Problem sheet 8

Solutions has to be uploaded into Moodle:
https://lernen.min.uni-hamburg.de/mod/assign/view.php?id=40729
until 20:00, February 11.

1. Let U be a finite set and $\mathcal{P}(U)$ be the metric space of all probability measures on U equipped with the total variation distance.

   (i) Let $|\nu|_{TV}$ denote the total variation of a signed measure[1] on U. Show that

   $$|\mu - \nu|_{TV} = \sum_{i=1}^{d} |\mu(\{u_i\}) - \nu(\{u_i\})|.$$

   Therefore, the convergence of a sequence $(\nu_n)_{n\geq 1}$ to ν in $\mathcal{P}(U)$ is equivalent to the convergence of $\nu_n(\{u_i\}) \to \nu(\{u_i\})$, $n \to \infty$, for each $i \in [d]$.

   (ii) Show that a sequence $(\nu_n)_{n\geq 1}$ converges in ν in $\mathcal{P}(U)$ if and only it $\nu_n \to \nu$ weakly.

   (iii) Prove that the space $\mathcal{P}(U)$ is complete and separable.

   (*Hint:* Use the isometry between $\mathcal{P}(U)$ and the simplex $\Delta = \left\{(x_1, \dots, x_d) \in \mathbb{R}^d : \ x_1 + \dots + x_d = 1\right\}$)

HW1 **[3 points]** Let $X_1, X_2, \dots$ be independent random variables taking values from a finite space U and have distribution μ. Set

$$\mu_n := \frac{1}{n}\sum_{k=1}^{n} \delta_{X_k} \quad n \geq 1.$$

Show that $\mu_n \to \mu$ in $\mathcal{P}(U)$ a.s.

(*Hint:* Use the previous exercise and the strong law of large numbers)

2. Let $H(\nu|\mu)$ be a relative entropy of ν given μ, where $\nu, \mu \in \mathcal{P}(U)$ and U be a finite space.

   (i) Show that the function $H(\cdot|\mu) : \mathcal{P}(U) \to \mathbb{R}$ is continuous.

   (ii) Prove that $H(\nu|\mu) > 0$ for every $\nu \neq \mu$ and $H(\mu|\mu) = 0$.

   (iii) Show that the function $H(\cdot|\mu)$ is good, that is, the level sets $\{\nu \in \mathcal{P}(U) : \ H(\nu|\mu) \leq \alpha\}$, $\alpha \geq 0$, are compact in $\mathcal{P}(U)$.

HW2 **[4 points]** Let $\xi_1, \xi_2, \dots$ be independent Bernoulli distributed random variables with parameter $p \in (0,1)$. Using Sanov's theorem and the contraction principle show that the family $(\frac{1}{n}S_n)_{n\geq 1}$ satisfies the large deviation principle with good rate function

$$I(x) = \begin{cases} x \ln \frac{x}{p} + (1-x)\ln\frac{1-x}{1-p} & \text{if } x \in [0,1], \\ +\infty & \text{otherwise,} \end{cases}$$

where $S_n = \xi_1 + \dots + \xi_n$.

[1] The total variation $|\nu|_{TV}$ of a signed measure $\nu \in \mathcal{P}(U)$ is defined as $|\nu|_{TV} = \sup\limits_{\pi} \sum_{A\in\pi} |\nu(A)|$, where is taken over all partitions π of the set U

3. Let f be a continuous and bounded above function from a metric space E to $\mathbb{R}$. Show that for every $n \geq 1$ there exists a family of closed subsets B_k, $k \in [m]$, of E such that $f \leq -n$ on $B_0 := \left(\bigcup_{k=1}^m B_k\right)^c$ and the oscillation of f on each B_k is at most $\frac{1}{n}$.

   Hint: Consider the sets $f^{-1}\left(\left[\frac{k-1}{n}, \frac{k}{n}\right]\right)$, $k \in \mathbb{Z}$.

HW3 [2 points] Let B be a subset of E and $f, g : A \to \mathbb{R}$ and $\inf_{x \in A} g(x) > -\infty$. Prove that

$$\inf_{x \in A} f(x) - \inf_{x \in A} g(x) \leq \sup_{x \in A} (f(x) - g(x)).$$