Problem sheet 2

Solutions has to be uploaded into Moodle:
https://lernen.min.uni-hamburg.de/mod/assign/view.php?id=30898
until 20:00, November 26.

HA1 **[3 points]** Let φ^* be the Fenchel-Legendre transform of the comulant generating function of a random variable ξ. Let also $\beta = \operatorname{ess\,sup} \xi < \infty$. Show that $\varphi^*(x) = +\infty$ for all $x > \beta$.

Hint: Show that $\lim_{\lambda \to +\infty}(\lambda x - \varphi(\lambda)) = +\infty$.

1. Let $\xi_1, \xi_2, \dots$ be independent identically distributed random variables. Consider a non-negative Borel measurable function $f : \mathbb{R} \to [0, \infty)$ such that $\mathbb{E}\, f(\xi_1) \in (0, \infty)$. Define the family of independent random variables $\eta_1, \eta_2, \dots$ with distribution

$$\mathbb{P}\left\{\eta_i \in B\right\} = \frac{1}{C}\mathbb{E}\left[f(\xi_i)\mathbb{I}_{\{\xi_i \in B\}}\right], \quad B \in \mathcal{B}(\mathbb{R}),$$

where $C = \mathbb{E}\, f(\xi_i)$ is the normalizing constant.

HA2 **[4 points]** Find the distribution of η_1, if ξ_1 has the exponential distribution with parameter $\lambda > 0$, and $f(x) = e^{-\alpha x}$, $x \in \mathbb{R}$, where $\alpha > -\lambda$ is a positive constants.

(a) Show that for every $n \in \mathbb{N}$ and $B_i \in \mathcal{B}(\mathbb{R})$

$$\mathbb{P}\left\{\eta_1 \in B_1, \dots, \eta_n \in B_n\right\} = \frac{1}{C^n}\mathbb{E}\left[f(\xi_1)\dots f(\xi_n)\mathbb{I}_{\{\xi_1 \in B_1, \dots, \xi_n \in B_n\}}\right].$$

2. Let $E = \mathbb{R}$ and $\xi \sim N(0,1)$. Show that the family $(\varepsilon\xi)_{\varepsilon>0}$ satisfies the LDP with rate function

$$I(x) = \begin{cases} +\infty & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Compare this result with the LDP for $(\sqrt{\varepsilon}\xi)_{\varepsilon>0}$.

3. Let $(\xi_\varepsilon)_{\varepsilon>0}$ satisfies the LDP in E with rate function I. Show that

a) if A is such that $\inf\limits_{x \in A^\circ} I(x) = \inf\limits_{x \in \bar{A}} I(x)$, then

$$\lim_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}\left\{\xi_\varepsilon \in A\right\} = -\inf_{x \in A} I(x);$$

b) $\inf\limits_{x \in E} I(x) = 0$.

4. Let a_n, b_n, $n \geq 1$, be positive real numbers. Show that

$$\varlimsup_{n \to \infty} \frac{1}{n}\ln(a_n + b_n) = \varlimsup_{n \to \infty} \frac{1}{n}\ln a_n \vee \varlimsup_{n \to \infty} \frac{1}{n}\ln b_n,$$

where $a \vee b$ denotes the maximum of the set $\{a, b\}$.

HA3 **[3 bonus points]** Let $\eta_1, \eta_2 \sim N(0,1)$. Let also for every $\varepsilon > 0$ a random variable ξ_ε have the distribution defined as follows

$$\mathbb{P}\left\{\xi_\varepsilon \in A\right\} = \frac{1}{2}\mathbb{P}\left\{-1 + \sqrt{\varepsilon}\eta_1 \in A\right\} + \frac{1}{2}\mathbb{P}\left\{1 + \sqrt{\varepsilon}\eta_2 \in A\right\}$$

for all Borel sets A. Show that the family $(\xi_\varepsilon)_{\varepsilon>0}$ satisfies the LDP with rate function $I(x) = \frac{1}{2}\min\left\{(x-1)^2, (x+1)^2\right\}$, $x \in \mathbb{R}$.