all $n \in \mathbb{N}$.
Proof of b): By induction over n.
1. Basis ($n = 1$):

$$\eta(1) = \lambda + \mu R_{\tilde{J}\tilde{J}} = \lambda + \mu_{S(1)\cap W} R_{S(1)\cap W\ \tilde{J}} + \mu_{U(1)\cup V} R_{U(1)\cup V\ \tilde{J}}, \tag{9}$$

$$\eta(2) = \lambda + \eta(2)_{S(1)\cap W} R_{S(1)\cap W\ \tilde{J}} + \mu_{U(1)\cup V} R_{U(1)\cup V\ \tilde{J}}, \tag{10}$$

so $\eta(1) \geq \eta(2)$ (component-wise) is equivalent to

$$\mu_{S(1)\cap W} R_{S(1)\cap W\ \tilde{J}} \geq \eta(2)_{S(1)\cap W} R_{S(1)\cap W\ \tilde{J}}.$$

Note that if $S(1) = \emptyset$ then $\eta(1) = \eta(2)$ follows directly. We therefore consider $S(1) \neq \emptyset$ for the remainder of the induction basis.
With (9) and (10) we have

$$\eta(1) = \eta(2) + \mu_{S(1)\cap W} R_{S(1)\cap W\ \tilde{J}} - \eta(2)_{S(1)\cap W} R_{S(1)\cap W\ \tilde{J}}$$

and from definition $\mu_{S(1)} > \eta(1)_{S(1)}$ holds component-wise, so $\mu_{S(1)\cap W} > \eta(1)_{S(1)\cap W}$ and

$$\begin{aligned} &\mu_{S(1)\cap W} > \eta(2)_{S(1)\cap W} + \mu_{S(1)\cap W} R_{S(1)\cap W\ S(1)\cap W} - \eta(2)_{S(1)\cap W} R_{S(1)\cap W\ S(1)\cap W} \\ \Leftrightarrow\ &\mu_{S(1)\cap W}(\mathbf{I} - R_{S(1)\cap W\ S(1)\cap W}) > \eta(2)_{S(1)}(\mathbf{I} - R_{S(1)\cap W\ S(1)\cap W}) \end{aligned}$$

Multiplying both sides of the last inequality from the right side with $(\mathbf{I} - R_{S(1)\cap W\ S(1)\cap W})^{-1}$ (which exists and is positive) yields

$$\mu_{S(1)\cap W} > \eta(2)_{S(1)\cap W} \ \Rightarrow\ \mu_{S(1)\cap W} R_{S(1)\cap W\ \tilde{J}} \geq \eta(2)_{S(1)\cap W} R_{S(1)\cap W\ \tilde{J}}.$$

2. Inductive step ($n \curvearrowright n+1$):
Induction hypothesis: For some $n \in \mathbb{N}_+$ holds $\eta(n-1) \geq \eta(n)$ ($\Rightarrow S(n-1) \subseteq S(n)$).
We show that $\eta(n) \geq \eta(n+1)$ holds under the induction hypothesis.
With $S' := S(n) \setminus S(n-1)$ we have

$$\begin{aligned} \eta(n) &= \lambda + \eta(n)_{S(n-1)\cap W} R_{S(n-1)\cap W\ \tilde{J}} + \mu_{U(n-1)\cup V} R_{U(n-1)\cup V\ \tilde{J}} \\ &= \lambda + \eta(n)_{S(n-1)\cap W} R_{S(n-1)\cap W\ \tilde{J}} + \mu_{S'\cap W} R_{S'\cap W\ \tilde{J}} + \mu_{U(n)\cup V} R_{U(n)\cup V\ \tilde{J}} \qquad (11) \\ \eta(n+1) &= \lambda + \eta(n+1)_{S(n)\cap W} R_{S(n)\cap W\ \tilde{J}} + \mu_{U(n)\cup V} R_{U(n)\cup V\ \tilde{J}} \\ &= \lambda + \eta(n+1)_{S(n-1)\cap W} R_{S(n-1)\cap W\ \tilde{J}} + \eta(n+1)_{S'\cap W} R_{S'\cap W\ \tilde{J}} + \mu_{U(n)\cup V} R_{U(n)\cup V\ \tilde{J}} \qquad (12) \end{aligned}$$

so $\eta(n) \geq \eta(n+1)$ (component-wise) is equivalent to

$$\eta(n)_{S(n-1)\cap W} R_{S(n-1)\cap W\ \tilde{J}} + \mu_{S'\cap W} R_{S'\cap W\ \tilde{J}} \geq \eta(n+1)_{S(n)\cap W} R_{S(n)\cap W\ \tilde{J}}. \tag{13}$$

Note that if $S' = \emptyset$ (i.e., $S(n-1) = S(n)$) then $\eta(n) = \eta(n+1)$ follows directly. We therefore consider the case $S' \neq \emptyset$ for the remainder of the induction step.
From (11) we have

$$\begin{aligned} \eta(n)_{S(n-1)\cap W} = \lambda_{S(n-1)\cap W} &+ \eta(n)_{S(n-1)\cap W} R_{S(n-1)\cap W\ S(n-1)\cap W} + \\ &+ \mu_{S'\cap W} R_{S'\cap W\ S(n-1)\cap W} + \mu_{U(n)\cup V} R_{U(n)\cup V\ S(n-1)\cap W}, \end{aligned}$$

and

$$\begin{aligned} \eta(n)_{S'\cap W} &= \lambda_{S'\cap W} + \eta(n)_{S(n-1)\cap W} R_{S(n-1)\cap W\ S'\cap W} + \\ &\quad + \mu_{S'\cap W} R_{S'\cap W\ S'\cap W} + \mu_{U(n)\cup V} R_{U(n)\cup V\ S'\cap W} \\ \Leftrightarrow\ \mu_{S'\cap W} &= \lambda_{S'\cap W} + \eta(n)_{S(n-1)\cap W} R_{S(n-1)\cap W\ S'\cap W} + \\ &\quad + \mu_{S'\cap W} R_{S'\cap W\ S'\cap W} + \mu_{U(n)\cup V} R_{U(n)\cup V\ S'\cap W} + \mu_{S'\cap W} - \eta(n)_{S'\cap W}. \end{aligned}$$

With $\eta^*(n)_{S(n)\cap W} := (\eta(n)_{S(n-1)\cap W}, \mu_{S'\cap W})$ and

$$\lambda^*_{S(n)\cap W} := (\lambda_{S(n-1)\cap W}, \lambda_{S'\cap W} + \mu_{S'\cap W} - \eta(n)_{S'\cap W})$$

we have

$$\eta^*(n)_{S(n)\cap W} = \lambda^*_{S(n)\cap W} + \eta^*(n)_{S(n)\cap W} R_{S(n)\cap W\ S(n)\cap W} + \mu_{U(n)\cup V} R_{U(n)\cup V\ S(n)\cap W}$$

and with the existence and positivity of $(\mathbf{I} - R_{S(n)\cap W\ S(n)\cap W})^{-1}$ we get

$$\eta^*(n)_{S(n)\cap W} = (\lambda^*_{S(n)\cap W} + \mu_{U(n)\cup V} R_{U(n)\cup V\ S(n)\cap W})(\mathbf{I} - R_{S(n)\cap W\ S(n)\cap W})^{-1}.$$

Similarly we get the solution of

$$\eta(n+1)_{S(n)\cap W} \overset{(12)}{=} \lambda_{S(n)\cap W} + \eta(n+1)_{S(n)\cap W} R_{S(n)\cap W\ S(n)\cap W} + \mu_{U(n)\cup V} R_{U(n)\cup V\ S(n)\cap W}$$

as

$$\eta(n+1)_{S(n)\cap W} = (\lambda_{S(n)\cap W} + \mu_{U(n)\cup V} R_{U(n)\cup V\ S(n)\cap W})(\mathbf{I} - R_{S(n)\cap W\ S(n)\cap W})^{-1}.$$

By definition holds $\mu_{S'} > \eta(n)_{S'}$, so $\mu_{S'\cap W} > \eta(n)_{S'\cap W}$ and therefore $\lambda^*_{S(n)\cap W} \geq \lambda_{S(n)\cap W}$. Thus

$$\begin{aligned}
&\eta^*(n)_{S(n)\cap W} \geq \eta(n+1)_{S(n)\cap W} \\
\Leftrightarrow\ & (\eta(n)_{S(n-1)\cap W}, \mu_{S'\cap W}) \geq (\eta(n+1)_{S(n-1)\cap W}, \eta(n+1)_{S'\cap W}) \\
\Leftrightarrow\ & \eta(n)_{S(n-1)\cap W} \geq \eta(n+1)_{S(n-1)\cap W}\ \wedge\ \mu_{S'\cap W} \geq \eta(n+1)_{S'\cap W} \\
\Rightarrow\ & \eta(n)_{S(n-1)\cap W} R_{S(n-1)\cap W\ \tilde{J}} \geq \eta(n+1)_{S(n-1)\cap W} R_{S(n-1)\cap W\ \tilde{J}} \\
& \wedge\ \mu_{S'\cap W} R_{S'\cap W\ \tilde{J}} \geq \eta(n+1)_{S'\cap W} R_{S'\cap W\ \tilde{J}}
\end{aligned}$$

which yields (13).
Existence of a solution of (5)**:** For the existence of a solution of the general traffic equations we need to show for all $n \in \tilde{J}$ that the $S(n) \cap W$-partition of the traffic equations, (7) and (8), has a solution. Transforming (7) into

$$\eta(n+1)_{S(n)\cap W} = (\lambda_{S(n)\cap W} + \mu_{U(n)\cup V} R_{U(n)\cup V\ S(n)\cap W})(\mathbf{I} - R_{S(n)\cap W\ S(n)\cap W})^{-1}$$

yields the unique solution of (7) and inserting this solution into equation (8) yields the unique solution of (8), but the transformation is possible if and only if $(\mathbf{I} - R_{S(n)\cap W\ S(n)\cap W})^{-1}$ exists and is positive. Because of the irreducibility of R, $(\mathbf{I} - R_{S(n)\cap W\ S(n)\cap W})^{-1}$ exists and is positive for all $n \in \tilde{J}$.
Uniqueness of a solution of (5)**:** Suppose η and $\hat{\eta}$ are both solutions of (5). Then for all nodes $i \in \tilde{J}$ holds

$$\begin{aligned}
\eta_i - \hat{\eta}_i &= \lambda_i - \lambda_i + \sum_{j\in W} (\min(\eta_j, \mu_j) - \min(\hat{\eta}_j, \mu_j)) r(j,i) + \sum_{j\in V} (\mu_j r(j,i) - \mu_j r(j,i)) \\
\Rightarrow |\eta_i - \hat{\eta}_i| &= \left| \sum_{j\in W} (\min(\eta_j, \mu_j) - \min(\hat{\eta}_j, \mu_j)) r(j,i) \right|.
\end{aligned}$$

Summing over all $i \in W$ yields:

$$\begin{aligned}
\sum_{i\in W} |\eta_i - \hat{\eta}_i| &= \sum_{i\in W} \left| \sum_{j\in W} (\min(\eta_j, \mu_j) - \min(\hat{\eta}_j, \mu_j)) r(j,i) \right| \\
&\overset{(*1)}{\leq} \sum_{i\in W} \sum_{j\in W} |(\min(\eta_j, \mu_j) - \min(\hat{\eta}_j, \mu_j))| \cdot |r(j,i)| \\
&= \sum_{j\in W} |(\min(\eta_j, \mu_j) - \min(\hat{\eta}_j, \mu_j))| \cdot \underbrace{\sum_{i\in W} |r(j,i)|}_{=1-r(j,0)-\sum_{i\in V} r(j,i)} \\
&\leq \sum_{j\in W} |(\min(\eta_j, \mu_j) - \min(\hat{\eta}_j, \mu_j))| \leq \sum_{j\in W} |\eta_j - \hat{\eta}_j|,
\end{aligned} \tag{14}$$

where $(*1)$ holds because of the triangle inequality. (14) yields

$$\sum_{i\in W} |\eta_i - \hat{\eta}_i| = \sum_{i\in W} |(\min(\eta_i, \mu_i) - \min(\hat{\eta}_i, \mu_i))|$$
$$\Leftrightarrow |\eta_i - \hat{\eta}_i| = |(\min(\eta_i, \mu_i) - \min(\hat{\eta}_i, \mu_i))| \ \forall\ i \in W,$$

because in any case $|\eta_i - \hat{\eta}_i| \geq |(\min(\eta_i, \mu_i) - \min(\hat{\eta}_i, \mu_i))| \ \forall\ i \in W$.
So $\{i \in W : \eta_i < \mu_i\} = \{i \in W : \hat{\eta}_i < \mu_i\} =: S \cap W$ and therefore η and $\hat{\eta}$ are the solutions of the same $S \cap W$-partition of the traffic equation (which has a unique solution, see above), which means $\eta = \hat{\eta}$. □

Whenever analyzing a Jackson network (with infinite supply), it is essential to detect the stable respectively unstable nodes, i.e., to determine $\min(\eta_i, \mu_i), i \in W$. Algorithm 1 provides this information in any case but the following modification will guide us in many cases to a short cut: When there is at most one "bottleneck", running Algorithm 1 is avoided. If the network is expected to be overloaded at many nodes, one may skip the first task of Algorithm 2 and start with Algorithm 1 right away. But in general, Algorithm 2 reduces the computational effort, because in cases when there is at most one "bottleneck", running Algorithm 1 is avoided.

Algorithm 2. *To determine which nodes are stable and which nodes are unstable and the appropriate traffic equations in a Jackson network with infinite supply at nodes in V. Let $W := \tilde{J} \setminus V$ denote the set of nodes without infinite supply.*

1. *Solve the standard traffic equations* (1). *Check if $\eta_i < \mu_i$ holds for all nodes $i \in \tilde{J}$.*
   - *If $\eta_i < \mu_i$ holds for all nodes $i \in \tilde{J}$, then all nodes are stable and* (1) *are the appropriate traffic equations.*
   - *If $\eta_i < \mu_i$ holds for all nodes in W and if the condition does not hold for at least one node with infinite supply ($\in V$), then all nodes in W are stable, but those nodes in V for which the condition does not hold are unstable. Nevertheless* (1) *are the appropriate traffic equations.*
   - *If there is only one node in W, say i_*, for which the condition $\eta_{i*} < \mu_{i*}$ does not hold, this node is unstable and the appropriate traffic equations are given by:*

   $$\eta_j = \lambda_j + \sum_{i\in W\setminus\{i_*\}} \eta_i r(i,j) + \sum_{i\in V\cup\{i_*\}} \mu_i r(i,j), \quad j \in \tilde{J}.$$

   - *If there is more than only one node i in W for which the condition $\eta_i < \mu_i$ does not hold, proceed to the following step.*
2. *Run Algorithm 1 to solve the general traffic equations* (5). *With the detected set $S := \{i : \eta_i < \mu_i\}$ of stable nodes and $U := \tilde{J} \setminus S$ of unstable nodes the appropriate traffic equations are then given by*

   $$\eta_j = \lambda_j + \sum_{i\in S\cap W} \eta_i r(i,j) + \sum_{i\in U\cup V} \mu_i r(i,j), \quad j \in \tilde{J}.$$

Result of the Algorithm: *The algorithm stops after at most J iterations and provides* OUTPUT $\eta = (\eta_i : i \in \tilde{J})$, THE OVERALL ARRIVAL RATES AT THE NODES AND $S \subseteq \tilde{J}$, THE SET OF STABLE NODES.

2.3 The Ergodic Case

In this section we prove some properties of Jackson networks with IVQs, which are complements and slight extensions of [Wei05].
Main parts of the next theorem are proved by Weiss in [Wei05] for ergodic Jackson networks with infinite supply where immediate feedback is not allowed. To fit to our later needs, we generalize and prove similar statements for Jackson networks where immediate feedback is allowed. We provide a detailed proof because central arguments will be reused later on in the non-ergodic case. These central arguments occur already in the sketch of the proof concerning ergodic networks in [Wei05], and are here extracted under the headings of a "Subnetwork Argument" and an "M/M/1 Argument", because we consider even in the ergodic case a slightly general setting. In our view the extension to the non-ergodic setting was not immediately to expect.

Theorem 7. (Local equilibrium analysis) *Consider a Jackson network with infinite supply as in Definition 1.Assume that $\eta_i < \mu_i$ holds for all nodes $i \in \tilde{J}$ where $\eta = (\eta_1, ..., \eta_J)$ is the unique solution of the traffic equations* (1)*. Denote by $X = ((X_1(t), ..., X_J(t)) : t \geq 0)$ the queue-length process on $\mathbb{N}^J$.*

(i) For nodes without infinite supply, the joint marginal limiting distribution is of product form:

$$\lim_{t\to\infty} P(X_i(t) = n_i : i \in W) = \prod_{i\in W} \left(1 - \frac{\eta_i}{\mu_i}\right)\left(\frac{\eta_i}{\mu_i}\right)^{n_i}, \tag{15}$$

for all $(n_i : i \in W) \in \mathbb{N}^{|W|}$ and this is a stationary distribution for the subnetwork on W as well.

(ii) If the system is started with an initial distribution which has (15) *as marginal joint queue lengths distribution on W, the arrival stream at $i \in V$ from $j \in W$ is a Poisson stream with rate $\eta_j r(j, i)$.*

(iii) If the system is started with an initial distribution which has (15) *as marginal joint queue lengths distribution on W, then the marginal limiting distribution for a node $i \in V$ with infinite supply which has no immediate feedback, i.e., $r(i, i) = 0$, is*

$$\lim_{t\to\infty} P(X_i(t) = n_i) = \left(1 - \frac{\eta_i}{\mu_i}\right)\left(\frac{\eta_i}{\mu_i}\right)^{n_i}, \tag{16}$$

for all $n_i \in \mathbb{N}$ and this is a one-dimensional stationary distribution as well.

Remark 8. *The main difference of Theorem 7 to Proposition 1 of Weiss in [Wei05] is the explicite condition "If the system is started with an initial distribution which has* (15) *as marginal joint queue lengths distribution on W" which is needed in (ii) as well as in (iii). Weiss implicitly uses this condition in his sketch of the proof. The point is that without this assumption customer streams from W to V, in general, are not Poisson. This will be evident in the following proof.*

Proof. of Theorem 7 (i): We start with a "subnetwork argument" which will be reused for several instances.
(Begin of Subnetwork Argument) Consider the subset W of nodes without infinite supply. We have the following information about the subnetwork W:

- All service times are exponentially distributed and the service discipline at all nodes is FCFS.
- Routing of customers is Markovian: A customer completing service at node $i \in W$ will either move to some node $j \in W$ with probability $r(i, j)$ or leave the subnetwork with probability $1 - \sum_{j\in W} r(i, j)$, which is non-zero for at least one $i \in W$ because of the routing matrix being irreducible for the global network on $\tilde{J}$.
- At each node $i \in W$, we have external Poisson arrival streams with rate $\lambda_i \geq 0$. Furthermore all streams from nodes $j \in V$ with infinite supply into nodes $i \in W$ are Poisson streams with rate $\mu_j r(j, i)$, see Theorem 2.

  All (inter-)arrival times from the source and from nodes in V into node $i \in W$ constitute a set of independent random variables. Thus all arrival streams from the outside of the subnetwork W into each node $i \in W$ constitute independent Poisson processes with rate $\lambda_i + \sum_{j\in V} \mu_j r(j, i)$.
- All service and inter-arrival times constitute a set of independent random variables.

These properties guarantee that the subnetwork W develops as a Jackson network with $|W|$ nodes where the source and sink represent $\{0\} \cup V$, see Definition 1. The corresponding queueing process

$$\tilde{X} := ((\tilde{X}_i(t) : i \in W) : t \in \mathbb{R}_+)$$

is a Markov process of its own. The traffic equations of the described subnetwork W are given by

$$\tilde{\eta}_i = \tilde{\lambda}_i + \sum_{j\in W} \tilde{\eta}_j r(j, i), \quad i \in W,$$

where

$$\tilde{\lambda}_i := \lambda_i + \sum_{j\in V} \mu_j r(j, i),$$

so $\eta_i = \tilde{\eta}_i$ holds for all $i \in W$. (<u>End of</u> Subnetwork Argument)
According to Jackson's theorem (see [Jac57]), $\tilde{X}$ has the unique stationary and limiting distribution (15) because $\eta_i < \mu_i$ for all $i \in W$ holds by assumption. Thus, even if the subnetwork V of nodes with infinite supply is not in equilibrium, the equilibrium on the subnetwork W of nodes without infinite supply is preserved, if the initial distribution has the joint marginal (15).
(ii): It is well known that ergodic Jackson networks with Poisson arrival streams from the source to node i with rate $\tilde{\lambda}_i$ have, in equilibrium, Poisson departure streams from node i to the sink with rate $\tilde{\eta}_i\tilde{r}(i,0)$, see [Mel79, Example 7.1]. From the proof of (i), we know that the subnetwork on W behaves like an ergodic Jackson network for its own with $\tilde{\lambda}_i := \lambda_i + \sum_{j\in V} \mu_j r(j,i)$ and

$$\tilde{\eta}_i\tilde{r}(i,0) = \eta_i\Big(1 - \sum_{j\in W} r(i,j)\Big) = \eta_i\Big(r(i,0) + \sum_{j\in V} r(i,j)\Big).$$

Hence, if the global network process is started with an initial distribution which has the marginal (15) on W, departures to the sink from nodes $i \in W$ are Poisson streams with rate $\eta_i r(i,0)$ and departures to any node $j \in V$ are also Poisson streams with rate $\eta_i r(i,j)$, because a portion $\frac{r(i,j)}{r(i,0)+\sum_{j\in V} r(i,j)}$ of the departure stream $\tilde{\eta}_i\tilde{r}(i,0)$ from node $i \in W$ is directed to $j \in V$. This holds even if the subnetwork V is not in equilibrium.
(iii): We start with an "M/M/1 argument" which will be used for several instances.
(<u>Begin of</u> M/M/1 Argument) Consider a node $i \in V$ with infinite supply and without immediate feedback (i.e., $r(i,i) = 0$):

- The node has exponential-μ_i distributed service, the service discipline is FCFS.
- Because of $r(i,i) = 0$, a customer being served at i, leaves node i with probability 1.
- The external arrival stream is Poisson with rate $\lambda_i \geq 0$. From (ii) it follows directly that, if the global network process is started with an initial distribution which has the marginal (15) on W, the arrival streams at node $i \in V$ from nodes $j \in W$ are Poisson with rate $\eta_j r(j,i)$. Arrival streams from nodes $j \in V \setminus \{i\}$ are Poisson streams with rate $\mu_j r(j,i)$, see Theorem 2. All these Poisson streams are independent. Thus the arrival stream at node $i \in V$ is a Poisson process with rate

$$\hat{\lambda}_i := \lambda_i + \sum_{j\in W} \eta_j r(j,i) + \sum_{j\in V\setminus\{i\}} \mu_j r(j,i) \overset{(1)}{=} \eta_i.$$

- All service and inter-arrival times constitute a set of independent random variables.

Thus, if the subnetwork W is in equilibrium and if $r(i,i) = 0$ holds, node $i \in V$ behaves as an $M/M/1-$ system of its own. The corresponding queue length process $\hat{X}$ is a birth-death process on state space $\mathbb{N}$ with birth rates $\hat{\lambda}_i = \eta_i$ and death rates μ_i. (<u>End of</u> M/M/1 Argument)
$\hat{X}$ has a stationary distribution $\pi_i(n) = \left(1 - \frac{\eta_i}{\mu_i}\right)\left(\frac{\eta_i}{\mu_i}\right)^n$, $n \in \mathbb{N}$, because $\eta_i < \mu_i$ was assumed. □

Remark 9. *Note that in Theorem 7 in (i) and (ii) immediate feedback is allowed at all nodes. Only in (iii) we required that the special node $i \in V$ under consideration has no immediate feedback. Necessity of this condition for the result can be seen from the balance equations as follows:*
Consider node $i \in V$ as in Theorem 7, but allow immediate feedback at all nodes. Then all facts utilized in the proof of (iii) hold except for:

- *If the subnetwork W is in equilibrium, node $i \in V$ behaves as an $M/M/1-$system with infinite supply and with immediate feedback of its own.*
- *The traffic equation then is*

$$\hat{\eta}_i = \lambda_i + \sum_{j\in W} \eta_j r(j,i) + \sum_{j\in V} \mu_j r(j,i), \tag{17}$$

  thus $\hat{\eta}_i = \eta_i$ holds, see (1).

- *The balance equations of $\hat{X}$ are for all $n \in \mathbb{N}$*

$$\pi_i(n)\Big(\lambda_i + \sum_{j\in W}\eta_j r(j,i) + \sum_{j\in V\setminus\{i\}}\mu_j r(j,i) + \mu_i r(i,i)1_{\{0\}}(n) + \mu_i(1-r(i,i))1_{\mathbb{N}_+}(n)\Big)$$
$$= \pi_i(n-1)\Big(\lambda_i + \sum_{j\in W}\eta_j r(j,i) + \sum_{j\in V\setminus\{i\}}\mu_j r(j,i)\Big)1_{\mathbb{N}_+}(n) +$$
$$+ \pi_i(n-1)\mu_i r(i,i)1_{\{1\}}(n) + \pi_i(n+1)\mu_i(1-r(i,i)). \tag{18}$$

Plugging (17) *into* (18) *yields*

$$\pi_i(n)\Big(\eta_i - \mu_i r(i,i) + \mu_i r(i,i)1_{\{0\}}(n) + \mu_i(1-r(i,i))1_{\mathbb{N}_+}(n)\Big)$$
$$= \pi_i(n-1)(\eta_i - \mu_i r(i,i))1_{\mathbb{N}_+}(n) + \pi_i(n-1)\mu_i r(i,i)1_{\{1\}}(n) + \pi_i(n+1)\mu_i(1-r(i,i)).$$

With $\pi_i(n) = \left(\frac{\eta_i}{\mu_i}\right)^n$ *this is equivalent to*

$$\eta_i - \mu_i r(i,i) + \mu_i r(i,i)1_{\{0\}}(n) + \mu_i(1-r(i,i))1_{\mathbb{N}_+}(n)$$
$$= \frac{\mu_i}{\eta_i}(\eta_i - \mu_i r(i,i))1_{\mathbb{N}_+}(n) + \frac{\mu_i}{\eta_i}\mu_i r(i,i)1_{\{1\}}(n) + \frac{\eta_i}{\mu_i}\mu_i(1-r(i,i)) \ \Leftrightarrow$$
$$-\mu_i r(i,i) + \mu_i r(i,i)1_{\{0\}}(n) - \mu_i r(i,i)1_{\mathbb{N}_+}(n) = -\frac{\mu_i}{\eta_i}\mu_i r(i,i)1_{\mathbb{N}_+}(n) + \frac{\mu_i}{\eta_i}\mu_i r(i,i)1_{\{1\}}(n) - \eta_i r(i,i)$$

With $r(i,i) > 0$ *the last equation holds if and only if*

$$\eta_i - \mu_i + \mu_i\Big(1_{\{0\}}(n) - \frac{\mu_i}{\eta_i}1_{\{1\}}(n)\Big) = \Big(1 - \frac{\mu_i}{\eta_i}\Big)\mu_i 1_{\mathbb{N}_+}(n). \tag{19}$$

- *In case of* $n = 0$ *equation* (19) *is reduced to* $\eta_i - \mu_i + \mu_i = 0 \ \Leftrightarrow\ \eta_i = 0$.
- *In case of* $n = 1$ *equation* (19) *reduces to* $\eta_i - \mu_i - \mu_i\frac{\mu_i}{\eta_i} = \left(1 - \frac{\mu_i}{\eta_i}\right)\mu_i \ \Leftrightarrow\ \eta_i = 2\mu_i$.
- *In case of* $n \geq 2$ *equation* (19) *is reduced to* $\eta_i - \mu_i = \left(1 - \frac{\mu_i}{\eta_i}\right)\mu_i \ \Leftrightarrow\ \eta_i = \mu_i$.

Thus, equation (19) *holds if and only if* $\mu_i = 0$ *holds for* $i \in V$ *which is a contradiction to the assumptions in the definition of a Jackson network with infinite supply.*

Remark 10. *In general, ergodic Jackson networks with infinite supply of work (even if* $r(i,i) = 0\ \forall i \in V$ *holds) do not have stationary distributions of product form. So, even in equilibrium, the queue lengths of the nodes with an infinite supply of work (*$i \in V$*) are at a fixed time instant neither independent of each other nor independent of the queue lengths of the nodes in* W*, although all flows between the nodes with infinite supply are Poisson.*
In contrary, the product form of (15) *says that in equilibrium the queue length processes of the subnetwork* W *at a fixed time instant are independent, although flows between these nodes are, in general, not Poisson.*

The negative statement of Remark 10 is founded by plugging $\pi(n_1,...,n_J) = \prod_{i\in\tilde{J}}\left(1-\frac{\eta_i}{\mu_i}\right)\left(\frac{\eta_i}{\mu_i}\right)^{n_i}$ into the global balance equations of the network process. Assuming $r(i,i) = 0\ \forall i \in V$, yields:

$$\sum_{i\in\tilde{J}}\lambda_i + \sum_{i\in\tilde{J}}\sum_{j\in V}\mu_j r(j,i)1_{\{0\}}(n_j) + \sum_{i\in\tilde{J}}\mu_i(1-r(i,i))1_{\mathbb{N}_+}(n_i)$$
$$= \sum_{i\in\tilde{J}}\frac{\mu_i}{\eta_i}\Big(\lambda_i + \sum_{j\in V}\mu_j r(j,i)1_{\{0\}}(n_j)\Big)1_{\mathbb{N}_+}(n_i) + \sum_{i\in\tilde{J}}\frac{\eta_i}{\mu_i}\mu_i r(i,0) + \sum_{i\in\tilde{J}}\sum_{j\in\tilde{J}\setminus\{i\}}\frac{\mu_j}{\eta_j}\frac{\eta_i}{\mu_i}\mu_i r(i,j)1_{\mathbb{N}_+}(n_j)$$

$$\Leftrightarrow \sum_{i\in\tilde{J}}\lambda_i + \sum_{i\in\tilde{J}}\sum_{j\in V}\mu_j r(j,i)1_{\{0\}}(n_j) + \sum_{i\in\tilde{J}}\mu_i(1-r(i,i))1_{\mathbb{N}_+}(n_i)$$
$$= \sum_{i\in\tilde{J}}\frac{\mu_i}{\eta_i}\Big(\underbrace{\lambda_i + \sum_{j\in V}\mu_j r(j,i)1_{\{0\}}(n_j) + \sum_{j\in\tilde{J}\setminus\{i\}}\eta_j r(j,i)}_{\overset{(*2)}{=}\eta_i(1-r(i,i)) - \sum_{j\in V}\mu_j r(j,i)1_{\mathbb{N}_+}(n_j) + \sum_{j\in V}\eta_j r(j,i)}\Big)1_{\mathbb{N}_+}(n_i) + \sum_{i\in\tilde{J}}\eta_i r(i,0)$$

$$\Leftrightarrow \sum_{i\in\tilde{J}}\lambda_i+\sum_{i\in\tilde{J}}\sum_{j\in V}\mu_j r(j,i)1_{\{0\}}(n_j)=\sum_{i\in\tilde{J}}\frac{\mu_i}{\eta_i}\sum_{j\in V}r(j,i)(\eta_j-\mu_j 1_{\mathbb{N}_+}(n_j))1_{\mathbb{N}_+}(n_i)+\sum_{i\in\tilde{J}}\eta_i r(i,0), \quad (20)$$

where $(*2)$ holds because of (1) and $r(i,i)=0\ \forall i\in V$. With

$$\sum_{i\in\tilde{J}}\lambda_i=\sum_{i\in W}\eta_i r(i,0)+\sum_{i\in V}\mu_i r(i,0)+\sum_{i\in V}(\eta_i-\mu_i)$$

(20) is equivalent to

$$\begin{aligned}
&\sum_{i\in V}\mu_i r(i,0)+\sum_{i\in V}(\eta_i-\mu_i)+\sum_{i\in\tilde{J}}\sum_{j\in V}\mu_j r(j,i)1_{\{0\}}(n_j)\\
&=\sum_{i\in\tilde{J}}\frac{\mu_i}{\eta_i}\sum_{j\in V}r(j,i)(\eta_j-\mu_j 1_{\mathbb{N}_+}(n_j))1_{\mathbb{N}_+}(n_i)+\sum_{i\in V}\eta_i r(i,0)\\
\Leftrightarrow &\sum_{i\in V}(\eta_i-\mu_i)\underbrace{(1-r(i,0))}_{=\sum_{j\in\tilde{J}}r(i,j)}+\sum_{i\in\tilde{J}}\sum_{j\in V}\mu_j r(j,i)1_{\{0\}}(n_j)=\sum_{i\in\tilde{J}}\frac{\mu_i}{\eta_i}\sum_{j\in V}r(j,i)(\eta_j-\mu_j 1_{\mathbb{N}_+}(n_j))1_{\mathbb{N}_+}(n_i)\\
\Leftrightarrow &\sum_{i\in V}(\eta_i-\mu_i 1_{\mathbb{N}_+}(n_i))\sum_{j\in\tilde{J}}r(i,j)=\sum_{i\in\tilde{J}}\frac{\mu_i}{\eta_i}\sum_{j\in V}r(j,i)(\eta_j-\mu_j 1_{\mathbb{N}_+}(n_j))1_{\mathbb{N}_+}(n_i)\\
\Leftrightarrow &\sum_{i\in V}(\eta_i-\mu_i 1_{\mathbb{N}_+}(n_i))\sum_{j\in\tilde{J}}r(i,j)\left(1-\frac{\mu_j}{\eta_j}1_{\mathbb{N}_+}(n_j)\right)=0.
\end{aligned}$$

The last equation is valid only if $r(i,j)=0$ holds for all $i\in V$ and $j\in\tilde{J}$. This yields the following

Corollary 11. *The only class of Jackson networks with infinite supply where the stationary queue lengths distribution is of product form is characterized by the following property: Customers departing from a node $i\in V$ leave the network directly to the sink with probability 1, i.e., $r(i,0)=1\ \forall i\in V$.*

Put it another way, independence of the queue lengths in the system (at a fixed time instant in equilibrium) is maintained only if the nodes with infinite supply do not interact with each other and if there are no customer streams from V to W. The only streams inside the network are from W to V or inside W. It is intriguing that all the departure streams from nodes with infinite supply are Poisson streams and exactly these seem to be the source of the dependence structure in equilibrium. The low priority customers from the infinite supply depot are then directed to the sink immediately after their first service and therefore they do not influence arrival rates in the network.

Example 3. *If in the Example 1 of Adan and Weiss [AW05, Wei05] under the conditions of Theorem 7 holds*

$$\eta_i=\mu_{3-i}r(3-i,i)<\mu_i,\quad i=1,2\,,$$

then the marginal stationary distribution of node i is geometric with success probability $1-(\mu_{3-i}r(3-i,i)/\mu_i)$, $i=1,2$, from Theorem 7 (16). In [AW05, Wei05] it is shown that the queue lengths in equilibrium are not independent for fixed t.

During our detailed analysis of the dependencies in the system, we came across with the following interesting fact. Whenever a Jackson network with infinite supply, $V\neq\emptyset$, features a subset $\emptyset\neq W\subset\tilde{J}$ (without infinite supply) with the property:
There is only one node, say $i_*=i_*(W)$, from which customers can move directly from W to V, and it holds $r(j,j)=0, r(j,i_*)>0,\ \forall\, j\in V$, then for any fixed $n\in\mathbb{N}$ and $t\geq 0$ in equilibrium the two events

- node i_* is empty at a time t,
- node $j\in V$ has a queue length of n customers at time t,

are independent of each other. The proof is rather involved and lengthy and can be found in [Myl13][Proposition 4.25].
We do not yet know whether this observation is an artefact, investigation of this is part of our ongoing research because it is in our view a striking fact because the independence of these events is part of the independence structure in space at a given fixed time instant.

Remark 12. *A similar independence property was found by Kopzon, Nazarathy, and Weiss in [KNW09][Theorem 1] in a two node network, which is related to Example 1 with the specific feature that customers' service time distribution and routing matrix are type dependent. In this case the only irreducible class of the state space is* $(\mathbb{N} \times \{0\}) \cup (\{0\} \times \mathbb{N})$. *Under ergodicity conditions for any fixed* $n \in \mathbb{N}$ *and* $t \geq 0$ *in equilibrium the two events*

- *node* i *is empty at a time* t,
- *node* $3 - i$ *has a queue length of* n *customers at time* t,

are independent of each other, $i = 1, 2$.

2.4 The Non-Ergodic Case

Jackson networks with unstable nodes cannot be ergodic in the classical sense, so there exists no steady-state distribution for the global network process, but in [GM84] it is proved that for the set of stable nodes a well defined limiting distribution of product form exists. The message of the next theorem is that for stable W the marginal limiting distribution on W exists similar to the result of Goodman and Massey and, moreover, it is a stationary distribution on W. The latter observation is surprising and might be compared with the result in Theorem 14 below, where as in the framework of Goodman and Massy [GM84] stationarity of the limiting distribution is not proposed. Recall from Definition 5 the distinction between stable and unstable nodes.

Theorem 13. (Local equilibrium analysis) *Consider a Jackson network where nodes in* $V \subseteq \tilde{J}$ *have an infinite supply of work as in Definitions 1. Denote by* $\eta = (\eta_1, ..., \eta_J)$ *the unique solution of the traffic equations* (5). *We assume that all nodes without IVQs are stable, i.e.* $U \cap W = \emptyset$. *Then the traffic equations* (5) *reduce to*

$$\eta_i = \lambda_i + \sum_{j \in W} \eta_j r(j,i) + \sum_{j \in V} \mu_j r(j,i), \quad i \in \tilde{J}. \tag{21}$$

Denote by $X = ((X_1(t), ..., X_J(t)) : t \geq 0)$ *the queue-length process on* $\mathbb{N}^J$.

(i) For nodes without infinite supply, the joint marginal limiting distribution is of product form:

$$\lim_{t \to \infty} P(X_i(t) = n_i : i \in W) = \prod_{i \in W} \left(1 - \frac{\eta_i}{\mu_i}\right) \left(\frac{\eta_i}{\mu_i}\right)^{n_i}, \tag{22}$$

for all $(n_i, i \in W) \in \mathbb{N}^{|W|}$, *and this is a stationary distribution on* W *as well.*

(ii) If the system is started with an initial distribution which has (22) *as marginal joint queue lengths distribution on* W, *the arrival stream from* $j \in W$ *to* $i \in V$ *is Poisson with rate* $\eta_j r(j,i)$. *All these Poisson streams are independent.*

(iii) If the system is started with an initial distribution which has (22) *as marginal joint queue lengths distribution on* W, *then for a stable node* $i \in V \cap S$ *with* $r(i,i) = 0$ *holds for all* $n_i \in \mathbb{N}$

$$\lim_{t \to \infty} P(X_i(t) = n_i) = \left(1 - \frac{\eta_i}{\mu_i}\right) \left(\frac{\eta_i}{\mu_i}\right)^{n_i}, \tag{23}$$

and this is a one-dimensional stationary distribution as well.

(iv) If the system is started with an initial distribution which has (22) *as marginal joint queue lengths distribution on* W, *then for unstable nodes with infinite supply,* $i \in U \subseteq V$, *and with* $r(i,i) = 0$, *the limit of the marginal queue length probability is for all* $n_i \in \mathbb{N}$

$$\lim_{t \to \infty} P(X_i(t) = n_i) = 0\,. \tag{24}$$

The message of the statements in (i) and (iii) is that instability of nodes with infinite supply does not matter neither for the limiting and stationary behavior of the joint distribution on W nor the local limiting and stationary distribution of stable nodes in V, which extends the Goodman-Massey results where no stationarity is proved for the stable part of the network. The statement of (iv) is what is expected from the respective results in [GM84].

Proof. (i): We start the proof with evocation of the SUBNETWORK ARGUMENT from p. 11, which yields in this case (21). According to Jackson's theorem [Jac57], $\tilde{X}$ has the unique stationary and limiting distribution (22) if and only if $\tilde{\eta}_i < \mu_i$ holds for all $i \in W$. This condition is equivalent to $\eta_i < \mu_i$ for all $i \in W$ which was assumed.
So the subnetwork on W is in equilibrium, if and only if the global network process on $\tilde{J}$ is started with an initial distribution which has the marginal (22).
(ii): It is well known that ergodic Jackson networks with Poisson arrival streams from the source to node i with rate $\tilde{\lambda}_i$ have, in equilibrium, Poisson departure streams from node i to the sink with some rate $\tilde{\eta}_i \tilde{r}(i, 0)$, see [Mel79, Example 7.1]. From the proof of (i), we know that the subset W behaves like an ergodic Jackson network of its own with $\tilde{\lambda}_i := \lambda_i + \sum_{j \in V} \mu_j r(j, i)$ and

$$\tilde{\eta}_i \tilde{r}(i, 0) = \eta_i \Big(1 - \sum_{j \in W} r(i, j)\Big) = \eta_i \Big(r(i, 0) + \sum_{j \in V} r(i, j)\Big).$$

Hence, if the subnetwork W is in equilibrium, departures to the sink from node $i \in W$ are a Poisson stream with rate $\eta_i r(i, 0)$ and departures to any node $j \in V$ are also a Poisson stream with rate $\eta_i r(i, j)$, because a portion

$$\frac{r(i, j)}{r(i, 0) + \sum_{j \in V} r(i, j)}$$

of the departure stream $\tilde{\eta}_i \tilde{r}(i, 0)$ from node $i \in W$ is directed to $j \in V$.

(iii)-(iv): We start the proof with evocation of the M/M/1 ARGUMENT showing that the resulting birth and death process $\hat{X}$ has a unique limiting and stationary distribution

$$\pi_i(n) = \left(1 - \frac{\eta_i}{\mu_i}\right) \left(\frac{\eta_i}{\mu_i}\right)^n, \quad n \in \mathbb{N},$$

if and only if $\eta_i < \mu_i$ holds. (Note, that customers who arrive from the infinite supply storage of this node are not counted by $\hat{X}$.) If $\eta_i \geq \mu_i$, the node is unstable and the limiting queue length distribution of $\hat{X}$ degenerates to a one-point distribution in ∞. □

The next theorem can be termed a "Goodman-Massey theorem" for a subnetwork of a generalized Jackson network when there are outside of the subnetwork nodes which have infinite supply. The intriguing observation is that (as Goodman and Massey) we can only prove results on limiting distributions. Note, if $V = \emptyset$ it is exactly the main result of Goodman and Massey [GM84].

Theorem 14. (LOCAL LIMITING ANALYSIS.) *Denote by $\eta = (\eta_1, ..., \eta_J)$ the unique solution of the traffic equations* (5)*. Then we have independent of the initial distribution for all $n_i \in \mathbb{N}$:*

$$\lim_{t \to \infty} P(X_i(t) = n_i : i \in S \cap W) = \prod_{i \in S \cap W} \left(1 - \frac{\eta_i}{\mu_i}\right) \left(\frac{\eta_i}{\mu_i}\right)^{n_i}, \tag{25}$$

$$\lim_{t \to \infty} P(X_i(t) = n_i) = 0 \quad \forall i \in U \cap W. \tag{26}$$

Proof. We start the proof with evocation of the SUBNETWORK ARGUMENT from p. 11. The traffic equations of the described subnetwork W now are given by

$$\tilde{\eta}_i = \tilde{\lambda}_i + \sum_{j \in W} \min(\tilde{\eta}_j, \mu_j) r(j, i), \quad i \in W,$$

where $\tilde{\lambda}_i := \lambda_i + \sum_{j \in V} \mu_j r(j, i)$, so $\eta_i = \tilde{\eta}_i$ holds for all $i \in W$, see (5). According to Theorem 1 in [GM84], $\tilde{X}$ has the limiting marginal joint queue length distribution (25) on $S \cap W$ and the limiting marginal queue length probabilities (26) for nodes in $U \cap W$. □

Remark 15. *In the general situation of Theorem 14 we cannot prove a statement about the marginal limiting distribution like* (23) *for stable nodes with infinite supply ($\in S \cap V$) with similar arguments as above. If $U \cap W \neq \emptyset$ holds, the queue length process of the subnetwork W is not ergodic, so the argument of Poisson departure streams from W into nodes in V does not apply. If $U \cap W = \emptyset$ holds, Theorem 13 applies.*

Example 4. *If in the Example 1 of Adan and Weiss [AW05, Wei05] under the initial conditions of Theorem 13 holds*

$$\eta_1 = \mu_2 r(2,1) < \mu_1, \quad \text{and} \quad \eta_2 = \mu_1 r(1,2) \geq \mu_2\,,$$

then the marginal limiting and stationary distribution of node 1 *is geometric with success probability* $1-(\mu_2 r(2,1)/\mu_1)$, *whereas the queue length of node* 2 *diverges from Theorem 13* (23) *and* (24).
In this example it is not possible that both nodes are unstable: $U \neq \tilde{J}$ *holds in any parameter setting.*

3 Applications

The most important performance metrics are the system oriented throughput of the network or parts of it, and the customer oriented passage or sojourn times. We will give prototypes of examples for both problems. Eventually, we discuss bottleneck analysis.

3.1 Evaluation of System Performance

We start with discussion of general quality assessment for networks from Sections 2.3 and 2.4 via long-time average returns (or long-time average costs).
(I) For ergodic systems long-time average returns can be approximated via stationary characteristics if these are explicitly known. For ergodic Jackson network with infinite supply, however, the stationary joint queue length distribution is not available. Nevertheless, exploiting the results of Section 2.3 it is possible for ergodic networks to approximate accumulated cost over a long time horizon a slightly weakened form. Take non-decreasing bounded cost functions $g_i : \mathbb{N} \to \mathbb{R}$ associated with queue length at node $i \in V$ and $g_W : \mathbb{N}^{|W|} \to \mathbb{R}$ associated with a queue lengths at nodes in W. The time average of accumulated costs of the system over time horizon $[0,T]$ is

$$d(T) = \sum_{i\in V} \frac{1}{T}\int_0^T g_i(X_i(t))\ dt + \frac{1}{T}\int_0^T g_W(X_i(t) : i \in W)\ dt.$$

If, for $i \in S$, $\eta_i < \mu_i$ holds for all $i \in \tilde{J}$, then from Theorem 7 (i) follows

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T g_W(X_i(t) : i \in W) dt = \sum_{(n_i : i\in W)\in\mathbb{N}^{|W|}} g_W(n_i : i \in W)\pi_W(n_i : i \in W).$$

If $\eta_i < \mu_i$ holds for all $i \in \tilde{J}$ and $r(i,i) = 0$ for all $i \in V$, then from Theorem 7*(iii)* follows

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T g_i(X_i(t))\ dt = \sum_{n_i\in\mathbb{N}} g_i(n_i)\pi_i(n_i)\ \forall i \in V.$$

So for large T time averaged costs can be approximated by state space averages in ergodic Jackson networks with infinite supply:

$$d(T) \approx \sum_{i\in V}\sum_{n_i\in\mathbb{N}} g_i(n_i)\left(1-\frac{\eta_i}{\mu_i}\right)\left(\frac{\eta_i}{\mu_i}\right)^{n_i} + \sum_{(n_i:i\in W)\in\mathbb{N}^{|W|}} g_W(n_i : i \in W)\prod_{i\in W}\left(1-\frac{\eta_i}{\mu_i}\right)\left(\frac{\eta_i}{\mu_i}\right)^{n_i}.$$

(II) In case of non-ergodic Jackson networks with infinite supply we clearly cannot apply the ergodic theorem to exchange time averages and space averages for to assess average long time accumulated costs in general. Due to the special structure of the network, in case $W \cap U = \emptyset$, at least the average accumulated costs of the subsystem W over a time horizon $[0,T]$ can be predicted. Utilizing the idea which leads to the "subnetwork argument" (see the proof of Theorem 7), for non-decreasing cost function $g_W : \mathbb{N}^{|W|} \to \mathbb{R}$ we have almost surely

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T g_W(X_i(t) : i \in W) dt = \sum_{(n_i : i\in W)\in\mathbb{N}^{|W|}} g_W(n_i : i \in W)\pi_W(n_i : i \in W).$$

So, even in *non-ergodic* Jackson networks with infinite supply, the path-wise evaluated time averages on the subnet W for a time horizon $[0,T]$ with large T can be estimated by state space averages:

$$\frac{1}{T}\int_0^T g_W(X_i(t) : i \in W)\ dt \approx \sum_{(n_i:i\in W)\in\mathbb{N}^{|W|}} g_W(n_i : i \in W)\prod_{i\in W}\left(1-\frac{\eta_i}{\mu_i}\right)\left(\frac{\eta_i}{\mu_i}\right)^{n_i}.$$

3.2 Throughput

We first compute the throughput in ergodic networks with infinite supply nodes. Subsequently, we deal with the non-ergodic case.

3.2.1 Throughput in ergodic Jackson networks with infinite supply ($S = \tilde{J}, V \neq \emptyset$)

Ergodic Jackson networks with infinite supply require for computation of throughput schemes which are slightly different from the case of classical networks. Recall that nodes $i \in W$ have no infinite supply, nodes $j \in V$ are fed by infinite supply. Due to ergodicity, the stationary throughput TH_i of node i exists, which is the mean number of departures from i per time unit, $i \in \tilde{J}$. It follows directly for $i \in W$

$$TH_i =: \sum_{(n_1,\ldots,n_J)\in\mathbb{N}^J} \pi(n_1, ..., n_J)\mu_i 1_{\mathbb{N}_+}(n_i) = \eta_i \,,$$

and for $j \in V$

$$TH = \sum_{i\in W} TH_i r(i,0) + \sum_{j\in V} TH_j r(j,0) = \mu_j \,.$$

The stationary total throughput TH of the network, i.e., the mean number of departures from the network to the outside (sink) is

$$TH = \sum_{j\in\tilde{J}} TH_j r(j,0) = \sum_{j\in W} \eta_j r(j,0) + \sum_{j\in V} \mu_j r(j,0) \overset{(1)}{=} \lambda - \sum_{j\in V} \underbrace{(\eta_j - \mu_j)}_{<0} > \lambda.$$

Here (1) follows from

$$\lambda = \sum_{i\in\tilde{J}} \lambda_i = \sum_{i\in W} \eta_i r(i,0) + \underbrace{\sum_{i\in V} \mu_i r(i,0)}_{(*)} + \sum_{i\in V} (\eta_i - \mu_i),$$

where in $(*)$ customers from infinite supply which immediately depart after service are counted as proper customers.

3.2.2 Throughput in non-ergodic Jackson networks with infinite supply ($S \subset \tilde{J}, V \neq \emptyset$)

Non-ergodic Jackson networks with infinite supply ($S \subset \tilde{J}, V \neq \emptyset$) surprisingly admit computation of "stationary throughput" by use of Theorem 13 with elaboration of the local equilibrium analysis. If nodes without infinite supply are stable, the stationary throughput TH_i of a node $i \in W$ as

$$TH_i := \sum_{(n_j:j\in W)\in\mathbb{N}^{|W|}} \pi_W(n_j : j \in W)\mu_i 1_{\mathbb{N}_+}(n_i) = \eta_i,$$

and if $r(j,j) = 0$ holds for all $j \in V \cap S$, the stationary throughput TH_j of a stable node $j \in V$ with infinite supply is

$$TH_j := \sum_{n_j\in\mathbb{N}} \pi_j(n_j)\mu_j = \mu_j.$$

If nodes without infinite supply are stable and if $r(j,j) = 0$ holds for all $j \in V \cap S$, the stationary throughput TH_S of the subnetwork of stable nodes, i.e., the mean number of departures from the subnetwork S to the outside (sink) is

$$TH_S = \sum_{i\in W} TH_i r(i,0) + \sum_{j\in V\cap S} TH_j r(j,0) \overset{(2)}{=} \lambda - \sum_{j\in V} (\eta_j - \mu_j) - \sum_{j\in V\cap U} \mu_j r(j,0)\,,$$

where (2) follows with the same arguments as (1) above.

3.3 Travel times

Travel times of customers inside a network or passage times for customers through the whole network are the most important performance indices from a customer's point of view. We will discuss in this section these topics and show that our results from Section 2 lay the ground for e.g.

1. finding feasible paths to traverse the network,
2. finding shortest feasible paths between two nodes, and
3. determining expected travel times.

3.3.1 Expected travel times and shortest paths

Paths are understood in the natural way, but we have for different problems to take care whether for measuring a customer's travel time includes the sojourn time at the initial node and at the end node of the path.

Definition 16. PATHS. *A path in the network is a sequence of connected nodes, where nodes may occur repeatedly, i.e., a sequence of nodes $\langle j_1, j_2, \ldots, j_m \rangle$ with the property $r(j_k, j_{k+1}) > 0, \forall k = 1, \ldots, m-1$. A path in the network is feasible if it does not contain unstable nodes.*

To elaborate on paths and their structure we naturally have to consider the directed transition graph of the network's routing chain $R = (r(i,j) : i, j \in \{0, 1, \ldots, J\})$, defined as $G_R = (V_R, E_R)$ with set $V_R = \{0, 1, \ldots, J\}$ of vertices and with edge set $E_R \subseteq V_R^2$ defined by $(i,j) \in E_R \leftrightarrow r(i,j) > 0$.
The most important technical devices in our evaluation will be Algorithms 1 and 2 which are developed to solve the traffic equations of the network. For a simpler presentation we will only refer to Algorithm 1 which in any case determines the solution of these equations.
Recall that the algorithm stops after at most J iterations and provides as OUTPUT $\eta = (\eta_i : i \in \tilde{J})$, the overall arrival rates of regular customers (of high priority) at the respective nodes and $S \subseteq \tilde{J}$, the set of stable nodes and $U = \tilde{J} \setminus S \subseteq \tilde{J}$, the set of unstable nodes. It therefore prepares directly to find feasible paths.

Problem 1. DETERMINE A FEASIBLE PATH TO CROSS THE NETWORK USING PRESCRIBED ENTRANCE AND EXIT SETS. *We consider a Jackson network with IVQs as described in Section 2 with a set $\emptyset \neq \tilde{I} \subseteq \tilde{J}$ of entrance nodes and a set $\emptyset \neq \tilde{O} \subseteq \tilde{J}$ of exit nodes. I.e. $j \in \tilde{I} \Rightarrow r(0,j) > 0$ and $j \in \tilde{O} \Rightarrow r(j,0) > 0$. The problem is to determine a feasible path from some $i \in \tilde{I}$ to some $j \in \tilde{O}$.*
Solution.
• *Run Algorithm 1. Take the directed transition graph $G_R = (V_R, E_R)$ of the network's routing chain.*
• *Delete all nodes in U and all ingoing and outgoing arcs from nodes in U to obtain a subgraph denoted by $G_R(-U)$.*
• *There exists a feasible path to cross the network iff in $G_R(-U)$ exist a path from some node in $\tilde{I} \setminus U$ to some node in $\tilde{O} \setminus U$.*

From the result of Algorithm 1 we can directly compute the utilization, i.e., the fraction of time the stable nodes are serving regular customers.

Definition 17. *Let node $i \in \tilde{J}$ be stable, i.e. it holds $\eta_i < \mu_i$. Then the utilization of node i by regular customers is $\rho_i := \frac{\eta_i}{\mu_i}$. If node $i \in \tilde{J}$ is unstable, i.e. it holds $\eta_i \geq \mu_i$ the utilization of node i by regular customers is defined to be $\rho_i := 1$.*

This definition is a little bit subtle. In classical ergodic Jackson networks the utilization of a node refers to the fraction of time the node is working, either in a stationary network or as a limiting fraction, which from ergodicity are identical. In our setting we usually do not have ergodic systems, but as shown in the previous sections, e.g. we may have in parts of the network W stationary and limiting distributions (Theorem 13) or in parts of the subnetwork V with IVQs we may find limiting distributions (Theorem 14). Definition 17 will apply in both situations and refer to stationary and to limiting distributions.
For investigation of travel time distributions, resp. expected travel times, we use the feature of "test customers." These are by definition customers who find at their arrival at some path $\langle j_1, j_2, \ldots, j_m \rangle$, i.e. when entering the tail of the queue at node j_1 the other customers in the network distributed according the asymptotic or stationary distribution. In simulation of response times (= travel times) in networks

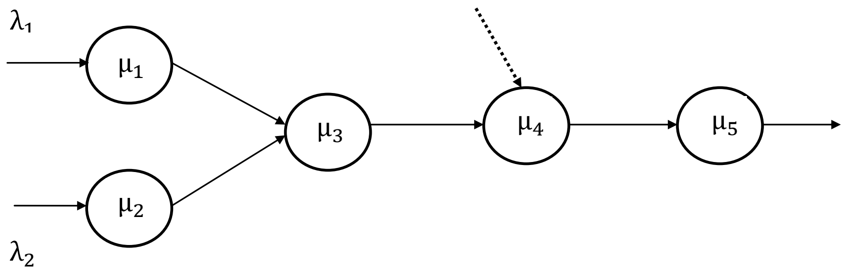

Figure 2: Identification of weak bottelnecks

this concept is known as "marked customer technique," for a description of the procedure in simulation framework see e.g. Chapter 4 in [IS80]. A framework for correctness of the concept using test customers is provided by Palm distributions from point process theory, for an in depth study with regard to sojourn time distributions in networks see [DS02].

Corollary 18. *For a stable node $i \in S$ the asymptotic or stationary sojourn time of a test customer at node i is exponentially distributed with parameter $\mu_i - \eta_i$.*
The travel time $tr\langle j_1, j_2, \dots, j_m\rangle$ for a test customer to travel path $\langle j_1, j_2, \dots, j_m\rangle$ of stable nodes has mean value

$$\mathbb{E}[tr\langle j_1, j_2, \dots, j_m\rangle] = \sum_{k=1}^{m} \frac{1}{\mu_{j_k} - \eta_{j_k}}\,. \tag{27}$$

Proof. Because a test customer finds the other customers at his entrance epoch distributed according to a geometrical distribution with success probability $1 - \eta_i/\mu_i$, the first statement follows from the node's structure as single server under FCFS. The second statement follows by additivity of expectations. □

Example 5. *Consider the queuing network depicted in Figure 2, where the additional dotted input arrow at node 4 indicates that node 4 is a infinite supply node. Let $\lambda_1 = \lambda_2 = 1$, $\mu_1 = 2$, $\mu_2 = 4$, $\mu_3 = 3$, $\mu_4 = 5/2$ and $\mu_5 = 3$. Then, $\eta_1 = \eta_2 = 1$, $\eta_3 = \eta_4 = 2$, and $\eta_5 = 5/2$. Hence, all nodes are stable and the mean waiting times at node i, denoted by W_i, is $W_1 = 1, W_2 = 1/3, W_3 = 1, W_4 = 2$, and $W_5 = 2$. By Little's law, the corresponding mean queue lengths, denoted by L_i, are given by $L_1 = 1, L_2 = 1/3, L_3 = 2, L_4 = 4$, and $L_5 = 5$.*

We now prepare the network to apply standard graph algorithms to compute fastest routes between prescribed nodes. This needs to construct a weighted directed graph $G_{RW} = (V_R, E_R, W)$ associated with the network. We start with the directed transition graph $G_R = (V_R, E_R)$ of the network's routing chain and associate with each edge $(i, j) \in E_R$ a weight

$$w(i,j) = \begin{cases} (\mu_i - \eta_i)^{-1}, & \text{if } i \in S\ ; \\ \infty, & \text{if } i \in U. \end{cases} \tag{28}$$

Problem 2. FIND SHORTEST PATHS BETWEEN TWO STABLE NODES. *We consider a Jackson network with IVQs as described in Section 2 and two stable nodes $i, j \in \tilde{J}$.*
The problem is to determine a path from i to j with shortest expected travel time for a test customer through all nodes of the path including i and j.
Solution.
- *Run Algorithm 1. Construct the weighted directed transition graph $G_{RW} = (V_R, E_R, W)$ of the network's routing chain.*
- *Apply Dijkstra's Algorithm [BJG09][p. 94 - 97] to determine a shortest path $sp(i, j) := \langle i =$*

$j_1, j_2, \ldots, j_m = j\rangle$ from i to j and the sum

$$w(sp(i,j)) = \sum_{k=1}^{m-1} w(j_k, j_{k+1}) \tag{29}$$

of the weights on this path.

- *If (29) yields $w(sp(i,j)) = \infty$, there is no feasible path from i to j.*
- *Otherwise: Add $(\mu_j - \eta_j)^{-1}$ to obtain*

$$\mathbb{E}[tr(sp(i,j))] = \sum_{k=1}^{m-1} w(j_k, j_{k+1}) + \frac{1}{(\mu_j - \eta_j)} = \mathbb{E}[tr\langle i, j_2, \ldots, j_{m-1}, j\rangle] = \sum_{k=1}^{m} \frac{1}{\mu_{j_k} - \eta_{j_k}},$$

which is the minimal expected travel time from i to j, including the time to pass the initial node and the destination node.

Note, that in the solution procedure of Problem 2 we may substitute the Dijkstra's Algorithm by any other algorithm which solves the shortest path finding problem for weighted directed graphs.

Remark 19. *The problem to determine characteristics other than mean values of passage times over general paths in Jackson networks is a challenging open problem even in classical Jackson networks, for a survey see [BD90]. Clearly, nodes with IVQ will pose even more difficulties in the solution of this problem because of the unknown correlations in steady state queue lengths vectors. This is part of our ongoing research in this area.*

3.4 Bottleneck analysis

Clearly, the unstable nodes in $U \subseteq \tilde{J}$ are natural bottlenecks of the system. Identification of these "strong" bottlenecks is easy with the aid of Algorithms 1 and 2, which output the set $S = \tilde{J} \setminus U$. Practical experience shows that even in stable networks ($U = \emptyset$ because of $\eta_j < \mu_j \ \forall j \in \tilde{J}$) "weak" bottlenecks emerge, which may have strong influence on the performance of the network. Heuristically, these "weak" bottlenecks are those nodes where the local queue lengths become large compared to the other nodes' queue lengths with respect to some stochastic order.

It is worth noting that Algorithm 1 and 2 allow for bottleneck analysis in this weak sense as well. To see this, consider an overall stable network and scale the service rates at all station by a common factor $\theta \in (0, 1]$. Solving the traffic equations and identifying the possible instable nodes in the reduced service rate setting, then identifies possible bottlenecks for the actual (stable) network. It is worth noting that this allows, for example, to identify the stability behavior of a network in the planning phase when the service rates not exactly known.

Consider a classical Jackson network, i.e., $V = \emptyset = U$. Then the mean queue length at node i is $L_i = \eta_i/(\mu_i - \eta_i)$ and the effect of a marginal increase in μ_i on the mean queue length is given by

$$\frac{d}{d\,\mu_i} L_i = -\frac{\eta_i}{(\mu_i - \eta_i)^2} = -\frac{L_i^2}{\eta_i}.$$

As it should be, a bottleneck may occur if the service rate at node i is decreased. As we will illustrate in the following example, this is different for the non-classical Jackson networks considered in this paper.

Example 6. *Revisit the queuing network presented in Example 5, see 2. Inspecting the mean-queue lengths as computed in Example 5, node 5 seems to be the candidate for creating a bottleneck. However, the set of unstable nodes as a mapping of θ, denoted by $U(\theta)$, is given by*

$$U(\theta) = \begin{cases} \emptyset, & 1 \geq \theta > 4/5 \\ \{4\}, & 4/5 \geq \theta > 2/3 \\ \{3,4\}, & 2/3 \geq \theta > 1/2 \\ \{1,3,4\}, & 1/2 \geq \theta > 1/4 \\ \{1,2,3,4\}, & 1/4 \geq \theta. \end{cases}$$

The above analysis shows that node 5 is stable for all values of θ, and that node 4 is the weak bottleneck of the system. Moreover, node 3 is the node that is next critical to node 4, and following this reasoning, the nodes can be ranked. Note that this analysis suggests that node 5 is not critical for stability of the network at all!

As the above example illustrates, mean-queue lengths and mean-waiting times depend in a more involved way on the service rates than in classical Jackson networks. The main reason for this phenomenon is that the traffic rates η_i are piecewise linear mappings of the μ_i's in our generalized networks whereas they are independent of some of the μ_i's in classical networks.

Conclusion

In this article we have developed a unified framework for two types of non standard Jackson networks: Networks where some nodes have an additional infinite supply (infinite virtual queue), and networks with unstable nodes. Both of these topics have been studied before alone, but obviously in practice there are networks where both phenomena occur in parallel, which makes the present work a desire.
We obtained closed-form analytical solutions of the steady-state queue length distribution at stable nodes and described the interplay of IVQ and instability within the network. For certain subnetworks we proved stationary distributions (if these exists), resp. asymptotic marginal distribution of product form.
A main contribution is an algorithm which elaborates on the modified traffic equations to obtain customer flows in the network, which detects stable and unstable nodes and allows to identify bottlenecks.
Our ongoing research encompasses problems of availability of service in these networks and connections to modeling supply chains of production systems, inventory control, and transportation.

References

[AW05] Ivo Adan and G. Weiss. A two-node Jackson network with infinite supply of work. *Probability in the Engineering and Informational Sciences*, 19:191 –212, 2005.

[BJG09] J. Bang-Jensen and G. Z. Gutin. *Digraphs*, 2nd Edition. Springer, London, 2009.

[BD90] O. J. Boxma and H. Daduna. Sojourn times in queueing networks. In H. Takagi, editor, *Stochastic Analysis of Computer and Communication Systems*, pages 401 – 450, Amsterdam, 1990. IFIP, North–Holland.

[CHT01] X. Chao, W. Henderson, and P. G. Taylor. State-Dependent Coupling in General Networks. *Queueing Syst. Theory Appl.*, 39(4): 337–348, 2001.

[CY01] H. Chen and D.D. Yao. *Fundamentals of Queueing Networks.* Springer, Berlin, 2001.

[DHS14] H. Daduna, B. Heidergott, and J. Sommer. Non-ergodic Jackson networks with infinite supply – local stabilization and local equilibrium analysis. Preprint, Schwerpunkt Mathematische Statistik und Stochastische Prozesse, Fachbereich Mathematik der Universität Hamburg, 2014.

[Dos90] B.T. Doshi. Single server queues with vacations. In H. Takagi, editor, *Stochastic Analysis of Computer and Communication Systems*, pages 217 – 267, Amsterdam, 1990. IFIP, North–Holland.

[DS02] H. Daduna and R. Szekli. Conditional job observer property for multitype closed queueing networks. *Journal of Applied Probability*, 39: 865–881, 2002.

[GM84] J. B. Goodman and W. A. Massey. The non–ergodic Jackson network. *Journal of Applied Probability*, 21:860– 869, 1984.

[Guo08] Y. Guo. Stability of generalized Jackson networks with infinite supply of work. *Journal of System Sciences and Complexity*, 21:283–295, 2008.

[GLN13] Y. Guo, E. Lefeber, Y. Nazarathy, G. Weiss, and H. Zhang. Stability of multi-class queueing networks with infinite virtual queues. *Queueing Systems*, pages 1–34, 2013. to appear.

[HaMe92] B. Haverkort and A. Meeuwissen. Sensitivity and uncertainty analysis in performance modelling. *Proceedings 11th Symposium on Reliable Distributed Systems*, IEEE Computer Society Press, 93-102, 1992.

[Hend03] S. Henderson. Input model uncertainty: why do we care and what should we do about it? *Proceedings of the 2003 Winter Simulation Conference* (S. Chick, P. J. Sanchez, D. Ferrin, and D. J. Morrice, eds.), 90-100, 2003.

[IS80] Iglehart,D. L. and Shedler, G. S. *Regenerative Simulation of Response Times in Networks of Queues.* Lecture Notes in Control and Information Sciences 26, Springer, Berlin 1980.

[Jac57] J.R. Jackson. Networks of waiting lines. *Operations Research*, 5:518–521, 1957.

[Kel79] F. P. Kelly. *Reversibility and Stochastic Networks.* Wiley, Chichester, 1979.

[KW02] A. Kopzon and G. Weiss. A push-pull queueing system. *Operations Research Letter*, 30:351–359, 2002.

[KNW09] A. Kopzon, Y. Nazarathy, and G. Weiss. A push-pull network with infinite supply of work. *Queueing Systems and their Applications*, 65:75–111, 2009.

[Kull59] S. Kullback. *Information Theory and Statistics.* Wiley, 1959.

[LY75] Y. Levy and U. Yechiali. Utilization of idle time in an M/G/1 queueing system. *Management Science*, 22:202–211, 1975.

[Myl13] J. Mylosz. Local stabilization of non-ergodic Jackson networks with unreliable nodes. *PhD Thesis*, Hamburg 2013.

[MD09] J. Mylosz and H. Daduna. On the behavior of stable subnetworks in non-ergodic networks with unreliable nodes. *Computer Networks*, 53(8):1249–1263, 2009.

[Mel79] B. Melamed. On Poisson traffic processes in discrete-state Markovian systems with applications to queueing theory. *Advances in Applied Probability*, 11(1):218–239, 1979.

[SD03] C. Sauer and H. Daduna. Availability formulas and performance measures for separable degradable networks. *Economic Quality Control*, 18:165–194, 2003.

[VWV00] N. Vandaele, T. Van Woensel, A. Verbruggen. A queuing based traffic flow model. *Transportation Research-D: Transport and environment*,5:121–135, 2000.

[Wei05] G. Weiss. Jackson-networks with unlimited supply of work. *Journal of Applied Probability*, 42:879–882, 2005.

[WV06] T. Van Woensel, N. Vandaele. Modelling traffic flows with queueing models: a review. *Asia-Pasific Journal of Operations Research*, 1–27, 2006