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MASTER THESIS

Period polynomials and multivariate extensions

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1. Introduction

This master thesis covers topics from number theory, commutative algebra and representation theory. The main objective of this work is to work out and present in an uniform way recent and mostly not yet published results concerning bi-period polynomials. These results were presented during the last years in either the Arithmetische Geometrie und Zahlentheorie Seminar or in courses on multiple zeta values by U. Kühn at the Universität Hamburg. This leads to the question for even more general multivariate period polynomials.

We will mainly study polynomials that satisfy certain relations. In the case of bivariate polynomials $f \in \mathbb{Q}[x, y]$, these relations are given by

$$\begin{aligned} f(x, y) + f(-y, x) &= 0 \\ f(x, y) + f(x - y, x) + f(-y, x - y) &= 0. \end{aligned} \tag{1.1}$$

We denote the vector space of all homogeneous polynomials of degree $k - 2$ that satisfy (1.1) by W_k. These *period polynomials* are classical in the theory of modular forms (see e. g. [Lan87], [Zag91], [GKZ06], [PP13] and [CPZ19]). A theorem by Eichler–Shimura–Manin shows that the space of modular forms is isomorphic to a certain subspace of period polynomials. Apart from modular forms, period polynomials also have applications in the theory of multiple zeta values where they can be used to describe certain $\mathbb{Q}$-linear relations (see [GKZ06], [Bro21], [Eca11] and [Sch15]). So overall, period polynomials are an interesting object of research with important applications.

The term period polynomial usually refers to polynomials in one variable or to their homogeneous analogues, i. e. homogeneous polynomials in two variables. Lately, however, variants of period polynomials in 4 variables were studied in the context of multiple q-zeta values (see [BBK20]). We will refer to them as *bi-period polynomials* in order to distinguish them from the regular ones. We will see that most statements on period polynomials also hold for bi-period polynomials.

This leads to the question of whether there is some generalization of period polynomials in n variables. To generalize the relations (1.1), we want to define an appropriate right action of $\mathrm{SL}_2(\mathbb{Z})$ on the homogeneous spaces of $\mathbb{Q}[x_1, \dots, x_n]$. We therefore consider the irreducible n-dimensional representations of the matrix Lie group $\mathrm{SL}_2(\mathbb{C})$. Via the inclusion $\mathrm{SL}_2(\mathbb{Z}) \subset \mathrm{SL}_2(\mathbb{C})$ we obtain our desired action. There are currently, however, no applications of these multivariate extensions of period polynomials known to the author.

This master thesis is organized as follows.

In chapter 2 we will first review the theory of period polynomials. We therefore begin by defining a right action of the group $\mathrm{GL}_2(\mathbb{Z})$ on the homogeneous spaces of $\mathbb{Q}[x, y]$ in section 2.1. We refer to this action as the *slash operator*. This $\mathrm{GL}_2(\mathbb{Z})$-action will be extended to an action of the group ring $\mathbb{Z}[\mathrm{GL}_2(\mathbb{Z})]$ in section 2.2 which is then used to define the space of period polynomials W_k. In section 2.3, we consider a non-degenerate pairing on the homogeneous spaces of $\mathbb{Q}[x, y]$. This pairing has the useful property of being invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$. We use this pairing in section 2.4 to describe the orthogonal complement of W_k. This allows us to compute the dimensions of W_k as well as the (anti)symmetric subspaces (w. r. t. swapping x and y). To conclude chapter 2, we discuss applications of period polynomials to modular forms and multiple zeta values. In section 2.5 we review some theory of modular forms and emphasize the connections to period polynomials. In the following sections we then discuss applications of multiple zeta values. In section 2.6 we begin by reviewing the definition of multiple zeta values. We use the previously established theory of period polynomials such as the invariant and non-degenerate pairing to compute linear relations amongst single and double zeta values. In section 2.7 we consider the linearized double shuffle Lie algebra $\mathfrak{ls}$ and describe connections between period polynomials and conjectural generators and relations of this Lie algebra.

In chapter 3 we review the theory of bi-period polynomials. This discussion is mostly analogous to the previous chapter to emphasize the similarities to period polynomials. We also begin in section 3.1 by defining an action of $\mathrm{GL}_2(\mathbb{Z})$ on the homogeneous spaces of $\mathbb{Q}[x_1, x_2, y_1, y_2]$ and refer to this as the *bi-slash operator*. We will then use this action to define the space of bi-period polynomials $\mathcal{W}_k$. As the quadratic form $q = x_1y_1 + x_2y_2$ will turn out to play a key role in this discussion, we extend the bi-slash operator to an action of its group of isometries Γ. In section 3.2 we will prove a special case of a theorem by Weitzenböck for polynomials in 4 variables. This theorem explicitly describes the finitely generated subalgebra that is invariant under an action of the additive group $\mathbb{G}_a(\mathbb{C})$. Similar to before, we will then define a non-degenerate pairing on the homogeneous spaces of $\mathbb{Q}[x_1, x_2, y_1, y_2]$ and show that this is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$ in section 3.3. Both the theorem of Weitzenböck and the pairing will then be used in section 3.4 to compute the dimensions of $\mathcal{W}_k$ as well as the (anti)symmetric subspaces of $\mathcal{W}_k$. Besides the invariant pairing, we also study another interesting structure that interacts well with both the bi-slash operator and the pairing, namely the differential operator that is associated to the quadratic form q. The study of this Laplacian operator is the content of section 3.5. One important implication of this theory is the fact that we obtain recursive decompositions of the homogeneous spaces. The idea of considering this differential operator was proposed by D. Zagier in a private conversation with U. Kühn. We conclude chapter 3 by discussing applications of bi-period polynomials to q-analogues of multiple zeta values. In section 3.6 we begin by reviewing multiple q-zeta values. We consider combinatorial multiple Eisenstein series as a spanning set for the vector space of multiple q-zeta values. Similar to section 2.6, we then use bi-period polynomials and established tools such as the invariant and non-degenerate pairing to obtain relations amongst combinatorial multiple Eisenstein series. In section 3.7 we consider the linearized balanced quasi shuffle algebra $\mathfrak{lbs}$. Similar to section 2.7, we use bi-period polynomials to describe conjectural generators and relations of this Lie algebra. In particular, we use the recursive decomposition that is induced by the Laplacian operator to compute the dimension of relations in depth 2.

In chapter 4 we review central aspects of the representation theory of the matrix Lie group $\mathrm{SL}_2(\mathbb{C})$. The main goal is to describe all finite-dimensional irreducible representations of $\mathrm{SL}_2(\mathbb{C})$ (up to isomorphism). To compute the irreducible representations of $\mathrm{SL}_2(\mathbb{C})$, it turns out to be more convenient to compute the representations of the corresponding Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ instead. The established representation theory then tells us that the respective representations are in one-to-one correspondence. This, however, is only true for simply connected matrix Lie groups. So after reviewing some fundamental concepts of matrix Lie groups and Lie algebras in section 4.1, we will prove that $\mathrm{SL}_2(\mathbb{C})$ is simply connected in section 4.2. In section 4.3, we give some necessary definitions and state some propositions that will allow us to focus on computing the irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ instead of $\mathrm{SL}_2(\mathbb{C})$. Since the proofs of these propositions use concepts that are not relevant for the rest of this master thesis, we will not prove them but instead refer to respective literature. We will then compute the finite-dimensional irreducible representations of $\mathrm{SL}_2(\mathbb{C})$ in section 4.4. Throughout this chapter, we assume that the reader has basic knowledge of topology.

In chapter 5, we begin in section 5.1 by using the n-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ from the previous chapter to define an analogue of the slash operator on $\mathbb{Q}[x_1, \dots, x_n]$. As this operator coincides with the slash operator for the case of $n = 2$, we will also refer to this as the *slash operator*. The action of this slash operator is obtained for some $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ by considering the transformation matrix of the respective representation of γ^{-1}. Using this action, we introduce a generalization of period polynomials in n variables which we will refer to as *generalized period polynomials*. In section 5.2 we are then interested in the question of whether the space of generalized period polynomials $W_k^{(n)}$ coincides with the kernel of a certain operator in the cases where $W_k^{(n)}$ is non-trivial. Since this *Lewis space* has been shown to equal the space of (bi-)period polynomials in the previous chapters, we expect a similar result here. In fact, empirical evidence suggests that this is true. However, it was not possible to prove this within the scope of this thesis. In section 5.3 we proceed by defining a non-degenerate pairing and prove in Theorem 5.21 that this pairing is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$. By similar means as before, we then want to use this pairing in section 5.4 to compute the dimension of $W_k^{(n)}$. However, it turns out that the space of $\mathrm{SL}_2(\mathbb{Z})$-invariant polynomials is not so easy to describe. We thus conclude the discussion on the dimension of $W_k^{(n)}$ with a formula that still depends on the dimension of the space of invariant polynomials.

This master thesis contains three appendices. In appendix A we give an exemplary discussion on the representation theory of finite groups using the example of the dihedral group of order 6. This is useful as we are often interested in groups generated by certain matrices of finite order. An important tool that is repeatedly used is Molien's theorem. This lets us compute the Hilbert-Poincaré series of the space of polynomials that are invariant under the action of a finite group.

In appendix B we use Molien's theorem to compute the Hilbert-Poincaré series for spaces of polynomials that are invariant under the action of certain matrices. These matrices are used to define the space of generalized period polynomials.

In appendix C we list the source code of some computer based calculations in the context of this master thesis. The code is written for the free and open-source computer algebra system SageMath [The22] and is not claiming to be efficient.

2. Period polynomials

In order to define period polynomials, we first need to define an action of the general linear group of degree 2 over the ring of integers $\mathrm{GL}_2(\mathbb{Z})$ on bivariate polynomials, the *slash operator*. This will be the goal of section 2.1. We will then recall some fundamental statements of the established theory of period polynomials. In section 2.2 we will define the space of period polynomials as well as the Lewis space which coincides with the period polynomials whenever the latter space is non-trivial. We will then define a non-degenerate pairing on the period polynomials in section 2.3 in order to compute their dimensions in section 2.4. Finally, we discuss some applications of period polynomials in sections 2.5, 2.6 and 2.7. Unless stated otherwise, we let $k \in \mathbb{N}$ be an integer with $k \geq 2$.

2.1. $\mathrm{GL}_2(\mathbb{Z})$-action via slash operator

Definition 2.1. For $k \geq 2$ we denote the set of homogeneous polynomials in 2 variables of degree $k-2$ over $\mathbb{Q}$ by

$$V_k \coloneqq \{ f = f(x,y) \in \mathbb{Q}[x,y] \mid f \text{ homogeneous}, \deg(f) = k-2 \}.$$

We further set $V_0 \coloneqq \{0\}$, $V_1 \coloneqq \{0\}$ and

$$V \coloneqq \bigoplus_{k=0}^{\infty} V_k.$$

Note that V is a graded vector space with homogeneous components given by V_k.

Remark 2.2. We have $\dim(V_k) = k-1$ since a basis is given by

$$x^{k-2}, x^{k-3}y, \ldots, xy^{k-3}, y^{k-2}.$$

Definition 2.3. We define the *slash operator* on the space V_k by

$$\begin{aligned} \mathrm{GL}_2(\mathbb{Z}) \times V_k &\longrightarrow V_k \\ (\gamma, f) &\longmapsto f\,|\gamma \coloneqq f\left((\gamma \cdot z)^{\mathsf{t}}\right) \end{aligned}$$

where $z = (x,y)^{\mathsf{t}}$.

Proposition 2.4. *The slash operator yields a right group action on V_k.*

Proof. Let $f \in V_k$ and $\gamma \in \mathrm{GL}_2(\mathbb{Z})$. We then have $f\,|\gamma \in V_k$ by the binomial theorem. Let $\gamma_1, \gamma_2 \in \mathrm{GL}_2(\mathbb{Z})$ with

$$\gamma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \quad \text{for } i \in \{1,2\}.$$

Then

$$\begin{aligned}(f\,|\gamma_1\,)\,|\gamma_2 &= f(a_2x+b_2y, c_2x+d_2y)\,|\gamma_1\\ &= f\big(a_1(a_2x+b_2y)+b_1(c_2x+d_2y), c_1(a_2x+b_2y)+d_1(c_2x+d_2y)\big)\\ &= f\big((a_1a_2+b_1c_2)x+(a_1b_2+b_1d_2)y, (a_2c_1+c_2d_1)x+(b_2c_1+d_1d_2)y\big)\\ &= f\,|(\gamma_1\cdot\gamma_2)\ .\end{aligned}$$ □

Lemma 2.5. *For even $k \in \mathbb{N}$ the space V_k is invariant under -1, i. e. for $f \in V_k$ we have*

$$f\,|(-1) = f.$$

Proof. For $f \in V_k$ we write

$$f(x,y) = \sum_{i=0}^{k-2} a_i\, x^i y^{k-2-i}. \tag{2.1}$$

Then $k-2$ is even and thus

$$f\,|(-1) = f(-x,-y) = (-1)^{k-2}\sum_{i=0}^{k-2} a_i\, x^i y^{k-2-i} = f.$$ □

Notation. We consider the $\mathrm{SL}_2(\mathbb{Z})$-matrices

$$S = \begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix}, \qquad U = \begin{pmatrix} 1 & -1\\ 1 & 0\end{pmatrix}$$

as well as the $\mathrm{GL}_2(\mathbb{Z})$-matrices

$$\epsilon = \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix} \qquad \text{and} \qquad \delta = \begin{pmatrix} -1 & 0\\ 0 & 1\end{pmatrix}. \tag{2.2}$$

Definition 2.6. We denote the eigenspaces of the operator $|\epsilon$ on V_k with eigenvalues 1 and -1, respectively, by

$$V_k^+ = \{f \in V_k \mid f(y,x) = f(x,y)\} \qquad \text{and} \qquad V_k^- = \{f \in V_k \mid f(y,x) = -f(x,y)\}$$

and the eigenspaces of the operator $|\delta$ on V_k with eigenvalues 1 and -1, respectively, by

$$V_k^{\mathsf{ev}} = \{f \in V_k \mid f(-x,y) = f(x,y)\} \quad \text{and} \quad V_k^{\mathsf{odd}} = \{f \in V_k \mid f(-x,y) = -f(x,y)\}\,.$$

Furthermore, for a subspace $W \subseteq V_k$ we set $W^\bullet := W \cap V_k^\bullet$ for all $\bullet \in \{+,-,\mathsf{ev},\mathsf{odd}\}$.

Remark 2.7.

i) Since ϵ and δ are diagonalizable, we obtain decompositions in eigenspaces

$$V_k = V_k^+ \oplus V_k^- \qquad \text{and} \qquad V_k = V_k^{\mathsf{ev}} \oplus V_k^{\mathsf{odd}}$$

with projections given by

$$\pi^+ \colon V_k \to V_k^+, \quad f \mapsto \frac{1}{2}(f + f\,|\epsilon) \qquad \pi^- \colon V_k \to V_k^-, \quad f \mapsto \frac{1}{2}(f - f\,|\epsilon)$$

and

$$\pi^{\mathsf{ev}} \colon V_k \to V_k^{\mathsf{ev}}, \quad f \mapsto \frac{1}{2}(f + f\,|\delta) \qquad \pi^{\mathsf{odd}} \colon V_k \to V_k^{\mathsf{odd}} \quad f \mapsto \frac{1}{2}(f - f\,|\delta).$$

We further denote

$$f^\bullet := \pi^\bullet(f)$$

for all $f \in V_k$ and $\bullet \in \{+, -, \mathsf{ev}, \mathsf{odd}\}$.

ii) Let k be even. The operators $|\epsilon$ and $|\delta$ on V_k commute in this case since $\epsilon\delta \equiv \delta\epsilon$ modulo ± 1 (cf. Lemma 2.5). This implies that $|\epsilon$ acts on V_k^{ev} and V_k^{odd} and $|\delta$ acts on V_k^+ and V_k^-. Hence we obtain a refined decomposition given by

$$V_k = V_k^{+,\mathsf{ev}} \oplus V_k^{+,\mathsf{odd}} \oplus V_k^{-,\mathsf{ev}} \oplus V_k^{-,\mathsf{odd}}$$

where

$$V_k^{+,\mathsf{ev}} := V_k^+ \cap V_k^{\mathsf{ev}}, \qquad V_k^{+,\mathsf{odd}} := V_k^+ \cap V_k^{\mathsf{odd}},$$
$$V_k^{-,\mathsf{ev}} := V_k^- \cap V_k^{\mathsf{ev}}, \qquad V_k^{-,\mathsf{odd}} := V_k^- \cap V_k^{\mathsf{odd}}.$$

Note that the projections π^+ and π^- commute pairwise with π^{ev} and π^{odd} for even k. These compositions yield projections onto $V_k^{+,\mathsf{ev}}$, $V_k^{+,\mathsf{odd}}$, $V_k^{-,\mathsf{ev}}$ and $V_k^{-,\mathsf{odd}}$ which we denote by

$$\pi^{+,\mathsf{ev}} := \pi^+ \circ \pi^{\mathsf{ev}}, \qquad \pi^{+,\mathsf{odd}} := \pi^+ \circ \pi^{\mathsf{odd}},$$
$$\pi^{-,\mathsf{ev}} := \pi^- \circ \pi^{\mathsf{ev}}, \qquad \pi^{-,\mathsf{odd}} := \pi^- \circ \pi^{\mathsf{odd}}.$$

We further denote

$$f^{\bullet,\circ} := \pi^{\bullet,\circ}(f)$$

for all $f \in V_k$, $\bullet \in \{+, -\}$ and $\circ \in \{\mathsf{ev}, \mathsf{odd}\}$.

Projections onto invariant subspaces such as π^+ and π^{ev} are generalized in the following definition.

Definition 2.8. Let $G \subset \mathrm{GL}_2(\mathbb{Z})$ be a finite group. The *Reynolds operator of* G on V_k is given by

$$\rho_G \colon f \mapsto \frac{1}{|G|} \sum_{\gamma \in G} f\,|\gamma\,.$$

Lemma 2.9. *Let $G \subset \mathrm{GL}_2(\mathbb{Z})$ be a finite group. The Reynolds operator of G is a surjective linear map $\rho_G \colon V_k \longrightarrow V_k^G$ where $V_k^G \subset V_k$ is the G-invariant subspace.*

Proof. The linearity of ρ_G follows immediately from the definition. For surjectivity note that ρ_G averages over the action of G. Hence $\rho_G(f) = f$ for all $f \in V_k^G$. Now let $\sigma \in G$ and $f \in V_k$. We then have by standard group theoretical arguments that

$$\rho_G\,(f)\,|\sigma \;=\; \frac{1}{|G|} \sum_{\gamma \in G} f\,|(\gamma\sigma) \;=\; \frac{1}{|G|} \sum_{\gamma' \in G} f\,\big|\gamma' \;=\; \rho_G(f). \qquad \square$$

Remark 2.10. Note that the operators $|S$ and $|U$ have order 2 and 3, respectively, with eigenvalues ± 1 and $\{1, \omega, \omega^2\}$ where $\omega = e^{\frac{2\pi i}{3}}$. Following [Zag00], we denote

- the eigenspace of $|S$ on V_k with eigenvalue 1 by A_k,
- the eigenspace of $|S$ on V_k with eigenvalue -1 by B_k,
- the eigenspace of $|U$ on V_k with eigenvalue 1 by C_k and
- the sum of the eigenspaces of $|U$ on V_k with eigenvalues ω and ω^2 by D_k.

We further denote the finite groups generated by S and U by G_S and G_U, respectively. We then have $A_k = \mathrm{im}(\rho_{G_S})$, $B_k = \ker(\rho_{G_S})$, $C_k = \mathrm{im}(\rho_{G_U})$ and $D_k = \ker(\rho_{G_U})$. Hence we obtain the decompositions[1]

$$V_k = A_k \oplus B_k \qquad \text{and} \qquad V_k = C_k \oplus D_k. \tag{2.3}$$

Note that we can compute the respective dimensions for all $k \in \mathbb{N}$ via Molien's theorem A.12. This yields

$$\dim A_k = 1 + 2\left\lfloor \frac{k-2}{4} \right\rfloor \qquad \text{and} \qquad \dim C_k = 1 + 2\left\lfloor \frac{k-2}{6} \right\rfloor. \tag{2.4}$$

[1]For more details on this, see appendix A. There is an exemplary discussion on the dihedral group D_6.

2.2. Group ring action and period polynomials

Definition 2.11. Let G be a group and R be a ring. The set of formal linear combinations

$$R[G] \coloneqq \left\{ \sum_{g\in G} r_g\, g \;\middle|\; r_g \in R, r_g = 0 \text{ except for finitely many } g \in G \right\}$$

equipped with addition

$$\left(\sum_{g\in G} a_g\, g\right) + \left(\sum_{g\in G} b_g\, g\right) \coloneqq \sum_{g\in G} (a_g + b_g)\, g$$

and multiplication

$$\left(\sum_{g\in G} a_g\, g\right) \cdot \left(\sum_{g\in G} b_g\, g\right) \coloneqq \sum_{g\in G} (a_g \cdot b_g)\, g$$

is the *group ring of G with coefficients in R.*

The notion of a group ring lets us extend Definition 2.3 to the group ring $\mathbb{Z}[\mathrm{GL}_2(\mathbb{Z})]$.

Definition 2.12. For $\sum_{\gamma\in \mathrm{GL}_2(\mathbb{Z})} r_\gamma\, \gamma \in \mathbb{Z}[\mathrm{GL}_2(\mathbb{Z})]$ and $f \in V_k$ we set

$$f \left| \sum_{\gamma\in \mathrm{GL}_2(\mathbb{Z})} r_\gamma\, \gamma \right. \coloneqq \sum_{\gamma\in \mathrm{GL}_2(\mathbb{Z})} r_\gamma\, (f\,|\gamma).$$

Remark 2.13. We can rewrite the spaces A_k, B_k, C_k and D_k from Remark 2.10 in terms of the group ring action as

$$A_k = \ker(1 - S) = \mathrm{im}(1 + S), \qquad B_k = \ker(1 + S) = \mathrm{im}(1 - S)$$

and

$$C_k = \ker(1 - U) = \mathrm{im}(1 + U + U^2), \qquad D_k = \ker(1 + U + U^2) = \mathrm{im}(2 - U - U^2).$$

The respective identities of kernels and images follow immediately via both-way inclusions.

We are now able to define the space of period polynomials.

Definition 2.14. For $k \geq 1$ the space of *period polynomials* is given by

$$W_k \coloneqq \left\{ f \in V_k \;\middle|\; f\,|1 + S\; = f\,\middle|1 + U + U^2\; = 0 \right\}.$$

For compatibility reasons, we further set $W_0 \coloneqq \mathbb{Q}$ (cf. Theorem 2.40).

Remark 2.15. For odd k we have $W_k = \{0\}$ since for $f \in V_k$ as in (2.1) $f \in \ker(1+S)$ implies

$$a_i = a_{k-2-i} \qquad \text{and} \qquad a_i = -a_{k-2-i}$$

for all $i \in \{0, \ldots, k-2\}$. Hence $f = 0$.

Example 2.16. For $k = 12$ we consider

$$\begin{aligned} p_{12}(x,y) &= x^{10} - y^{10} \\ r_\Delta^{\mathsf{ev}}(x,y) &= x^8y^2 - x^2y^8 - 3\,(x^6y^4 - x^4y^6) \\ r_\Delta^{\mathsf{odd}}(x,y) &= 4\,x^9y - 25\,x^7y^3 + 42\,x^5y^5 - 25\,x^3y^7 + 4\,xy^9. \end{aligned}$$

Note that p_{12} and r_Δ^{ev} are simultaneously contained in W_k^- and W_k^{ev}. Similarly, r_Δ^{odd} is contained in W_k^+ and W_k^{odd}. This is no coincidence as we will see in Proposition 2.17.

Similar to Definition 2.14, we set $W_0^+ \coloneqq \{0\}$, $W_0^- \coloneqq \mathbb{Q}$, $W_0^{\mathsf{ev}} \coloneqq \mathbb{Q}$ and $W_0^{\mathsf{odd}} \coloneqq \{0\}$ for compatibility reasons.

Proposition 2.17. *For $k \in \mathbb{N}_0$ we have*

$$W_k^+ = W_k^{\mathsf{odd}} \qquad \textit{and} \qquad W_k^- = W_k^{\mathsf{ev}}.$$

Proof. The claim follows by definition for the case $k = 0$. Now let $k \in \mathbb{N}$ and $f \in W_k$. By Remark 2.15 the claim is trivial for odd k. So without loss of generality we assume that k is even. We need to show that

$$f \in \ker(1 \mp \epsilon) \iff f \in \ker(1 \pm \delta).$$

Note that $\delta \equiv \epsilon S$ modulo ± 1 and $\epsilon = \delta S$. Now if $f \in \ker(1 \mp \epsilon)$ then

$$f\,|1 \pm \delta\ = f\,|1 \pm \epsilon S\ = f\,|(S \pm \epsilon)\ |S\ = f\,|(1 + S - (1 \mp \epsilon))\ |S\ = 0.$$

And if $f \in \ker(1 \pm \delta)$ we have

$$f\,|1 \mp \epsilon\ = f\,|1 \mp \delta S\ = f\,|(S \mp \delta)\ |S\ = f\,|(1 + S - (1 \pm \delta))\ |S\ = 0. \qquad \square$$

Definition 2.18. We set $T' = \epsilon T \epsilon$. The action of $1 - T - T'$ is called the *Lewis operator*. The kernel of the Lewis operator

$$L_k \coloneqq \ker(1 - T - T') \subseteq V_k$$

is the *Lewis space*.

Proposition 2.19 ([GKZ06])**.** *Let $k \geq 4$ be even. Then*

$$W_k = L_k. \tag{2.5}$$

Table 1: Dimensions of W_k, $W_k^\pm$ and L_k for $k \in \{3, \dots, 20\}$.[2]

k	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\dim W_k$	0	1	0	1	0	1	0	1	0	3	0	1	0	3	0	3	0	3
$\dim W_k^+$	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	1	0	1
$\dim W_k^-$	0	1	0	1	0	1	0	1	0	2	0	1	0	2	0	2	0	2
$\dim L_k$	1	1	1	1	1	1	1	1	1	3	1	1	1	3	1	3	1	3

Proof. For $f \in V_k$ we have

$$f \,\big|1 - T - T' \,|S = f \,\big|1 + S - (1 + U + U^2)\,. \tag{2.6}$$

If $f \in \ker(1 - T - T')$ then $f\,|1 + S = f\,\big|1 + U + U^2$. This polynomial is then invariant under both $|S$ and $|U$ and is thus invariant under the slash operator of $U \cdot S = -T$. Due to the invariance under $|(-1)$ (Lemma 2.5) this implies invariance under $|T$. Since the only non-zero $|T$-invariant polynomials in V_k are of the form $a \cdot y^{k-2}$, for some $a \in \mathbb{Q}$, we obtain $f \in W_k$ as y^{k-2} is not invariant under $|S$. Now if $f \in W_k$ then the right-hand side of (2.6) vanishes. Applying $|S$ again yields $f \in \ker(1 - T - T')$ since $S^2 \equiv 1$ modulo ± 1. □

Remark 2.20. Equation (2.5) is called the *Lewis equation*.

Corollary 2.21. Let $k \geq 4$ be even. Then

$$W_k^\pm = \ker(1 - T \mp T\epsilon). \tag{2.7}$$

Proof. For $f \in V_k$ assume $f\,|\epsilon = \pm f$. Then $f\,|\epsilon T \epsilon = \pm f\,|T\epsilon$ and hence

$$f\,|1 - T \mp T\epsilon = f\,\big|1 - T - T'\,.$$

The identities in (2.7) therefore follow immediately from (2.5). □

Remark 2.22. Note that the Lewis space L_k is, unlike W_k, non-trivial for odd k. The respective dimensions for small values of k can be found in table 1.

2.3. An invariant pairing

In order to compute the dimension of W_k we construct a direct sum decomposition of V_k that contains W_k as a summand. To do so, we want to compute an orthogonal complement of W_k in V_k and therefore consider a pairing on the space V_k that is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$.

[2] The source code that was used to compute these dimensions can be found in appendix C.1.

Definition 2.23. For $r, s, m, n \geq 1$ we set

$$\left\langle x^{r-1}y^{s-1}, x^{m-1}y^{n-1}\right\rangle \coloneqq \frac{(-1)^{r-1}}{\binom{k-2}{m-1}}\delta_{(r,s),(n,m)}$$

and linearly extend this to a non-degenerate pairing $\langle \cdot, \cdot \rangle$ on V.

Example 2.24. We consider $f, g \in V_8$ with $f(x,y) = 22x^5y + 146x^4y^2 + 25x^3y^3$ and $g(x,y) = 82x^3y^3 + 11x^2y^4 + 13xy^5$. Then we have

$$\langle f, g \rangle = -\frac{431}{10}.$$

The pairing from Definition 2.23 can also be described in terms of partial derivatives.

Definition 2.25. For $f, g \in V$ we set

$$\langle f, g \rangle_\partial \coloneqq \frac{1}{(k-2)!} f\left(-\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right)\Big(g\,(x,y)\Big)(0).$$

Proposition 2.26. *For $f, g \in V$ we have*

$$\langle f, g \rangle = \langle f, g \rangle_\partial \,.$$

Proof. Since both pairings are linear, it suffices to proof the claim for monomials $f \in V_{k_1}$ and $g \in V_{k_2}$. If $k_1 \neq k_2$ then both pairings vanish. So we assume without loss of generality that $f = x^{r-1}y^{s-1}$ and $g = x^{m-1}y^{n-1}$ for some $r, s, m, n \geq 1$ with $r + s = k = m + n$. Now both pairings vanish unless $r = n$ and $s = m$. In this case we have

$$\begin{aligned}\langle f, g \rangle_\partial &= \frac{(-1)^{r-1}}{(k-2)!}\left(\frac{\partial^{r-1}}{\partial y^{r-1}}\frac{\partial^{s-1}}{\partial x^{s-1}}\right)\left(x^{s-1}y^{r-1}\right)\\ &= \frac{(-1)^{r-1}(r-1)!(s-1)!}{(k-2)!}\\ &= \frac{(-1)^{r-1}}{\binom{k-2}{r-1}} = \langle f, g \rangle \,. \end{aligned}$$

□

Considering the inclusion morphism $\mathrm{SL}_2(\mathbb{Z}) \hookrightarrow \mathrm{GL}_2(\mathbb{Z})$, the $\mathrm{GL}_2(\mathbb{Z})$-action from Definition 2.3 induces an $\mathrm{SL}_2(\mathbb{Z})$-action on V_k.

Proposition 2.27. *The pairing $\langle \cdot, \cdot \rangle$ is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$, i. e. for $f, g \in V_k$ we have*

$$\langle f\,|\gamma\,, g\,|\gamma \rangle = \langle f, g \rangle \quad \textit{for all } \gamma \in \mathrm{SL}_2(\mathbb{Z}). \tag{2.8}$$

Furthermore, if k is even then the pairing $\langle \cdot, \cdot \rangle$ on V_k is invariant under the action of $\mathrm{GL}_2(\mathbb{Z})$.

Proof. Due to the linearity of $\langle \cdot, \cdot \rangle$ we assume without loss of generality that $f = x^{r-1}y^{s-1}$ and $g = x^{m-1}y^{n-1}$ for some $r, s, m, n \geq 1$ with $r + s = k = m + n$. Since the group $\mathrm{SL}_2(\mathbb{Z})$ is generated by S and T, it suffices to check that (2.8) holds for for $\gamma \in \{S, T\}$. This is equivalent to showing

$$\langle f\,|S\,, g\rangle = \left\langle f, g\,\middle|S^{-1}\right\rangle \tag{2.9a}$$

and

$$\langle f\,|T\,, g\rangle = \left\langle f, g\,|T^{-1}\right\rangle. \tag{2.9b}$$

The left-hand side of (2.9a) yields

$$\langle f\,|S\,, g\rangle = (-1)^{r-1}\left\langle x^{s-1}y^{r-1}, x^{m-1}y^{n-1}\right\rangle = \frac{(-1)^{k-2}}{\binom{k-2}{m-1}}\delta_{(s,r),(n,m)}$$

while the right-hand side is

$$\left\langle f, g\,\middle|S^{-1}\right\rangle = (-1)^{n-1}\left\langle x^{r-1}y^{s-1}, x^{n-1}y^{m-1}\right\rangle = \frac{(-1)^{n+r-2}}{\binom{k-2}{n-1}}\delta_{(s,r),(n,m)}.$$

Both vanish unless $f = g$. If $f = g$ both sides equal $(-1)^{k-2}\binom{k-2}{m-1}^{-1}$.

The left-hand side of (2.9b) yields

$$\begin{aligned}
\langle f\,|T\,, g\rangle &= \left\langle (x+y)^{r-1}y^{s-1}, x^{m-1}y^{n-1}\right\rangle \\
&= \sum_{i=0}^{r-1}\binom{r-1}{i}\left\langle x^i y^{k-2-i}, x^{m-1}y^{n-1}\right\rangle \\
&= \binom{r-1}{n-1}\frac{(-1)^{n-1}}{\binom{k-2}{m-1}}
\end{aligned}$$

while the right-hand side is

$$\begin{aligned}
\left\langle f, g\,\middle|T^{-1}\right\rangle &= \left\langle x^{r-1}y^{s-1}, (x-y)^{m-1}y^{n-1}\right\rangle \\
&= \sum_{i=0}^{m-1}\binom{m-1}{i}(-1)^{m-1-i}\left\langle x^{r-1}y^{s-1}, x^i y^{k-2-i}\right\rangle \\
&= \binom{m-1}{s-1}\frac{(-1)^{m+r-s+1}}{\binom{k-2}{s-1}} = \binom{m-1}{s-1}\frac{(-1)^{n-1}}{\binom{k-2}{r-1}}.
\end{aligned}$$

First note that the left-hand side vanishes if and only if $n > r$ and the right-hand side vanishes if and only if $s > m$. Since $n + m = r + s$ these conditions are equivalent. Now assume $r \geq n$. This is equivalent to $m \geq s$. By using $r - n = m - s$ we obtain that

$$\begin{aligned}
\binom{r-1}{n-1}\frac{(-1)^{n-1}}{\binom{k-2}{m-1}} &= (-1)^{n-1}\frac{(r-1)!(m-1)!(n-1)!}{(n-1)!(r-n)!(k-2)!} \\
&= (-1)^{n-1}\frac{(r-1)!(m-1)!(s-1)!}{(s-1)!(m-s)!(k-2)!} \\
&= \binom{m-1}{s-1}\frac{(-1)^{n-1}}{\binom{k-2}{r-1}}.
\end{aligned}$$

This proves the first claim.

Now let k be even. Since the group $\mathrm{GL}_2(\mathbb{Z})$ is generated by $\{\epsilon, \delta, T\}$ it suffices to show that

$$\langle f\,|\epsilon\,, g\rangle = \langle f, g\,|\epsilon\rangle \tag{2.10a}$$

and

$$\langle f\,|\delta\,, g\rangle = \langle f, g\,|\delta\rangle\,. \tag{2.10b}$$

The left-hand side of (2.10a) yields

$$\langle f\,|\epsilon\,, g\rangle = \left\langle x^{s-1}y^{r-1}, x^{m-1}y^{n-1}\right\rangle = \frac{(-1)^{s-1}}{\binom{k-2}{m-1}}\delta_{(s,r),(n,m)}$$

while the right-hand side is

$$\langle f, g\,|\epsilon\rangle = \left\langle x^{r-1}y^{s-1}, x^{n-1}y^{m-1}\right\rangle = \frac{(-1)^{r-1}}{\binom{k-2}{n-1}}\delta_{(s,r),(n,m)}.$$

Both sides vanish unless $f = g$. If $f = g$ then both sides agree since $s + r$ is even by assumption and therefore $s \equiv r \mod 2$.

The left-hand side of (2.10b) yields

$$\langle f\,|\delta\,, g\rangle = (-1)^{r-1}\,\langle f, g\rangle$$

while the right-hand side is

$$\langle f, g\,|\delta\rangle = (-1)^{m-1}\,\langle f, g\rangle\,.$$

Both sides vanish unless $r = n$ and $s = m$. In this case we have $r+m$ is even by assumption, hence $r \equiv m \mod 2$ and both sides agree. □

Corollary 2.28. Recall the spaces A_k, B_k, C_k and D_k from Remark 2.10. We then have $\langle f, g\rangle = 0$ for

1. $f \in A_k$, $g \in B_k$ and
2. $f \in C_k$, $g \in D_k$.

Proof. This follows immediately from Proposition 2.27 since for $M \in \{S, U\}$ we have

$$\langle f, g\rangle = \langle f\,|M\,, g\,|M\rangle = \langle f, \lambda_M \cdot g\rangle = \lambda_M\,\langle f, g\rangle$$

where $\lambda_S = -1$ and $\lambda_U = \omega^i$ for some $i \in \{1, 2\}$. In particular we have $\lambda_M \neq 1$, hence the claim follows. □

2.4. Dimensions

In this section, we compute the dimensions of W_k, W_k^+ and W_k^- by following [Zag00]. We recall that $W_0 = \mathbb{Q}$ (see Definition 2.14) and that $W_0^+ = \{0\}$ and $W_0^- = \mathbb{Q}$.

Proposition 2.29.

i) Let $k \geq 4$ be even. The dimension of W_k is explicitly given by

$$\dim W_k = k - 3 - 2\left(\left\lfloor \frac{k-2}{4} \right\rfloor + \left\lfloor \frac{k-2}{6} \right\rfloor\right).$$

ii) The generating series of W_k is given by

$$\sum_{k=0}^{\infty} \dim W_k \, x^k = \frac{1+x^{12}}{(1-x^4)(1-x^6)}.$$

Proof. Let $k \geq 4$ be even. Corollary 2.28 shows that the decompositions (2.3) are in fact orthogonal, i. e.

$$A_k \perp B_k = V_k = C_k \perp D_k. \tag{2.11}$$

Now since we have $W_k = B_k \cap D_k$ by definition, the orthogonality in (2.11) implies that $W_k = (A_k + C_k)^\perp$. However, the space $A_k \cap C_k$ is trivial as we saw in the proof of Proposition 2.19. We thus obtain $A_k + C_k = A_k \oplus C_k$ and therefore

$$V_k = W_k \oplus A_k \oplus C_k \tag{2.12}$$

which implies dimension-wise that

$$\dim W_k = \dim V_k - \dim A_k - \dim C_k. \tag{2.13}$$

For i) we use the dimensions of V_k form Remark 2.2 and the dimensions of A_k and C_k from (2.4). Then (2.13) implies that

$$\dim W_k = k - 3 - 2\left(\left\lfloor \frac{k-2}{4} \right\rfloor + \left\lfloor \frac{k-2}{6} \right\rfloor\right).$$

For ii) we first observe that the definition of V_k immediately yields since $V_0 = V_1 = \{0\}$ that

$$\sum_{k=0}^{\infty} \dim V_k \, x^k = \frac{x^2}{(1-x)^2}.$$

Note that V_k is non-trivial for odd k. However, since (2.13) only holds for even k we first observe that

$$\sum_{\substack{k=0 \\ k \text{ even}}}^{\infty} \dim V_k \, x^k = \frac{x^2\left(1+x^2\right)}{(1-x^2)^2}.$$

We further obtain the generating series of A_k and C_k via Molien's theorem A.12 which yields

$$\sum_{k=0}^{\infty} \dim A_k\, x^k = \frac{x^2\left(1+x^4\right)}{(1-x^4)(1-x^2)}$$
$$\sum_{k=0}^{\infty} \dim C_k\, x^k = \frac{x^2(1+x^6)}{(1-x^6)(1-x^2)}.$$

The generating series yield that both A_k and C_k are trivial for odd k. Since (2.13) only holds for $k \geq 4$ we still have to account for the cases $k = 0$ and $k = 2$. We have $\dim W_0 = 1$ and $\dim W_2 = 0$. The right-hand side of (2.13), however, yields 0 for $k = 0$ and -1 for $k = 2$. Hence, we obtain the generating series as

$$\begin{aligned}\sum_{k=0}^{\infty} \dim W_k\, x^k &= \frac{x^2\left(1+x^2\right)}{(1-x^2)^2} - \frac{x^2\left(1+x^4\right)}{(1-x^4)(1-x^2)} - \frac{x^2(1+x^6)}{(1-x^6)(1-x^2)} + x^2 + 1 \\ &= \frac{1+x^{12}}{(1-x^4)(1-x^6)}.\end{aligned}$$ □

Proposition 2.30.

i) Let $k \geq 4$ be even. The dimension of W_k^+ is explicitly given by

$$\dim W_k^+ = \frac{k}{2} - 2 - \left(\left\lfloor \frac{k-2}{4} \right\rfloor + \left\lfloor \frac{k-2}{6} \right\rfloor\right).$$

ii) The generating series of W_k^+ is given by

$$\sum_{k=0}^{\infty} \dim W_k^+\, x^k = \frac{x^{12}}{(1-x^4)(1-x^6)}.$$

Proof. Since

$$\epsilon S \equiv S\epsilon, \qquad \epsilon U \equiv U^2\epsilon, \qquad \epsilon U^2 \equiv U\epsilon$$

hold modulo ± 1 it follows that $|\epsilon$ acts on the spaces A_k, C_k and W_k. We thus obtain for even $k \geq 4$ from (2.12) that

$$V_k^+ = W_k^+ \oplus A_k^+ \oplus C_k^+$$

which implies dimension-wise that

$$\dim W_k^+ = \dim V_k^+ - \dim A_k^+ - \dim C_k^+. \tag{2.14}$$

We can, again, compute the Hilbert-Poincaré series of V_k^+, A_k^+ and C_k^+ with Molien's theorem A.12 since the groups generated by $\{\epsilon\}$, $\{\epsilon, S\}$ and $\{\epsilon, U\}$ are finite respectively. This yields

$$\dim V_k^+ = \frac{k}{2} \qquad \dim A_k^+ = 1 + \left\lfloor \frac{k-2}{4} \right\rfloor \qquad \dim C_k^+ = 1 + \left\lfloor \frac{k-2}{6} \right\rfloor$$

where the generating series are given by

$$\sum_{k=0}^{\infty} \dim V_k^+ \, x^k = \frac{x^2}{(1-x^2)(1-x)},$$
$$\sum_{k=0}^{\infty} \dim A_k^+ \, x^k = \frac{x^2}{(1-x^4)(1-x^2)},$$
$$\sum_{k=0}^{\infty} \dim C_k^+ \, x^k = \frac{x^2}{(1-x^6)(1-x^2)}.$$

Now i) follows immediately from (2.14) and the respective dimensions.

For ii) we first note that V_k^+ is, again, non-trivial for odd k. So we observe that

$$\sum_{\substack{k=0 \\ k \text{ even}}}^{\infty} \dim V_k^+ \, x^k = \frac{x^2}{(1-x^2)^2}.$$

By adjusting for the $k = 2$ case in (2.14) we obtain

$$\begin{aligned} \sum_{k=0}^{\infty} \dim W_k^+ \, x^k &= \frac{x^2}{(1-x^2)^2} - \frac{x^2}{(1-x^4)(1-x^2)} - \frac{x^2}{(1-x^6)(1-x^2)} + x^2 \\ &= \frac{x^{12}}{(1-x^4)(1-x^6)}. \end{aligned}$$

□

Corollary 2.31.

i) Let $k \geq 4$ be even. The dimension of W_k^- is explicitly given by

$$\dim W_k^- = \frac{k}{2} - 1 - \left(\left\lfloor \frac{k-2}{4} \right\rfloor + \left\lfloor \frac{k-2}{6} \right\rfloor \right).$$

ii) The generating series of W_k^- is given by

$$\sum_{k=0}^{\infty} \dim W_k^- \, x^k = \frac{1}{(1-x^4)(1-x^6)}.$$

Proof. Let $k \geq 4$ be even. Since $|\epsilon$ acts on W_k we also have the decomposition

$$W_k = W_k^+ \oplus W_k^-.$$

Propositions 2.29 and 2.30 therefore yield

$$\dim W_k^- = \dim W_k - \dim W_k^+ = \frac{k}{2} - 1 - \left(\left\lfloor \frac{k-2}{4} \right\rfloor + \left\lfloor \frac{k-2}{6} \right\rfloor \right)$$

and

$$\sum_{k=0}^{\infty} \dim W_k^- \, x^k = \frac{1+x^{12}}{(1-x^4)(1-x^6)} - \frac{x^{12}}{(1-x^4)(1-x^6)}$$
$$= \frac{1}{(1-x^4)(1-x^6)}. \qquad \square$$

Remark 2.32. Since $W_k^+ = \{0\}$ and $W_k^- = \mathbb{Q}$, Proposition 2.30 and Corollary 2.31 yield in particular for all even $k \in \mathbb{N}_0$ with $k \neq 2$ that

$$\dim W_k^+ = \dim W_k^- - 1.$$

By Proposition 2.17 we also have

$$\dim W_k^{\mathsf{odd}} = \dim W_k^{\mathsf{ev}} - 1.$$

The dimensions of W_k, W_k^+ and W_k^- for small values of k can be found in table 1.

2.5. Applications 1: Modular forms

We denote the upper complex half plane by

$$\mathbb{H} := \{z \in \mathbb{C} \mid \operatorname{im}(z) > 0\}.$$

Definition 2.33. Let $k \in \mathbb{N}$ be an integer. A *modular form of weight k* (w. r. t. $\mathrm{SL}_2(\mathbb{Z})$) is a holomorphic function $f \colon \mathbb{H} \to \mathbb{C}$ satisfying for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $z \in \mathbb{H}$ that

$$f(z) = (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) \tag{2.15}$$

with Fourier series expansion

$$f(z) = \sum_{n \geq 0} a_n q^n \tag{2.16}$$

where $q = e^{2\pi i z}$. Furthermore, a modular form f of weight k is called a *cusp form of weight k* if the constant coefficient a_0 in (2.16) vanishes.

We denote the space of modular forms of weight k by M_k and the space of cusp forms of weight k by S_k.

Remark 2.34. For odd k we have $M_k = \{0\}$ since condition (2.15) implies for $\gamma = -1$ that all $f \in M_k$ satisfy

$$f(z) = (-1)^{-k} f(z).$$

Definition 2.35. Let $f \in S_k$. Then

$$r_f(x) \coloneqq \int_0^{i\infty} f(z)(z-x)^{k-2}\,\mathrm{d}z$$

is called the *period polynomial of* f.

Definition 2.36. For $f \in S_k$ we call

$$r_n(f) \coloneqq \int_0^{i\infty} f(z) z^n\,\mathrm{d}z$$

the nth *period of* f.

Proposition 2.37. *Definition 2.35 actually yields a polynomial of degree $k-2$. To be precise, for $f \in S_k$ we have*

$$r_f(x) = \sum_{n=0}^{k-2} (-1)^n \binom{k-2}{n} r_n(f) x^{k-2-n} \in \mathbb{C}[x].$$

Proof. By Remark 2.34 we assume without loss of generality that k is even. We then have

$$\begin{aligned} r_f(x) &= \int_0^{i\infty} f(z)(z-x)^{k-2}\,\mathrm{d}z \\ &= \sum_{n=0}^{k-2} \binom{k-2}{n} \int_0^{i\infty} f(z) z^n (-x)^{k-2-n}\,\mathrm{d}z \\ &= \sum_{n=0}^{k-2} (-1)^n \binom{k-2}{n} r_n(f) x^{k-2-n}. \end{aligned}$$ □

Example 2.38. For $z \in \mathbb{H}$ let

$$\Delta(z) = q \cdot \prod_{n=1}^{\infty} (1-q^n)^{24} = q - 24\,q^2 + 252\,q^3 \mp \ldots$$

Then Δ is a cusp form of weight 12. Using the decomposition $V_k = V_k^+ \oplus V_k^-$ from Remark 2.7, we have

$$\begin{aligned} r_\Delta^+ &= \omega_\Delta^+ \cdot r_\Delta^{\mathsf{odd}} \\ r_\Delta^- &= \omega_\Delta^- \cdot \left(\frac{36}{691} p_{12} - r_\Delta^{\mathsf{ev}} \right) \end{aligned}$$

with p_{12}, r_Δ^{ev} and r_Δ^{odd} from Example 2.16 and complex constants

$$\omega_\Delta^+ = 0.00926927\ldots \qquad \text{and} \qquad \omega_\Delta^- = i \cdot 0.114379\ldots$$

For more details, we refer to [Zag91].

Theorem 2.40 below states a fundamental correspondence between cusp forms and their respective period polynomials. They can be described as either uni- or bivariate polynomials.

Remark 2.39. There is a natural bijection

$$\Phi\colon \left\{f \in \mathbb{Q}[x] \mid \deg(f) \leq k-2\right\} \xrightarrow{\sim} \left\{F \in \mathbb{Q}[x,y] \mid F \text{ homogeneous}, \deg(F) = k-2\right\}$$

given by

$$\Phi(f) = y^{k-2} \cdot f\left(\frac{x}{y}\right)$$
$$\Phi^{-1}(F) = F(x,1).$$

The Eichler–Shimura–Manin theory gives an important correspondence between the space of cusp forms S_k and the spaces W_k^{ev} and W_k^{odd}. Using the decomposition $W_k = W_k^{\mathsf{ev}} \oplus W_k^{\mathsf{odd}}$ and the notation from Remark 2.7, we write $p = p^{\mathsf{ev}} + p^{\mathsf{odd}}$ for $p \in W_k$. We will state the theorem without giving a proof. For more details, we refer to [Zag00] and [MR05].

Theorem 2.40 (Eichler–Shimura–Manin)**.** *The maps*

$$r^{\mathsf{odd}}\colon S_k \to W_k^{\mathsf{odd}} \otimes \mathbb{C}$$
$$f \mapsto r_f^{\mathsf{odd}}$$

and

$$r^{\mathsf{ev}}\colon S_k \to W_k^{\mathsf{ev}}/(\mathbb{Q}\cdot p_k) \otimes \mathbb{C}$$
$$f \mapsto r_f^{\mathsf{ev}}$$

with $p_k = x^{k-2} - y^{k-2}$ *are isomorphisms of vector spaces.*

The proof of Theorem 2.40 makes use of a Hermitian scalar product on S_k that has a connection to the pairing on V_k. For $f, g \in S_k$ the Petersson scalar product is given by

$$\langle f, g\rangle := \int_{\mathcal{F}} f(z)\overline{g(z)} y^k \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$$

where $z = x + iy$ and $\mathcal{F} = \{z \in \mathbb{H} \mid |\mathrm{Re}(z)| \leq \frac{1}{2}, |z| \geq 1\}$ is the fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$. Now consider the natural extension of the pairing $\langle\cdot,\cdot\rangle$ on V to a Hermitian form on $\mathbb{C}[x,y]$ by applying the complex conjugation to the second argument. We then have for $f, g \in S_k$ that

$$\langle f, g\rangle = c_k \cdot \langle r_f \left| T - T^{-1} \right., \overline{r_g}\rangle$$

where $c_k \in \mathbb{C}$ is a complex constant depending on the weight k.

Remark 2.41. Zagier shows in [Zag91, p. 453] that the polynomial $p_k = x^{k-2} - y^{k-2}$ can be naturally considered as the even period polynomial of the Eisenstein series

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \quad q = e^{2\pi i z}$$

with the $(k-1)$th divisor sum

$$\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$$

and the kth Bernoulli number given by

$$\sum_{k=0}^{\infty} \frac{B_k}{k!} x^k = \frac{x}{e^x - 1}.$$

Remark 2.42. The generating series of S_k is known to be

$$\sum_{k=0}^{\infty} \dim S_k \, x^k = \frac{x^{12}}{(1-x^4)(1-x^6)}.$$

Note that this coincides with the series of $W_k^+ = W_k^{\mathsf{odd}}$ from Proposition 2.30. By Remark 2.32 we further obtain from Theorem 2.40 for even $k \geq 4$ that $\dim W_k = 2 \cdot \dim S_k + 1$. We have

$$\frac{x^4}{1-x^2} = \sum_{\substack{k=4 \\ k \text{ even}}}^{\infty} x^k$$

and the generating series of W_k from Proposition 2.29 can indeed be computed after accounting for $\dim W_0 = 1$ as

$$\frac{2\,x^{12}}{(1-x^4)(1-x^6)} + \frac{x^4}{1-x^2} + 1 = \frac{1+x^{12}}{(1-x^4)(1-x^6)}.$$

2.6. Applications 2: Exotic relations for double zeta values

The theory of multiple zeta values is concerned with real numbers of the form

$$\zeta(s_1, \ldots, s_l) := \sum_{n_1 > \cdots > n_l \geq 1} \frac{1}{n_1^{s_1}, \ldots, n_l^{s_l}}$$

where $(s_1, \ldots, s_l) \in \mathbb{N}^l$ with $s_1 > 1$. In particular, one is interested in relations amongst them. For example, we have

$$\zeta(3) = \zeta(2, 1).$$

For an introduction to multiple zeta values, see [BGF17].

An application of period polynomials in this theory was studied by Gangl, Kaneko and Zagier in [GKZ06] to compute $\mathbb{Q}$-linear relations amongst them. Our presentation of their results follows the notes of a talk by U. Kühn [Küh20].

Consider the generating series of single and double zeta values

$$\mathcal{Z}_1(x) = \sum_{s \geq 2} \zeta(s)\, x^{s-1}$$
$$\mathcal{Z}_2(x,y) = \sum_{\substack{s_1 \geq 2 \\ s_2 \geq 1}} \zeta(s_1, s_2)\, x^{s_1-1} y^{s_2-1}.$$

The multiple zeta values satisfy the *double shuffle relations*

$$\begin{aligned} \zeta(s_1)\cdot\zeta(s_2) &= \zeta(s_1,s_2) + \zeta(s_2,s_1) + \zeta(s_1+s_2) \\ &= \sum_{r=1}^{s_1+s_2-1} \left(\binom{r-1}{s_1-1} + \binom{r-1}{s_2-1} \right) \zeta(r, s_1+s_2-r), \end{aligned}$$

for $s_1, s_2 \geq 2$. Hence

$$\mathcal{Z}_2(x,y) + \mathcal{Z}_2(y,x) + \frac{\mathcal{Z}_1(x) - \mathcal{Z}_1(y)}{x-y} = \mathcal{Z}_2(x+y,x) + \mathcal{Z}_2(x+y,y) + \zeta(2). \tag{2.17}$$

By extending our definition of the slash operator to power series in variables x, y we can rewrite (2.17) as

$$\mathcal{Z}_2(x,y)\,|(T-1)(1+\epsilon) = \frac{\mathcal{Z}_1(x) - \mathcal{Z}_1(y)}{x-y} - \zeta(2) \tag{2.18}$$

since $(T-1)(1+\epsilon) = T + T\epsilon - 1 - \epsilon$.

We now extend the non-degenerate pairing $\langle \cdot, \cdot \rangle$ to a duality pairing $\mathbb{Q}[x,y] \times \mathbb{Q}[\![x,y]\!] \to \mathbb{Q}$. This is still invariant under the $\mathrm{SL}_2(\mathbb{Z})$ action and allows us to associate relations amongst multiple zeta values to certain polynomials.

Proposition 2.43. *Let $k \geq 4$ be even. Assume* (2.18) *encodes all non-trivial linear relations of the form*

$$\sum_{\substack{s_1+s_2=k \\ s_1 \geq 2 \\ s_2 \geq 1}} \lambda_{s_1,s_2} \zeta(s_1,s_2) = \lambda_k \zeta(k) \tag{2.19}$$

with $\lambda_{s_1,s_2}, \lambda_k \in \mathbb{Q}$. Then there exists a $f \in V_k$ such that

$$\lambda_k \zeta(k) = \left\langle f(x,y), \frac{\mathcal{Z}_1(x) - \mathcal{Z}_1(y)}{x-y} - \zeta(2) \right\rangle$$
$$\sum_{\substack{s_1+s_2=k \\ s_1 \geq 2 \\ s_2 \geq 1}} \lambda_{s_1,s_2} \zeta(s_1,s_2) = \left\langle f(x,y)\,\middle|(1+\epsilon)(T^{-1}-1)\,, \mathcal{Z}_2(x,y) \right\rangle.$$

Proof. Let $k \geq 4$ be even. The coefficient of a fixed monomial $x^i y^j \in V_k$ on both sides of (2.18) must be equal. This yields a relation

$$\sum_{\substack{s_1+s_2=k \\ s_1 \geq 2 \\ s_2 \geq 1}} \lambda^{i,j}_{s_1,s_2} \zeta(s_1, s_2) = \zeta(k). \tag{2.20}$$

By our previous considerations, we obtain (2.20) by applying the linear map $\langle x^j y^i, \cdot \rangle$ to (2.18). By assumption, any linear relation of the form (2.19) is a linear combination of the linear relations given by (2.20). We therefore find a $f \in V_k$ such that $\langle f(x,y), \cdot \rangle$ applied to (2.18) gives (2.19), i. e.

$$\lambda_k \zeta(k) = \left\langle f(x,y), \frac{\mathcal{Z}_1(x) - \mathcal{Z}_1(y)}{x-y} - \zeta(2) \right\rangle$$
$$\sum_{s_1+s_2=k} \lambda_{s_1,s_2} \zeta(s_1,s_2) = \langle f(x,y), \mathcal{Z}_2(x,y) \,|(T-1)(1+\epsilon) \rangle .$$

By Proposition 2.27 the pairing $\langle \cdot, \cdot \rangle$ is invariant under the action of $\mathrm{GL}_2(\mathbb{Z})$ since k is even. So adjoints are given by inverses. Observe that the summands in $(1+\epsilon)(T^{-1}-1) = T^{-1} + \epsilon T^{-1} - 1 - \epsilon$ are pairwise inverse to $(T-1)(1+\epsilon)$ from above. So the two operators are adjoints to one another and the claim follows. □

Remark 2.44. It is still an open question whether the multiple zeta values satisfy $\mathbb{Q}$-linear relations that cannot be derived from the algebraic double shuffle relations. So instead of considering the single and double zeta values, [GKZ06] consider the *formal double zeta space* $\mathcal{D}_k$ instead. This space is generated by formal symbols $Z_{r,s}$, $P_{r,s}$ and Z_k that satisfy the double shuffle relations

$$\begin{aligned} Z_{r,s} + Z_{s,r} &= P_{r,s} - Z_k \quad \text{for } r,s \geq 2 \text{ with } r+s=k \\ \sum_{r=2}^{k-1} \left[\binom{r-1}{j-1} + \binom{r-1}{k-j-1} \right] Z_{r,k-r} &= P_{j,k-j} \qquad \text{for } 2 \leq j \leq \frac{k}{2} \end{aligned} \tag{2.21}$$

and no other linear relations. So we have

$$\mathcal{D}_k = \frac{\{\mathbb{Q}\text{-linear combinations of } Z_k, Z_{r,s}, P_{r,s}\}}{\langle \text{relations (2.21)} \rangle_{\mathbb{Q}}}.$$

Proposition 2.43 then holds without the additional assumption by replacing $\zeta(k)$ with Z_k, $\zeta(s_1,s_2)$ with Z_{s_1,s_2} and $\zeta(s_1)\zeta(s_2)$ with P_{s_1,s_2}.

Proposition 2.45. *If $f, f' \in V_k$ determine the same relation via applying the linear maps $\langle f(x,y), \cdot \rangle$ and $\langle f'(x,y), \cdot \rangle$, respectively, to* (2.18) *then their projections onto V_k^+ agree.*

Proof. Observe that

$$\frac{\mathcal{Z}_1(x) - \mathcal{Z}_1(y)}{x-y} = \sum_{k \geq 2} \zeta(k) \frac{x^{k-1} - y^{k-1}}{x-y} = \sum_{k \geq 2} \zeta(k) \sum_{r=1}^{k-1} x^{r-1} y^{k-r-1}$$

is an infinite sum of symmetric polynomials. Now recall the orthogonal decomposition $V_k = V_k^+ \oplus V_k^-$ and the projection π^+ onto V_k^+ from Remark 2.7. So for any $f \in V_k$ we obtain

$$\left\langle f(x,y), \frac{\mathcal{Z}_1(x) - \mathcal{Z}_1(y)}{x-y} \right\rangle = \left\langle \pi^+(f)(x,y), \frac{\mathcal{Z}_1(x) - \mathcal{Z}_1(y)}{x-y} \right\rangle.$$

For the other side we consider the adjoint operators $\Delta = (T-1)(1+\epsilon)$ and $\Delta^* = (1+\epsilon)(T^{-1}-1)$ from above and obtain that

$$\langle f(x,y), \mathcal{Z}_2(x,y)\,|\Delta \rangle = \langle f(x,y)\,|\Delta^*\,, \mathcal{Z}_2(x,y)\rangle = 2\left\langle \pi^+(f)\,\middle|(T^{-1}-1)\,, \mathcal{Z}_2(x,y)\right\rangle.$$

Now if $f, f' \in V_k$ give the same relations, then

$$\left\langle \pi^+(f-f')\,\middle|(T^{-1}-1)\,, \mathcal{Z}_2(x,y)\right\rangle = 0.$$

Since the pairing is non-degenerate and $\zeta(s_1,s_2) > 0$ for all $s_1 \geq 2$, $s_2 \geq 1$ this implies that $\pi^+(f-f')\,\big|(T^{-1}-1) = 0$. Hence $\pi^+(f-f')$ is invariant under $|\epsilon$ and $|T$. But this space is trivial and thus $\pi^+(f-f') = 0$. □

Example 2.46. Let $k = 4$. We consider the symmetric polynomials $f_1, f_2 \in V_4^+$ with $f_1(x,y) = x^2 + y^2$ and $f_2(x,y) = xy$. Since

$$\left\langle f_1(x,y), \frac{\mathcal{Z}_1(x) - \mathcal{Z}_1(y)}{x-y} - \zeta(2) \right\rangle = 2\zeta(4)$$
$$\left\langle f_1(x,y)\,\middle|(1+\epsilon)(T^{-1}-1)\,, \mathcal{Z}_2(x,y)\right\rangle = 2\zeta(3,1) + 2\zeta(2,2)$$

and

$$\left\langle f_2(x,y), \frac{\mathcal{Z}_1(x) - \mathcal{Z}_1(y)}{x-y} - \zeta(2) \right\rangle = -\frac{1}{2}\zeta(4)$$
$$\left\langle f_2(x,y)\,\middle|(1+\epsilon)(T^{-1}-1)\,, \mathcal{Z}_2(x,y)\right\rangle = -2\zeta(3,1)$$

Proposition 2.43 yields the relations

$$\zeta(3,1) + \zeta(2,2) = \zeta(4) \quad \text{and} \quad 4\zeta(3,1) = \zeta(4).$$

Combining these relations, we further obtain

$$\zeta(2,2) = 3\zeta(3,1).$$

Proposition 2.47. *Let $f \in W_k^{\mathsf{ev}}$. Then the relation* (2.19) *induced by applying the linear map $\langle f\,|T\,, \cdot\rangle$ to* (2.18) *is symmetric in $\zeta(\mathsf{ev},\mathsf{ev})$, i. e. $\lambda_{s_1,s_2} = \lambda_{s_2,s_1}$ for all even $s_1, s_2 \in \mathbb{N}$.*

Proof. Without loss of generality, we assume that k is even since $W_k = \{0\}$ for odd k.

Recall the decompositions of V_k and the corresponding projections from Remark 2.7. Now let $s_1, s_2 \in \mathbb{N}$ with $s_1 + s_2 = k$ and $s_1 \geq 2$. Then

$$\left\langle x^{s_2-1}y^{s_1-1}, \mathcal{Z}_2(x,y)\right\rangle = \lambda_{s_1,s_2}\zeta(s_1,s_2)$$

for some $\lambda_{s_1,s_2} \in \mathbb{Q}$. So for $p \in V_k$ we see that

$$\langle p(x,y), \mathcal{Z}_2(x,y)\rangle$$

is symmetric in $\zeta(\mathsf{ev},\mathsf{ev})$ if $p^{\mathsf{odd}} \in V_k^+$. Now let $f \in W_k^{\mathsf{ev}}$ and set $g := f\,|T$. By our previous considerations, it suffices to show that $(g\,|\Delta^*)^{\mathsf{odd}} \in V_k^+$ where $\Delta^* = (1+\epsilon)(T^{-1}-1)$ is the operator from Proposition 2.43.

To show that $(g\,|\Delta^*)^{\mathsf{odd}} \in V_k^+$ we first observe that

$$2g^- = g\,|(1-\epsilon) = f\,|(T - T\epsilon) = f$$

where the last equality holds by assumption since we have $f \in W_k^-$ (cf. Proposition 2.17) and satisfies the refined Lewis equation (Corollary 2.21). Applying π^{ev} and π^{odd} to this identity yields, respectively, that $2\,g^{-,\mathsf{ev}} = f$ and $g^{-,\mathsf{odd}} = 0$. Since $g^{\mathsf{odd}} = g^{+,\mathsf{odd}} + g^{-,\mathsf{odd}}$ we further obtain that $g^{+,\mathsf{odd}} = g^{\mathsf{odd}} \in V_k^+$. Our goal is to show that $(g\,|\Delta^*)^{\mathsf{odd}} = -g^{\mathsf{odd}}$ which will imply the claim.

Observe that $g^{-,\mathsf{odd}} = 0$ implies

$$g\,|S = g\,|\delta\epsilon = (g^{\mathsf{ev}} - g^{\mathsf{odd}})\,|\epsilon = (g^{+,\mathsf{ev}} - g^{-,\mathsf{ev}} - g^{+,\mathsf{odd}}).$$

Now recall the matrix $T' = \epsilon T \epsilon$ from Definition 2.18 and note that $ST^{-1} = T'S$. Since S is self-inverse modulo ± 1 and commutes with $(1+\epsilon)$ we have

$$\begin{aligned}
g\,|\Delta^* &= (g\,|S)\,|S\Delta^* = \left(g^{+,\mathsf{ev}} - g^{-,\mathsf{ev}} - g^{+,\mathsf{odd}}\right)|S\Delta^* \\
&= \left(g^{+,\mathsf{ev}} - g^{+,\mathsf{odd}}\right)\big|(1+\epsilon)S(T^{-1}-1) - g^{-,\mathsf{ev}}\big|(1+\epsilon)S(T^{-1}-1) \\
&= 2\left(g^{+,\mathsf{ev}} - g^{+,\mathsf{odd}}\right)\big|S(T^{-1}-1) \\
&= 2\left(g^{+,\mathsf{ev}} - g^{+,\mathsf{odd}}\right)\big|(T'-1)S \\
&= 2\left(g^{+,\mathsf{ev}} - g^{-,\mathsf{ev}} - g^{+,\mathsf{odd}}\right)\big|(T'-1)S + 2g^{-,\mathsf{ev}}\big|(T'-1)S\,.
\end{aligned}$$

By using $TST' = S$ and $f \in W_k^{\mathsf{ev}} = W_k^-$, we further compute that

$$\left(g^{+,\mathsf{ev}} - g^{-,\mathsf{ev}} - g^{+,\mathsf{odd}}\right)\big|T' = g\,\big|ST' = f\,\big|TST' = f\,|S = f\,|\delta\epsilon = -f \tag{2.22}$$

$$2\,g^{-,\mathsf{ev}}\big|T' = f\,\big|T' = -f\,|T\epsilon = -g\,|\epsilon = -g^{+,\mathsf{ev}} + g^{-,\mathsf{ev}} - g^{+,\mathsf{odd}}. \tag{2.23}$$

By combining (2.22) and (2.23) with $f = 2\,g^{-,\mathsf{ev}}$ and the formula for $g\,|\Delta^*$ from above, we obtain that

$$\begin{aligned}
g\,|\Delta^* &= \left(-2\,f - 2\left(g^{+,\mathsf{ev}} - g^{-,\mathsf{ev}} - g^{+,\mathsf{odd}}\right) - g^{+,\mathsf{ev}} + g^{-,\mathsf{ev}} - g^{+,\mathsf{odd}} - 2\,g^{-,\mathsf{ev}}\right)|S \\
&= \left(-3\,g^{+,\mathsf{ev}} - 3\,g^{-,\mathsf{ev}} + g^{+,\mathsf{odd}}\right)|S = \left(-3\,g^{\mathsf{ev}} + g^{\mathsf{odd}}\right)|S \\
&= \left(-3\,g^{\mathsf{ev}} - g^{\mathsf{odd}}\right)|\epsilon = -3\,g^{\mathsf{ev}}\,|\epsilon - g^{\mathsf{odd}}.
\end{aligned}$$

This shows that $(g\,|\Delta^*)^{\mathsf{odd}} = -g^{\mathsf{odd}}$ since $g^{\mathsf{ev}}\,|\epsilon$ is even. □

Examples 2.48.

i) Consider the polynomial $p_6(x,y) = x^4 - y^4 \in W_6^{\mathsf{ev}}$. This yields

$$\left\langle p_6(x,y)\,|T\,,\frac{\mathcal{Z}_1(x)-\mathcal{Z}_1(y)}{x-y}\right\rangle = 0$$
$$\langle p_6(x,y)\,|T\Delta^*\,,\mathcal{Z}_2(x,y)\rangle = \zeta(4,2)+\zeta(2,4)-3\big(\zeta(5,1)+\zeta(3,3)\big)$$

and we have the relation

$$3\big(\zeta(5,1)+\zeta(3,3)\big) = \zeta(4,2)+\zeta(2,4).$$

ii) Consider the polynomial

$$r_\Delta^{\mathsf{ev}}(x,y) = x^8y^2 - x^2y^8 - 3\left(x^6y^4 - x^4y^6\right) \in W_{12}^{\mathsf{ev}}$$

from Example 2.16. This yields

$$\left\langle r_\Delta^{\mathsf{ev}}(x,y)\,|T\,,\frac{\mathcal{Z}_1(x)-\mathcal{Z}_1(y)}{x-y}\right\rangle = -\frac{1}{105}\zeta(12),$$
$$\langle r_\Delta^{\mathsf{ev}}(x,y)\,|T\Delta^*\,,\mathcal{Z}_2(x,y)\rangle = \frac{1}{15}\big(\zeta(8,4)+\zeta(4,8)\big)+\frac{19}{126}\zeta(6,6) - \left(\frac{1}{15}\zeta(9,3)+\frac{5}{14}\zeta(7,5)+\frac{2}{5}\zeta(5,7)\right)$$

and we have the relation

$$-\frac{1}{105}\zeta(12) = \frac{1}{15}\big(\zeta(8,4)+\zeta(4,8)\big)+\frac{19}{126}\zeta(6,6)-\left(\frac{1}{15}\zeta(9,3)+\frac{5}{14}\zeta(7,5)+\frac{2}{5}\zeta(5,7)\right).$$

For the formal double zeta space from Remark 2.44, [GKZ06, Theorem 3] show, in the formulation of [Küh20], the following theorem.

Theorem 2.49. *There is an isomorphism of vector spaces*

$$W_k^{\mathsf{ev}} \cong \frac{\langle \textit{relations in } \mathcal{D}_k \textit{ which are symmetric in } Z_{\mathsf{ev,ev}}\rangle_{\mathbb{Q}}}{\langle \textit{relations in } \mathcal{D}_k \textit{ which are symmetric in } Z_{\mathsf{ev,ev}} \textit{ without } Z_{\mathsf{odd,odd}}\rangle_{\mathbb{Q}}}.$$

2.7. Applications 3: Quadratic relations and depth 4 generators for $\mathfrak{ls}$

Another occurrence of period polynomials in the theory of multiple zeta values can be found in the study of the *linearized double shuffle* Lie algebra $\mathfrak{ls}$. In this master thesis, we only need some fragments of this theory. For more details, we refer to [Bro21] and [Sch15] as well as to [MT18] for a comparison. We have a decomposition

$$\mathfrak{ls} = \bigoplus_{k,d=0}^{\infty} \mathfrak{ls}_{k,d} \subset \prod_{d=0}^{\infty} \mathbb{Q}[x_1,\dots,x_d]$$

where $\mathfrak{ls}_{k,d}$ contains certain homogeneous polynomials in d variables of degree $k-d$. In fact, the pair $(\mathfrak{ls},\{\,,\,\})$, where $\{\,,\,\}$ denotes the Ihara bracket, is a bigraded Lie algebra. There are two occurrences of period polynomials in the theory of $\mathfrak{ls}$.

1.) The subspace in depth $d = 1$ is

$$\mathfrak{ls}_1 = \bigoplus_{k=1}^{\infty} \mathbb{Q}\, \sigma_{2k+1}$$

where

$$\sigma_{2k+1}(x_1) = x_1^{2k} \in \mathbb{Q}[x_1] \tag{2.24}$$

is an even polynomial and the bracket

$$\{\ ,\ \}\colon \mathfrak{ls}_1 \times \mathfrak{ls}_1 \longrightarrow \mathfrak{ls}_2$$

is explicitly given for $f, g \in \mathfrak{ls}_1$ by

$$\{f, g\}(x_1, x_2) = \big(f(x_1)g(x_2) - f(x_2)g(x_1)\big)\,\big|1 + U + U^2\ . \tag{2.25}$$

The relations of $\mathfrak{ls}$ in depth 2 are given by the kernel of $\{\ ,\ \}\colon \mathfrak{ls}_1 \times \mathfrak{ls}_1 \longrightarrow \mathfrak{ls}_2$. Note that the bracket factors through the exterior product $\mathfrak{ls}_1 \wedge \mathfrak{ls}_1$, i. e. we have a short exact sequence

$$0 \longrightarrow \ker(\{\ ,\ \}) \longrightarrow \mathfrak{ls}_1 \wedge \mathfrak{ls}_1 \longrightarrow \mathfrak{ls}_2 \longrightarrow 0\ .$$

Since $\mathfrak{ls}_1$ is spanned by polynomials of the form (2.24), we obtain that $\mathfrak{ls}_1 \wedge \mathfrak{ls}_1$ is isomorphic to the space that is spanned by homogeneous polynomials of the form

$$\Sigma(x_1, x_2) := \sigma_{2k_1+1}(x_1)\sigma_{2k_2+1}(x_2) - \sigma_{2k_1+1}(x_2)\sigma_{2k_2+1}(x_1)$$

for $k_1, k_2 \in \mathbb{N}$. A non-zero Σ has degree $2(k_1 + k_2)$ and we have $\Sigma \in \ker(1 + S)$. Since σ_{2k+1} is an even polynomial, we observe that

$$\Sigma(-x_1, x_2) = \Sigma(x_1, x_2) = \Sigma(x_1, -x_2).$$

We therefore obtain that

$$\Sigma \in \Big\{ f \in V_k \ \Big|\ f\,|1 + S = 0,\ f\,|\delta = f \Big\}.$$

By (2.25), the bracket on $\mathfrak{ls}_1$ is given by applying the operator $\big|1 + U + U^2$. Hence we obtain for the weight k component of the kernel that

$$\ker(\{\ ,\ \})_k \cong W_k^{\mathsf{ev}}.$$

For example, in weight $k = 12$ we have the relation (cf. [Bro21, Example 7.2])

$$\mathrm{rel}_\Delta\colon 3\{\sigma_5, \sigma_7\} - \{\sigma_3, \sigma_9\} = 0$$

which corresponds to the even period polynomial r_Δ^{ev} from Example 2.16 of the cusp form Δ from Example 2.38 by replacing $\{\sigma_{2a+1}, \sigma_{2b+1}\}$ with $x_1^{2a}x_2^{2b} - x_1^{2b}x_2^{2a}$.

2.) Let $f \in W_k^{\mathsf{ev}} / \langle x^{k-2} - y^{k-2} \rangle_{\mathbb{Q}}$ be an even period polynomial with $f(x,0)=0$. Then Brown [Bro21, § 8] associates the element

$$\mathsf{e}_f := \sum_{\mathbb{Z}/5\mathbb{Z}} \left(f_1(y_4-y_3, y_2-y_1) + (y_0-y_1) f_0(y_2-y_3, y_4-y_3) \right)$$

to it, where we sum over all cyclic permutations of $(y_0,\dots,y_4) \mapsto (y_1,\dots,y_5,y_0)$ with

$$f_0(x,y) = \frac{f(x,y)}{xy(x-y)} \qquad \text{and} \qquad f_1(x,y) = (x-y) f_0(x,y).$$

Note that $f_0(x,y) \in \mathbb{Q}[x,y]$ since f vanishes at $x=0$, $y=0$ and $x-y=0$. Now define $\bar{\mathsf{e}}_f \in \mathbb{Q}[x_1,\dots,x_4]$ by setting $y_0=0$ and $y_i = x_1+\dots+x_i$, for all $i \in \{1,2,3,4\}$ in e_f. Brown has shown that the elements $\bar{\mathsf{e}}_f$ satisfy the linearized double shuffle relations. An alternative construction for these elements is given by Ecalle in [Eca11] by using a refinement of 1.).

Conjecturally, the Lie algebra $\mathfrak{ls}$ is generated by the elements σ_{2k+1} from above and $\bar{\mathsf{e}}_f$ from 2.) and all relations are of the form described in 1.). This would imply the Broadhurst-Kreimer conjecture.

3. Bi-period polynomials

We now consider a slight generalization of period polynomials in 4 variables. Similar to chapter 2, we are going to define an action of $\mathrm{GL}_2(\mathbb{Z})$ first and again consider $\ker(1+S) \cap \ker(1+U+U^2)$. Before doing so, however, we will study the Laplacian operator of a particular quadratic form. This will give us a new decomposition of the space of polynomials which will turn out useful for computing the dimension of certain subspaces. Throughout this chapter we let, again, $k \in \mathbb{N}$ be an integer with $k \geq 2$ unless stated otherwise.

3.1. Bi-slash operator

Definition 3.1. For $k \geq 2$ we denote the set of homogeneous polynomials in 4 variables of degree $k-2$ over $\mathbb{Q}$ by

$$\mathcal{V}_k \coloneqq \{f = f(x_1, x_2, y_1, y_2) \in \mathbb{Q}[x_1, x_2, y_1, y_2] \mid f \text{ homogeneous}, \deg(f) = k-2\}.$$

We further set $\mathcal{V}_0 \coloneqq \{0\}$, $\mathcal{V}_1 \coloneqq \{0\}$ and

$$\mathcal{V} \coloneqq \bigoplus_{k=0}^{\infty} \mathcal{V}_k.$$

Note that $\mathcal{V}$ is a graded vector space with homogeneous components given by $\mathcal{V}_k$.

Lemma 3.2. *We have*

$$\dim \mathcal{V}_k = \binom{k+1}{3}.$$

Proof. The definition of $\mathcal{V}_k$ yields immediately that

$$\sum_{k=0}^{\infty} \dim \mathcal{V}_k \, x^k = \frac{x^2}{(1-x)^4}.$$

Note that

$$(1-x)^4 \cdot \left(\sum_{k=2}^{\infty} \binom{k+1}{3} x^k \right) = (x^4 - 4x^3 + 6x^2 - 4x + 1) \cdot \left(\sum_{k=2}^{\infty} \binom{k+1}{3} x^k \right) = x^2$$

since the first few coefficients are easily checked and a straight forward computation shows that

$$\binom{k+1}{3} - 4\binom{k}{3} + 6\binom{k-1}{3} - 4\binom{k-2}{3} + \binom{k-3}{3} = 0.$$

We thus have

$$\sum_{k=0}^{\infty} \dim \mathcal{V}_k \, x^k = \frac{x^2}{(1-x)^4} = \sum_{k=0}^{\infty} \binom{k+1}{3} x^k. \qquad \square$$

Definition 3.3. For $\gamma \in \mathrm{GL}_2(\mathbb{Z})$ we set $\gamma^{-\mathsf{t}} := \left(\gamma^{-1}\right)^{\mathsf{t}}$ and define the *bi-slash operator* on the space $\mathcal{V}_k$ by

$$\begin{aligned} \mathrm{GL}_2(\mathbb{Z}) \times \mathcal{V}_k &\longrightarrow \mathcal{V}_k \\ (\gamma, f) &\longmapsto f\,|\gamma := f\left((\gamma \cdot z_1)^{\mathsf{t}}, \left(\gamma^{-\mathsf{t}} \cdot z_2\right)^{\mathsf{t}}\right) \end{aligned}$$

where $z_1 = (x_1, x_2)^{\mathsf{t}}$ and $z_2 = (y_1, y_2)^{\mathsf{t}}$.

Remark 3.4. The bi-slash operator on $\mathcal{V}_k$ is essentially given by applying the slash operator from Definition 2.3 twice. It thus follows by a similar calculation as in Proposition 2.4 that this in fact yields a right group action on $\mathcal{V}_k$.

Lemma 3.5. *For even $k \in \mathbb{N}$ the space $\mathcal{V}_k$ is invariant under -1, i. e. for $f \in \mathcal{V}_k$ we have*

$$f\,|(-1) = f.$$

Proof. Note that -1 is self-inverse and fixed under transposing. The proof thus follows analogously to Lemma 2.5. □

For the following definition, we recall the $\mathrm{GL}_2(\mathbb{Z})$-matrices ϵ and δ from (2.2).

Definition 3.6. We denote the eigenspaces of the operator $|\epsilon$ on $\mathcal{V}_k$ with eigenvalues 1 and -1, respectively, by

$$\begin{aligned} \mathcal{V}_k^{+} &= \{f \in \mathcal{V}_k \mid f(x_2, x_1, y_2, y_1) = f(x_1, x_2, y_1, y_2)\} \\ \mathcal{V}_k^{-} &= \{f \in \mathcal{V}_k \mid f(x_2, x_1, y_2, y_1) = -f(x_1, x_2, y_1, y_2)\} \end{aligned}$$

and the eigenspaces of the operator $|\delta$ on $\mathcal{V}_k$ with eigenvalues 1 and -1, respectively, by

$$\begin{aligned} \mathcal{V}_k^{\mathsf{ev}} &= \{f \in \mathcal{V}_k \mid f(-x_1, x_2, -y_1, y_2) = f(x_1, x_2, y_1, y_2)\} \\ \mathcal{V}_k^{\mathsf{odd}} &= \{f \in \mathcal{V}_k \mid f(-x_1, x_2, -y_1, y_2) = -f(x_1, x_2, y_1, y_2)\}. \end{aligned}$$

Furthermore, for a subspace $\mathcal{W} \subseteq \mathcal{V}_k$ we set $\mathcal{W}^{\bullet} := \mathcal{W} \cap \mathcal{V}_k^{\bullet}$ for all $\bullet \in \{+, -, \mathsf{ev}, \mathsf{odd}\}$.

Remark 3.7.

i) Analogously to Remark 2.7 we have orthogonal decompositions in eigenspaces

$$\mathcal{V}_k = \mathcal{V}_k^{+} \oplus \mathcal{V}_k^{-} \qquad \text{and} \qquad \mathcal{V}_k = \mathcal{V}_k^{\mathsf{ev}} \oplus \mathcal{V}_k^{\mathsf{odd}}$$

with projections given by

$$\begin{aligned} \pi^{+} \colon \mathcal{V}_k &\to \mathcal{V}_k^{+}, & \pi^{-} \colon \mathcal{V}_k &\to \mathcal{V}_k^{-}, \\ f &\mapsto \frac{1}{2}(f + f\,|\epsilon) & f &\mapsto \frac{1}{2}(f - f\,|\epsilon) \end{aligned}$$

and

$$\pi^{\mathsf{ev}}\colon \mathcal{V}_k \to \mathcal{V}_k^{\mathsf{ev}}, \qquad \pi^{\mathsf{odd}}\colon \mathcal{V}_k \to \mathcal{V}_k^{\mathsf{odd}}$$
$$f \mapsto \frac{1}{2}(f + f\,|\delta) \qquad f \mapsto \frac{1}{2}(f - f\,|\delta).$$

We further denote

$$f^\bullet := \pi^\bullet(f)$$

for all $f \in \mathcal{V}_k$ and $\bullet \in \{+,-,\mathsf{ev},\mathsf{odd}\}$.

ii) Let k be even. The operators $|\epsilon$ and $|\delta$ on $\mathcal{V}_k$ commute in this case since $\epsilon\delta \equiv \delta\epsilon$ modulo ± 1 (cf. Lemma 3.5). This implies that $|\epsilon$ acts on $\mathcal{V}_k^{\mathsf{ev}}$ and $\mathcal{V}_k^{\mathsf{odd}}$ and $|\delta$ acts on $\mathcal{V}_k^+$ and $\mathcal{V}_k^-$. Hence we obtain a refined decomposition given by

$$\mathcal{V}_k = \mathcal{V}_k^{+,\mathsf{ev}} \oplus \mathcal{V}_k^{+,\mathsf{odd}} \oplus \mathcal{V}_k^{-,\mathsf{ev}} \oplus \mathcal{V}_k^{-,\mathsf{odd}}$$

where

$$\mathcal{V}_k^{+,\mathsf{ev}} := \mathcal{V}_k^+ \cap \mathcal{V}_k^{\mathsf{ev}}, \qquad \mathcal{V}_k^{+,\mathsf{odd}} := \mathcal{V}_k^+ \cap \mathcal{V}_k^{\mathsf{odd}},$$
$$\mathcal{V}_k^{-,\mathsf{ev}} := \mathcal{V}_k^- \cap \mathcal{V}_k^{\mathsf{ev}}, \qquad \mathcal{V}_k^{-,\mathsf{odd}} := \mathcal{V}_k^- \cap \mathcal{V}_k^{\mathsf{odd}}.$$

Note that the projections π^+ and π^- commute pairwise with π^{ev} and π^{odd} for even k. These compositions yield projections onto $\mathcal{V}_k^{+,\mathsf{ev}}$, $\mathcal{V}_k^{+,\mathsf{odd}}$, $\mathcal{V}_k^{-,\mathsf{ev}}$ and $\mathcal{V}_k^{-,\mathsf{odd}}$ which we denote by

$$\pi^{+,\mathsf{ev}} := \pi^+ \circ \pi^{\mathsf{ev}}, \qquad \pi^{+,\mathsf{odd}} := \pi^+ \circ \pi^{\mathsf{odd}},$$
$$\pi^{-,\mathsf{ev}} := \pi^- \circ \pi^{\mathsf{ev}}, \qquad \pi^{-,\mathsf{odd}} := \pi^- \circ \pi^{\mathsf{odd}}.$$

We further denote

$$f^{\bullet,\circ} := \pi^{\bullet,\circ}(f)$$

for all $f \in \mathcal{V}_k$, $\bullet \in \{+,-\}$ and $\circ \in \{\mathsf{ev},\mathsf{odd}\}$.

Remark 3.8. Let $\gamma \in \mathrm{GL}_2(\mathbb{Z})$. Then there are different ways to describe the bi-slash operator from Definition 3.3.

1. We set

$$\bar{\gamma} = \begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-\mathsf{t}} \end{pmatrix}$$

and consider the natural analogue of the slash operator from chapter 2, i. e.

$$\mathrm{GL}_4(\mathbb{Z}) \times \mathcal{V}_k \longrightarrow \mathcal{V}_k$$
$$(\sigma, f) \longmapsto f\left((\sigma \cdot z)^{\mathsf{t}}\right)$$

where $z = (x_1, x_2, y_1, y_2)^{\mathsf{t}}$. Then the bi-slash operator $f\,|\gamma$ is given by $f\left((\bar{\gamma} \cdot z)^{\mathsf{t}}\right)$.

2. Note for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ that

$$\begin{aligned}(x_1', x_2', y_1', y_2') &= \left((\gamma \cdot z_1)^{\mathsf{t}}, (\gamma^{-\mathsf{t}} \cdot z_2)^{\mathsf{t}}\right) \\ &= (ax_1 + bx_2, cx_1 + dx_2, dy_1 - cy_2, ay_2 - by_1)\end{aligned}$$

and

$$\begin{aligned}\begin{pmatrix} x_1' & x_2' \\ -y_2' & y_1' \end{pmatrix} &= \begin{pmatrix} x_1 & x_2 \\ -y_2 & y_1 \end{pmatrix} \cdot \gamma^{\mathsf{t}} \\ &= \begin{pmatrix} ax_1 + bx_2 & cx_1 + dx_2 \\ -(ay_2 - by_1) & dy_1 - cy_2 \end{pmatrix}.\end{aligned}$$

Now set

$$Z := \begin{pmatrix} x_1 & x_2 \\ -y_2 & y_1 \end{pmatrix}$$

and consider $f \in \mathcal{V}_k$ as a function on 2×2-matrices via

$$f \colon Z \longmapsto f(Z) := f(x_1, x_2, y_1, y_2).$$

This yields

$$f\,|\gamma = f\left(Z \cdot \gamma^{\mathsf{t}}\right).$$

Definition 3.9. Let $q \in \mathcal{V}_4$ be the polynomial given by

$$q(Z) = \det(Z) = x_1 y_1 + x_2 y_2.$$

The following observation is now trivial.

Proposition 3.10. *The polynomial q is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$, i. e. for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ we have*

$$q\,|\gamma = q.$$

Definition 3.11. Let $\Gamma \subset \mathrm{GL}_4(\mathbb{Q})$ be the group of isometries of q, i. e.

$$\Gamma := \left\{\gamma \in \mathrm{GL}_4(\mathbb{Q}) \mid q((\gamma \cdot z)^{\mathsf{t}}) = q(z^{\mathsf{t}})\right\}$$

where $z = (x_1, x_2, y_1, y_2)^{\mathsf{t}}$.

Remark 3.12. Since $\mathrm{GL}_4(\mathbb{Q})$ acts naturally on $\mathcal{V}_k$ we obtain an action of Γ on $\mathcal{V}_k$. By Remark 3.8 we also have an embedding $\mathrm{GL}_2(\mathbb{Z}) \hookrightarrow \Gamma$ via

$$\gamma \longmapsto \begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-\mathsf{t}} \end{pmatrix}.$$

Example 3.13. Consider the matrices

$$\tau = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad \nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \in \mathrm{GL}_4(\mathbb{Z}).$$

Since $q((\tau \cdot z)^{\mathrm{t}}) = q(z^{\mathrm{t}})$ and $q((\nu \cdot z)^{\mathrm{t}}) = q(z^{\mathrm{t}})$ for $z = (x_1, x_2, y_1, y_2)^{\mathrm{t}}$ we have $\tau, \nu \in \Gamma$. By using the notation of Remark 3.8, we have for $f \in \mathcal{V}_k$ that

$$f\,|\tau \coloneqq f(x_1, -y_2, y_1, -x_2) = f(Z^{\mathrm{t}})$$

and

$$f\,|\nu \coloneqq f(x_1, -x_2, y_1, -y_2) = f(\delta Z \delta^{\mathrm{t}}).$$

Remark 3.14. In fact, [Sie36, p. 259] shows that the action of Γ is generated by the embedded actions

$$\begin{aligned} &(1, \mathrm{SL}_2(\mathbb{Z})) \hookrightarrow \Gamma, && (1, \gamma) \mapsto \big(f \mapsto f(Z \cdot \gamma^{\mathrm{t}})\big) \\ &(\mathrm{SL}_2(\mathbb{Z}), 1) \hookrightarrow \Gamma, && (\gamma, 1) \mapsto (f \mapsto f(\gamma \cdot Z)) \end{aligned}$$

as well as $f\,|\tau$ and $f\,|\nu$ from Example 3.13.

Remark 3.15. We extend the Γ-action on $\mathcal{V}_k$ to an action of the group ring $\mathbb{Z}[\Gamma]$ on $\mathcal{V}_k$ analogously to Definition 2.12.

We will now define the central objects of this chapter and show two basic results.

Definition 3.16. For $k \geq 1$ the space of *bi-period polynomials* is given by

$$\mathcal{W}_k \coloneqq \{f \in \mathcal{V}_k \mid f\,|1 + S = f\,\big|1 + U + U^2 = 0\}.$$

We further set $W_0 \coloneqq \mathbb{Q}$.

Lemma 3.17. *For odd k we have $\mathcal{W}_k = \{0\}$.*

Proof. Let $f \in \mathcal{V}_k$ with

$$f(x_1, x_2, y_1, y_2) = \sum_{\substack{i,j,m,n \geq 0, \\ i+j+m+n=k-2}} a_{i,j,m,n}\, x_1^i\, x_2^j\, y_1^m\, y_2^n.$$

Fix some $i_1, i_2, j_1, j_2 \geq 0$ with $i_1 + i_2 + j_1 + j_2 = k - 2$. Since $S^{-\mathrm{t}} = S$, the coefficient of $x_1^{i_1} x_2^{i_2} y_1^{j_1} y_2^{j_2}$ in $f\,|S$ is given by $(-1)^{i_2+j_2} a_{i_2,i_1,j_2,j_1}$. Then $f \in \ker(1 + S)$ implies

$$(-1)^{i_2+j_2} a_{i_2,i_1,j_2,j_1} + a_{i_1,i_2,j_1,j_2} = 0 \qquad \text{and} \qquad (-1)^{i_1+j_1} a_{i_1,i_2,j_1,j_2} + a_{i_2,i_1,j_2,j_1} = 0.$$

Since $k - 2$ is odd, we have $(-1)^{i_2+j_2} \neq (-1)^{i_1+j_1}$, hence $f = 0$. $\square$

Recall the spaces $\mathcal{W}_k^+$, $\mathcal{W}_k^-$, $\mathcal{W}_k^{\mathsf{ev}}$ and $\mathcal{W}_k^{\mathsf{odd}}$ from Definition 3.6. Analogously to chapter 2, we also set $\mathcal{W}_0^+ := \{0\}$, $\mathcal{W}_0^- := \mathbb{Q}$, $\mathcal{W}_0^{\mathsf{ev}} := \mathbb{Q}$ and $\mathcal{W}_0^{\mathsf{odd}} := \{0\}$.

Note that the proof of Proposition 2.17 only relies on invariance under $|{-1}$ and matrix identities. Hence we obtain the following lemma by a similar proof.

Lemma 3.18. *For $k \in \mathbb{N}_0$ we have*

$$\mathcal{W}_k^+ = \mathcal{W}_k^{\mathsf{odd}} \quad \text{and} \quad \mathcal{W}_k^- = \mathcal{W}_k^{\mathsf{ev}}.$$

3.2. A special case of Weitzenböck's theorem

Recall the calculation for $\dim(W_k)$ in section 2.4. First, we considered the orthogonal complement of W_k in V_k. We were then able to compute the dimension of that complement since there are no $\mathrm{SL}_2(\mathbb{Z})$-invariant polynomials in V_k. We want to pursue the same approach to compute the dimension of $\mathcal{W}_k$ in the next section. However, we already saw that $q = x_1y_1 + x_2y_2$ is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$. The goal of this section is therefore to describe the space

$$\mathcal{E}_k := \big\{ f \in \mathcal{V}_k \;\big|\; f = f\,|\gamma \;\text{ for all } \gamma \in \mathrm{SL}_2(\mathbb{Z}) \big\}.$$

Remark 3.19. Since the group $\mathrm{SL}_2(\mathbb{Z})$ is generated by S and T, the space $\mathcal{E}_k$ consists of all polynomials that are invariant under both $|S$ and $|T$, i. e.

$$\mathcal{E}_k = \ker(1 - S) \cap \ker(1 - T).$$

Note that the space $\ker(1 - S)$ is invariant under a finite group. So we can compute its dimension via Molien's theorem A.12. The group generated by T, however, is infinite. Our first goal will therefore be to describe $\ker(1 - T)$ more explicitly.

Remark 3.20. For $\lambda \in \mathbb{Z}$ we have

$$T^\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}.$$

We thus obtain a group homomorphism

$$\begin{aligned} (\mathbb{Z}, +) &\longrightarrow (\mathrm{SL}_2(\mathbb{Z}), \cdot) \\ \lambda &\longmapsto T^\lambda \end{aligned}$$

and note for $f \in \mathbb{Q}[x_1, x_2, y_1, y_2]$ that

$$f\,|T^\lambda = f(x_1 + \lambda x_2, x_2, y_1, y_2 - \lambda y_1).$$

So by considering $\lambda \in \mathbb{C}$ and $f \in \mathbb{C}[x_1, x_2, y_1, y_2]$ instead, we can define an action of the additive group $\mathbb{G}_a(\mathbb{C})$ on $\mathcal{V} \otimes \mathbb{C} = \mathbb{C}[x_1, x_2, y_1, y_2]$ via

$$\lambda * f(x_1, x_2, y_1, y_2) := f(x_1 + \lambda x_2, x_2, y_1, y_2 - \lambda y_1).$$

We further observe that $q = x_1y_1 + x_2y_2$ is invariant under this action since

$$\begin{aligned} q(x_1+\lambda x_2, x_2, y_1, y_2-\lambda y_1) &= (x_1+\lambda x_2)y_1 + x_2(y_2-\lambda y_1) \\ &= q(x_1,x_2,y_1,y_2) + \lambda(x_2y_1 - x_2y_1) = q(x_1,x_2,y_1,y_2). \end{aligned}$$

A theorem by Weitzenböck [Wei32] states that the algebra of $\mathbb{G}_a(\mathbb{C})$-invariant polynomials in n variables is finitely generated. The following theorem is a special case of this.

Theorem 3.21. *We have*

$$\mathbb{C}[x_1,x_2,y_1,y_2]^{\mathbb{G}_a(\mathbb{C})} = \mathbb{C}[x_1y_1+x_2y_2, x_2, y_1].$$

Proof. This proof follows [FR17, Example 7.4.10]. Since the polynomials $x_1y_1 + x_2y_2$, x_2 and y_1 are invariant under $\mathbb{G}_a(\mathbb{C})$, we immediately obtain that

$$\mathbb{C}[x_1y_1+x_2y_2, x_2, y_1] \subseteq \mathbb{C}[x_1,x_2,y_1,y_2]^{\mathbb{G}_a(\mathbb{C})}.$$

To show the converse inclusion, let

$$f(x_1,x_2,y_1,y_2) = \sum_{\substack{i_1,i_2,j_1,j_2\geq 0,\\ i_1+i_2+j_1+j_2=k-2}} a_{i_1,i_2,j_1,j_2}\, x_1^{i_1}x_2^{i_2}y_1^{j_1}y_2^{j_2} \in \mathcal{V}_k$$

with $f = \lambda * f$, i. e.

$$f(x_1,x_2,y_1,y_2) = f(x_1+\lambda x_2, x_2, y_1, y_2-\lambda y_1).$$

Differentiating this equation w. r. t. λ and evaluating at $\lambda = 0$ yields due to the product rule

$$\begin{aligned} 0 &= \sum_{\substack{i_1,i_2,j_1,j_2\geq 0,\\ i_1+i_2+j_1+j_2=k-2}} \frac{\mathrm{d}}{\mathrm{d}\lambda} a_{i_1,i_2,j_1,j_2}\, (x_1+\lambda x_2)^{i_1}x_2^{i_2}y_1^{j_1}(y_2-\lambda y_1)^{j_2}\Big|_{\lambda=0} \\ &= \sum_{\substack{i_1,i_2,j_1,j_2\geq 0,\\ i_1+i_2+j_1+j_2=k-2}} a_{i_1,i_2,j_1,j_2}\left(i_1\cdot x_1^{i_1-1}x_2^{i_2+1}y_1^{j_1}y_2^{j_2} - j_2\cdot x_1^{i_1}x_2^{i_2}y_1^{j_1+1}y_2^{j_2-1}\right) \\ &= x_2\cdot\frac{\partial f}{\partial x_1} - y_1\cdot\frac{\partial f}{\partial y_2}. \end{aligned}$$

Now consider the substitutions

$$\begin{aligned} X_1 &= x_1y_1+x_2y_2, & X_2 &= x_2, & Y_1 &= y_1, & Y_2 &= y_2 \\ X_1' &= x_1, & X_2' &= x_2, & Y_1' &= y_1, & Y_2' &= x_1y_1+x_2y_2 \end{aligned}$$

and set

$$\begin{aligned} p_1(X_1,X_2,Y_1,Y_2,Y_1^{-1}) &\coloneqq f\left(\frac{X_1-X_2Y_2}{Y_1}, X_2, Y_1, Y_2\right) \in \mathbb{C}[X_1,X_2,Y_1,Y_2,Y_1^{-1}] \\ p_2(X_1',X_2',Y_1',Y_2',X_2'^{-1}) &\coloneqq f\left(X_1',X_2',Y_1',\frac{Y_2'-X_1'Y_1'}{X_2'}\right) \in \mathbb{C}[X_1',X_2',Y_1',Y_2',X_2'^{-1}]. \end{aligned}$$

Note that $\frac{X_1 - X_2 Y_2}{Y_1} = x_1$ and $\frac{Y_2' - X_1' Y_1'}{X_2'} = y_2$. So after resubstitution we have $p_1, p_2 \in \mathbb{C}[x_1, x_2, y_1, y_2]$ with

$$p_1(x_1, x_2, y_1, y_2) = f(x_1, x_2, y_1, y_2) = p_2(x_1, x_2, y_1, y_2). \tag{3.1}$$

The product rule now implies that

$$\begin{aligned}\frac{\partial p_1}{\partial Y_2} &= \sum_{\substack{i_1, i_2, j_1, j_2 \geq 0, \\ i_1 + i_2 + j_1 + j_2 = k-2}} \frac{\partial}{\partial Y_2} a_{i_1, i_2, j_1, j_2} \left(\frac{X_1 - X_2 Y_2}{Y_1} \right)^{i_1} X_2^{i_2} Y_1^{j_1} Y_2^{j_2} \\ &= - \sum_{\substack{i_1, i_2, j_1, j_2 \geq 0, \\ i_1 + i_2 + j_1 + j_2 = k-2}} a_{i_1, i_2, j_1, j_2}\, i_1 \cdot \left(\frac{X_1 - X_2 Y_2}{Y_1} \right)^{i_1 - 1} X_2^{i_2 + 1} Y_1^{j_1 - 1} Y_2^{j_2} \\ &\quad + \sum_{\substack{i_1, i_2, j_1, j_2 \geq 0, \\ i_1 + i_2 + j_1 + j_2 = k-2}} a_{i_1, i_2, j_1, j_2}\, j_2 \cdot \left(\frac{X_1 - X_2 Y_2}{Y_1} \right)^{i_1} X_2^{i_2} Y_1^{j_1} Y_2^{j_2 - 1}.\end{aligned}$$

By resubstituting we see that

$$\frac{\partial p_1}{\partial Y_2} = -\frac{1}{y_1} \left(x_2 \frac{\partial f}{x_1} - y_1 \frac{\partial f}{\partial y_2} \right) = 0.$$

Similarly, we have

$$\frac{\partial p_2}{\partial X_1'} = \frac{1}{x_2} \left(x_2 \frac{\partial f}{\partial x_1} - y_1 \frac{\partial f}{\partial y_2} \right) = 0.$$

This shows $p_1 \in \mathbb{C}[X_1, X_2, Y_1, Y_1^{-1}]$ and $p_2 \in \mathbb{C}[X_2', Y_1', Y_2', X_2'^{-1}]$. Since p_1 and p_2 consist of finitely many monomials, we find minimal numbers $n, m \in \mathbb{N}_0$ such that

$$\begin{aligned} q_1(X_1, X_2, Y_1) &:= Y_1^n \cdot p_1(X_1, X_2, Y_1, Y_1^{-1}) \in \mathbb{C}[X_1, X_2, Y_1] \\ q_2(X_2', Y_1', Y_2') &:= (X_2')^m \cdot p_2(X_2', Y_1', Y_2', X_2'^{-1}) \in \mathbb{C}[X_2', Y_1', Y_2']. \end{aligned}$$

After resubstituting we have $p_1 = p_2$ by (3.1) and hence

$$x_2^m \cdot q_1(x_1 y_1 + x_2 y_2, x_2, y_1) = y_1^n \cdot q_2(x_2, y_1, x_1 y_1 + x_2 y_2).$$

Now assume $n > 0$. We then obtain for $y_1 = 0$ that $q_1(x_2 y_2, x_2, 0) = 0$. In particular, we have $q_1(y_2, 1, 0) = 0$ for $x_2 = 1$. This implies that y_1 still divides q_1, i. e. we find some $\tilde{q}_1 \in \mathbb{C}[x_1 y_1 + x_2 y_2, x_2, y_1]$ with $y_1 \cdot \tilde{q}_1 = q_1$. But then

$$\tilde{q}_1 = y_1^{n-1} \cdot p_1$$

contradicts the minimality of n. Hence $n = 0$ and $f \in \mathbb{C}[x_1 y_1 + x_2 y_2, x_2, y_1]$. □

Corollary 3.22. Let $k \geq 4$ be even. Then the space $\mathcal{E}_k$ is spanned by a power of q, i. e.

$$\mathcal{E}_k = \left\langle q^{\frac{k-2}{2}} \right\rangle_{\mathbb{Q}}$$

where $q = x_1 y_1 + x_2 y_2$ and the generating series of $\mathcal{E}_k$ is given by

$$\sum_{k=0}^{\infty} \dim \mathcal{E}_k\, x^k = \frac{x^2}{1 - x^2}.$$

Proof. The inclusion $\left\langle q^{\frac{k-2}{2}} \right\rangle_{\mathbb{Q}} \subseteq \mathcal{E}_k$ is clear.

For the converse inclusion let $f \in \mathcal{E}_k$. By Remark 3.19 we have $f \in \mathcal{E}_k$ if and only if $f \in \ker(1-S)$ and $f \in \ker(1-T)$. Theorem 3.21 implies for the latter condition that $f \in \mathbb{Q}[x_1y_1 + x_2y_2, x_2, y_1]$. But since

$$(x_2)\,|S = x_1 \qquad\qquad (y_1)\,|S = -y_2$$

the invariance under $|S$ implies $f \in \mathbb{Q}[x_1y_1+x_2y_2]$. The generating series follows immediately from the fact that $\mathcal{E}_k$ is one-dimensional for even k and trivial for odd k. □

3.3. An invariant pairing

As we want to use the same approach to compute the dimension of $\mathcal{W}_k$ as we did for W_k in section 2.4, we need a pairing on $\mathcal{V}_k$ that is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$ in order to describe the orthogonal complement of $\mathcal{W}_k$.

Definition 3.23. For $(i_1, i_2, j_1, j_2), (m_1, m_2, n_1, n_2) \in \mathbb{N}_0^4$ with we set

$$\left\langle x_1^{i_1} x_2^{i_2} y_1^{j_1} y_2^{j_2}, x_1^{m_1} x_2^{m_2} y_1^{n_1} y_2^{n_2} \right\rangle := \frac{(-1)^{i_1+j_1}}{\binom{k-2}{i_1,i_2,j_1,j_2}} \delta_{(i_1,i_2,j_1,j_2),(m_2,m_1,n_2,n_1)}$$

and linearly extend this to a pairing $\langle \cdot, \cdot \rangle$ on $\mathcal{V}$.

Definition 3.24. For $p, q \in \mathcal{V}$ we set $\partial(p) := p\left(-\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1}, -\frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_1}\right)$ and

$$\langle p, q \rangle_\partial := \frac{1}{(k-2)!}\big(\partial(p)\big)(q)(0).$$

Lemma 3.25. *The pairing $\langle \cdot, \cdot \rangle_\partial$ is a non-degenerate bilinear form on $\mathcal{V}$.*

Proof. Since ∂ is essentially a substitution of variables, we immediately obtain that ∂ is a linear map on $\mathcal{V}_k$. Hence $\langle \cdot, \cdot \rangle_\partial$ is linear in the first argument. So without loss of generality let $p = a \cdot x_1^{i_1} x_2^{i_2} y_1^{j_1} y_2^{j_2}$ for some $a \in \mathbb{Q}$ and $i_1, i_2, j_1, j_2 \in \mathbb{N}_0$. Then for a polynomial $r \in \mathcal{V}$ of the form

$$r = \sum_{i,j,m,n} b_{i,j,m,n} \cdot x_1^i x_2^j y_1^m y_2^n$$

we have

$$\langle p, r \rangle_\partial = \frac{(-1)^{i_1+j_1}}{\binom{k-2}{i_1,i_2,j_1,j_2}} \cdot a \cdot b_{i_2,i_1,j_2,j_1}.$$

Hence $\langle \cdot, \cdot \rangle_\partial$ is also linear in the second argument. So $\langle \cdot, \cdot \rangle_\partial$ is a bilinear form.

The form is non-degenerate since for any non-zero $P \in \mathcal{V}$ let p as above denote one of its monomials, then for $p_0 = x_1^{i_2} x_2^{i_1} y_1^{j_2} y_2^{j_1}$ we have

$$\langle P, p_0 \rangle_\partial = \frac{(-1)^{i_1+j_1}}{\binom{k-2}{i_1,i_2,j_1,j_2}} \cdot a \neq 0.$$ □

Similarly to Proposition 2.26 we have the following lemma.

Lemma 3.26. *The pairings on $\mathcal{V}_k$ agree, i. e. we have $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_\partial$.*

Proof. By linearity of both pairings it suffices to prove the claim for monomials $f \in \mathcal{V}_{k_1}$ and $g \in \mathcal{V}_{k_2}$. If $k_1 \neq k_2$ then both pairings vanish. So assume without loss of generality that $f(x_1,x_2,y_1,y_2) = x_1^{i_1}x_2^{i_2}y_1^{j_1}y_2^{j_2}$ and $g(x_1,x_2,y_1,y_2) = x_1^{m_1}x_2^{m_2}y_1^{n_1}y_2^{n_2}$ with $i_1+i_2+j_1+j_2 = k-2$ and $m_1+m_2+n_1+n_2 = k-2$ for some $k \geq 2$. Then

$$\begin{aligned}\langle f,g\rangle_\partial &= \frac{(-1)^{i_1+j_1}}{(k-2)!}\left(\frac{\partial^{i_2}}{\partial x_1^{i_2}}\right)\left(\frac{\partial^{i_1}}{\partial x_2^{i_1}}\right)\left(\frac{\partial^{n_2}}{\partial y_1^{n_2}}\right)\left(\frac{\partial^{n_1}}{\partial y_2^{n_1}}\right)(x_1^{m_1}x_2^{m_2}y_1^{n_1}y_2^{n_2})\\ &= \frac{(-1)^{i_1+j_1}}{(k-2)!}(i_1)!\,(i_2)!\,(j_1)!\,(j_2)!\,\delta_{(i_1,i_2,j_1,j_2),(m_2,m_1,n_2,n_1)}\\ &= \frac{(-1)^{i_1+j_1}}{\binom{k-2}{i_1,i_2,j_1,j_2}}\delta_{(i_1,i_2,j_1,j_2),(m_2,m_1,n_2,n_1)} = \langle f,g\rangle\,.\end{aligned}$$

□

Theorem 3.27. *The pairing $\langle\cdot,\cdot\rangle$ is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$, i. e. for all $f,g \in \mathcal{V}_k$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ we have*

$$\langle f\,|\gamma\,, g\,|\gamma\rangle = \langle f,g\rangle\,. \tag{3.2}$$

Proof. By linearity of the pairing it suffices to prove the claim for monomials $f,g \in \mathcal{V}_k$. So let $f(x_1,x_2,y_1,y_2) = x_1^{i_1}x_2^{i_2}y_1^{j_1}y_2^{j_2}$ and $g(x_1,x_2,y_1,y_2) = x_1^{m_1}x_2^{m_2}y_1^{n_1}y_2^{n_2}$. Note that (3.2) is equivalent to

$$\langle f\,|\gamma\,, g\rangle = \left\langle f, g\,\middle|\gamma^{-1}\right\rangle$$

since this implies

$$\langle f\,|\gamma\,, g\,|\gamma\rangle = \left\langle f, g\,\middle|\gamma\cdot\gamma^{-1}\right\rangle = \langle f,g\rangle\,.$$

Since the group $\mathrm{SL}_2(\mathbb{Z})$ is generated by S and T, it suffices to show that

$$\langle f\,|S\,, g\rangle = \left\langle f, g\,\middle|S^{-1}\right\rangle \tag{3.3}$$

$$\langle f\,|T\,, g\rangle = \left\langle f, g\,\middle|T^{-1}\right\rangle. \tag{3.4}$$

We have for the left-hand side of (3.3) that

$$\langle f\,|S\,, g\rangle = \frac{(-1)^{k-2}}{\binom{k-2}{i_1,i_2,j_1,j_2}}\delta_{(i_2,i_1,j_2,j_1),(m_2,m_1,n_2,n_1)}$$

and for the right-hand side that

$$\left\langle f, g\,\middle|S^{-1}\right\rangle = \frac{(-1)^{i_1+j_1+m_2+n_2}}{\binom{k-2}{i_1,i_2,j_1,j_2}}\delta_{(i_1,i_2,j_1,j_2),(m_1,m_2,n_1,n_2)}.$$

So both sides vanish unless $f = g$ in which case we have $i_1+j_1+m_2+n_2 = k-2$.

To compute both sides of (3.4) we first note that

$$\begin{aligned} f\,|T &= f(x_1+x_2, x_2, y_1, y_2-y_1) \\ &= \left(\sum_{l_1=0}^{i_1}\binom{i_1}{l_1}x_1^{i_1-l_1}x_2^{i_2+l_1}\right)\cdot\left(\sum_{l_2=0}^{j_2}(-1)^{l_2}\binom{j_2}{l_2}y_1^{j_1+l_2}y_2^{j_2-l_2}\right) \end{aligned}$$

and

$$\begin{aligned} g\,|T^{-1} &= g(x_1-x_2, x_2, y_1, y_1+y_2) \\ &= \left(\sum_{l_1=0}^{m_1}(-1)^{l_1}\binom{m_1}{l_1}x_1^{m_1-l_1}x_2^{m_2+l_2}\right)\cdot\left(\sum_{l_2=0}^{n_2}\binom{n_2}{l_2}y_1^{n_1+l_2}y_2^{n_2-l_2}\right). \end{aligned}$$

So we obtain that the left-hand side of (3.4) is

$$\langle f\,|T\,, g\rangle = \frac{(-1)^{j_1+m_2}}{\binom{k-2}{m_1,m_2,n_1,n_2}}\binom{i_1}{m_2}\binom{j_2}{n_1}\delta_{(i_1+i_2,j_1+j_2),(m_1+m_2,n_1+n_2)}$$

while the right-hand side yields

$$\langle f, g\,|T^{-1}\rangle = \frac{(-1)^{i_1+i_2+j_1+m_1}}{\binom{k-2}{i_1,i_2,j_1,j_2}}\binom{m_1}{i_2}\binom{n_2}{j_1}\delta_{(i_1+i_2,j_1+j_2),(m_1+m_2,n_1+n_2)}.$$

So both sides vanish unless $i_1+i_2=m_1+m_2$ and $j_1+j_2=n_1+n_2$. In this case, $i_1 \geq m_2$ is equivalent to $m_1 \geq i_2$ and $j_2 \geq n_1$ is equivalent to $n_2 \geq j_1$. Hence the binomial coefficients on both sides vanish simultaneously. The conditions further imply that

$$(-1)^{i_1+i_2+j_1+m_1} = (-1)^{j_1+m_2}.$$

The equality of both sides of (3.4) then follows from

$$\begin{aligned} \binom{i_1}{m_2}\binom{j_2}{n_1}(m_1)!\,(m_2)!\,(n_1)!\,(n_2)! &= \frac{(i_1)!\,(j_2)!\,(m_1)!\,(m_2)!\,(n_1)!\,(n_2)!}{(i_1-m_2)!\,(m_2)!\,(j_2-n_1)!\,(n_1)!} \\ &= \frac{(i_1)!\,(j_2)!\,(m_1)!\,(n_2)!}{(m_1-i_2)!\,(n_2-j_1)!} \\ &= \binom{m_1}{i_2}\binom{n_2}{j_1}(i_1)!\,(i_2)!\,(j_1)!\,(j_2)!\,. \end{aligned}$$

□

Remark 3.28. Similar to the second part of Proposition 2.27 it follows analogously that the pairing $\langle\cdot,\cdot\rangle$ is invariant under the action of $\mathrm{GL}_2(\mathbb{Z})$ if k is even.

Remark 3.29. The pairing $\langle\cdot,\cdot\rangle$ is not invariant under the action of Γ. E. g., consider $f, g \in \mathcal{V}_8$ with $f(x_1,x_2,y_1,y_2) = x_1x_2^2y_1y_2^2$ and $g(x_1,x_2,y_1,y_2) = x_1^2x_2y_1^2y_2$. We then have

$$\langle f(Z), g(Z)\rangle = \frac{1}{180} \neq \frac{8}{45} = \langle f(T\cdot Z), g(T\cdot Z)\rangle$$

where $f(T\cdot Z) = f((\gamma\cdot z)^{\mathsf{t}})$ with

$$\gamma = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma.$$

3.4. Dimensions

Now that we have established an invariant pairing by Theorem 3.27, we are able to compute the dimension of $\mathcal{W}_k$ by considering similar decompositions as we did for W_k in section 2.4. We recall that $\mathcal{W}_0 = \mathbb{Q}$ (see Definition 3.16) and that $\mathcal{W}_0^+ = \{0\}$ and $\mathcal{W}_0^- = \mathbb{Q}$. We begin with a definition that will be convenient for denoting the respective dimensions.

Definition 3.30. For $\lambda \in \mathbb{R}$ we set

$$[\lambda] := \begin{cases} \lfloor \lambda \rfloor, & \text{if } \lambda + \frac{1}{2} < \lceil \lambda \rceil \\ \lceil \lambda \rceil, & \text{if } \lambda + \frac{1}{2} \geq \lceil \lambda \rceil. \end{cases}$$

This definition essentially rounds a number $\lambda \in \mathbb{R}$ to its closest integer with the convention that $\left[\frac{1}{2}\right] = 1$.

Theorem 3.31.

i) Let $k \geq 4$ be even. The dimension of $\mathcal{W}_k$ is explicitly given by

$$\dim \mathcal{W}_k = \frac{1}{6}\binom{k+1}{3} + (-1)^{\frac{k}{2}}\frac{k}{4} + \frac{2}{3}\left(\varrho_k \cdot k - \left[\frac{k}{3}\right]\right) + 1$$

where

$$\varrho_k = \begin{cases} 1, & \text{if } k \equiv 0 \mod 3 \\ 0, & \text{else.} \end{cases}$$

ii) The generating series of $\mathcal{W}_k$ is given by

$$\sum_{k=0}^{\infty} \dim \mathcal{W}_k \, x^k = \frac{1 + x^4 + 6\,x^6 + 10\,x^8 + 6\,x^{10} + 15\,x^{12} + 10\,x^{14} + x^{16} - 2\,x^{18}}{(1-x^4)^2(1-x^6)^2}.$$

Proof. For even $k \geq 4$ the space V_k is invariant under $|{-1}$ (Lemma 3.5). The operators $|S$ and $|U$ thus have order 2 and 3, respectively. Hence analogously to Remark 2.10 we obtain the decompositions

$$\mathcal{V}_k = \mathcal{A}_k \oplus \mathcal{B}_k \quad \text{and} \quad \mathcal{V}_k = \mathcal{C}_k \oplus \mathcal{D}_k \tag{3.5}$$

where

$$\begin{aligned} \mathcal{A}_k &= \ker(1 - S), & \mathcal{B}_k &= \ker(1 + S), \\ \mathcal{C}_k &= \ker(1 - U), & \mathcal{D}_k &= \ker(1 + U + U^2). \end{aligned}$$

Since the pairing $\langle \cdot, \cdot \rangle$ on $\mathcal{V}_k$ is invariant under $\mathrm{SL}_2(\mathbb{Z})$ by Proposition 3.27 we obtain that the decompositions (3.5) are in fact orthogonal (cf. Corollary 2.28), i. e.

$$\mathcal{A}_k \perp \mathcal{B}_k = \mathcal{V}_k = \mathcal{C}_k \perp \mathcal{D}_k. \tag{3.6}$$

Now $\mathcal{W}_k = \mathcal{B}_k \cap \mathcal{D}_k$ and (3.6) imply that $\mathcal{W}_k = (\mathcal{A}_k + \mathcal{C}_k)^\perp$. With $\mathcal{E}_k = \mathcal{A}_k \cap \mathcal{C}_k$ as before, we obtain

$$\mathcal{V}_k = \mathcal{W}_k \oplus \mathcal{A}_k/\mathcal{E}_k \oplus \mathcal{C}_k/\mathcal{E}_k \oplus \mathcal{E}_k. \tag{3.7}$$

Since $\mathcal{E}_k$ is 1-dimensional by Corollary 3.22 this yields dimension-wise that

$$\dim \mathcal{W}_k = \dim \mathcal{V}_k - \dim \mathcal{A}_k - \dim \mathcal{C}_k + 1. \tag{3.8}$$

Since $\bar{S}$ and $\bar{U}$ generate finite groups (cf. Remark 3.8 for the notation), we can compute the dimensions of $\mathcal{A}_k$ and $\mathcal{C}_k$ via Molien's theorem A.12. This yields

$$\begin{aligned}\dim(\mathcal{A}_k) &= \frac{1}{2}\left(\binom{k+1}{3} + (-1)^{\frac{k}{2}+1}\cdot\frac{k}{2}\right)\\ \dim(\mathcal{C}_k) &= \frac{1}{3}\left(\binom{k+1}{3} + 2\left[\frac{k}{3}\right] - \varrho_k\cdot 2k\right)\end{aligned}$$

where the generating series are given by

$$\begin{aligned}\sum_{k=0}^{\infty}\dim \mathcal{A}_k\, x^k &= \frac{x^2\left(1+2\,x^2+10\,x^4+2\,x^6+x^8\right)}{(1-x^4)^2(1-x^2)^2}\\ \sum_{k=0}^{\infty}\dim \mathcal{C}_k\, x^k &= \frac{x^2\left(1+2\,x^2+2\,x^4+14\,x^6+2\,x^8+2\,x^{10}+x^{12}\right)}{(1-x^6)^2(1-x^2)^2}.\end{aligned}$$

Recall that $\dim(\mathcal{V}_k) = \binom{k+1}{3}$ by Lemma 3.2. We now obtain i) from (3.8) and the respective dimensions above as this implies

$$\dim \mathcal{W}_k = \frac{1}{6}\binom{k+1}{3} + (-1)^{\frac{k}{2}}\frac{k}{4} + \frac{2}{3}\left(\varrho_k\cdot k - \left[\frac{k}{3}\right]\right) + 1.$$

To show ii) we also use (3.8). Recall from the proof of Lemma 3.2 that the generating series of $\mathcal{V}_k$ is given by

$$\sum_{k=0}^{\infty}\dim \mathcal{V}_k\, x^k = \frac{x^2}{(1-x)^4}.$$

Note that $\mathcal{V}_k$ is non-trivial for odd k. However, since (3.8) only holds for even k, we first observe that

$$\sum_{\substack{k=0\\ k \text{ even}}}^{\infty}\dim \mathcal{V}_k\, x^k = \frac{x^2\left(1+6\,x^2+x^4\right)}{(1-x^2)^4}.$$

By Corollary 3.22 we have

$$\sum_{k=0}^{\infty}\dim \mathcal{E}_k\, x^k = \frac{x^2}{1-x^2}.$$

Note that (3.8) also holds for $k=2$ since $\mathcal{W}_2$ is trivial and the right-hand side yields in fact 0 as all spaces are 1-dimensional is this case. So by adjusting for $\dim \mathcal{W}_0 = 1$ the claim

Table 2: Dimensions of $\mathcal{W}_k$, $\mathcal{W}_k^{\pm}$ and $\mathcal{L}_k$ for $k \in \{3, \dots, 14\}$.[3]

k	3	4	5	6	7	8	9	10	11	12	13	14
$\dim \mathcal{W}_k$	0	3	0	8	0	15	0	24	0	57	0	70
$\dim \mathcal{W}_k^+$	0	1	0	3	0	6	0	10	0	26	0	32
$\dim \mathcal{W}_k^-$	0	2	0	5	0	9	0	14	0	31	0	38
$\dim \mathcal{L}_k$	2	3	6	8	12	15	20	24	30	57	42	70

follows from

$$\begin{aligned}\sum_{k=0}^{\infty} \dim \mathcal{W}_k\, x^k &= \frac{x^2\left(1+6\,x^2+x^4\right)}{(1-x^2)^4} - \frac{x^2\left(1+2\,x^2+10\,x^4+2\,x^6+x^8\right)}{(1-x^4)^2(1-x^2)^2} \\ &\quad - \frac{x^2\left(1+2\,x^2+2\,x^4+14\,x^6+2\,x^8+2\,x^{10}+x^{12}\right)}{(1-x^6)^2(1-x^2)^2} + \frac{x^2}{1-x^2} + 1 \\ &= \frac{1+x^4+6\,x^6+10\,x^8+6\,x^{10}+15\,x^{12}+10\,x^{14}+x^{16}-2\,x^{18}}{(1-x^4)^2(1-x^6)^2}. \end{aligned}$$ □

Proposition 3.32.

i) Let $k \geq 4$ be even. The dimension of $\mathcal{W}_k^+$ is explicitly given by

$$\dim \mathcal{W}_k^+ = \binom{k+1}{3}\left(\frac{k-4}{4(k-1)} - \frac{1}{6}\right) + \frac{k}{2}\left(1 + \left\lfloor \frac{k}{4} \right\rfloor\right) + \frac{2}{3}\left[\frac{k}{3}\right] - \varrho_k^+ + 1$$

where

$$\varrho_k^+ = \begin{cases} \frac{k}{2}, & \text{if } k \equiv 0 \mod 3 \\ 5 \cdot \left[\frac{k}{6}\right] - 2, & \text{if } k \equiv 1 \mod 3 \\ 5 \cdot \left[\frac{k}{6}\right] + 2, & \text{if } k \equiv 2 \mod 3. \end{cases}$$

ii) The generating series of $\mathcal{W}_k^+$ is given by

$$\sum_{k=0}^{\infty} \dim \mathcal{W}_k^+\, x^k = \frac{x^4\left(1+3\,x^2+4\,x^4+2\,x^6+9\,x^8+7\,x^{10}+x^{12}-2\,x^{14}-x^{16}\right)}{(1-x^4)^2(1-x^6)^2}.$$

Proof. Recall from Remark 3.7 that we have decompositions

$$\mathcal{V}_k = \mathcal{V}_k^+ \oplus \mathcal{V}_k^- \quad \text{and} \quad \mathcal{V}_k = \mathcal{V}_k^{\mathsf{ev}} \oplus \mathcal{V}_k^{\mathsf{odd}}.$$

The operator $|\epsilon$ acts on the spaces $\mathcal{W}_k$, $\mathcal{A}_k$, $\mathcal{C}_k$ and $\mathcal{E}_k$ since the identities $\epsilon S \equiv S\epsilon$, $\epsilon U \equiv U^2\epsilon$ and $\epsilon U^2 \equiv U\epsilon$ hold modulo ± 1. Since ϵ is diagonalizable, we obtain splittings in

[3]The source code that was used to compute these dimensions can be found in appendix C.2.

the eigenspaces with eigenvalues 1 and -1, respectively. Corollary 3.22 yields $\mathcal{E}_k^+ = \mathcal{E}_k$ and $\mathcal{E}_k^- = \{0\}$. We thus obtain by (3.7) that

$$\mathcal{V}_k^+ = \mathcal{W}_k^+ \oplus \mathcal{A}_k^+/\mathcal{E}_k \oplus \mathcal{C}_k^+/\mathcal{E}_k \oplus \mathcal{E}_k.$$

Since $\dim \mathcal{E}_k = 1$ this yields dimension-wise that

$$\dim \mathcal{W}_k^+ = \dim \mathcal{V}_k^+ - \dim \mathcal{A}_k^+ - \dim \mathcal{C}_k^+ + 1. \tag{3.9}$$

We can compute the generating series of $\mathcal{V}_k^+$, $\mathcal{A}_k^+$ and $\mathcal{C}_k^+$ with Molien's theorem A.12 since the groups generated by $\{\bar{\epsilon}\}$, $\{\bar{\epsilon}, \bar{S}\}$ and $\{\bar{\epsilon}, \bar{U}\}$ are finite, respectively. This yields for even $k \geq 4$ that

$$\begin{aligned}
\dim \mathcal{V}_k^+ &= \frac{1}{2}\left(\binom{k+1}{3} + \frac{k}{2}\right)\\
\dim \mathcal{A}_k^+ &= \frac{1}{2}\left(\frac{k(\frac{k}{2}+1)(k+1)}{6} - k\left\lfloor\frac{k}{4}\right\rfloor\right)\\
\dim \mathcal{C}_k^+ &= \frac{1}{6}\left(\binom{k+1}{3} - 4\left[\frac{k}{3}\right] - \frac{3}{2}k\right) + \varrho_k^+
\end{aligned}$$

where the generating series are given by

$$\begin{aligned}
\sum_{k=0}^{\infty} \dim \mathcal{V}_k^+\, x^k &= \frac{x^2\left(1+x^2\right)}{(1-x^2)^2(1-x)^2}\\
\sum_{k=0}^{\infty} \dim \mathcal{A}_k^+\, x^k &= \frac{x^2\left(1+x^2+4\,x^4+x^6+x^8\right)}{(1-x^4)^2(1-x^2)^2}\\
\sum_{k=0}^{\infty} \dim \mathcal{C}_k^+\, x^k &= \frac{x^2\left(1+x^2+x^4+6\,x^6+x^8+x^{10}+x^{12}\right)}{(1-x^6)^2(1-x^2)^2}.
\end{aligned}$$

Now i) follows from (3.9) and the respective dimensions since

$$\frac{1}{2}\left(\binom{k+1}{3} - \frac{k(\frac{k}{2}+1)(k+1)}{6}\right) = \binom{k+1}{3}\frac{k-4}{4(k-1)}.$$

For ii) we first note that $\mathcal{V}_k^+$ is non-trivial for odd k. So we first observe that

$$\sum_{\substack{k=0\\ k \text{ even}}}^{\infty} \dim \mathcal{V}_k^+\, x^k = \frac{x^2\left(1+x^2\right)^2}{(1-x^2)^4}.$$

By using the generating series of $\mathcal{E}_k$ from Corollary 3.22 we thus obtain from (3.9) that

$$\begin{aligned}
\sum_{k=0}^{\infty} \dim \mathcal{W}_k^+\, x^k &= \frac{x^2\left(1+x^2\right)^2}{(1-x^2)^4} - \frac{x^2\left(1+x^2+4\,x^4+x^6+x^8\right)}{(1-x^4)^2(1-x^2)^2}\\
&\quad - \frac{x^2\left(1+x^2+x^4+6\,x^6+x^8+x^{10}+x^{12}\right)}{(1-x^6)^2(1-x^2)^2} + \frac{x^2}{1-x^2}\\
&= \frac{x^4\left(1+3\,x^2+4\,x^4+2\,x^6+9\,x^8+7\,x^{10}+x^{12}-2\,x^{14}-x^{16}\right)}{(1-x^4)^2(1-x^6)^2}. \qquad \square
\end{aligned}$$

Corollary 3.33.

i) Let $k \geq 4$ be even. The dimension of $\mathcal{W}_k^-$ is explicitly given by

$$\dim \mathcal{W}_k^- = \binom{k+1}{3}\left(\frac{1}{3} - \frac{k-4}{4(k-1)}\right) - \frac{k}{4}\left(2\left[\frac{k}{4}\right]+1\right) - \frac{4}{3}\left[\frac{k}{3}\right] + \varrho_k \frac{2k}{3} + \varrho_k^+$$

where

$$\varrho_k = \begin{cases} 1, & \text{if } k \equiv 0 \mod 3 \\ 0, & \text{else} \end{cases}$$

and

$$\varrho_k^+ = \begin{cases} \frac{k}{2}, & \text{if } k \equiv 0 \mod 3 \\ 5 \cdot \left[\frac{k}{6}\right] - 2, & \text{if } k \equiv 1 \mod 3 \\ 5 \cdot \left[\frac{k}{6}\right] + 2, & \text{if } k \equiv 2 \mod 3. \end{cases}$$

ii) The generating series of $\mathcal{W}_k^-$ is given by

$$\sum_{k=0}^{\infty} \dim \mathcal{W}_k^- \, x^k = \frac{1 + 3\,x^6 + 6\,x^8 + 4\,x^{10} + 6\,x^{12} + 3\,x^{14} + x^{20}}{(1-x^4)^2(1-x^6)^2}.$$

Proof. In the proof of Proposition 3.32 we saw that $|\epsilon$ acts on $\mathcal{W}_k$. We thus obtain a decomposition in eigenspaces with eigenvalues 1 and -1, respectively, i. e.

$$\mathcal{W}_k = \mathcal{W}_k^+ \oplus \mathcal{W}_k^-.$$

Theorem 3.31 and Proposition 3.32 therefore yield for even $k \geq 4$ that

$$\begin{aligned} \dim \mathcal{W}_k^- &= \dim \mathcal{W}_k - \dim \mathcal{W}_k^+ \\ &= \binom{k+1}{3}\left(\frac{1}{3} - \frac{k-4}{4(k-1)}\right) - \frac{k}{4}\left(2\left[\frac{k}{4}\right]+1\right) - \frac{4}{3}\left[\frac{k}{3}\right] + \varrho_k \frac{2k}{3} + \varrho_k^+ \end{aligned}$$

and

$$\begin{aligned} \sum_{k=0}^{\infty} \dim \mathcal{W}_k^- \, x^k &= \sum_{k=0}^{\infty} \dim \mathcal{W}_k \, x^k - \sum_{k=0}^{\infty} \dim \mathcal{W}_k^+ \, x^k \\ &= \frac{1 + x^4 + 6\,x^6 + 10\,x^8 + 6\,x^{10} + 15\,x^{12} + 10\,x^{14} + x^{16} - 2\,x^{18}}{(1-x^4)^2(1-x^6)^2} \\ &\quad - \frac{x^4\left(1 + 3\,x^2 + 4\,x^4 + 2\,x^6 + 9\,x^8 + 7\,x^{10} + x^{12} - 2\,x^{14} - x^{16}\right)}{(1-x^4)^2(1-x^6)^2} \\ &= \frac{1 + 3\,x^6 + 6\,x^8 + 4\,x^{10} + 6\,x^{12} + 3\,x^{14} + x^{20}}{(1-x^4)^2(1-x^6)^2}. \end{aligned}$$

□

The dimensions of $\mathcal{W}_k$, $\mathcal{W}_k^+$ and $\mathcal{W}_k^-$ for small values of k can be found in table 2.

3.5. Laplacian operator

After the previous discussions were analogous to chapter 2 for the most part, we are now going to introduce the Laplacian operator associated with the quadratic form q from Definition 3.9. The idea of considering this operator was proposed by D. Zagier in a private conversation with U. Kühn.

Definition 3.34. We set the *Laplacian operator (associated with q)* on $\mathcal{V}$ to be

$$\Delta := \frac{\partial^2}{\partial x_1 \partial y_1} + \frac{\partial^2}{\partial x_2 \partial y_2}.$$

Remark 3.35. The Laplacian maps homogeneous polynomials to homogeneous polynomials. So for all $k \in \mathbb{N}$, $k \geq 2$, we have a linear map

$$\Delta|_{\mathcal{V}_k} : \mathcal{V}_k \to \mathcal{V}_{k-2}$$

and we write $\ker(\Delta)_k := \ker(\Delta) \cap \mathcal{V}_k$.

Lemma 3.36. *There is a decomposition*

$$\mathcal{V}_k = \ker(\Delta)_k \oplus (x_1 y_1 + x_2 y_2) \cdot \mathcal{V}_{k-2}.$$

Proof. Let $p, q, r \in \mathcal{V}$. First note that $\partial(qr) = \partial(q)\partial(r) = \partial(r)\partial(q)$. This is immediate if we think of ∂ as a formal substitution of variables. We thus have

$$(k-2)! \langle qr, p \rangle = \partial(qr)(p)(0) = \partial(r)(\partial(q)(p))(0) = (k-2)! \langle r, \partial(q)p \rangle .$$

Hence applying $\partial(q)$ is adjoint to multiplication with q. In particular, for $q(x_1, x_2, y_1, y_2) = x_1 y_1 + x_2 y_2$, $r \in \mathcal{V}_{k-2}$ and $p \in \mathcal{V}_k$ we have that the kernel of $\partial(q)$ in $\mathcal{V}_k$ is the orthogonal complement to $q \cdot \mathcal{V}_{k-2}$. The claimed decomposition follows since $\Delta = \partial(q)$. □

Corollary 3.37. We have

$$\dim(\Delta)_k = (k-1)^2.$$

Proof. Lemma 3.36 yields that

$$\dim(\Delta)_k = \dim(\mathcal{V}_k) - \dim(\mathcal{V}_{k-2})$$

and by Lemma 3.2 we have $\dim(\mathcal{V}_k) = \binom{k+1}{3}$. Hence

$$\begin{aligned} \dim(\Delta)_k &= \binom{k+1}{3} - \binom{k-1}{3} = (k-1) \cdot \frac{(k+1)k - (k-2)(k-3)}{6} \\ &= (k-1) \cdot \frac{6k-6}{6} = (k-1)^2. \end{aligned}$$

□

Lemma 3.38. *Let $a, b, c, d \in \mathbb{C}$ be complex numbers. Then for all $k \in \mathbb{N}$ we have*

$$\Delta\Big((-bdx_1 + bcx_2 + acy_1 + ady_2)^k\Big) = 0.$$

Proof. For $r = -bdx_1 + bcx_2 + acy_1 + ady_2$ the multinomial theorem yields

$$r^k = \sum_{i+j+m+n=k} \binom{k}{i, j, m, n} (-1)^i \, (bdx_1)^i \, (bcx_2)^j \, (acy_1)^m \, (ady_2)^n.$$

Now fix some $i_1, i_2, j_1, j_2 \in \mathbb{N}_0$. The coefficient of $x_1^{i_1} x_2^{i_2} y_1^{j_1} y_2^{j_2}$ in $\Delta(r^k)$ is then given by

$$\binom{k}{i_1, i_2, j_1, j_2} (-1)^{i_1} \, (bd)^{i_1} \, (bc)^{i_2} \, (ac)^{j_1} \, (ad)^{j_2} \, \big(-(bd)(ac) + (bc)(ad) \big) = 0. \qquad \square$$

Remark 3.39. For $i_1, i_2, j_1, j_2 \in \mathbb{N}_0$ with $i_1 + i_2 = j_1 + j_2$ we consider the system of linear equations

$$i_1 = m_1 + m_2, \qquad i_2 = n_1 + n_2, \qquad j_1 = m_1 + n_2, \qquad j_2 = m_2 + n_1 \tag{3.10}$$

with $m_1, m_2, n_1, n_2 \in \mathbb{N}_0$. Since this yields

$$i_1 = j_1 + j_2 - i_2, \qquad i_2 = j_1 + j_2 - i_1, \qquad j_1 = i_1 + i_2 - j_2, \qquad j_2 = i_1 + i_2 - j_1$$

the solution has one degree of freedom. A solution (m_1, m_2, n_1, n_2) to (3.10) is given by

$$\begin{cases} (j_1, i_1 - j_1, i_2, 0) & \text{if } i_1 \geq j_1 \\ (i_1 - j_2, j_2, 0, i_2) & \text{if } i_1 \geq j_2 \\ (0, i_1, i_2 - j_1, j_1) & \text{if } i_2 \geq j_1 \\ (i_1, 0, j_2, i_2 - j_2) & \text{if } i_2 \geq j_2 \end{cases}$$

Furthermore, if (m_1, m_2, n_1, n_2) solves (3.10), then $(m_1 - \lambda, m_2 + \lambda, n_1 - \lambda, n_2 + \lambda) \in \mathbb{N}_0^4$ is a solution for all suitable $\lambda \in \mathbb{Z}$.

Lemma 3.40. *The map*

$$\begin{aligned} \varphi \colon \mathcal{V}_k &\longrightarrow V_k \otimes V_k \\ f(x_1, x_2, y_1, y_2) &\longmapsto f(ac, ad, -bd, bc) \end{aligned}$$

is well-defined.

Proof. Since φ is linear, it suffices to show the claim for monomials. Let $x_1^{i_1} x_2^{i_2} y_1^{j_1} y_2^{j_2} \in \mathcal{V}_k$ then

$$\varphi(x_1^{i_1} x_2^{i_2} y_1^{j_1} y_2^{j_2}) = (-1)^{j_1} \, a^{i_1+i_2} b^{j_1+j_2} \cdot c^{i_1+j_2} d^{i_2+j_1} \in V_k \otimes V_k$$

which is clear if we identify $V_k \otimes V_k$ with the $\mathcal{V}$-subspaces of bi-homogeneous polynomials in 4 variables, i. e.

$$\langle f \cdot g \mid f \in \mathbb{Q}[a, b], g \in \mathbb{Q}[c, d] \text{ homogeneous monomials}, \deg(f) = \deg(g) = k - 2 \rangle_{\mathbb{Q}}. \qquad \square$$

Remark 3.41. Using the description

$$f(x_1, x_2, y_1, y_2) = f\left(\begin{pmatrix} x_1 & x_2 \\ -y_2 & y_1 \end{pmatrix}\right)$$

from Remark 3.8, we get

$$\varphi(f) = f\left(\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} c & d \end{pmatrix}\right),$$

i. e. we evaluate f at a rank 1 matrix.

Theorem 3.42. *The restriction* $\varphi|_{\ker(\Delta)_k} \colon \ker(\Delta)_k \to V_k \otimes V_k$ *of the map* φ *from Lemma 3.40 is an isomorphism of vector spaces, i. e.*

$$\ker(\Delta)_k \cong V_k \otimes V_k.$$

The inverse map is given by

$$\begin{aligned} \psi \colon V_k \otimes V_k &\longrightarrow \ker(\Delta)_k \\ f(a, b; c, d) &\longmapsto \frac{(-1)^k}{(k-2)!} \left\langle f(a, b; c, d), r(a, b, c, d)^{k-2} \right\rangle \end{aligned}$$

where $r(a, b, c, d) = -bdx_1 + bcx_2 + acy_1 + ady_2 \in (V_k \otimes V_k) \otimes \mathcal{V}_k$.

Proof. Let $\varphi \colon \mathcal{V}_k \longrightarrow V_k \otimes V_k$ be the linear map from Lemma 3.40. Since $\varphi(x_1y_1 + x_2y_2) = 0$ the decomposition from Lemma 3.36 implies that φ can naturally be considered as a map $\ker(\Delta)_k \longrightarrow V_k \otimes V_k$ instead.

To show that the map ψ is well-defined, let $r = -bdx_1 + bcx_2 + acy_1 + ady_2 \in (V_k \otimes V_k) \otimes \mathcal{V}_k$ and observe that for $f(a, b; c, d) \in V_k \otimes V_k$ we have

$$\begin{aligned} \Delta\left(\left\langle f(a, b; c, d), r(a, b, c, d)^{k-2} \right\rangle_\partial\right) &= \Delta\left(\frac{1}{(k-2)!}\, \partial(f)(r^{k-2})\Big|_{(a,b,c,d)=(0,0,0,0)}\right) \\ &= \frac{1}{(k-2)!}\, \partial(f)\left(\Delta\left(r^{k-2}\right)\right)\Big|_{(a,b,c,d)=(0,0,0,0)} \\ &\overset{3.38}{=} 0. \end{aligned}$$

We first show $\varphi \circ \psi = \mathrm{id}_{V_k \otimes V_k}$. Let $f(a, b; c, d) = a^{i_1} b^{i_2} \cdot c^{j_1} d^{j_2}$ with $i_1 + i_2 = k - 2 = j_1 + j_2$ and assume without loss of generality that $i_1 \geq j_1$. We then have

$$\begin{aligned} \psi(f) &= \frac{(-1)^k}{(k-2)!} \left\langle f(a, b; c, d), r(a, b, c, d)^{k-2} \right\rangle_\partial \\ &= \frac{(-1)^{i_2+j_1}}{(k-2)!^2} \left(\frac{\partial^{i_2}}{\partial a^{i_2}} \frac{\partial^{i_1}}{\partial b^{i_1}} \frac{\partial^{j_2}}{\partial c^{j_2}} \frac{\partial^{j_1}}{\partial d^{j_1}}\right) \\ &\quad \left(\sum_{\substack{m_1, m_2, n_1, n_2 \geq 0, \\ m_1+m_2+n_1+n_2 = k-2}} (-1)^{m_1} \binom{k-2}{m_1, m_2, n_1, n_2} a^{n_1+n_2} b^{m_1+m_2} c^{m_2+n_1} d^{m_1+n_2} x_1^{m_1} x_2^{m_2} y_1^{n_1} y_2^{n_2}\right) \\ &= \sum_{\lambda=0}^{\min(i_2, j_1)} \frac{(-1)^{i_2-\lambda}}{(k-2)!^2} \binom{k-2}{j_1-\lambda, i_1-j_1+\lambda, i_2-\lambda, \lambda} (i_1)!(i_2)!(j_1)!(j_2)!\, x_1^{j_1-\lambda} x_2^{i_1-j_1+\lambda} y_1^{i_2-\lambda} y_2^{\lambda} \end{aligned}$$

where the last equality follows from Remark 3.39. Applying φ to this yields

$$\begin{aligned}\varphi(\psi(f)) &= \sum_{\lambda=0}^{\min(i_2,j_1)} \frac{1}{(k-2)!^2}\binom{k-2}{j_1-\lambda, i_1-j_1+\lambda, i_2-\lambda, \lambda}(i_1)!(i_2)!(j_1)!(j_2)!a^{i_1}b^{i_2}c^{j_1}d^{j_2} \\ &= a^{i_1}b^{i_2}c^{j_1}d^{j_2}\end{aligned}$$

where the last equality holds follows from

$$\begin{aligned}&\sum_{\lambda=0}^{\min(i_2,j_1)} \frac{1}{(k-2)!^2}\binom{k-2}{j_1-\lambda, i_1-j_1+\lambda, i_2-\lambda, \lambda}(i_1)!(i_2)!(j_1)!(j_2)! \\ &= \sum_{\lambda=0}^{\min(i_2,j_1)} \frac{(i_1)!(i_2)!}{(j_1-\lambda)!(i_1-j_1+\lambda)!(i_2-\lambda)!\lambda!}\binom{k-2}{j_1}^{-1} \\ &= \left(\sum_{\lambda=0}^{\min(i_2,j_1)} \binom{i_2}{\lambda}\binom{i_1}{j_1-\lambda}\right)\binom{k-2}{j_1}^{-1} \\ &= 1\end{aligned}$$

since $i_1 = k-2-i_2$ and the Chu–Vandermonde identity yields for $K, M, N \in \mathbb{N}_0$ that

$$\sum_{\lambda=0}^{\min(M,K)} \binom{M}{\lambda}\binom{N-M}{K-\lambda} = \binom{N}{K}.$$

Note that $\ker(\Delta)_k$ and $V_k \otimes V_k$ are finite-dimensional. It thus suffices to show that their dimensions agree. We have $\dim V_k = k-1$ by Remark 2.2, hence $\dim(V_k \otimes V_k) = (k-1)^2$ and Corollary 3.37 yields that $\dim(\ker(\Delta)_k) = (k-1)^2$. □

Lemma 3.43. *The Laplacian operator commutes with the action of Γ, i. e. for $f \in \mathcal{V}_k$ and $\gamma \in \Gamma$ we have*

$$\Delta(f\,|\gamma) = \Delta(f)\,|\gamma\,. \tag{3.11}$$

Proof. Since the Laplacian is linear, assume without loss of generality that $f = x_1^{i_1}x_2^{i_2}y_1^{j_1}y_2^{j_2}$. By Remark 3.14 it suffices to show that (3.11) holds for all $\gamma \in \Gamma$ such that

$$f\,|\gamma = f(\gamma_1 \cdot Z \cdot \gamma_2^{\mathrm{t}})$$

with $(\gamma_1, \gamma_2) \in \{(1, S), (S, 1), (1, T), (T, 1)\}$ and for $\gamma \in \{\tau, \nu\}$ with τ and ν from Example 3.13. The respective actions yield

$$\begin{aligned} f(Z \cdot S^{\mathrm{t}}) &= f(-x_2, x_1, -y_2, y_1) \\ f(S \cdot Z) &= f(y_2, -y_1, x_2, -x_1) \\ f(Z \cdot T^{\mathrm{t}}) &= f(x_1 + x_2, x_2, y_1, y_2 - y_1) \\ f(T \cdot Z) &= f(x_1 - y_2, x_2 + y_1, y_1, y_2) \\ f\,|\tau &= f(x_1, -y_2, y_1, -x_2) \\ f\,|\nu &= f(x_1, -x_2, y_1, -y_2) \end{aligned}$$

and we have

$$\Delta(f) = i_1 j_1\, x_1^{i_1-1} x_2^{i_2} y_1^{j_1-1} y_2^{j_2} + i_2 j_2\, x_1^{i_1} x_2^{i_2-1} y_1^{j_1} y_2^{j_2-1}. \tag{3.12}$$

Let $(\gamma_1, \gamma_2) = (1, S)$. It is straight forward to verify that both sides of (3.11) yield

$$i_1 j_1\, x_1^{i_2} (-x_2)^{i_1-1} y_1^{j_2} (-y_2)^{j_1-1} + i_2 j_2\, x_1^{i_2-1} (-x_2)^{i_1} y_1^{j_2-1} (-y_2)^{j_1}.$$

Let $(\gamma_1, \gamma_2) = (S, 1)$. Then both sides of (3.11) are equal to

$$i_1 j_1\, (-x_1)^{j_2}\, {x_2}^{j_1-1}\, (-y_1)^{i_2}\, {y_2}^{i_1-1} + i_2 j_2\, (-x_1)^{j_2-1}\, {x_2}^{j_1}\, (-y_1)^{i_2-1}\, {y_2}^{i_1}.$$

Let $(\gamma_1, \gamma_2) = (1, T)$. Since

$$\Delta((x_1 + x_2)^{i_1} (y_2 - y_1)^{j_2}) = 0$$

we obtain by the product rule that

$$\begin{aligned}
\Delta(f\,|\gamma) &= \Delta((x_1+x_2)^{i_1} x_2^{i_2} y_1^{j_1} (y_2 - y_1)^{j_2}) \\
&= i_1 j_1\, (x_1+x_2)^{i_1-1} x_2^{i_2} y_1^{j_1-1} (y_2-y_1)^{j_2} + i_2 j_2\, (x_1+x_2)^{i_1} x_2^{i_2-1} y_1^{j_1} (y_2-y_1)^{j_2-1} \\
&= \Delta(f)\,|\gamma
\end{aligned}$$

where the last equality follows from (3.12).

Let $(\gamma_1, \gamma_2) = (T, 1)$. Since

$$\Delta((x_1 - y_2)^{i_1} (x_2 + y_1)^{i_2}) = 0$$

we have by the product rule that

$$\begin{aligned}
\Delta(f\,|\gamma) &= \Delta((x_1 - y_2)^{i_1} (x_2 + y_1)^{i_2} y_1^{j_1} y_2^{j_2}) \\
&= i_1 j_1\, (x_1 - y_2)^{i_1-1} (x_2 + y_1)^{i_2} {y_1}^{j_1-1} {y_2}^{j_2} + i_2 j_2\, (x_1 - y_2)^{i_1} (x_2 + y_1)^{i_2-1} {y_1}^{j_1} {y_2}^{j_2-1} \\
&= \Delta(f)\,|\gamma
\end{aligned}$$

where the last equality follows, again, from (3.12).

Let $\gamma = \tau$. It is straight forward to verify that both sides of (3.11) yield

$$i_1 j_1\, x_1^{i_1-1} (-x_2)^{j_2} y_1^{j_1-1} (-y_2)^{i_2} + i_2 j_2\, x_1^{i_1} (-x_2)^{j_2-1} y_1^{j_1} (-y_2)^{i_2-1}.$$

Let $\gamma = \nu$. Then both sides of (3.11) are equal to

$$i_1 j_1\, {x_1}^{i_1-1}\, (-x_2)^{i_2}\, {y_1}^{j_1-1}\, (-y_2)^{j_2} + i_2 j_2\, {x_1}^{i_1}\, (-x_2)^{i_2-1}\, {y_1}^{j_1}\, (-y_2)^{j_2-1}\,.$$

This finishes the proof. □

We can now use our previous results to gain new insights on the space of bi-period polynomials. In particular, we obtain the following decomposition for $\mathcal{W}_k$ which is analogue to the decomposition of $\mathcal{V}_k$ from Lemma 3.36.

Proposition 3.44. *Let $k \geq 4$ be even. Then there is a decomposition*

$$\mathcal{W}_k = \ker\left(\Delta|_{\mathcal{W}_k}\right) \oplus (x_1y_1 + x_2y_2) \cdot \mathcal{W}_{k-2}.$$

Proof. Let $f \in \mathcal{W}_k$. The decomposition from Lemma 3.36 yields $f = f_1 + q \cdot f_2$ with unique $f_1 \in \ker(\Delta)_k$, $f_2 \in \mathcal{V}_{k-2}$ and $q(x_1, x_2, y_1, y_2) = x_1y_1 + x_2y_2$. We need to show that $f_1 \in \ker\left(\Delta|_{\mathcal{W}_k}\right)$ and $f_2 \in \mathcal{W}_{k-2}$. Since q is invariant under the bi-slash operator and Δ commutes with $|S$ and $|U$ (Lemma 3.43) we have

$$0 = f\,|1+S = (f_1 + q \cdot f_2)\,|1+S\,. \tag{3.13}$$

Applying Δ yields

$$0 = \Delta\left((q \cdot f_2)\,|1+S\right) = \Delta\left(q \cdot (f_2\,|1+S)\right).$$

But $\ker(\Delta)_k \cap q \cdot \mathcal{V}_{k-2} = \{0\}$ which implies $f_2 \in \ker(1+S)$. An analogue calculation shows $f_2 \in \ker(1+U+U^2)$, hence $f_2 \in \mathcal{W}_{k-2}$. Now (3.13) shows immediately that $f_1 \in \ker(1+S)$ and analogously that $f_1 \in \ker(1+U+U^2)$, hence $f_1 \in \ker(\Delta|_{\mathcal{W}_k})$. □

Theorem 3.45. *Let $k \geq 4$ be even. We then have*

$$\ker(\Delta|_{\mathcal{W}_k}) \cong V_k \otimes W_k.$$

Proof. By Theorem 3.42 we have

$$\ker(\Delta)_k \cong V_k \otimes V_k = \varphi(\mathcal{V}_k).$$

For $f \in \mathcal{V}_k$ and $\gamma \in \mathrm{GL}_2(\mathbb{Z})$ we have by Remark 3.8 that

$$\varphi(f\,|\gamma) = f\left(\begin{pmatrix} a \\ b \end{pmatrix}\left(\begin{pmatrix} c & d \end{pmatrix} \cdot \gamma^{\mathrm{t}}\right)\right).$$

This implies for $\varphi(f) = g_1 \otimes g_2$ that

$$\varphi(f\,|1+S) = 0 \iff g_2\,|1+S = 0$$

and similarly for $\left|1+U+U^2\right.$. Hence

$$\ker(\Delta|_{\mathcal{W}_k}) \cong V_k \otimes W_k.$$ □

Combining Proposition 3.44 and Theorem 3.45 lets us compute the dimension of $\mathcal{W}_k$ recursively. In particular, the dimension formula is equal to the formula from Theorem 3.31.

Corollary 3.46.

i) Let $k \geq 4$ be even. The dimension of $\mathcal{W}_k$ is explicitly given by

$$\dim \mathcal{W}_k = \binom{k}{3} - \frac{k-2}{2} - 2 \sum_{\substack{n=4,\\ n \text{ even}}}^{k} \left(\left\lfloor \frac{n-2}{4} \right\rfloor + \left\lfloor \frac{n-2}{6} \right\rfloor \right) (n-1).$$

ii) The generating series of $\mathcal{W}_k$ is given by

$$\sum_{k=0}^{\infty} \dim \mathcal{W}_k \, x^k = 1 + \frac{x^2}{1-x^2} \cdot \frac{\mathrm{d}}{\mathrm{d}x} \Big(\sum_{k=1}^{\infty} \dim W_k \, x^{k-1} \Big).$$

Proof. Since $\mathcal{W}_2 = \{0\}$ we obtain from Proposition 3.44 that

$$\sum_{k=1}^{\infty} \dim \mathcal{W}_k \, x^k = \frac{1}{1-x^2} \sum_{k=1}^{\infty} \dim(\ker(\Delta|_{\mathcal{W}_k}))\, x^k. \tag{3.14}$$

Now let $k \geq 4$ be even. For claim i) we recall from Proposition 2.29 that

$$\dim W_k = k - 3 - 2 \left(\left\lfloor \frac{k-2}{4} \right\rfloor + \left\lfloor \frac{k-2}{6} \right\rfloor \right).$$

Hence Theorem 3.45 yields

$$\begin{aligned} \dim(\ker(\Delta|_{\mathcal{W}_k})) &= \left(k - 3 - 2 \left(\left\lfloor \frac{k-2}{4} \right\rfloor + \left\lfloor \frac{k-2}{6} \right\rfloor \right) \right) \cdot (k-1) \\ &= (k-2)^2 - 1 - 2 \left(\left\lfloor \frac{k-2}{4} \right\rfloor + \left\lfloor \frac{k-2}{6} \right\rfloor \right) (k-1). \end{aligned}$$

We then obtain by using (3.14) that

$$\begin{aligned} \dim \mathcal{W}_k &= \sum_{\substack{n=4,\\ n \text{ even}}}^{k} (n-2)^2 - 1 - 2 \left(\left\lfloor \frac{n-2}{4} \right\rfloor + \left\lfloor \frac{n-2}{6} \right\rfloor \right) (n-1) \\ &= \binom{k}{3} - \frac{k-2}{2} - 2 \sum_{\substack{n=4,\\ n \text{ even}}}^{k} \left(\left\lfloor \frac{n-2}{4} \right\rfloor + \left\lfloor \frac{n-2}{6} \right\rfloor \right) (n-1). \end{aligned}$$

For claim ii) we recall from Remark 2.2 that $\dim V_k = k - 1$. By Theorem 3.45 we further have $\ker(\Delta|_{\mathcal{W}_k}) \cong V_k \otimes W_k$. Hence

$$\begin{aligned} \sum_{k=1}^{\infty} \dim(\ker(\Delta|_{\mathcal{W}_k}))\, x^k &= \sum_{k=1}^{\infty} \dim V_k \cdot \dim W_k \, x^k \\ &= x^2 \cdot \sum_{k=1}^{\infty} (k-1) \dim W_k \, x^{k-2} \\ &= x^2 \cdot \frac{\mathrm{d}}{\mathrm{d}x} \Big(\sum_{k=1}^{\infty} \dim W_k \, x^{k-1} \Big). \end{aligned}$$

Now with $\dim \mathcal{W}_0 = 1$ we obtain from (3.14) that

$$\sum_{k=0}^{\infty} \dim \mathcal{W}_k \, x^k = 1 + \frac{1}{1-x^2} \sum_{k=1}^{\infty} \dim(\ker(\Delta|_{\mathcal{W}_k}))\, x^k$$
$$= 1 + \frac{x^2}{1-x^2} \cdot \frac{\mathrm{d}}{\mathrm{d}x} \Big(\sum_{k=1}^{\infty} \dim W_k \, x^{k-1} \Big). \qquad \square$$

Remark 3.47. Recall from Proposition 2.29 that the generating series of W_k is given by

$$\sum_{k=0}^{\infty} \dim W_k \, x^k = \frac{1+x^{12}}{(1-x^4)(1-x^6)}$$

and therefore

$$\sum_{k=1}^{\infty} \dim W_k \, x^{k-1} = \frac{1}{x} \left(\frac{1+x^{12}}{(1-x^4)(1-x^6)} - 1 \right).$$

So Corollary 3.46 yields that

$$\sum_{k=0}^{\infty} \dim \mathcal{W}_k \, x^k = 1 + \frac{x^2}{1-x^2} \cdot \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{1+x^{12}}{x(1-x^4)(1-x^6)} - \frac{1}{x} \right)$$
$$= \frac{1 + x^4 + 6\,x^6 + 10\,x^8 + 6\,x^{10} + 15\,x^{12} + 10\,x^{14} + x^{16} - 2\,x^{18}}{(1-x^4)^2(1-x^6)^2}.$$

Note that this indeed coincides with the generating series of $\mathcal{W}_k$ from Theorem 3.31.

We now consider an analogue of the Lewis space from Definition 2.18.

Definition 3.48. We denote the kernel of the operator $1 - T - T'$ by

$$\mathcal{L}_k := \ker(1 - T - T') \subseteq \mathcal{V}_k.$$

Lemma 3.49. *Let $k \geq 4$ be even. Then there is an isomorphism of vector spaces*

$$\mathcal{W}_k \cong \mathcal{L}_k.$$

Proof. The proof of Proposition 3.44 also yields that

$$\mathcal{L}_k = \ker(\Delta|_{\mathcal{L}_k}) \oplus (x_1 y_1 + x_2 y_2) \cdot \mathcal{L}_{k-2} \tag{3.15}$$

and the proof of Theorem 3.45 also shows that

$$\ker(\Delta|_{\mathcal{L}_k}) \cong V_k \otimes L_k. \tag{3.16}$$

We therefore obtain that

$$\mathcal{L}_k \underset{(3.15)}{=} \bigoplus_{\substack{i=0,\\ i \text{ even}}}^{k-4} \ker(\Delta|_{\mathcal{L}_{k-i}}) \underset{(3.16)}{\cong} \bigoplus_{\substack{i=0,\\ i \text{ even}}}^{k-4} V_{k-i} \otimes L_{k-i}$$

$$\underset{2.19}{=} \bigoplus_{\substack{i=0,\\ i \text{ even}}}^{k-4} V_{k-i} \otimes W_{k-i} \underset{3.45}{\cong} \bigoplus_{\substack{i=0,\\ i \text{ even}}}^{k-4} \ker(\Delta|_{\mathcal{W}_{k-i}}) \underset{3.44}{=} \mathcal{W}_k. \qquad \square$$

Remark 3.50. Observe that we have $\mathcal{W}_k \subseteq \mathcal{L}_k$. For $f \in \mathcal{W}_k$ we have $f\,|1+S = 0$ and $f\,\big|1+U+U^2 = 0$. By subtracting the latter from the first equation we obtain

$$f\,\big|S - U - U^2 = 0$$

and applying $|S$ yields

$$f\,\big|1 - T - T' = 0.$$

This shows that the isomorphism in Lemma 3.49 is in fact an equality of vector spaces, i. e.

$$\mathcal{W}_k = \mathcal{L}_k.$$

Corollary 3.51. Let $k \geq 4$ be even. Then

$$\mathcal{W}_k^{\pm} = \ker(1 - T \mp T\epsilon).$$

Proof. Follows analogously to Corollary 2.21. $\square$

We conclude this section by considering another subspace of $\mathcal{V}_k$ and use the Laplacian Δ to describe this space in terms of even and odd period polynomials.

Definition 3.52. For even $k \geq 4$ we set

$$\begin{aligned} \mathcal{W}_k^{\tau} &:= \{f \in \mathcal{W}_k \mid f(Z) = f(Z^{\mathsf{t}})\} \\ \mathcal{W}_k^{\nu} &:= \{f \in \mathcal{W}_k \mid f(Z) = f(\delta Z \delta^{\mathsf{t}})\} \\ \mathcal{W}_k^{\tau,\nu} &:= \{f \in \mathcal{W}_k \mid f(Z) = f(Z^{\mathsf{t}}) = f(\delta Z \delta^{\mathsf{t}})\} \end{aligned}$$

where $Z = \begin{pmatrix} x_1 & x_2 \\ -y_2 & y_1 \end{pmatrix}$ (cf. Remark 3.8).

Recall from Example 3.13 that $f(Z^{\mathsf{t}})$ and $f(\delta Z \delta^{\mathsf{t}})$ are given by the action of $|\tau$ and $|\nu$, respectively.

Proposition 3.53. *Let $k \geq 4$ be even.*

i) There is a decomposition

$$\mathcal{W}_k^{\tau} = \ker(\Delta|_{\mathcal{W}_k^{\tau}}) \oplus (x_1y_1 + x_2y_2) \cdot \mathcal{W}_{k-2}^{\tau}.$$

ii) There is a decomposition

$$\mathcal{W}_k^{\nu} = \ker(\Delta|_{\mathcal{W}_k^{\nu}}) \oplus (x_1y_1 + x_2y_2) \cdot \mathcal{W}_{k-2}^{\nu}.$$

iii) There is a decomposition

$$\mathcal{W}_k^{\tau,\nu} = \ker(\Delta|_{\mathcal{W}_k^{\tau,\nu}}) \oplus (x_1y_1 + x_2y_2) \cdot \mathcal{W}_{k-2}^{\tau,\nu}.$$

Proof. With $q(x_1, x_2, y_1, y_2) = x_1y_1 + x_2y_2$ we have $\mathcal{W}_k = \ker(\Delta|_{\mathcal{W}_k}) \oplus q \cdot \mathcal{W}_{k-2}$ by Proposition 3.44. So for each $f \in \mathcal{W}_k$ we obtain unique $f_1 \in \ker(\Delta|_{\mathcal{W}_k})$ and $f_2 \in \mathcal{W}_{k-2}$ with

$$f = f_1 + q \cdot f_2. \tag{3.17}$$

For claim i) let $f \in \mathcal{W}_k^{\tau}$. Applying $|\tau$ to (3.17) yields

$$f\,|\tau = f_1\,|\tau + (q \cdot f_2)\,|\tau\,. \tag{3.18}$$

Recall that the action of τ commutes with the Laplacian by Lemma 3.43. So applying Δ to (3.17) yields

$$\Delta(f) = \Delta(q \cdot f_2)$$

while applying Δ to (3.18) yields

$$\Delta(f\,|\tau) = \Delta((q \cdot f_2)\,|\tau).$$

Since $f = f\,|\tau$ and $q = q\,|\tau$ we thus obtain

$$\Delta(q \cdot f_2) = \Delta(q \cdot (f_2\,|\tau)).$$

Hence $q \cdot (f_2 - f_2\,|\tau) \in \ker(\Delta|_{\mathcal{W}_k}) \cap q \cdot \mathcal{W}_{k-2}$. But this space is trivial, hence

$$q \cdot (f_2 - f_2\,|\tau) = 0$$

which shows $f_2 = f_2\,|\tau$. Now applying $|\tau$ to (3.17) immediately yields $f_1 = f_1\,|\tau$. Hence $f_1 \in \ker(\Delta|_{\mathcal{W}_k^{\tau}})$ and $f_2 \in \mathcal{W}_{k-2}^{\tau}$ which proves the first claim.

For claim ii) let $f \in \mathcal{W}_k^{\nu}$. Note that Lemma 3.43 also holds for ν. So by replacing τ with ν in the proof of part i), we similarly obtain that $\Delta(q \cdot f_2) = \Delta(q \cdot (f_2\,|\nu))$. Hence $q \cdot (f_2 - f_2\,|\nu) \in \ker(\Delta|_{\mathcal{W}_k}) \cap q \cdot \mathcal{W}_{k-2} = \{0\}$. This shows $f_2 = f_2\,|\nu$. Applying ν to (3.17) then implies $f_1 = f_1\,|\nu$ which proves the second claim.

The statements i) and ii) directly imply claim iii). □

Theorem 3.54. *Let $k \geq 4$ be even.*

i) We have

$$\ker\left(\Delta|_{\mathcal{W}_k^\tau}\right) \cong \langle f(a,b)\otimes g(c,d) + g(a,b)\otimes f(c,d) \mid f,g \in W_k\rangle_{\mathbb{Q}}.$$

ii) We have

$$\ker\left(\Delta|_{\mathcal{W}_k^\nu}\right) \cong \left(V_k^{\mathsf{ev}} \otimes W_k^{\mathsf{ev}}\right) \oplus \left(V_k^{\mathsf{odd}} \otimes W_k^{\mathsf{odd}}\right).$$

iii) We have

$$\ker\left(\Delta|_{\mathcal{W}_k^{\tau,\nu}}\right) \cong \left\langle f(a,b)\otimes g(c,d) + g(a,b)\otimes f(c,d) \mid f,g \in W_k^{\mathsf{ev}} \text{ or } f,g \in W_k^{\mathsf{odd}}\right\rangle_{\mathbb{Q}}.$$

Proof. Recall the map $\varphi\colon \mathcal{V}_k \to V_k \otimes V_k$ from Lemma 3.40 which is given by

$$\varphi(f(x_1,x_2,y_1,y_2)) = f(ac, ad, -bd, bc).$$

The restriction to $\varphi|_{\ker(\Delta)_k}$ yields an isomorphism $\ker(\Delta)_k \cong V_k \otimes V_k$ by Theorem 3.42. We further used this map in Theorem 3.45 to show that $\ker(\Delta|_{\mathcal{W}_k}) \cong V_k \otimes W_k$. Let $f \in \mathcal{W}_k$. We then find $g_1 \in V_k$ and $g_2 \in W_k$ such that $\varphi(f) = g_1(a,b)\otimes g_2(c,d)$.

For claim i) assume $f \in \mathcal{W}_k^\tau$. Recall that $f\,|\tau = f(Z^{\mathsf{t}})$. By Remark 3.41 we have

$$\varphi(f\,|\tau) = f\left(\begin{pmatrix} c \\ d\end{pmatrix}\begin{pmatrix} a & b\end{pmatrix}\right)$$

and therefore

$$g_1 \otimes g_2 = \varphi(f) = \varphi(f\,|\tau) = g_2 \otimes g_1.$$

We observe that $f \in \mathcal{W}_k$ then implies that $g_1 \in W_k$ (cf. proof of Theorem 3.45) and since $g_1 \otimes g_2 = g_2 \otimes g_1$, the claim follows.

For claim ii) assume $f \in \mathcal{W}_k^\nu$. Recall that $f\,|\nu = f(\delta Z \delta^{\mathsf{t}})$. Since $\delta^{\mathsf{t}} = \delta$ we obtain by Remarks 3.8 and 3.41 that

$$\varphi(f\,|\nu) = f\left(\delta\begin{pmatrix} a \\ b\end{pmatrix}\left(\begin{pmatrix} c & d\end{pmatrix}\delta\right)\right).$$

Hence

$$g_1 \otimes g_2 = \varphi(f) = \varphi(f\,|\nu) = g_1\,|\delta \otimes g_2\,|\delta$$

where $g_i\,|\delta$, $i = 1,2$, denotes the slash operator from chapter 2. We thus have

$$\begin{aligned}\mathcal{W}_k^\nu &\cong \left\{g_1 \otimes g_2 \in V_k \otimes W_k \;\middle|\; g_1\,|\delta \otimes g_2\,|\delta = g_1 \otimes g_2\right\} \\ &\cong \left(V_k^{\mathsf{ev}} \otimes W_k^{\mathsf{ev}}\right) \oplus \left(V_k^{\mathsf{odd}} \otimes W_k^{\mathsf{odd}}\right).\end{aligned}$$

The statements i) and ii) now directly imply iii). ∎

Corollary 3.55. We have

$$\sum_{\substack{k=4,\\ k \text{ even}}}^{\infty} \dim \mathcal{W}_k^{\tau,\nu}\, x^k = \frac{1}{1-x^2} \sum_{\substack{k=4,\\ k \text{ even}}}^{\infty} \dim(\ker(\Delta|_{\mathcal{W}_k^{\tau,\nu}}))\, x^k$$

which yields for all even $k \geq 4$ that

$$\dim \mathcal{W}_k^{\tau,\nu} = \sum_{\substack{n=4,\\ n \text{ even}}}^{k} \dim(W_n^{\mathsf{ev}})^2.$$

Proof. The first statement follows immediately from part iii) of Proposition 3.53. So it suffices to show that

$$\dim(\ker(\Delta|_{\mathcal{W}_k^{\tau,\nu}})) = \dim(W_k^{\mathsf{ev}})^2.$$

Using the basis for $\ker(\Delta|_{\mathcal{W}_k^{\tau,\nu}})$ from part iii) of Theorem 3.54 we observe for $f, g \in W_k^{\mathsf{odd}}$ that there are

$$\binom{\dim(W_k^{\mathsf{odd}})+1}{2}$$

ways of choosing (possibly equal) $f, g \in W_k^{\mathsf{odd}}$. For $f, g \in W_k^{\mathsf{ev}}$ we similarly have

$$\binom{\dim(W_k^{\mathsf{ev}})+1}{2}$$

ways of choosing $f, g \in W_k^{\mathsf{ev}}$. By Remark 2.32 we have $\dim(W_k^{\mathsf{ev}}) = \dim(W_k^{\mathsf{odd}}) + 1$. Hence

$$\begin{aligned}
\dim(\ker(\Delta|_{\mathcal{W}_k^{\tau,\nu}})) &= \binom{\dim(W_k^{\mathsf{ev}})}{2} + \binom{\dim(W_k^{\mathsf{ev}})+1}{2} \\
&= \frac{(\dim(W_k^{\mathsf{ev}}))(\dim(W_k^{\mathsf{ev}})-1) + (\dim(W_k^{\mathsf{ev}})+1)(\dim(W_k^{\mathsf{ev}}))}{2} \\
&= \dim(W_k^{\mathsf{ev}}) \cdot \frac{2\dim(W_k^{\mathsf{ev}})}{2} \\
&= \dim(W_k^{\mathsf{ev}})^2.
\end{aligned}$$

□

3.6. Applications 1: Exotic relations for double q-zeta values

The theory of multiple q-zeta values is concerned with q-analogues of multiple zeta values, i. e. q-series ζ_q such that we retrieve multiple zeta values by taking the limit as $q \to 1$.

In general, we consider for $s_1 \geq 1$, $s_2, \ldots, s_l \geq 0$ and polynomials $Q_1(t) \in t\mathbb{Q}[t]$ and $Q_2, \ldots, Q_l(t) \in \mathbb{Q}[t]$ the power series

$$\zeta_q(s_1, \ldots, s_l; Q_1, \ldots, Q_l) = \sum_{n_1 > \cdots > n_l > 0} \frac{Q_1(q^{n_1}) \ldots Q_l(q^{n_l})}{(1-q^{n_1})^{s-1} \ldots (1-q^{n_l})^{s_l}} \in \mathbb{Q}[\![q]\!].$$

We leave out some technical details but remark that multiplying by $(1-q)^{s_1+\dots+s_l}$ and then taking the limit as $q \to 1$ yields the multiple zeta value $\zeta(s_1, \dots, s_l)$ as long as $s_1 \geq 2$ and $s_2, \dots, s_l \geq 1$. Now consider

$$\mathcal{Z}_q := \operatorname{span}_{\mathbb{Q}}\{\zeta_q(s_1, \dots, s_l; Q_1, \dots, Q_l) \mid l \geq 0, \deg(Q_j) \leq s_j\}$$

where we set $\zeta_q(\emptyset; \emptyset) = 1$.

The vector space $\mathcal{Z}_q$ has different spanning sets. For details on this topic as well as an overview over some of the common spanning sets we refer to [BK20], [Zha20], [Bac20] and [Bri21]. We will use combinatorial multiple Eisenstein series as a spanning set. For $r \geq 1$, $k_1, \dots, k_r \geq 1$ and $d_1, \dots, d_r \geq 0$ we denote the combinatorial multiple Eisenstein series in depth r by

$$G\binom{k_1, \dots, k_r}{d_1, \dots, d_r} \in \mathbb{Q}[\![q]\!].$$

For $k_1 \geq d_1 + 2$ and $k_i \geq d_i + 1$, $i = 2, \dots, r$, we have

$$\lim_{q \to 1} (1-q)^{k_1+\dots+k_r} G\binom{k_1, \dots, k_r}{d_1, \dots, d_r} = \zeta(k_1 - d_1, \dots, k_r - d_r).$$

For further details on this we refer to [BB22].

Recall our discussion in section 2.6. In Propositions 2.43 and 2.47 we used the non-degenerate pairing to obtain that certain $\mathbb{Q}$-linear relations amongst single and double zeta values can be computed by applying the linear map $\langle f, \cdot \rangle$ to an identity of generating series for single and double zeta values. We will pursue a similar approach in this section.

Consider the generating series of combinatorial multiple Eisenstein series in depth 1 and 2 which are given by

$$\begin{aligned}
\mathfrak{G}_1\binom{x}{y} &= \sum_{\substack{k \geq 1 \\ d \geq 0}} G\binom{k}{d} x^{k-1} \frac{y^d}{d!}, \\
\mathfrak{G}_2\binom{x_1, x_2}{y_1, y_2} &= \sum_{\substack{k_1, k_2 \geq 1 \\ d_1, d_2 \geq 0}} G\binom{k_1, k_2}{d_1, d_2} x_1^{k_1-1} x_2^{k_2-1} \frac{y_1^{d_1}}{d_1!} \frac{y_2^{d_2}}{d_2!}.
\end{aligned}$$

In [BB22, Proposition 6.7] it is shown that the combinatorial multiple Eisenstein series satisfy an analogue of the double shuffle relation from section 2.6, i. e.

$$\begin{aligned}
G\binom{k_1}{d_1} \cdot G\binom{k_2}{d_2} &= G\binom{k_1, k_2}{d_1, k_2} + G\binom{k_2, k_1}{d_2, d_1} + G\binom{k_1 + k_2}{d_1 + d_2} \\
&= \sum_{\substack{l_1 + l_2 = k_1 + k_2 \\ e_1 + e_2 = d_1 + d_2 \\ l_1, l_2 \geq 1, e_1, e_2 \geq 0}} \left(\binom{l_1 - 1}{k_1 - 1}\binom{d_1}{e_1}(-1)^{d_1 - e_1} + \binom{l_1 - 1}{k_2 - 1}\binom{d_2}{e_1}(-1)^{d_2 - e_1} \right) G\binom{l_1, l_2}{e_1, e_2} \\
&\quad + \frac{d_1! d_2!}{(d_1 + d_2 + 1)!}\binom{k_1 + k_2 - 2}{k_1 - 1} G\binom{k_1 + k_2 - 1}{d_1 + d_2 + 1}.
\end{aligned} \tag{3.19}$$

Similar to the double shuffle relations in (2.17), by setting

$$\mathfrak{R}^*\binom{x_1,x_2}{y_1,y_2} := \frac{\mathfrak{G}_1\binom{x_1}{y_1+y_2} - \mathfrak{G}_1\binom{x_2}{y_1+y_2}}{x_1-x_2},$$

$$\mathfrak{R}^{\sqcup\!\sqcup}\binom{x_1,x_2}{y_1,y_2} := \frac{\mathfrak{G}_1\binom{x_1+x_2}{y_1} - \mathfrak{G}_1\binom{x_1+x_2}{y_2}}{y_1-y_2}$$

we can rewrite (3.19) as

$$\begin{aligned}\mathfrak{G}_1\binom{x_1}{y_1}\cdot\mathfrak{G}_1\binom{x_2}{y_2} &= \mathfrak{G}_2\binom{x_1,x_2}{y_1,y_2} + \mathfrak{G}_2\binom{x_2,x_1}{y_2,y_1} + \mathfrak{R}^*\binom{x_1,x_2}{y_1,y_2} \\ &= \mathfrak{G}_2\binom{x_1+x_2,x_2}{y_1,y_2-y_1} + \mathfrak{G}_2\binom{x_1+x_2,x_1}{y_2,y_1-y_2} + \mathfrak{R}^{\sqcup\!\sqcup}\binom{x_1,x_2}{y_1,y_2}.\end{aligned} \tag{3.20}$$

By extending the bi-slash operator to power series in variables x_1, x_2, y_1, y_2, we can rewrite the second identity in (3.20) as

$$\mathfrak{G}_2\,|(T-1)(1+\epsilon) = \mathfrak{R}^*\binom{x_1,x_2}{y_1,y_2} - \mathfrak{R}^{\sqcup\!\sqcup}\binom{x_1,x_2}{y_1,y_2}. \tag{3.21}$$

We cannot expect, similar to Proposition 2.43, that (3.21) encodes all non-trivial linear relations amongst combinatorial multiple Eisenstein series since the generating series $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are also invariant under the swap-involution (see [BB22, Theorem 6.5]), i. e.

$$\mathfrak{G}_1\binom{x}{y} = \binom{y}{x}, \qquad \mathfrak{G}_2\binom{x_1,x_2}{y_1,y_2} = \mathfrak{G}_2\binom{y_1+y_2,y_1}{x_2,x_1-x_2}.$$

Therefore, we consider the formal double Eisenstein space. This space was introduced by [BBK20] and [BKM21] in an attempt of generalizing the work of [GKZ06]. We also refer to [Bac21] for further details.

Definition 3.56. For an integer $K \geq 1$, the *formal double Eisenstein space* of weight K is

$$\mathcal{E}_K = \left\langle Z\binom{k}{d}, Z\binom{k_1,k_2}{d_1,d_2}, P\binom{k_1,k_2}{d_1,d_2} \;\middle|\; \substack{k+d=k_1+k_2+d_1+d_2=K \\ k,k_1,k_2\geq 1, d,d_1,d_2\geq 0} \right\rangle_{\mathbb{Q}} \Big/ (3.22)$$

where we divide out the following relations

$$\begin{aligned}P\binom{k_1,k_2}{d_1,d_2} &= Z\binom{k_1,k_2}{d_1,k_2} + Z\binom{k_2,k_1}{d_2,d_1} + Z\binom{k_1+k_2}{d_1+d_2} \\ &= \sum_{\substack{l_1+l_2=k_1+k_2 \\ e_1+e_2=d_1+d_2 \\ l_1,l_2\geq 1, e_1,e_2\geq 0}} \left(\binom{l_1-1}{k_1-1}\binom{d_1}{e_1}(-1)^{d_1-e_1} + \binom{l_1-1}{k_2-1}\binom{d_2}{e_1}(-1)^{d_2-e_1}\right) Z\binom{l_1,l_2}{e_1,e_2} \\ &\quad + \frac{d_1!d_2!}{(d_1+d_2+1)!}\binom{k_1+k_2-2}{k_1-1} Z\binom{k_1+k_2-1}{d_1+d_2+1}.\end{aligned} \tag{3.22}$$

We also consider the generating series

$$\mathfrak{Z}_1\binom{x}{y} = \sum_{\substack{k\geq 1\\ d\geq 0}} Z\binom{k}{d} x^{k-1}\frac{y^d}{d!},$$

$$\mathfrak{Z}_2\binom{x_1,x_2}{y_1,y_2} = \sum_{\substack{k_1,k_2\geq 1\\ d_1,d_2\geq 0}} Z\binom{k_1,k_2}{d_1,d_2} x_1^{k_1-1}x_2^{k_2-1}\frac{y_1^{d_1}}{d_1!}\frac{y_2^{d_2}}{d_2!},$$

$$\mathfrak{P}\binom{x_1,x_2}{y_1,y_2} = \sum_{\substack{k_1,k_2\geq 1\\ d_1,d_2\geq 0}} P\binom{k_1,k_2}{d_1,d_2} x_1^{k_1-1}x_2^{k_2-1}\frac{y_1^{d_1}}{d_1!}\frac{y_2^{d_2}}{d_2!}.$$

We can then rewrite the defining relations (3.22) for $\mathcal{E}_K$ analogously to above as

$$\begin{aligned}\mathfrak{P}\binom{x_1,x_2}{y_1,y_2} &= \mathfrak{Z}_2\binom{x_1,x_2}{y_1,y_2} + \mathfrak{Z}_2\binom{x_2,x_1}{y_2,y_1} + \mathfrak{R}_f^{*}\binom{x_1,x_2}{y_1,y_2}\\ &= \mathfrak{Z}_2\binom{x_1+x_2,x_2}{y_1,y_2-y_1} + \mathfrak{Z}_2\binom{x_1+x_2,x_1}{y_2,y_1-y_2} + \mathfrak{R}_f^{\sqcup\!\sqcup}\binom{x_1,x_2}{y_1,y_2}\end{aligned} \tag{3.23}$$

where

$$\mathfrak{R}_f^{*}\binom{x_1,x_2}{y_1,y_2} := \frac{\mathfrak{Z}_1\binom{x_1}{y_1+y_2} - \mathfrak{Z}_1\binom{x_2}{y_1+y_2}}{x_1-x_2},$$

$$\mathfrak{R}_f^{\sqcup\!\sqcup}\binom{x_1,x_2}{y_1,y_2} := \frac{\mathfrak{Z}_1\binom{x_1+x_2}{y_1} - \mathfrak{Z}_1\binom{x_1+x_2}{y_2}}{y_1-y_2}.$$

Now rewrite the second identity in (3.23) as

$$\mathfrak{Z}_2\,|(T-1)(1+\epsilon) = \mathfrak{R}_f^{*}\binom{x_1,x_2}{y_1,y_2} - \mathfrak{R}_f^{\sqcup\!\sqcup}\binom{x_1,x_2}{y_1,y_2}. \tag{3.24}$$

By extending the non-degenerate pairing $\langle\cdot,\cdot\rangle$ to a duality pairing $\mathbb{Q}[x_1,x_2,y_1,y_2]\times\mathbb{Q}[\![x_1,x_2,y_1,y_2]\!]\longrightarrow\mathbb{Q}$, we have the following statement which is analogue to Proposition 2.43.

Proposition 3.57. *Let $K\geq 4$ be even. Then for all non-trivial linear relations of the form*

$$\sum_{\substack{k_1+k_2+d_1+d_2=K\\ k_1,k_2\geq 1\\ d_1,d_2\geq 0}} \lambda_{\substack{k_1,k_2\\ d_1,d_2}} Z\binom{k_1,k_2}{d_1,d_2} = \sum_{\substack{k+d=K\\ k\geq 1, d\geq 0}} \lambda_{\substack{k\\ d}} Z\binom{k}{d} \tag{3.25}$$

with $\lambda_{\substack{k_1,k_2\\d_1,d_2}}, \lambda_{\substack{k\\d}} \in \mathbb{Q}$ *there exists a* $f \in \mathcal{V}_k$ *such that*

$$\sum_{\substack{k+d=K\\k\geq 1, d\geq 0}} \lambda_{\substack{k\\d}} Z\binom{k}{d} = \left\langle f(x_1,x_2,y_1,y_2), \mathfrak{R}_f^{*}\binom{x_1,x_2}{y_1,y_2} - \mathfrak{R}_f^{ш}\binom{x_1,x_2}{y_1,y_2}\right\rangle$$

$$\sum_{\substack{k_1+k_2+d_1+d_2=K\\k_1,k_2\geq 1\\d_1,d_2\geq 0}} \lambda_{\substack{k_1,k_2\\d_1,d_2}} Z\binom{k_1,k_2}{d_1,d_2} = \left\langle f(x_1,x_2,y_1,y_2)\left|(1+\epsilon)(T^{-1}-1)\right., \mathfrak{Z}_2\binom{x_1,x_2}{y_1,y_2}\right\rangle.$$

The proof of Proposition 3.57 follows analogously to the proof of Proposition 2.43 since we only used properties of the group ring and the pairing $\langle\cdot,\cdot\rangle$ which are still true in this case.

Analogue to section 2.6, we write $\Delta^* = (1+\epsilon)(T^{-1}-1)$.

Example 3.58. Consider the polynomial $q(x_1,x_2,y_1,y_2) = x_1y_1 + x_2y_2$ from Definition 3.9. By applying the linear map $\langle q(x_1,x_2,y_1,y_2),\cdot\rangle$ to (3.21) we have

$$\langle q, \mathfrak{R}^* - \mathfrak{R}^{ш}\rangle = \frac{1}{2}\left(G\binom{3}{1} - G\binom{2}{2}\right)$$
$$\langle q\,|\Delta^*, \mathfrak{G}_2\rangle = 0$$

where the last equality follows from $q\,|\Delta^* = 0$. Hence we obtain the relation

$$G\binom{3}{1} = G\binom{2}{2}.$$

Proposition 3.59. *If* $f, f' \in \mathcal{V}_K$ *determine the same relation via applying the linear maps* $\langle f(x_1,x_2,y_1,y_2),\cdot\rangle$ *and* $\langle f'(x_1,x_2,y_1,y_2),\cdot\rangle$*, respectively, to* (3.24) *then their projections onto* $\mathcal{V}_K^+$ *differ by a multiple of* $q^{\frac{K-2}{2}}$.

Proof. Recall the proof of Proposition 2.45. Analogously we obtain that $\pi^+(f-f')$ is invariant under $|\epsilon$ and $|T$. This vector space is spanned by $q^{\frac{K-2}{2}}$ by Theorem 3.21 and since $(x_2)\,|\epsilon = x_1$ and $(y_1)\,|\epsilon = y_2$. □

Example 3.60. Let $K=4$. We consider the symmetric polynomials $f_1, f_2 \in \mathcal{V}_4^+$ with $f_1(x_1,x_2,y_1,y_2) = x_1^2 + x_2^2 + y_1^2 + y_2^2$ and $f_2(x_1,x_2,y_1,y_2) = x_1y_2 + x_2y_1$. Since

$$\langle f_1, \mathfrak{R}^* - \mathfrak{R}^{ш}\rangle = 2\,G\binom{4}{0} - 2\,G\binom{3}{1} + G\binom{2}{2} - \frac{1}{3}G\binom{1}{3}$$
$$\langle f_1\,|\Delta^*, \mathfrak{G}_2\rangle = 2\,G\binom{3,1}{0,0} + 2\,G\binom{2,2}{0,0} - 2\,G\binom{1,1}{1,1} + G\binom{1,1}{0,2}$$

and

$$\langle f_2, \mathfrak{R}^* - \mathfrak{R}^{\sqcup\!\sqcup} \rangle = -G\binom{3}{1} + \frac{1}{2}G\binom{2}{2}$$
$$\langle f_2 \,|\Delta^*\,, \mathfrak{G}_2 \rangle = G\binom{2,1}{0,1} - G\binom{2,1}{1,0} + G\binom{1,2}{0,1}$$

we obtain the relations

$$2\,G\binom{4}{0} - 2\,G\binom{3}{1} + G\binom{2}{2} - \frac{1}{3}G\binom{1}{3} = 2\,G\binom{3,1}{0,0} + 2\,G\binom{2,2}{0,0} - 2\,G\binom{1,1}{1,1} + G\binom{1,1}{0,2}$$

and

$$-2\,G\binom{3}{1} + G\binom{2}{2} = 2\,G\binom{2,1}{0,1} - 2\,G\binom{2,1}{1,0} + 2\,G\binom{1,2}{0,1}.$$

Combining these relations, we further yields that

$$\begin{aligned} 2\,G\binom{4}{0} - \frac{1}{3}G\binom{1}{3} = {} & 2\,G\binom{3,1}{0,0} + 2\,G\binom{2,2}{0,0} + 2\,G\binom{2,1}{1,0} - 2\,G\binom{2,1}{0,1} \\ & - 2\,G\binom{1,2}{0,1} - 2\,G\binom{1,1}{1,1} + G\binom{1,1}{0,2}. \end{aligned}$$

Notation. We say that a relation of the form (3.25)

- is *symmetric in* $Z(\mathsf{ev},\mathsf{ev})$ if $k_1 + d_1$ and $k_2 + d_2$ even implies that

$$\lambda_{\substack{k_1,k_2\\d_1,d_2}} = \lambda_{\substack{k_2,k_1\\d_2,d_1}}$$

- *contains no* $Z(\mathsf{odd},\mathsf{odd})$ *terms* if $k_1 + d_1$ and $k_2 + d_2$ odd implies that

$$\lambda_{\substack{k_1,k_2\\d_1,d_2}} = 0.$$

Proposition 3.61. *Let* $f \in \mathcal{W}_k^{\mathsf{ev}}$*. Then the relation* (3.25) *induced by applying the linear map* $\langle f \,|T\,, \cdot \rangle$ *to* (3.24) *is symmetric in* $Z(\mathsf{ev},\mathsf{ev})$.

Proof. Follows analogously to the proof of Proposition 2.47. □

Example 3.62. We consider the generic polynomial in $\mathcal{W}_6^{\mathsf{ev}}$ for $a_1, \ldots, a_5 \in \mathbb{Q}$ given by

$$\begin{aligned} p(x_1,x_2,y_1,y_2) = {} & a_1\, f_1(x_1,x_2,y_1,y_2) + a_2\, f_2(x_1,x_2,y_1,y_2) + a_3\, f_3(x_1,x_2,y_1,y_2) \\ & + a_4\, f_4(x_1,x_2,y_1,y_2) + a_5\, f_5(x_1,x_2,y_1,y_2) \end{aligned}$$

where

$$f_1(x_1,x_2,y_1,y_2) = x_1^4 - x_2^4, \qquad f_2(x_1,x_2,y_1,y_2) = x_1^3y_1 - x_1x_2^2y_1 + x_1^2x_2y_2 - x_2^3y_2,$$
$$f_3(x_1,x_2,y_1,y_2) = x_2^2y_1^2 - x_1^2y_2^2, \quad f_4(x_1,x_2,y_1,y_2) = x_1y_1^3 + x_2y_1^2y_2 - x_1y_1y_2^2 - x_2y_2^3$$
$$f_5(x_1,x_2,y_1,y_2) = y_1^4 - y_2^4$$

is a basis of $\mathcal{W}_6^{\mathsf{ev}}$. We then have

$$\begin{aligned}\langle p\,|T\ ,\mathfrak{R}^* - \mathfrak{R}^{\sqcup\!\sqcup}\rangle = a_1 G\binom{5}{1} + \frac{a_2-a_3}{4} G\binom{4}{2} - \Big(\frac{a_3}{36}+\frac{a_4}{4}\Big) G\binom{3}{3} \\ + \Big(\frac{a_4}{36} - \frac{5a_5}{8}\Big) G\binom{2}{4} + \frac{a_5}{30} G\binom{1}{5}\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}\langle p\,|T\Delta^*\ ,\mathfrak{G}_2\rangle = {} & a_1\left(-3\,G\binom{5,1}{0,0} + G\binom{4,2}{0,0} - 3\,G\binom{3,3}{0,0} + G\binom{2,4}{0,0}\right) \\ & + a_2\left(-\frac{3}{4}G\binom{4,1}{1,0} + \frac{1}{6}G\binom{3,2}{1,0} - \frac{1}{4}G\binom{3,2}{0,1} + \frac{1}{6}G\binom{2,3}{0,1}\right) \\ & + a_3\left(\frac{1}{4}G\binom{3,1}{2,0} + \frac{1}{6}G\binom{3,1}{1,1} + \frac{1}{4}G\binom{3,1}{0,2} - \frac{1}{12}G\binom{2,2}{2,0}\right. \\ & \left. - \frac{1}{2}G\binom{2,2}{1,1} - \frac{1}{12}G\binom{2,2}{0,2} + \frac{1}{6}G\binom{1,3}{1,1} + \frac{1}{4}G\binom{1,3}{0,2}\right) \\ & + a_4\left(\frac{1}{12}G\binom{2,1}{2,1} + \frac{1}{8}G\binom{2,1}{1,2} + \frac{1}{12}G\binom{1,2}{1,2} + \frac{1}{8}G\binom{1,2}{0,3}\right) \\ & + a_5\left(\frac{1}{6}G\binom{1,1}{3,1} + \frac{3}{4}G\binom{1,1}{2,2} + \frac{1}{6}G\binom{1,1}{1,3} + \frac{1}{8}G\binom{1,1}{0,4}\right).\end{aligned} \tag{3.27}$$

For any choice of $a_1,\dots,a_5 \in \mathbb{Q}$, this yields a relation between the right-hand sides of (3.26) and (3.27) that is symmetric in $Z(\mathsf{ev},\mathsf{ev})$.

Relations that are symmetric in $Z(\mathsf{ev},\mathsf{ev})$ that arise from applying the linear map $\langle f,\cdot\rangle$, for some $f \in \mathcal{V}_K$, to (3.24) modulo such relations that contain no $Z(\mathsf{odd},\mathsf{odd})$ are isomorphic to $\mathcal{W}_K^{\mathsf{ev}}$. Analogously to Theorem 2.49 we obtain the following theorem.

Theorem 3.63. *There is a isomorphism of vector spaces*

$$\mathcal{W}_K^{\mathsf{ev}} \cong \frac{\langle \textit{relations in } \mathcal{E}_K \textit{ which are symmetric in } Z(\mathsf{ev},\mathsf{ev})\rangle_{\mathbb{Q}}}{\langle \textit{relations in } \mathcal{E}_K \textit{ which are symmetric in } Z(\mathsf{ev},\mathsf{ev}) \textit{ and contain no } Z(\mathsf{odd},\mathsf{odd})\rangle_{\mathbb{Q}}}.$$

Remark 3.64. In other words, the space of polynomials which give symmetric relations in $Z(\mathsf{ev},\mathsf{ev})$ is isomorphic to $\mathcal{W}_k^{\mathsf{ev}} \oplus U_k$ where $f \in U_k$ yields relations via $\langle f,\cdot\rangle$ which are symmetric in $Z(\mathsf{ev},\mathsf{ev})$ but contain no $Z(\mathsf{odd},\mathsf{odd})$.

3.7. Applications 2: Quadratic relations in the Lie algebra $\mathfrak{lbs}$

Another occurrence of bi-period polynomials in the theory of multiple q-zeta values can be found in study of the *linearized balanced quasi shuffle Lie algebra* $\mathfrak{lbs}$. In this thesis we only need fragments of this theory. For more details, we refer to the forthcoming thesis of A. Burmester [Bur22].

We have a decomposition

$$\mathfrak{lbs} = \bigoplus_{k,r=0}^{\infty} \mathfrak{lbs}_{k,r} \subset \prod_{r=0}^{\infty} \mathbb{Q}[x_1, y_1, \dots, x_r, y_r]$$

where $\mathfrak{lbs}_{k,r}$ contains certain homogeneous polynomials in r variables of degree $k-r$. In fact, the pair $(\mathfrak{lbs}, \{\ ,\ \})$, where $\{\ ,\ \}$ can considered as a generalization of the Ihara bracket, is a bigraded Lie algebra. There are two occurrences of period polynomials in the theory of $\mathfrak{lbs}$.

1.) The subspace in depth $r=1$ is

$$\mathfrak{lbs}_1 = \bigoplus_{\substack{r\geq 1 \\ s\geq 0 \\ r+s \text{ odd}}} \mathbb{Q}\cdot \xi_{\binom{r}{s}}$$

where

$$\xi_{\binom{r}{s}}(x,y) = x^{r-1}y^s + x^s y^{r-1}, \qquad r+s \text{ odd}, \tag{3.28}$$

are even polynomials and the Lie bracket

$$\{\ ,\ \} \colon \mathfrak{lbs}_1 \times \mathfrak{lbs}_1 \longrightarrow \mathfrak{lbs}_2$$

is explicitly given for $f, g \in \mathfrak{lbs}_1$ by

$$\{f,g\}(x_1,x_2,y_1,y_2) = \big(f(x_1,y_1)g(x_2,y_2) - f(x_2,y_2)g(x_1,y_1)\big)\,\big|1+U+U^2\ . \tag{3.29}$$

The relations of $\mathfrak{lbs}$ in depth 2 are given by the kernel of $\{\ ,\ \} \colon \mathfrak{lbs}_1 \times \mathfrak{lbs}_1 \longrightarrow \mathfrak{lbs}_2$. We remark that the bracket factors through the exterior product $\mathfrak{lbs}_1 \wedge \mathfrak{lbs}_1$, i. e. we have a short exact sequence

$$0 \longrightarrow \ker(\{\ ,\ \}) \longrightarrow \mathfrak{lbs}_1 \wedge \mathfrak{lbs}_1 \longrightarrow \mathfrak{lbs}_2 \longrightarrow 0\ .$$

Since $\mathfrak{lbs}_1$ is spanned by polynomials of the form (3.28), we obtain that $\mathfrak{lbs}_1 \wedge \mathfrak{lbs}_1$ is isomorphic to the space that is spanned by homogeneous polynomials of the form

$$\Xi\begin{pmatrix} x_1, x_2 \\ y_1, y_2 \end{pmatrix} := \xi_{\binom{r_1}{s_1}}(x_1,y_1)\xi_{\binom{r_2}{s_2}}(x_2,y_2) - \xi_{\binom{r_1}{s_1}}(x_2,y_2)\xi_{\binom{r_2}{s_2}}(x_1,y_1)$$

with r_1, r_2, s_1, s_2 odd. A non-zero Ξ has degree $K-2$ where $K = r_1+r_2+s_1+s_2$ is even. We have $\Xi \in \ker(1+S)$ and since $\xi_{\binom{r}{s}}$ is an even polynomial, we observe that

$$\Xi\begin{pmatrix} -x_1, x_2 \\ -y_1, y_2 \end{pmatrix} = \Xi\begin{pmatrix} x_1, x_2 \\ y_1, y_2 \end{pmatrix} = \Xi\begin{pmatrix} x_1, -x_2 \\ y_1, -y_2 \end{pmatrix}. \tag{3.30}$$

Since $\xi_{\binom{r}{s}}(x,y) = \xi_{\binom{r}{s}}(y,x)$ we further have

$$\Xi\begin{pmatrix} x_1, y_2 \\ y_1, x_2 \end{pmatrix} = \Xi\begin{pmatrix} x_1, x_2 \\ y_1, y_2 \end{pmatrix} = \Xi\begin{pmatrix} y_1, x_2 \\ x_1, y_2 \end{pmatrix}. \tag{3.31}$$

Now we can deduce from (3.30) and (3.31) that

$$\Xi = \Xi\,|\nu = \Xi\begin{pmatrix} x_1, -x_2 \\ y_1, -y_2 \end{pmatrix} \tag{3.32}$$

and

$$\Xi = \Xi\,|\tau = \Xi\begin{pmatrix} x_1, -y_2 \\ y_1, -x_2 \end{pmatrix}. \tag{3.33}$$

On the other hand, the identities (3.30) and (3.31) also follow from (3.32) and (3.33) since $\Xi \in \ker(1+S)$ and the degree of Ξ is even. So we obtain that

$$\Xi \in \Big\{ f \in \mathcal{V}_K \;\Big|\; f\,|1+S = 0,\; f\,|\nu = f\,|\tau = f \Big\}.$$

By (3.29), the bracket on $\mathfrak{lbs}_1$ is given by applying the operator $\big|1+U+U^2$. Hence we obtain for the weight K component of the kernel that

$$\ker(\{\,,\,\})_K \cong \big\{ \Xi \in \mathcal{W}_K \;\big|\; \Xi = \Xi\,|\tau\,,\; \Xi = \Xi\,|\nu \,\big\} = \mathcal{W}_K^{\tau,\nu}.$$

Now Corollary 3.55 yields for even $K \geq 4$ that

$$\dim(\ker(\{\,,\,\})_K) = \dim \mathcal{W}_K^{\tau,\nu} = \sum_{\substack{n=4,\\ n \text{ even}}}^{K} \dim(M_n)^2$$

with $\dim(M_K) = \dim(W_K^-)$ where M_K denotes the space of modular forms of weight K (cf. section 2.5).

2.) Recall the discussion on generators in depth 4 from section 2.7. Ecalle's construction of these elements can be extended to construct generators in depth 4 for $\mathfrak{lbs}$. The explicit construction, however, is too extensive for this thesis, so we refer to [Bur22] for details. But we mention that the number of these generators is essentially given by $\dim(S_K)^2$ where S_K is the space of cusp forms of weight K (cf. section 2.5).

Conjecturally, the Lie algebra $\mathfrak{lbs}$ is generated by the elements $\xi_{\binom{r}{s}}$ from part 1.) and the generators in depth 4 we mentioned in part 2.). All relations are expected to be in depth 2 and 5. This would imply the dimension conjecture for $\mathcal{Z}_q$ by Bachmann–Kühn [BK20].

4. Representation theory for $\mathrm{SL}_2(\mathbb{C})$

This section follows [Hal15]. Throughout the section we let $n \in \mathbb{N}$ be an arbitrary natural number with $n \geq 2$ (unless stated otherwise).

Representation theory is mainly about describing algebraic structures, e. g. finite groups, Lie groups and Lie algebras, as linear transformations of vector spaces. Since we are interested in finite-dimensional representations of the group $\mathrm{SL}_2(\mathbb{C})$, we will focus on the theory of matrix Lie groups and restrict the discussion to finite-dimensional representations.

4.1. Matrix Lie groups and Lie algebras

We begin our discussion by defining some fundamental concepts.

Definition 4.1. A *matrix Lie group* is a subgroup $G \subseteq \mathrm{GL}_n(\mathbb{C})$ that is closed in $\mathrm{GL}_n(\mathbb{C})$, i. e. if a sequence in G converges entrywise to some matrix $A \in \mathrm{GL}_n(\mathbb{C})$ then $A \in G$.

Examples 4.2.

- The special linear group

$$\mathrm{SL}_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid \det(A) = 1\}$$

  is a matrix Lie group. In fact, $\mathrm{SL}_n(\mathbb{C})$ is closed under limits since the determinant is a continuous function.

- Let V be a finite-dimensional $\mathbb{C}$-vector space. By choosing a basis for V, we can identify the group of invertible linear transformations of V, denoted $\mathrm{GL}(V)$, with $\mathrm{GL}_n(\mathbb{C})$ where $n = \dim_{\mathbb{C}}(V)$. Note that the induced topology on $\mathrm{GL}(V)$, however, is independent of the choice of basis. Hence, $\mathrm{GL}(V)$ has a matrix Lie group structure.

Definition 4.3. A *finite-dimensional complex Lie algebra* is a finite-dimensional $\mathbb{C}$-vector space $\mathfrak{g}$ equipped with a map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called *bracket*, such that $[\cdot,\cdot]$

1. is bilinear,
2. is skew-symmetric, i. e. $[X, Y] = -[Y, X]$ for all $X, Y \in \mathfrak{g}$ and
3. satisfies the *Jacobi identity*

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

   for all $X, Y, Z \in \mathfrak{g}$.

A well-known example for Lie algebras are associative algebras.

Proposition 4.4. *For an associative algebra A define*

$$[X,Y] \coloneqq XY - YX$$

for all $X, Y \in A$. This yields a Lie algebra structure on A.

Remark 4.5. The bracket from Proposition 4.4 is called *commutator bracket*. The properties of a Lie algebra are immediate to verify. The associativity of the algebra is essential for the Jacobi identity to hold.

Example 4.6. Let V be a finite-dimensional $\mathbb{C}$-vector space. The space of endomorphisms on V, denoted $\mathrm{End}(V)$, is an associative $\mathbb{C}$-algebra. Hence we obtain a Lie algebra by Proposition 4.4. To emphasize this Lie algebra structure we write

$$\mathfrak{gl}(V) \coloneqq \mathrm{End}(V).$$

We now introduce an important map in the theory of matrix Lie groups. We therefore set $X^0 = 1$ for all $X \in M_n(\mathbb{C})$.

Lemma 4.7. *Let $X \in M_n(\mathbb{C})$ be a square matrix. Then the series*

$$\sum_{m=0}^{\infty} \frac{X^m}{m!}$$

converges absolutely.

Proof. Consider the Hilbert-Schmidt norm

$$\|X\| = \left(\sum_{1 \le i,j \le n} |X_{ij}|^2 \right)^{\frac{1}{2}}$$

which satisfies $\|XY\| \le \|X\| \, \|Y\|$ for all $X, Y \in M_n(\mathbb{C})$ due to the Cauchy-Schwarz inequality. For $m \in \mathbb{N}$ we thus have $\|X^m\| \le \|X\|^m$ and hence

$$\sum_{m=0}^{\infty} \left\| \frac{X^m}{m!} \right\| \le \sum_{m=0}^{\infty} \frac{\|X\|^m}{m!} < \infty.$$ □

Definition 4.8. For $X \in M_n(\mathbb{C})$ we define the *exponential of X* by

$$e^X \coloneqq \sum_{m=0}^{\infty} \frac{X^m}{m!}.$$

Lemma 4.9. *Let $X, Y \in M_n(\mathbb{C})$ be commuting matrices, i. e. $XY = YX$. We then have*

$$e^{X+Y} = e^X e^Y = e^Y e^X.$$

Proof. If X and Y commute we have

$$(X+Y)^m = \sum_{i=0}^{m} \binom{m}{i} X^i Y^{m-i}.$$

Since the series converge absolutely, we have on the other hand

$$e^X e^Y = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{i=0}^{m} \binom{m}{i} X^i Y^{m-i} = e^Y e^X$$

which implies the claim. □

Corollary 4.10. For $X \in M_n(\mathbb{C})$ we have $e^X \in \mathrm{GL}_n(\mathbb{C})$ with inverse given by e^{-X}.

Lemma 4.11 ([Hal15, Theorem 2.12])**.** *For any $X \in M_n(\mathbb{C})$ we have*

$$\det(e^X) = e^{\mathrm{trace}(X)}.$$

Given a matrix Lie group G, the matrix exponential allows us to associate a Lie algebra to it. For details, including a proof that this is in fact a Lie algebra, see [Hal15, Section 3.3].

Definition 4.12. Let G be a matrix Lie group. The *Lie algebra of G* is then given by

$$\mathfrak{g} = \left\{ X \in M_n(\mathbb{C}) \;\middle|\; \text{for all } t \in \mathbb{R} : e^{tX} \in G \right\}$$

equipped with the commutator bracket $[X, Y] = XY - YX$ for $X, Y \in \mathfrak{g}$.

Example 4.13. Consider the matrix Lie group $\mathrm{SL}_n(\mathbb{C})$ from Example 4.2. For $X \in M_n(\mathbb{C})$ we have $\det(e^X) = e^{\mathrm{trace}(X)}$ by Lemma 4.11. So if $e^{tX} \in \mathrm{SL}_n(\mathbb{C})$ for all $t \in \mathbb{R}$ we have

$$e^{t \cdot \mathrm{trace}(X)} = 1.$$

Taking the first derivative w. r. t. t and evaluating in $t = 0$ on both sides yields

$$\mathrm{trace}(X) = 0.$$

The Lie algebra of $\mathrm{SL}_n(\mathbb{C})$ is thus given by

$$\mathfrak{sl}_n(\mathbb{C}) := \{X \in M_n(\mathbb{C}) \mid \mathrm{trace}(X) = 0\}.$$

4.2. Simple connectedness of $\mathrm{SL}_2(\mathbb{C})$

Note that matrix Lie groups inherit a subspace topology of the standard topology on $M_n(\mathbb{C}) \cong \mathbb{C}^{n^2}$. This allows for a topological approach to matrix Lie groups. The topological properties connectedness and simple connectedness will turn out to be of particular interest for our discussion on representations of matrix Lie groups and their Lie algebras in section 4.3. In Propositions 4.31 and 4.32 we will see that the irreducible representations of simply connected matrix Lie groups are determined by their Lie algebra. This is useful since the irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ can be computed with basic linear algebra.

Definition 4.14. A matrix Lie group G is

- *connected* if $G = G_1 \cup G_2$ for disjoint open sets $G_1, G_2 \subset G$ implies that $G_1 = \emptyset$ or $G_2 = \emptyset$,
- *path connected* if for all $A, B \in G$ there is a continuous function $\gamma\colon [0,1] \to G$ with $\gamma(0) = A$ and $\gamma(1) = B$. We say that γ is a *path from A to B*.

Remark 4.15. The property of being path connected defines an equivalence relation on G. To show that a space is path connected, it thus suffices to check if all points are connected to one fixed point in the space. Another well-known topological fact is that path connected spaces are connected. In fact, a matrix Lie group is connected if and only if it is path connected. See [Hal15, Section 3.8] for more details on this.

Lemma 4.16. *The matrix Lie group $\mathrm{SL}_n(\mathbb{C})$ is path connected.*

Proof. Let $A \in \mathrm{SL}_n(\mathbb{C})$ be an arbitrary matrix. Since all complex square matrices are triangularisable we find $P \in \mathrm{GL}_n(\mathbb{C})$ and an upper triangular matrix B such that $A = PBP^{-1}$. We denote the values on the diagonal of B by $\lambda_1, \dots, \lambda_n$. Since $A \in \mathrm{SL}_n(\mathbb{C})$ we have $\lambda_1 \cdots \lambda_n = 1$. Set $D = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$. Then B is path connected to D (e. g., obtain a path by multiplying each entry of B except for the diagonal by $t \in [0,1]$), thus A is path connected to PDP^{-1}. Now since the multiplicative group $\mathbb{C}^\times$ is path connected, we find paths $\gamma_i(t)$, $t \in [0,1]$, from λ_i to 1 for each $i \in \{1, \dots, n-1\}$ and set $\gamma_n(t) = (\gamma_1(t) \cdots \gamma_{n-1}(t))^{-1}$. This yields a path from PDP^{-1} to the identity via

$$\gamma(t) \coloneqq P \cdot \mathrm{diag}(\gamma_1(t), \dots, \gamma_n(t)) \cdot P^{-1}.$$

This path indeed lies in $\mathrm{SL}_n(\mathbb{C})$ since for any $t \in [0,1]$ we have

$$\det(\gamma(t)) = \det(P) \cdot \gamma_1(t) \cdots \gamma_n(t) \cdot \det(P)^{-1} = 1. \qquad \square$$

Definition 4.17. Let G be a matrix Lie group. A *loop in G* is a path $\gamma\colon [0,1] \to G$ with $\gamma(0) = \gamma(1)$. The group G is *simply connected* if it is path connected and every loop in G is contractible, i. e. there is a continuous map $H\colon [0,1]^2 \to G$, called *homotopy*, such that for all $s, t \in [0,1]$ we have

$$H(s,0) = H(s,1), \qquad H(0,t) = \gamma(t) \qquad \text{and} \qquad H(1,t) = H(1,0).$$

Some interesting examples for simply connected spaces are given by spheres. The unit circle $S^1 \subset \mathbb{R}^2$, however, is a loop itself and thus not simply connected. For further details as well as a proof of the following lemma we refer to [Man15, Section 11.4].

Lemma 4.18. *The unit n-sphere*

$$S^n := \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$$

with standard norm $\|\cdot\|$ *is simply connected for all* $n \geq 2$.

Definition 4.19. A *homeomorphism* is a continuous and bijective map between topological spaces such that the inverse map is also continuous. Two spaces are *homeomorphic* if there exists a homeomorphism between them.

Remark 4.20. Homeomorphic spaces satisfy the same topological properties. This follows essentially from the definition. In particular, two homeomorphic spaces are simply connected if and only if either of the spaces is simply connected.

Definition 4.21. Let $n \in \mathbb{N}$. The *special unitary group of degree n* is given by

$$\mathrm{SU}(n) := \{A \in M_n(\mathbb{C}) \mid \det(A) = 1, A^* = A^{-1}\}$$

where $A^* := \overline{A^{\mathrm{t}}}$ is the *adjoint matrix to A*.

Lemma 4.22. *The group* SU(2) *is simply connected.*

Proof. It suffices to show that

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \in M_2(\mathbb{C}) \;\middle|\; |\alpha|^2 + |\beta|^2 = 1 \right\}$$

since $\mathbb{C} \cong \mathbb{R}^2$ then implies that SU(2) is homeomorphic to S^3 and hence simply connected by Lemma 4.18 and Remark 4.20. An explicit homeomorphism $\mathrm{SU}(2) \longrightarrow S^3$ is then given by

$$\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \longmapsto (\mathrm{Re}(\alpha), \mathrm{Im}(\alpha), \mathrm{Re}(\beta), \mathrm{Im}(\beta)).$$

For $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$, it is straight forward to verify that $\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$ has determinant 1 with inverse given by the adjoint matrix.

Now let $A \in \mathrm{SU}(2)$ be an arbitrary matrix and write

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We then have $A^* = A^{-1}$, i. e.

$$\begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

which implies $d = \bar{a}$ and $b = -\bar{c}$. By setting $\alpha := a$ and $\beta := c$ we obtain that A is of the claimed form. Now $\det(A) = 1$ implies that $|\alpha|^2 + |\beta|^2 = 1$. □

Definition 4.23. Let X be a topological space. A subspace $Y \subset X$ is a *deformation retract of* X if there is a continuous map $R\colon X \times [0,1] \longrightarrow X$ such that

$$R(x,0) = x, \qquad R(x,1) \in Y, \qquad R(y,t) = y$$

for all $x \in X$, $y \in Y$ and $t \in [0,1]$. The map R is a *deformation of* X *into* Y.

Lemma 4.24. *The matrix Lie group* $\mathrm{SL}_2(\mathbb{C})$ *is simply connected.*

Proof. By Lemma 4.24 we only need to show that loops in $\mathrm{SL}_2(\mathbb{C})$ are contractible. To do that, we show that $\mathrm{SU}(2)$ is a deformation retract of $\mathrm{SL}_2(\mathbb{C})$. This is sufficient, since any loop in $\mathrm{SL}_2(\mathbb{C})$ can then be continuously mapped into the simply connected space $\mathrm{SU}(2)$.

To obtain a deformation of $\mathrm{SL}_2(\mathbb{C})$ into $\mathrm{SU}(2)$ we use the Gram-Schmidt process. We therefore consider the projection operator $\mathrm{proj}_u(v) := \frac{\langle u,v\rangle}{\langle u,u\rangle} \cdot u$ for $u, v \in \mathbb{C}^2$ with the standard scalar product $\langle \cdot, \cdot \rangle$ and write $A = (a_1, a_2)$ with column vectors $a_1, a_2 \in \mathbb{C}^2$. Now consider the continuous maps

$$\begin{aligned} r_t\colon \mathrm{SL}_2(\mathbb{C}) &\longrightarrow \mathrm{SL}_2(\mathbb{C}), & A &\mapsto (a_1, a_2 - t \cdot \mathrm{proj}_{a_1}(a_2)) \\ p_t\colon \mathrm{SL}_2(\mathbb{C}) &\longrightarrow \mathrm{SL}_2(\mathbb{C}), & A &\mapsto \left(\frac{a_1}{\|a_1\|^t}, \|a_1\|^t \cdot \big(a_2 - \mathrm{proj}_{a_1}(a_2)\big) \right) \end{aligned}$$

for all $t \in [0,1]$. To see that $r_t(A) \in \mathrm{SL}_2(\mathbb{C})$ note that elementary column operations do not change the determinant. Now set

$$\begin{aligned} R\colon \mathrm{SL}_2(\mathbb{C}) \times [0,1] &\longrightarrow \mathrm{SL}_2(\mathbb{C}) \\ (A,t) &\longmapsto \begin{cases} r_{2t}(A), & t < \frac{1}{2} \\ p_{2t-1}(A), & t \geq \frac{1}{2}. \end{cases} \end{aligned}$$

We claim that R is a deformation of $\mathrm{SL}_2(\mathbb{C})$ into $\mathrm{SU}(2)$. First note that $r_1(A) = p_0(A)$ and $r_0 = \mathrm{id}_{\mathrm{SL}_2(\mathbb{C})}$. If $A \in \mathrm{SU}(2)$ then $\langle a_1, a_2 \rangle = 0$ and $\|a_1\| = 1$, hence $R(A,t) = A$ for all $t \in [0,1]$. To prove that

$$R(A,1) = \left(\frac{a_1}{\|a_1\|}, \|a_1\| \cdot \big(a_2 - \mathrm{proj}_{a_1}(a_2)\big) \right)$$

is unitary for all $A \in \mathrm{SL}_2(\mathbb{C})$, it suffices to show that the column vectors yield an orthonormal basis of $\mathbb{C}^2$. The vectors are orthogonal since

$$\left\langle \frac{a_1}{\|a_1\|}, \|a_1\| \cdot \big(a_2 - \mathrm{proj}_{a_1}(a_2)\big) \right\rangle = \big\langle a_1, a_2 - \mathrm{proj}_{a_1}(a_2) \big\rangle = \langle a_1, a_2 \rangle - \frac{\langle a_1, a_2 \rangle}{\langle a_1, a_1 \rangle} \langle a_1, a_1 \rangle = 0.$$

And since the vectors are orthogonal, the determinant is 1 and $\frac{a_1}{\|a_1\|}$ is normalized, we also obtain that the second vector has norm 1. □

4.3. Representations of matrix Lie groups and Lie algebras

Definitions 4.25.

- Let G and H be matrix Lie groups. A continuous group homomorphism $\Pi\colon G \to H$ is called *Lie group homomorphism.*
- Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras. A *Lie algebra homomorphism* is a linear map $\pi\colon \mathfrak{g} \to \mathfrak{h}$ such that
$$\pi([X,Y]) = [\pi(X), \pi(Y)]$$
for all $X, Y \in \mathfrak{g}$.

Definition 4.26. Let V be a finite-dimensional $\mathbb{C}$-vector space.

- Let G be a matrix Lie group. A Lie group homomorphism
$$\Pi\colon G \to \mathrm{GL}(V)$$
is called *a (finite-dimensional) representation of G.*
- Let $\mathfrak{g}$ be a Lie algebra. A Lie algebra homomorphism
$$\pi\colon \mathfrak{g} \to \mathfrak{gl}(V)$$
is called *a (finite-dimensional) representation of $\mathfrak{g}$.*

In either case, the dimension of V is called the *dimension* of the representation.

Definition 4.27. Let $V_1, \dots, V_n$ be finite-dimensional $\mathbb{C}$-vector spaces.

- Let G be a matrix Lie group and $\Pi_1, \dots, \Pi_n$ be representations of G acting on $V_1, \dots, V_n$, respectively. The *direct sum of* $\Pi_1, \dots, \Pi_n$ is given by
$$\begin{aligned} \Pi_1 \oplus \cdots \oplus \Pi_n \colon G &\longrightarrow \mathrm{GL}(V_1 \oplus \cdots \oplus V_n) \\ A &\longmapsto ((v_1, \dots, v_n) \mapsto (\Pi_1(A)v_1, \dots, \Pi_n(A)v_n)) \,. \end{aligned}$$
- Let $\mathfrak{g}$ be a Lie algebra and $\pi_1, \dots, \pi_n$ be representations of $\mathfrak{g}$ acting on $V_1, \dots, V_n$, respectively. The *direct sum of* $\pi_1, \dots, \pi_n$ is given by
$$\begin{aligned} \pi_1 \oplus \cdots \oplus \pi_n \colon \mathfrak{g} &\longrightarrow \mathfrak{gl}(V_1 \oplus \cdots \oplus V_n) \\ X &\longmapsto ((v_1, \dots, v_n) \mapsto (\pi_1(X)v_1, \dots, \pi_n(X)v_n)) \,. \end{aligned}$$

It follows immediately from the definition that the direct sum of representations is again a representation.

Example 4.28. The slash operator from Definition 2.3 can be thought of as a representation of $\mathrm{SL}_2(\mathbb{C})$.[4] Let $\Pi\colon \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}(V_k)$ denote the slash operator. Then the bi-slash operator from Definition 3.3 is a direct sum of two slash operators, namely

$$\Pi(\gamma) \oplus \Pi(\gamma^{-\mathsf{t}}).$$

Definitions 4.29.

- Let G be a matrix Lie group and $\Pi\colon G \to \mathrm{GL}(V)$ be a finite-dimensional representation of G. A subspace $W \subseteq V$ is called *invariant* if

$$\Pi(A)(w) \in W \quad \text{for all } A \in G, w \in W.$$

- Let $\mathfrak{g}$ be a Lie algebra and $\pi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a finite-dimensional representation of $\mathfrak{g}$. A subspace $W \subseteq V$ is called *invariant* if

$$\pi(X)(w) \in W \quad \text{for all } X \in \mathfrak{g}, w \in W.$$

Furthermore, a proper subspace $W \subsetneqq V$ is *nontrivial* if $W \neq \{0\}$. A representation of a Lie algebra or matrix Lie group, respectively, is *irreducible* if it has no nontrivial invariant subspaces.

Proposition 4.30 ([Hal15, Proposition 4.4])**.** *Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$, V be a finite-dimensional $\mathbb{C}$-vector space and $\Pi\colon G \to \mathrm{GL}(V)$ be a representation of G. Then there is a unique representation $\pi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ such that*

$$\Pi(e^X) = e^{\pi(X)}$$

for all $X \in \mathfrak{g}$. The representation π can be explicitly computed as

$$\pi(X) = \frac{\mathrm{d}}{\mathrm{d}t}\Pi(e^{tX})\bigg|_{t=0}.$$

For a representation Π of a matrix Lie group we call the Lie algebra representation π from Proposition 4.30 the *associated representation to* Π. The following proposition gives strong connections between associated representations for connected matrix Lie groups.

Proposition 4.31 ([Hal15, Proposition 4.5])**.** *Let G be a connected matrix Lie group with Lie algebra $\mathfrak{g}$.*

1. *Let Π be a representation of G and π the associated representation of $\mathfrak{g}$. Then Π is irreducible if and only if π is irreducible.*

[4] One needs to replace $\gamma \cdot z$ with $\gamma^{-1} \cdot z$ in the definition since representations naturally induce left instead of right actions. See Definition 4.35 and Lemma 4.36 for details.

2. *Let Π_1 and Π_2 be representations of G and let π_1 and π_2 be the associated Lie algebra representations. Then π_1 and π_2 are isomorphic if and only if Π_1 and Π_2 are isomorphic.*

By Proposition 4.30 we can always associate a Lie algebra representation to a given Lie group representation. On the other hand, it is in general not possible to associate a Lie group representation to a given Lie algebra representation. The following proposition, however, states that this works if the matrix Lie group is simply connected.

Proposition 4.32 ([Hal15, Theorem 5.6])**.** *Let G and H be matrix Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively, and let $\pi\colon \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism. If G is simply connected, there exists a unique Lie group homomorphism $\Pi\colon G \to H$ such that*

$$\Pi\left(e^X\right) = e^{\pi(X)} \quad \textit{for all } X \in \mathfrak{g}.$$

We now established some connections between representations of Lie groups and of the associated Lie algebras. In addition to that, we want also to describe connections between different representations of the same Lie group or Lie algebra. We therefore consider the following definition.

Definition 4.33. Let V and W be a finite-dimensional $\mathbb{C}$-vector spaces.

- Let G be a matrix Lie group, $\Pi_1\colon G \to \mathrm{GL}(V)$ and $\Pi_2\colon G \to \mathrm{GL}(W)$ be representations of G. A linear map $\phi\colon V \to W$ such that

  $$\phi(\Pi_1(A)(v)) = \Pi_2(A)(\phi(v)) \qquad \text{for all } A \in G,\ v \in V$$

  is called an *intertwining map*.

- Let $\mathfrak{g}$ be a Lie algebra, $\pi_1\colon \mathfrak{g} \to \mathfrak{gl}(V)$ and $\pi_2\colon \mathfrak{g} \to \mathfrak{gl}(W)$ be representations of g. A linear map $\phi\colon V \to W$ such that

  $$\phi(\pi_1(X)(v)) = \pi_2(X)(\phi(v)) \qquad \text{for all } X \in \mathfrak{g},\ v \in V$$

  is called an *intertwining map*.

In either case, an *isomorphism of representations* is an intertwining map ϕ that is a vector space isomorphism. If there exists such an isomorphism, the representations are *isomorphic*.

4.4. Finite-dimensional irreducible representations of $\mathrm{SL}_2(\mathbb{C})$

In this section we want to characterize the irreducible representations of $\mathrm{SL}_2(\mathbb{C})$ up to isomorphism. To do so, we will first define a n-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$ for each $n \in \mathbb{N}$. To show the irreducibility of these representations, we first use Proposition 4.30 to compute the associated representations of $\mathfrak{sl}_2(\mathbb{C})$. Since $\mathrm{SL}_2(\mathbb{C})$ is simply connected by Lemma 4.24, it then suffices by Proposition 4.31 to show that the associated representations of $\mathfrak{sl}_2(\mathbb{C})$ are irreducible. To characterize the irreducible representations, we will finally show in Theorem 4.43 that irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ are uniquely determined (up to isomorphism) by their dimension. This will be sufficient for $\mathrm{SL}_2(\mathbb{C})$ due to Proposition 4.31.

Remark 4.34. For $n \geq 2$, we denote the space of homogeneous polynomials of degree $n-2$ over $\mathbb{C}$ by

$$V_{n,\mathbb{C}} = \{f = f(x,y) \in \mathbb{C}[x,y] \mid f \text{ homogeneous}, \deg(f) = n-2\}.$$

Definition 4.35. We define $\Pi_n\colon \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}(V_{n+1,\mathbb{C}})$ which acts on $f \in V_{n+1,\mathbb{C}}$ via

$$A \mapsto \left(f \mapsto f\left((A^{-1}\cdot z)^{\mathsf{t}}\right)\right)$$

where $z = (x,y)^{\mathsf{t}}$.

Lemma 4.36. *The map Π_n is an n-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$.*

Proof. By Remark 2.2 we have $\dim(V_{n+1,\mathbb{C}}) = n$. Let $A, B \in \mathrm{SL}_2(\mathbb{C})$ and $f \in V_{n+1,\mathbb{C}}$. Note that $f(A^{-1}\cdot z)$ is again a homogeneous polynomial of degree $n-1$ (cf. Proposition 2.4). And since

$$\begin{aligned}\Pi_n(A)(\Pi_n(B)(f)) &= \Pi_n(B)(f((A^{-1}\cdot z)^{\mathsf{t}})) = f\left(((B^{-1}A^{-1}\cdot z)^{\mathsf{t}})\right)\\ &= f\left(((AB)^{-1}\cdot z)^{\mathsf{t}}\right) = \Pi_n(AB)(f)\end{aligned}$$

the map Π_n is in fact a group homomorphism. □

Remark 4.37. We denote the associated representation to Π_n from Proposition 4.30 by $\pi_n\colon \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(V_{n+1,\mathbb{C}})$. For some $X \in \mathfrak{sl}_2(\mathbb{C})$ this representation acts on $f \in V_{n+1,\mathbb{C}}$ via

$$\pi_n(X)f(z) = \frac{\mathrm{d}}{\mathrm{d}t} f\left(e^{-tX}\cdot z\right)\Big|_{t=0}. \tag{4.1}$$

In order to work with π_n, we will first describe the action (4.1) more explicitly.

Lemma 4.38. *For $X \in \mathfrak{sl}_2(\mathbb{C})$ with*

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the action of $\pi_n(X)$ is given by

$$\pi_n(X) = -\frac{\partial}{\partial x}\cdot(ax+by) - \frac{\partial}{\partial y}\cdot(cx+dy).$$

Proof. Consider the curve $z(t) = e^{-tX} \cdot z$ and denote the coordinates by $z(t) = (x(t), y(t))$. Since each entry $(e^{tX})_{i,j}$ is given by a absolutely convergent power series in t we can compute the derivative term by term, hence

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{-tX} = \sum_{m=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t} t^m \frac{(-X)^m}{m!} = \sum_{m=1}^{\infty} t^{m-1} \frac{(-X)^m}{(m-1)!} = (-X) \cdot \sum_{m=0}^{\infty} \frac{(-tX)^m}{m!} = (-X) \cdot e^{-tX}.$$

This yields

$$\frac{\mathrm{d}}{\mathrm{d}t} z(t)\bigg|_{t=0} = -X \cdot z.$$

For some $f \in V_{n+1,\mathbb{C}}$ the chain rule now implies that

$$\pi_n(X) f = \frac{\mathrm{d}}{\mathrm{d}t} f(z(t))\bigg|_{t=0} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}\bigg|_{t=0} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}\bigg|_{t=0} = -\frac{\partial f}{\partial x} \cdot (ax+by) - \frac{\partial f}{\partial y} \cdot (cx+dy). \quad \square$$

Our next goal is to show that each π_n is irreducible. We will therefore make use of the vector space structure on $\mathfrak{sl}_2(\mathbb{C})$. Recall from Example 4.13 that the underlying set of $\mathfrak{sl}_2(\mathbb{C})$ consists of 2×2-matrices with vanishing trace. A basis is thus given by

$$H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The commutator bracket on $\mathfrak{sl}_2(\mathbb{C})$ yields the relations

$$[H, X] = 2X, \qquad [H, Y] = -2Y, \qquad [X, Y] = H. \tag{4.2}$$

Lemma 4.38 then yields

$$\begin{aligned} \pi_n(H) &= -x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \\ \pi_n(X) &= -y\frac{\partial}{\partial x} \\ \pi_n(Y) &= -x\frac{\partial}{\partial y} \end{aligned}$$

Now recall the standard basis on $V_{n+1,\mathbb{C}}$ from Remark 2.2 of the form $x^{n-1-i}y^i$ for $i \in \{0, \ldots, n-1\}$. By applying the operators above to this basis we obtain

$$\pi_n(H)(x^{n-1-i}y^i) = (-n+1+2i) \cdot x^{n-1-i}y^i \tag{4.3}$$
$$\pi_n(X)(x^{n-1-i}y^i) = -(n-1-i) \cdot x^{n-i-2}y^{i+1} \tag{4.4}$$
$$\pi_n(Y)(x^{n-1-i}y^i) = -i \cdot x^{n-i}y^{i-1}. \tag{4.5}$$

In particular, (4.3) shows that each basis element $x^{n-1-i}y^i$ is an eigenvector for $\pi_n(H)$ with eigenvalue $(-n+1+2i)$. The operators $\pi_n(X)$ and $\pi_n(Y)$ shift the exponent up or down by 1, respectively. We are now ready to prove the irreducibility of π_n.

Proposition 4.39. *The representation π_n is irreducible.*

Proof. Let $W \subseteq V_{n+1,\mathbb{C}}$ be an invariant subspace with $W \neq \{0\}$. It suffices to show that $W = V_{n+1,\mathbb{C}}$. Now let $f \in W$ be a non-zero element with

$$f(x,y) = a_0 x^{n-1} + a_1 x^{n-2} y + \cdots + a_{n-1} y^{n-1}$$

for some $a_i \in \mathbb{C}$.

Since $f \neq 0$ there exists some $i_0 = \min\{i \in \{0, \dots, n-1\} \mid a_i \neq 0\}$. Applying $\pi_n(X)^{n-1-i_0}$ to f yields by (4.4) a non-zero multiple of y^{n-1}. This is because $n-1-i$ vanishes if and only if $i = n-1$ but if $i_0 = n-1$ we already have $f = a_{n-1} y^{n-1}$ with $a_{n-1} \neq 0$. Since W is assumed to be invariant, we thus have $y^{n-1} \in W$.

Now for all $i \in \{1, \dots, n-1\}$ we obtain by (4.5) that $\pi_n(Y)^i(y^{n-1})$ is a non-zero multiple of $x^i y^{n-1-i}$. The invariance of W again yields that $x^i y^{n-1-i} \in W$. Since $W \subseteq V_{n+1,\mathbb{C}}$ therefore contains a basis of $V_{n+1,\mathbb{C}}$, we have $W = V_{n+1,\mathbb{C}}$ and the proof follows. □

Our next and final goal for this section is to show that any given irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ is isomorphic to some π_n. From now on, let $\pi\colon \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(V)$ be an arbitrary n-dimensional irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$. We will first prove two lemmas that will be needed to prove Theorem 4.43.

Lemma 4.40. *Let $u \in V$ be an eigenvector of $\pi(H)$ with eigenvalue $\lambda \in \mathbb{C}$. We then have*

$$\pi(H)\pi(X)u = (\lambda + 2) \cdot \pi(X)u$$
$$\pi(H)\pi(Y)u = (\lambda - 2) \cdot \pi(Y)u.$$

Proof. Since π is a Lie algebra homomorphism we have due to (4.2) that

$$[\pi(H), \pi(X)] = \pi([H,X]) = 2\pi(X)$$
$$[\pi(H), \pi(Y)] = \pi([H,Y]) = -2\pi(Y).$$

Hence

$$\pi(H)\pi(X)u = \pi(X)\pi(H)u + 2\pi(X)u = (\lambda+2) \cdot \pi(X)u$$
$$\pi(H)\pi(Y)u = \pi(Y)\pi(H)u - 2\pi(Y)u = (\lambda-2) \cdot \pi(Y)u.$$ □

Remark 4.41. Note that Lemma 4.40 implies that $\pi(X)u$ and $\pi(Y)u$ are eigenvectors of $\pi(H)$, unless they vanish, with eigenvalues $\lambda + 2$ and $\lambda - 2$, respectively.

Lemma 4.42. *Let $u \in V$ be an eigenvector of $\pi(H)$ with eigenvalue $\lambda \in \mathbb{C}$, $N \in \mathbb{N}_0$ such that $u_0 := \pi(X)^N u \neq 0$ but $\pi(X)^{N+1} u = 0$ and set $u_k := \pi(Y)^k u_0$ for all $k \geq 1$. We then have for all $k \geq 1$ that*

$$\pi(X)u_k = k\big((\lambda + 2N) - (k-1)\big)u_{k-1}. \tag{4.6}$$

Proof. For $\tilde{\lambda} = \lambda + 2N$ we obtain by applying Lemma 4.40 repeatedly that

$$\pi(H)u_0 = \tilde{\lambda}u_0$$

and

$$\pi(H)u_k = (\tilde{\lambda} - 2k)u_k.$$

Now observe that (4.2) yields $[\pi(X), \pi(Y)] = \pi(H)$ and thus

$$\pi(X)\pi(Y) = \pi(H) + \pi(Y)\pi(X).$$

We now prove (4.6) by induction on k.

The claimed identity holds for $k = 1$ since

$$\pi(X)u_1 = \pi(X)\pi(Y)u_0 = (\pi(H) + \pi(Y)\pi(X))u_0 = \pi(H)u_0 = \tilde{\lambda}u_0.$$

Now assume (4.6) holds for some $k \geq 1$. We then have

$$\begin{aligned} \pi(X)u_{k+1} &= \pi(X)\pi(Y)u_k = (\pi(H) + \pi(Y)\pi(X))u_k \\ &\overset{\text{IH}}{=} \pi(H)u_k + k\big(\tilde{\lambda} - (k-1)\big)\pi(Y)u_{k-1} \\ &= (\tilde{\lambda} - 2k)u_k + k\big(\tilde{\lambda} - (k-1)\big)u_k \\ &= (k+1)\big(\tilde{\lambda} - k)u_k. \end{aligned}$$

□

Theorem 4.43. *Let π be an irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$. Then π is isomorphic to π_n from* (4.1) *for some $n \in \mathbb{N}$.*

Proof. Let π act on the finite-dimensional vector space V. Since $\mathbb{C}$ is algebraically closed the operator $\pi(H)$ has an eigenvector u and we denote the corresponding eigenvalue by λ. By Lemma 4.40 we have

$$\pi(H)\pi(X)^k u = (\lambda + 2k) \cdot \pi(X)^k u$$

for all $k \in \mathbb{N}$. Since V is finite-dimensional, the operator $\pi(H)$ only has finitely many eigenvalues. So we have $\pi(X)^k u = 0$ for almost all $k \in \mathbb{N}$. But since $\pi(H)u$ does not vanish, we find in particular some $N \geq 0$ such that

$$\pi(X)^N u \neq 0 \quad \text{and} \quad \pi(X)^{N+1}u = 0.$$

We denote $u_0 \coloneqq \pi(X)^N u$ and have $\pi(H)u_0 = (\lambda + 2N) \cdot u_0$. Now define $u_k \coloneqq \pi(Y)^k u_0$ for all $k \geq 0$. Applying Lemma 4.40 to the eigenvector u_0 of $\pi(H)$ yields that

$$\pi(H)u_k = (\lambda + 2N - 2k)u_k.$$

Again, since $\pi(H)$ only has finitely many eigenvalues we find some $M \in \mathbb{N}_0$ such that $u_k \neq 0$ for all $k \leq M$ but $u_{M+1} = 0$. By Lemma 4.42 we now have

$$0 = \pi(X)u_{M+1} = (M+1)(\lambda + 2N - M)u_M.$$

Since $M + 1 > 0$ we obtain $\lambda + 2N = M$. So in conclusion, we have non-zero vectors $u_0, \dots, u_M$ with

$$\begin{aligned} \pi(H)u_k &= (M-2k)u_k \quad \text{for all } k \le M \\ \pi(Y)u_k &= \begin{cases} u_{k+1}, & \text{if } k < M \\ 0, & \text{if } k = M \end{cases} \\ \pi(X)u_k &= \begin{cases} k(M-(k-1))u_{k-1}, & \text{if } k > 0 \\ 0, & \text{if } k = 0 \end{cases} \end{aligned} \tag{4.7}$$

Now consider the span V' of $u_0, \dots, u_M$. Since they are eigenvectors of $\pi(H)$ with distinct eigenvalues, they are linearly independent. So we have $\dim V' = M + 1$ and clearly $V' \subseteq V$. But V' is invariant under $\pi(H)$, $\pi(X)$ and $\pi(Y)$. Since H, X, Y is a basis for $\mathfrak{sl}_2(\mathbb{C})$, the space is in fact an invariant subspace of V. This yields by our assumption that $V' = V$.

So if π_1 and π_2 are any irreducible representations acting on vector spaces V_1 and V_2 with $\dim(V_1) = n = \dim(V_2)$ then we find bases $\{u_0, \dots, u_{n-1}\}$ and $\{v_0, \dots, v_{n-1}\}$ for V_1 and V_2, respectively. Since both representations are uniquely described by (4.7) w. r. t. their respective basis, we obtain an isomorphism between π_1 and π_2 via $u_i \mapsto v_i$. □

Combining Proposition 4.39 and Theorem 4.43 implies that there is (up to isomorphism) exactly one n-dimensional irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ for each $n \in \mathbb{N}$. This also follows for the irreducible representations of $\mathrm{SL}_2(\mathbb{C})$ by Proposition 4.31. In particular, any (finite-dimensional) irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ is isomorphic to Π_n from Definition 4.35 for some $n \in \mathbb{N}$.

5. Generalized period polynomials

In this chapter, we introduce multivariate extensions of period polynomials. To define an analogue of the slash operator, we use the irreducible n-dimensional representation Π_n from Definition 4.35. We generally proceed similarly to chapters 2 and 3. After defining the space of generalized period polynomials $W_k^{(n)}$ in section 5.1 we will discuss the analogue of the Lewis space in section 5.2. We conjecture that the spaces agree in all cases where $W_k^{(n)}$ is non-trivial. This conjecture could not be proven in the scope of this master thesis as it remains unclear how to describe the space $E_k^{(n)}$ of $\mathrm{SL}_2(\mathbb{Z})$-invariant polynomials. In section 5.3 we introduce a non-degenerate pairing on the space of homogeneous polynomials in n variables. This pairing can be seen as a natural generalization of the pairing from section 2.3. We will prove that this pairing is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$ (Theorem 5.21). In section 5.4 we attempt to compute the dimension of $W_k^{(n)}$ similarly to sections 2.4 and 3.4. However, we can only provide a dimension formula that depends on the dimension of $E_k^{(n)}$.

5.1. Slash operator

Definition 5.1. For $n, k \in \mathbb{N}$ with $k \geq n$ we denote the space of homogeneous polynomials in n variables by

$$V_k^{(n)} := \{f = f(x_1, \dots, x_n) \in \mathbb{Q}[x_1, \dots, x_n] \mid f \text{ homogeneous}, \deg(f) = k - n\}.$$

If k and n are clear from the context, we also use the shorthand notation $d = k - n$.

Remark 5.2. Note that $V_k = V_k^{(2)}$ and $\mathcal{V}_k = V_{k+2}^{(4)}$.

Lemma 5.3. *For $k \geq n$ we have*

$$\dim V_k^{(n)} = \binom{k-1}{n-1}.$$

Proof. Observe that a basis for $V_k^{(n)}$ is given by

$$\left\{ x_1^{i_1} \cdots x_n^{i_n} \;\middle|\; (i_1, \dots, i_n) \in \mathbb{N}_0^n, i_1 + \cdots + i_n = k - n \right\}.$$

We can use the combinatorial method of *stars and bars* to count the number of basis vectors. Each tuple $(i_1, \dots, i_n) \in \mathbb{N}_0^n$ with $i_1 + \cdots + i_n = k - n$ can be uniquely depicted by placing $n-1$ bars ($|$) amongst $k-n$ stars ($\star$) via

$$\underbrace{\star \cdots \star}_{i_1} \mid \underbrace{\star \cdots \star}_{i_2} \mid \cdots \mid \underbrace{\star \cdots \star}_{i_n}. \tag{5.1}$$

Since there are $k-n$ stars it is clear that $i_1+\cdots+i_n=k-n$ and by possibly placing bars at the beginning, the end and next to one another we also have $(i_1,\ldots,i_n)\in\mathbb{N}_0^n$. Observe that (5.1) contains a total of $k-1$ objects. Hence the number of possible arrangements is given by $\binom{k-1}{n-1}$. □

Corollary 5.4. Since

$$\binom{k-1}{n-1}=\binom{k-1}{k-n}$$

there is an isomorphism of vector spaces

$$V_k^{(n)}\cong V_k^{(k-n+1)}.$$

To define an analogue of the slash operator from Definition 2.3, we use the irreducible representations Π_n from Definition 4.35 and consider the inclusion $\mathrm{SL}_2(\mathbb{Z})\hookrightarrow\mathrm{SL}_2(\mathbb{C})$. Since representations naturally induce left group actions, we will instead consider the corresponding right action. So for $\gamma\in\mathrm{SL}_2(\mathbb{Z})$ this right action is given by $\Pi_n(\gamma^{-1})$, i. e.

$$\gamma\mapsto\left(f(x,y)\mapsto f\left(\left(\gamma\cdot\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)\right)^{\mathsf{t}}\right)\right)$$

where $f\in V_{n+1,\mathbb{C}}$.

Definition 5.5. For $\gamma\in\mathrm{SL}_2(\mathbb{Z})$ we denote the transformation matrix of $\Pi_n(\gamma^{-1})\in\mathrm{GL}_n(\mathbb{Q})$ by $\tilde{\gamma}^{(n)}$. If n is clear from the context, we may omit the superscript.

Example 5.6. For $n=6$ we have

$$\tilde{S}=\begin{pmatrix}0&0&0&0&0&-1\\0&0&0&0&1&0\\0&0&0&-1&0&0\\0&0&1&0&0&0\\0&-1&0&0&0&0\\1&0&0&0&0&0\end{pmatrix}\qquad\text{and}\qquad\tilde{U}=\begin{pmatrix}1&-5&10&-10&5&-1\\1&-4&6&-4&1&0\\1&-3&3&-1&0&0\\1&-2&1&0&0&0\\1&-1&0&0&0&0\\1&0&0&0&0&0\end{pmatrix}.$$

The source code that was used to compute these matrices can be found in appendix C.3.

Definition 5.7. We define the *slash operator* on the space $V_k^{(n)}$ by

$$\begin{aligned}\mathrm{SL}_2(\mathbb{Z})\times V_k^{(n)}&\longrightarrow V_k^{(n)}\\(\gamma,f)&\longmapsto f\,|\tilde{\gamma}:=f\left((\tilde{\gamma}\cdot z)^{\mathsf{t}}\right)\end{aligned}$$

where $z=(x_1,\ldots,x_n)^{\mathsf{t}}$.

Recall from Lemmas 2.5 and 3.5 that homogeneous polynomials are invariant under the action of $|-1$ in all cases where the space of (bi-)period polynomials is non-trivial. The following lemma specifies the cases for which $V_k^{(n)}$ is invariant under $|-1$. We will later see that these are again the non-trivial cases we will be interested in (see Lemma 5.12).

Lemma 5.8. *For $n, k \in \mathbb{N}$ with $k \geq n$ such that n even implies that k is even, the space $V_k^{(n)}$ is invariant under $|-1$, i. e. for $f \in V_k^{(n)}$ we have*

$$f\,|(-1) = f.$$

Proof. For $f \in V_k^{(n)}$ we write

$$f = \sum_{\substack{\alpha_1,\dots,\alpha_n \geq 0, \\ \alpha_1+\dots+\alpha_n = d}} a_{\alpha_1,\dots,\alpha_n} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in V_k^{(n)}. \tag{5.2}$$

Since

$$\Pi_n(-\operatorname{id}_2) = (-1)^{n+1} \operatorname{id}_n$$

acts trivially for odd n, this case is clear. If n is even, then k is even by assumption. In this case, the degree $d = k - n$ is even and we have

$$f|(-1) = \sum_{\substack{\alpha_1,\dots,\alpha_n \geq 0, \\ \alpha_1+\dots+\alpha_n = d}} a_{\alpha_1,\dots,\alpha_n} (-x_1)^{\alpha_1} \cdots (-x_n)^{\alpha_n} = (-1)^d \cdot f = f. \qquad \square$$

Remark 5.9. In chapters 2 and 3 we considered an action of the general linear group $\mathrm{GL}_2(\mathbb{Z})$ instead of $\mathrm{SL}_2(\mathbb{Z})$. Note that Definition 5.5 also allows us to compute actions of $\mathrm{GL}_2(\mathbb{Z})$-matrices.
We therefore extend the $\mathrm{SL}_2(\mathbb{Z})$-action on $V_k^{(n)}$ first to an action of $\mathrm{GL}_2(\mathbb{Z})$ and then to an action of the group ring $\mathbb{Z}[\mathrm{GL}_2(\mathbb{Z})]$ on $V_k^{(n)}$ analogously to Definition 2.12.

Definition 5.10. We denote the eigenspaces of the operator $|\tilde{\epsilon}$ on $V_k^{(n)}$ with eigenvalues 1 and -1, respectively, by

$$V_k^{(n),+} := \left\{ f \in V_k^{(n)} \;\middle|\; f\,|\tilde{\epsilon} = f \right\} \qquad \text{and} \qquad V_k^{(n),-} := \left\{ f \in V_k^{(n)} \;\middle|\; f\,|\tilde{\epsilon} = -f \right\}$$

and the eigenspaces of the operator $\Big|\tilde{\delta}$ on $V_k^{(n)}$ with eigenvalues 1 and -1, respectively, by

$$V_k^{(n),\mathsf{ev}} := \left\{ f \in V_k^{(n)} \;\middle|\; f\,\Big|\tilde{\delta} = f \right\} \qquad \text{and} \qquad V_k^{(n),\mathsf{odd}} := \left\{ f \in V_k^{(n)} \;\middle|\; f\,\Big|\tilde{\delta} = -f \right\}.$$

Furthermore, for a subspace $W \subseteq V_k^{(n)}$ we set $W^\bullet := W \cap V_k^{(n),\bullet}$ for all $\bullet \in \{+, -, \mathsf{ev}, \mathsf{odd}\}$.

Definition 5.11. The space of *generalized period polynomials* is given by

$$W_k^{(n)} := \left\{ f \in V_k^{(n)} \;\middle|\; f\,\Big|1 + \tilde{S} = f\,\Big|1 + \tilde{U} + \tilde{U}^2 = 0 \right\}.$$

Table 3: Dimensions of $L_k^{(n)}$. Cases where $W_k^{(n)} = \{0\}$ (i. e. n even and k odd) are typeset in gray.[5]

$n \backslash k$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
2	1	1	1	1	1	1	1	1	1	3	1	1	1	3	1
3	–	1	1	2	2	5	3	8	6	11	9	16	12	21	17
4	–	–	1	2	3	4	6	14	10	24	15	48	21		
5	–	–	–	1	2	4	11	19	30	53	77	111			
6	–	–	–	–	1	5	6	19	20	80	49				
7	–	–	–	–	–	1	3	14	30	80	143				
8	–	–	–	–	–	–	1	8	10	53	49				
9	–	–	–	–	–	–	–	1	6	24	77				
10	–	–	–	–	–	–	–	–	1	11	15	111			
11	–	–	–	–	–	–	–	–	–	3	9	48			

Lemma 5.12. *For even n and odd k we have $W_k^{(n)} = \{0\}$.*

Proof. Let $f \in V_k^{(n)}$ as in (5.2). Then $f \in \ker(1 + \tilde{S})$ implies for all coefficients that

$$a_{\alpha_1,\dots,\alpha_n} + a_{\alpha_n,\dots,\alpha_1} = 0 \quad \text{and} \quad a_{\alpha_1,\dots,\alpha_n} = a_{\alpha_n,\dots,\alpha_1}.$$

Hence $f = 0$. □

5.2. Generalized Lewis space

Definition 5.13. We denote the kernel of the operator $1 - \tilde{T} - \tilde{T}'$ by

$$L_k^{(n)} := \ker(1 - \tilde{T} - \tilde{T}') \subseteq V_k^{(n)}$$

and refer to it as the *generalized Lewis space.*

We know from Proposition 2.19 that $W_k^{(2)} = L_k^{(2)}$ for even k. We now want to work out for what kind of pairs $(n, k) \in \mathbb{N}^2$ we also have

$$W_k^{(n)} = L_k^{(n)}. \tag{5.3}$$

[5] The source code that was used to compute these dimensions can be found in appendix C.4.

Remark 5.14. Recall that we have $W_k^{(n)} = \{0\}$ for even n and odd k (Lemma 5.12). However, the space $L_k^{(n)}$ is non-trivial in this case. E. g., the polynomial $f = x_1^d - x_n^d \in V_k^{(n)}$ suffices $f|1 - \tilde{T} - \tilde{T}' = 0$ since

$$f\Big|1-\tilde{T}-\tilde{T}' = x_1^d - x_n^d - \left(\left(\sum_{i=1}^{n}\binom{n-1}{i-1}x_i\right)^d - x_n^d\right) - \left(x_1^d - \left(\sum_{i=1}^{n}\binom{n-1}{i-1}x_i\right)^d\right) = 0.$$

Apart from the exception in Remark 5.14, the identity (5.3) seems to hold.

Conjecture 5.15. *For $n, k \in \mathbb{N}$ with $k \geq n$ and such that n even implies that k is even we have*

$$W_k^{(n)} = L_k^{(n)}.$$

Remark 5.16. Conjecture 5.15 has been tested for all cases in table 3. The symmetry along the columns follows from Remark 5.4.

One of the inclusions stated in Conjecture 5.15 holds for all $n, k \in \mathbb{N}$ with $k \geq n$.

Lemma 5.17. *For $n, k \in \mathbb{N}$ with $k \geq n$ we have*

$$W_k^{(n)} \subseteq L_k^{(n)}.$$

Proof. Let $f \in W_k^{(n)}$. Then

$$\begin{aligned} 0 &= f|\tilde{S} - \tilde{U} - \tilde{U}^2|\tilde{S} = f|(-1) - (-\tilde{T}) - (-\tilde{T}') \\ &= f|(-1)|1 - \tilde{T} - \tilde{T}'. \end{aligned}$$

The proof of Lemma 5.8 shows more generally that $f|(-1) = \pm f$. So $f \in L_k^{(n)}$. ☐

In the well-known case for $n = 2$, the other inclusion of Conjecture 5.15 makes usually use of the fact that non-trivial polynomials are not invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$ in this case. However, this is no longer true in general for polynomials in multiple variables.

Example 5.18. For $f(x_1, x_2, x_3) = x_2^2 - x_1 x_3 \in V_5^{(3)}$ we have

$$f\Big|\tilde{S}^{(3)} = f, \qquad f\Big|\tilde{U}^{(3)} = f \qquad \text{and} \qquad f\Big|\tilde{T}^{(3)} = f.$$

5.3. An invariant pairing

We recall the pairing on $V_k^{(2)}$ from Definition 2.23 which is $\mathrm{SL}_2(\mathbb{Z})$-invariant (Proposition 2.27). In this section, we discuss a generalization of this pairing for each $n \in \mathbb{N}$, $n \geq 2$, on the spaces $V_k^{(n)}$. By a slight abuse of notation, we also denote these pairings by $\langle \cdot, \cdot \rangle$. This is not ambiguous as the generalized version will coincide with the previous pairing for the case $n = 2$.

Definition 5.19. We define a pairing on monomials in $V_k^{(n)}$ by

$$\left\langle x_1^{\alpha_1} \cdots x_n^{\alpha_n}, x_1^{\beta_1} \cdots x_n^{\beta_n} \right\rangle := \frac{(-1)^{\sum_{i=1}^n i \cdot \alpha_i}}{\binom{\alpha_1 + \cdots + \alpha_n}{\alpha_1, \ldots, \alpha_n} \cdot \prod_{i=1}^n \binom{n-1}{i-1}^{\alpha_i}} \cdot \delta_{(\alpha_1, \ldots, \alpha_n), (\beta_n, \ldots, \beta_1)} \tag{5.4}$$

and extend this linearly to a pairing on $V_k^{(n)}$.

Remark 5.20. The additional factor $\Pi_{i=1}^n \binom{n-1}{i-1}^{-\alpha_i}$ is essential for the desired invariance property, as we will see in the proof of Theorem 5.21. We will have a further discussion on this at the end of this section.

Using the inclusion morphism $\mathrm{SL}_2(\mathbb{Z}) \hookrightarrow \mathrm{SL}_2(\mathbb{C})$, we obtain an induced action of $\mathrm{SL}_2(\mathbb{Z})$ on $V_k^{(n)}$.

Theorem 5.21. *The pairing from Definition 5.19 is invariant under the action of* $\mathrm{SL}_2(\mathbb{Z})$, *i. e. for all* $f, g \in V_k^{(n)}$ *and* $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ *we have*

$$\langle f\,|\tilde{\gamma}\,, g\,|\tilde{\gamma} \rangle = \langle f, g \rangle \,.$$

In order to prove Theorem 5.21, it suffices to show that it is invariant under actions of the $\mathrm{SL}_2(\mathbb{Z})$-generators S and T for monomials $f, g \in V_k^{(n)}$. To do so, we will introduce some further notation and lemmas first.

Definition 5.22. Let $m \in \mathbb{N}$ be a natural number. We denote the set of its *weak compositions* of length ℓ for some $\ell \in \mathbb{N}$ by

$$P_\ell(m) := \{(a_1, \ldots, a_\ell) \in \mathbb{N}_0^\ell \mid a_1 + \cdots + a_\ell = m\}.$$

For a multi index $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}_0^n$ we set

$$\mathcal{P}(\beta) := P_1(\beta_1) \times \cdots \times P_n(\beta_n).$$

We call $m \in \mathcal{P}(\beta)$ a *weak composition of* β and write $m^{(i)} = (m_1^{(i)}, \ldots, m_i^{(i)}) \in P_i(\beta_i)$ for the ith entry in m.

Further, let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ be another multi index. We set

$$\mathcal{P}_\alpha(\beta) := \left\{ m \in \mathcal{P}(\beta) \;\middle|\; \forall i \in \{1, \ldots, n\} : \sum_{j=i}^n m_i^{(j)} = \alpha_i \right\}$$

and call $m \in \mathcal{P}_\alpha(\beta)$ an *admissible weak composition of* β *with respect to* α.

Example 5.23. For $\alpha = (2,2,4,1)$ and $\beta = (0,3,3,3)$ we have

$$\mathcal{P}_\alpha(\beta) = \big\{\big((0),(2,1),(0,1,2),(0,0,2,1)\big), \big((0),(2,1),(0,0,3),(0,1,1,1)\big),\\ \big((0),(1,2),(1,0,2),(0,0,2,1)\big), \big((0),(1,2),(0,0,3),(1,0,1,1)\big)\big\}.$$

Notation. Let $n \in \mathbb{N}$ and $\alpha = \big(\alpha_1, \dots, \alpha_n\big) \in \mathbb{N}_0^n$ be a multi index. We then denote its inverted multi index by

$$\bar{\alpha} \coloneqq \big(\alpha_n, \dots, \alpha_1\big) \in \mathbb{N}_0^n.$$

Lemma 5.24. *Let $n \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{N}_0^n$ be multi indices. Then there is a bijection between $\mathcal{P}_\alpha(\bar{\beta})$ and $\mathcal{P}_\beta(\bar{\alpha})$.*

Proof. If both sets are empty, the claim is trivial. So assume without loss of generality that $\mathcal{P}_\alpha(\bar{\beta}) \neq \emptyset$. By symmetry, it suffices to show that there is an injective map $\varphi\colon \mathcal{P}_\alpha(\bar{\beta}) \longrightarrow \mathcal{P}_\beta(\bar{\alpha})$. For $m = \big(m^{(1)}, \dots, m^{(n)}\big) \in \mathcal{P}_\alpha(\bar{\beta})$ define

$$p_i^{(j)} \coloneqq m_{n+1-j}^{(n+1-i)} \qquad \forall\, 1 \leq i \leq j \leq n \tag{5.5}$$

and set

$$\varphi\left(\big(m^{(1)}, \dots, m^{(n)}\big)\right) \coloneqq \left(p^{(1)}, \dots, p^{(n)}\right).$$

Since (5.5) also yields the inverse relation $m_i^{(j)} = p_{n+1-j}^{(n+1-i)}$, the map φ is injective. So it suffices to show that this defines a map $\mathcal{P}_\alpha(\bar{\beta}) \longrightarrow \mathcal{P}_\beta(\bar{\alpha})$.

For $i \in \{1, \dots, n\}$ the ith entry in $\varphi(m)$ is given by

$$p^{(i)} = \left(m_{n+1-i}^{(n)}, \dots, m_{n+1-i}^{(n+1-i)}\right).$$

So the ith entry is indeed of length i. Now, since m is an admissible weak composition of $\bar{\beta}$ w. r. t. α we have for all $i \in \{1, \dots, n\}$ that

$$m_1^{(i)} + \dots + m_i^{(i)} = \beta_{n+1-i} \tag{5.6}$$

and

$$m_i^{(i)} + \dots + m_i^{(n)} = \alpha_i. \tag{5.7}$$

This implies that

$$p_i^{(i)} + \dots + p_i^{(n)} = m_1^{(n+1-i)} + \dots + m_{n+1-i}^{(n+1-i)} \overset{(5.6)}{=} \beta_i$$

and

$$p_1^{(i)} + \dots + p_i^{(i)} = m_{n+1-i}^{(n+1-i)} + \dots + m_{n+1-i}^{(n)} \overset{(5.7)}{=} \alpha_{n+1-i}$$

which shows $\varphi(m) \in \mathcal{P}_\beta(\bar{\alpha})$. □

Lemma 5.25. *Let $\alpha, \beta \in \mathbb{N}_0^n$ be multi indices such that $|\alpha| = |\beta|$. Then the coefficient of $x_1^{\beta_1} \cdots x_n^{\beta_n}$ in $x_1^{\alpha_1} \cdots x_1^{\alpha_n} \Big| \tilde{T}^{(n)}$ is given by*

$$\sum_{m \in \mathcal{P}_\alpha(\beta)} \prod_{i=1}^n \binom{\alpha_i}{m_i^{(i)}, \dots, m_i^{(n)}} \cdot \prod_{j=i}^n \binom{n-i}{j-i}^{m_i^{(j)}} .$$

Proof. Since

$$x_1^{\alpha_1} \dots x_n^{\alpha_n} \Big| \tilde{T}^{(n)} = \prod_{i=1}^n \left(\sum_{j=i}^n \binom{n-i}{j-i} x_j \right)^{\alpha_i} , \tag{5.8}$$

each summand corresponds to a choice from the n factors in (5.8). For $j \in \{1, \dots, n\}$ there are precisely the first j factors that contain x_j. Let $m^{(j)} = \big(m_1^{(j)}, \dots, m_j^{(j)}\big)$ be a weak composition of β_j, i.e. $m_1^{(j)} + \dots + m_j^{(j)} = \beta_j$ and $m_i^{(j)} \in \{0, \dots, \beta_j\}$ for all $i \in \{1, \dots, j\}$. This choice of $m^{(j)}$ yields exactly one possible way to obtain the factor $x_j^{\beta_j}$ in (5.8).

Now fix a weak composition $m = \big(m^{(1)}, \dots, m^{(n)}\big)$ for β such that x_j is chosen $m_i^{(j)}$ times from the ith factor in (5.8). For $i \in \{1, \dots, n\}$ there are thus

$$\binom{\alpha_i}{m_i^{(i)}, \dots, m_i^{(n)}}$$

possible ways to choose the respective number of factors from the ith factor in (5.8). Note that this multinomial coefficient vanishes, unless $m_i^{(i)} + \dots + m_i^{(n)} = \alpha_i$. It therefore suffices to restrict the weak compositions of β to admissible weak compositions w.r.t. α. Finally, the factor

$$\prod_{j=i}^n \binom{n-i}{j-i}^{m_i^{(j)}}$$

accounts for the respective coefficients of $x_i, \dots, x_n$ from (5.8). Hence summing over all admissible weak compositions of β w.r.t. α yields the desired coefficient. □

Corollary 5.26. Let $\alpha, \beta \in \mathbb{N}_0^n$ be multi indices such that $|\alpha| = |\beta|$. Then the coefficient of $x_1^{\beta_1} \cdots x_n^{\beta_n}$ in $x_1^{\alpha_1} \cdots x_n^{\alpha_n} \Big| \tilde{T}^{-1}$ is given by

$$(-1)^{\sum_{i=1}^n i \cdot (\alpha_i + \beta_i)} \sum_{m \in \mathcal{P}_\alpha(\beta)} \prod_{i=1}^n \binom{\alpha_i}{m_i^{(i)}, \dots, m_i^{(n)}} \cdot \prod_{j=i}^n \binom{n-i}{j-i}^{m_i^{(j)}} .$$

Proof. Since

$$x_1^{\alpha_1} \dots x_n^{\alpha_n} \Big| \tilde{T}^{-1} = \prod_{i=1}^n \left(\sum_{j=i}^n (-1)^{i+j} \binom{n-i}{j-i} x_j \right)^{\alpha_i} ,$$

Lemma 5.25 yields that

$$\sum_{m\in\mathcal{P}_\alpha(\beta)}\prod_{i=1}^{n}\binom{\alpha_i}{m_i^{(i)},\ldots,m_i^{(n)}}\cdot\prod_{j=i}^{n}\left((-1)^{i+j}\cdot\binom{n-i}{j-i}\right)^{m_i^{(j)}}. \tag{5.9}$$

The sign of each summand in (5.9) is given by

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=i}^{n}(i+j)\cdot m_i^{(j)} &= \sum_{i=1}^{n} i\cdot\sum_{j=i}^{n}m_i^{(j)}+\sum_{j=1}^{n}j\cdot\sum_{i=1}^{j}m_i^{(j)}\\ &=\sum_{i=1}^{n}i\cdot\alpha_i+\sum_{j=1}^{n}j\cdot\beta_j\\ &=\sum_{i=1}^{n}i\cdot(\alpha_i+\beta_i).\end{aligned}$$

□

We are now able to prove Theorem 5.21.

Proof of Theorem 5.21. Since the group $\mathrm{SL}_2(\mathbb{Z})$ is generated by S and T, it suffices to show

$$\left\langle x_1^{\alpha_1}\cdots x_n^{\alpha_n}\,\middle|\tilde{S}\,,x_1^{\beta_1}\cdots x_n^{\beta_n}\right\rangle=\left\langle x_1^{\alpha_1}\cdots x_n^{\alpha_n},x_1^{\beta_1}\cdots x_n^{\beta_n}\,\middle|\tilde{S}^{-1}\right\rangle \tag{5.10}$$

and

$$\left\langle x_1^{\alpha_1}\cdots x_n^{\alpha_n}\,\middle|\tilde{T}\,,x_1^{\beta_1}\cdots x_n^{\beta_n}\right\rangle=\left\langle x_1^{\alpha_1}\cdots x_n^{\alpha_n},x_1^{\beta_1}\cdots x_n^{\beta_n}\,\middle|\tilde{T}^{-1}\right\rangle. \tag{5.11}$$

The left-hand side of (5.10) is

$$(-1)^{\sum_{i=1}^{n}(i+1)\cdot\alpha_{n+1-i}}\left\langle x_1^{\alpha_n}\cdots x_n^{\alpha_1},x_1^{\beta_1}\cdots x_n^{\beta_n}\right\rangle \tag{5.12a}$$

while the right-hand side is

$$(-1)^{\sum_{i=1}^{n}(i+1)\cdot\beta_i}\left\langle x_1^{\alpha_1}\cdots x_n^{\alpha_n},x_1^{\beta_n}\cdots x_n^{\beta_1}\right\rangle. \tag{5.12b}$$

Both (5.12a) and (5.12b) vanish unless $\alpha=\beta$. If $\alpha=\beta$, then (5.12a) is

$$\frac{(-1)^{\sum_{i=1}^{n}(i+1)\cdot\alpha_{n+1-i}+i\cdot\alpha_{n+1-i}}}{\binom{\alpha_1+\cdots+\alpha_n}{\alpha_1,\ldots,\alpha_n}\prod_{i=1}^{n}\binom{n-1}{i-1}^{\alpha_{n+1-i}}}=\frac{(-1)^{\alpha_1+\cdots+\alpha_n}}{\binom{\alpha_1+\cdots+\alpha_n}{\alpha_1,\ldots,\alpha_n}\prod_{i=1}^{n}\binom{n-1}{i-1}^{\alpha_i}}$$

and (5.12b) is

$$\frac{(-1)^{\sum_{i=1}^{n}(i+1)\cdot\alpha_i+i\cdot\alpha_i}}{\binom{\alpha_1+\cdots+\alpha_n}{\alpha_1,\ldots,\alpha_n}\prod_{i=1}^{n}\binom{n-1}{i-1}^{\alpha_i}}=\frac{(-1)^{\alpha_1+\cdots+\alpha_n}}{\binom{\alpha_1+\cdots+\alpha_n}{\alpha_1,\ldots,\alpha_n}\prod_{i=1}^{n}\binom{n-1}{i-1}^{\alpha_i}}.$$

The left-hand side of (5.11) is due to Lemma 5.25 and the Kronecker delta in (5.4) given by

$$\frac{(-1)^{\sum_{i=1}^n i\cdot\beta_{n+1-i}}}{\binom{\beta_1+\dots+\beta_n}{\beta_1,\dots,\beta_n}\cdot\prod_{i=1}^n\binom{n-1}{i-1}^{\beta_{n+1-i}}}\sum_{m\in\mathcal{P}_\alpha(\bar\beta)}\prod_{i=1}^n\binom{\alpha_i}{m_i^{(i)},\dots,m_i^{(n)}}\cdot\prod_{j=i}^n\binom{n-i}{j-i}^{m_i^{(j)}}$$

$$=\frac{(-1)^{(n+1)\cdot d+\sum_{i=1}^n i\cdot\beta_i}}{|\beta|!\cdot\prod_{i=1}^n\binom{n-1}{i-1}^{\beta_{n+1-i}}}\alpha_1!\cdots\alpha_n!\cdot\beta_1!\cdots\beta_n!\sum_{m\in\mathcal{P}_\alpha(\bar\beta)}\prod_{i=1}^n\prod_{j=i}^n\frac{\binom{n-i}{j-i}^{m_i^{(j)}}}{m_i^{(j)}!}$$

while the right-hand side of (5.11) is due to Corollary 5.26 given by

$$\frac{(-1)^{\sum_{i=1}^n i\cdot(\alpha_i+\alpha_{n+1-i}+\beta_i)}}{\binom{\alpha_1+\dots+\alpha_n}{\alpha_1,\dots,\alpha_n}\cdot\prod_{i=1}^n\binom{n-1}{i-1}^{\alpha_i}}\sum_{m\in\mathcal{P}_\beta(\bar\alpha)}\prod_{i=1}^n\binom{\beta_i}{m_i^{(i)},\dots,m_i^{(n)}}\cdot\prod_{j=i}^n\binom{n-i}{j-i}^{m_i^{(j)}}$$

$$=\frac{(-1)^{(n+1)\cdot d+\sum_{i=1}^n i\cdot\beta_i}}{|\alpha|!\cdot\prod_{i=1}^n\binom{n-1}{i-1}^{\alpha_i}}\cdot\alpha_1!\cdots\alpha_n!\cdot\beta_1!\cdots\beta_n!\sum_{m\in\mathcal{P}_\beta(\bar\alpha)}\prod_{i=1}^n\prod_{j=i}^n\frac{\binom{n-i}{j-i}^{m_i^{(j)}}}{m_i^{(j)}!}.$$

We have

$$\prod_{j=1}^n\binom{n-1}{j-1}^{-\beta_{n+1-j}}\sum_{m\in\mathcal{P}_\alpha(\bar\beta)}\prod_{i=1}^n\prod_{j=i}^n\frac{\binom{n-i}{j-i}^{m_i^{(j)}}}{m_i^{(j)}!}=\sum_{m\in\mathcal{P}_\alpha(\bar\beta)}\prod_{i=2}^n\prod_{j=i}^n\left(\frac{\binom{n-i}{j-i}}{\binom{n-1}{j-1}}\right)^{m_i^{(j)}}\left(m_i^{(j)}!\right)^{-1}$$

and similarly for the sum over $\mathcal{P}_\beta(\bar\alpha)$ since $\prod_{i=1}^n\binom{n-1}{i-1}^{\alpha_i}=\prod_{i=1}^n\binom{n-1}{i-1}^{\alpha_{n+1-i}}$. To prove the claimed equality, we use the bijection between $\mathcal{P}_\beta(\bar\alpha)$ and $\mathcal{P}_\alpha(\bar\beta)$ from Lemma 5.24. So for every $p\in\mathcal{P}_\beta(\bar\alpha)$ there is a $m\in\mathcal{P}_\alpha(\bar\beta)$ such that $m_i^{(j)}=p_{n+1-j}^{(n+1-i)}$ and vise versa for all $1\le i\le j\le n$. Now fix some $m\in\mathcal{P}_\alpha(\bar\beta)$ and let $p\in\mathcal{P}_\beta(\bar\alpha)$ denote its corresponding admissible weak composition. We then have

$$\prod_{i=2}^n\prod_{j=i}^n\left(\frac{\binom{n-i}{j-i}}{\binom{n-1}{j-1}}\right)^{m_i^{(j)}}\left(m_i^{(j)}!\right)^{-1}=\prod_{i=2}^n\prod_{j=i}^n\left(\frac{(n-i)!(j-1)!}{(n-1)!(j-i)!}\right)^{m_i^{(j)}}\left(m_i^{(j)}!\right)^{-1}$$
$$=\prod_{i=2}^n\prod_{j=i}^n\left(\frac{(n-i)!(j-1)!}{(n-1)!(j-i)!}\right)^{p_{n+1-j}^{(n+1-i)}}\left(p_{n+1-j}^{(n+1-i)}!\right)^{-1}$$

which coincides with the respective summand of $p\in\mathcal{P}_\beta(\bar\alpha)$ in the second sum since

$$\frac{(n-i)!(j-1)!}{(n-1)!(j-i)!}$$

is invariant under the substitution $(i,j)\mapsto(n+1-j,n+1-i)$. Hence the sums agree summand-wise and are thus equal as claimed. This finishes the proof. $\square$

We conclude this section with a discussion on the alternative Definition 2.25 of $\langle\cdot,\cdot\rangle$ on $V_k^{(2)}$ in terms of partial derivatives. We will first give a generalized definition for all $n\in\mathbb{N}$ on the spaces $V_k^{(n)}$ and then compare it to the pairing from Definition 5.19. Recall the shorthand notation $d=k-n$.

Definition 5.27. For $f, g \in V_k^{(n)}$ we set

$$\langle f, g \rangle' := \frac{1}{d!} \cdot f\left(-\frac{\partial}{\partial x_n}, \dots, (-1)^i \frac{\partial}{\partial x_{n+1-i}}, \dots, (-1)^n \frac{\partial}{\partial x_1}\right) \Big(g(x_1, \dots, x_n)\Big).$$

The pairing on $V_k^{(n)}$ from Definition 5.27 no longer has the desired invariance property for $n > 2$.

Example 5.28. We consider $f, g \in V_9^{(3)}$ with $f(x_1, x_2, x_3) = x_1^2 x_2 x_3^3$ and $g(x_1, x_2, x_3) = x_1^3 x_2^2 x_3$. Then we have

$$\langle f, g \rangle' = 0 \quad \text{and} \quad \left\langle f \left|\tilde{T}\right., g \left|\tilde{T}\right. \right\rangle' = \frac{1}{30}.$$

However, for monomials $f, g \in V_k^{(n)}$ the pairings on $V_k^{(n)}$ agree up to a factor.

Proposition 5.29. *Let $f(x_1, \dots, x_n) = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $g(x_1, \dots, x_n) = x_1^{\beta_1} \cdots x_n^{\beta_n}$ be monomials in $V_k^{(n)}$. Then*

$$\prod_{i=1}^{n} \binom{n-1}{i-1}^{\alpha_i} \cdot \langle f, g \rangle = \langle f, g \rangle'.$$

Proof. Note that both pairings vanish unless $\alpha_i = \beta_{n+1-i}$ for all $i \in \{1, \dots, n\}$. In this case

$$\begin{aligned} \prod_{i=1}^{n} \binom{n-1}{i-1}^{\alpha_i} \langle f, g \rangle &= (-1)^{\sum_{i=1}^{n} i \cdot \alpha_i} \cdot \frac{\alpha_1! \cdots \alpha_n!}{d!} \\ &= \langle f, g \rangle'. \end{aligned}$$

□

This implies together with Theorem 5.21 that there is a slight modification of $\langle \cdot, \cdot \rangle'$ that makes the pairing invariant.

Corollary 5.30. Let $f, g \in V_k^{(n)}$ be monomials as in Proposition 5.29. Setting

$$\langle f, g \rangle'' := \prod_{i=1}^{n} \binom{n-1}{i-1}^{-\alpha_i} \langle f, g \rangle'$$

and extending this linearly yields a pairing on $V_k^{(n)}$ that coincides with $\langle \cdot, \cdot \rangle$ from Definition 5.19 and is thus invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$.

5.4. Dimensions

Now let $n, k \in \mathbb{N}$ with $k \geq n$ and such that n even implies that k is even. The $\mathrm{SL}_2(\mathbb{Z})$-action on $V_k^{(n)}$ is invariant under $|-1$ in this case (see Lemma 5.8). We thus have

$$V_k^{(n)} = A_k^{(n)} \oplus B_k^{(n)} \quad \text{and} \quad V_k^{(n)} = C_k^{(n)} \oplus D_k^{(n)} \tag{5.13}$$

where

$$A_k^{(n)} = \ker(1-\tilde{S}) = \operatorname{im}(1+\tilde{S}), \qquad B_k^{(n)} = \ker(1+\tilde{S}) = \operatorname{im}(1-\tilde{S})$$

and

$$C_k^{(n)} = \ker(1-\tilde{U}) = \operatorname{im}(1+\tilde{U}+\tilde{U}^2), \quad D_k^{(n)} = \ker(1+\tilde{U}+\tilde{U}^2) = \operatorname{im}(2-\tilde{U}-\tilde{U}^2).$$

We have by definition that $W_k^{(n)} = B_k^{(n)} \cap D_k^{(n)}$. The splittings in (5.13) are orthogonal by Theorem 5.21 (cf. Corollary 2.28), hence $W_k^{(n)} = \left(A_k^{(n)} + C_k^{(n)}\right)^{\perp}$. However, the space

$$E_k^{(n)} := A_k^{(n)} \cap C_k^{(n)}$$

is non-trivial in general (see Example 5.18). We therefore obtain

$$V_k^{(n)} = A_k^{(n)}/E_k^{(n)} \oplus C_k^{(n)}/E_k^{(n)} \oplus E_k^{(n)} \oplus W_k^{(n)}$$

which implies dimension-wise that

$$\dim W_k^{(n)} = \dim V_k^{(n)} - \dim A_k^{(n)} - \dim C_k^{(n)} + \dim E_k^{(n)}. \tag{5.14}$$

We have $\dim(V_k^{(n)}) = \binom{k-1}{n-1}$ by Lemma 5.3. Since the spaces $A_k^{(n)}$ and $C_k^{(n)}$ are invariant under the finite groups generated by $\tilde{S}$ and $\tilde{U}$, respectively, their dimensions can be computed via Molien's theorem A.12. We therefore introduce some further notation.

Notation 5.31. Let $M \in \mathrm{GL}_2(\mathbb{Z})$. We denote

- the algebra of $\tilde{M}^{(n)}$-invariant polynomials in n variables over $\mathbb{Q}$ by

$$\mathbb{Q}[x_1, \ldots, x_n]^{\tilde{M}} := \left\{ f \in \mathbb{Q}[x_1, \ldots, x_n] \;\middle|\; f \middle| \tilde{M} = f \right\},$$

- the space of homogeneous polynomials of degree k in $\mathbb{Q}[x_1, \ldots, x_n]^{\tilde{M}}$ by $\mathbb{Q}[x_1, \ldots, x_n]_k^{\tilde{M}}$ and
- the Hilbert-Poincaré series of $\mathbb{Q}[x_1, \ldots, x_n]^{\tilde{M}}$ by $P_M^{(n)}(x)$, i. e.

$$P_M^{(n)}(x) = \sum_{k=0}^{\infty} \dim_{\mathbb{Q}} \left(\mathbb{Q}[x_1, \ldots, x_n]_k^{\tilde{M}} \right) x^k.$$

Proposition 5.32. *Let $n, k \in \mathbb{N}$ with $n \geq 2$ and $k \geq n$.*

i) For even n and odd k we have $\dim A_k^{(n)} = \dim C_k^{(n)} = 0$. Otherwise we have

$$\dim A_k^{(n)} = \frac{1}{2}\left(\binom{k-1}{n-1} + a_{n,k} \cdot \binom{\left[\frac{k}{2}\right]-1}{\left[\frac{n}{2}\right]-1}\right)$$
$$\dim C_k^{(n)} = \frac{1}{3}\left(\binom{k-1}{n-1} + c_{n,k} \cdot 2\binom{\left[\frac{k+1}{3}\right]-1}{\left[\frac{n+1}{3}\right]-1}\right)$$

where

$$a_{n,k} = \begin{cases} (-1)^{\frac{k-n}{2}}, & n \text{ even} \\ (-1)^{\frac{n-1}{2}\cdot(k+1)}, & n \text{ odd} \end{cases}$$
$$c_{n,k} = \begin{cases} \frac{\omega^{2k}+\omega^k+1}{3}, & n \equiv 0 \mod 3 \\ 1, & n \equiv 1 \mod 3 \\ \left\lceil\frac{k}{3}\right\rceil - \left\lfloor\frac{k}{3}\right\rfloor + \left\lfloor\frac{k+1}{3}\right\rfloor - \left\lceil\frac{k+1}{3}\right\rceil, & n \equiv 2 \mod 3 \end{cases}$$

and $\omega = e^{\frac{2\pi i}{3}}$ is a third root of unity.

ii) For even n the generating series of $A_k^{(n)}$ and $C_k^{(n)}$ are given by

$$\sum_{k=0}^{\infty} \dim A_k^{(n)}\, x^k = x^n \frac{(1+x)^n(1+x^2)^{\frac{n}{2}} + (1-x)^n(1+x^2)^{\frac{n}{2}} + 2(1-x)^n(1+x)^n}{4(1-x^4)^{\frac{n}{2}}(1-x^2)^{\frac{n}{2}}}$$
$$\begin{aligned}\sum_{k=0}^{\infty} \dim C_k^{(n)}\, x^k = {} & \frac{x^n}{6(1-x^6)^{\left[\frac{n}{3}\right]}(1-x^2)^{n-\left[\frac{n}{3}\right]}} \\ & \cdot \Big((1+x)^n(1+x+x^2)^{\left[\frac{n}{3}\right]}(1-x+x^2)^{\left[\frac{n}{3}\right]} \\ & + (1-x)^n(1+x+x^2)^{\left[\frac{n}{3}\right]}(1-x+x^2)^{\left[\frac{n}{3}\right]} \\ & + 2(1-x)^{2\left[\frac{n}{3}\right]}(1+x)^n(1-x+x^2)^{\left[\frac{n}{3}\right]} \\ & + 2(1-x)^n(1+x)^{2\left[\frac{n}{3}\right]}(1+x+x^2)^{\left[\frac{n}{3}\right]}\Big).\end{aligned}$$

For odd n the generating series of $A_k^{(n)}$ and $C_k^{(n)}$ are given by

$$\sum_{k=0}^{\infty} \dim A_k^{(n)}\, x^k = x^n \cdot \frac{(1+x)^{2\left[\frac{n}{4}\right]} + (1-x)^{2\left[\frac{n}{4}\right]}}{2(1-x^2)^{2\left[\frac{n}{4}\right]}(1-x)^{n-2\left[\frac{n}{4}\right]}}$$
$$\sum_{k=0}^{\infty} \dim C_k^{(n)}\, x^k = x^n \cdot \frac{(1+x+x^2)^{\left[\frac{n}{3}\right]} + 2(1-x)^{2\left[\frac{n}{3}\right]}}{(1-x^3)^{\left[\frac{n}{3}\right]}(1-x)^{n-\left[\frac{n}{3}\right]}}.$$

Proof. The respective Hilbert-Poincaré series are computed in appendix B. The claimed identities in ii) follow from expanding the respective rational functions and multiplying by x^n to account for the degree $k-n$.

For even n we observe that the numerator and denominator of the generating series from ii) are even polynomials. We thus obtain that $A_k^{(n)}$ and $C_k^{(n)}$ are trivial in this case for odd k. To prove i), we will consider each of the remaining cases.

We first consider $A_k^{(n)}$. For even n appendix B.1 immediately yields that

$$P_S^{(n)}(x) = \frac{1}{2} \sum_{\substack{k=0,\\ k \text{ even}}}^{\infty} \left(\binom{k+n-1}{k} + (-1)^{\frac{k}{2}} \binom{\frac{k}{2}+\frac{n}{2}-1}{\frac{k}{2}} \right) x^k. \tag{5.15}$$

For odd n we have

$$P_S^{(n)}(x) = \frac{1}{2} \left(\sum_{k=0}^{\infty} \binom{k+n-1}{k} x^k + \sum_{\substack{k=0,\\ k \text{ even}}}^{\infty} \binom{\frac{k}{2}+2\left[\frac{n}{4}\right]-1}{\frac{k}{2}} x^k \cdot \frac{1}{(1-x)^{n-4\left[\frac{n}{4}\right]}} \right).$$

Since $n - 4\left[\frac{n}{4}\right] = (-1)^{\frac{n-1}{2}}$ for odd n, we consider the residue class of n modulo 4.

First assume $n \equiv 1 \mod 4$. We then have $n - 4\left[\frac{n}{4}\right] = 1$ and hence

$$\frac{1}{(1-x)^{n-4\left[\frac{n}{4}\right]}} = \sum_{k=0}^{\infty} x^k.$$

We thus obtain

$$\begin{aligned} \sum_{\substack{k=0,\\ k \text{ even}}}^{\infty} \binom{\frac{k}{2}+2\left[\frac{n}{4}\right]-1}{\frac{k}{2}} x^k \cdot \frac{1}{(1-x)^{n-4\left[\frac{n}{4}\right]}} &= \sum_{k=0}^{\infty} \left(\sum_{\substack{i=0,\\ i \text{ even}}}^{k} \binom{\frac{i}{2}+2\left[\frac{n}{4}\right]-1}{\frac{i}{2}} \right) x^k \\ &= \sum_{k=0}^{\infty} \left(\sum_{i=0}^{\left\lfloor\frac{k}{2}\right\rfloor} \binom{i+2\left[\frac{n}{4}\right]-1}{i} \right) x^k \\ &= \sum_{k=0}^{\infty} \binom{\left\lfloor\frac{k}{2}\right\rfloor+2\left[\frac{n}{4}\right]}{\left\lfloor\frac{k}{2}\right\rfloor} x^k \end{aligned}$$

where the last equality follows from the so-called hockey-stick identity

$$\sum_{i=0}^{K} \binom{N+i}{i} = \binom{K+N+1}{K} \tag{5.16}$$

for $N, K \in \mathbb{N}$. So we have

$$P_S^{(n)}(x) = \frac{1}{2} \sum_{k=0}^{\infty} \left(\binom{k+n-1}{k} + \binom{\left\lfloor\frac{k}{2}\right\rfloor+2\left[\frac{n}{4}\right]}{\left\lfloor\frac{k}{2}\right\rfloor} \right) x^k. \tag{5.17}$$

Now assume $n \equiv 3 \mod 4$. We then have $n - 4\left[\frac{n}{4}\right] = -1$ and hence

$$\frac{1}{(1-x)^{n-4\left[\frac{n}{4}\right]}} = 1 - x.$$

Since

$$\sum_{\substack{k=0,\\ k \text{ even}}}^{\infty} \binom{\frac{k}{2} + 2\left[\frac{n}{4}\right] - 1}{\frac{k}{2}} x^k \cdot (1-x) = \sum_{k=0}^{\infty} (-1)^k \binom{\left\lfloor\frac{k}{2}\right\rfloor + 2\left[\frac{n}{4}\right] - 1}{\left\lfloor\frac{k}{2}\right\rfloor} x^k$$

we have

$$P_S^{(n)}(x) = \frac{1}{2}\sum_{k=0}^{\infty}\left(\binom{k+n-1}{k} + (-1)^k \binom{\left\lfloor\frac{k}{2}\right\rfloor + 2\left[\frac{n}{4}\right] - 1}{\left\lfloor\frac{k}{2}\right\rfloor}\right) x^k. \tag{5.18}$$

We now consider $C_k^{(n)}$. First assume n is even. Appendix B.2 yields that

$$\begin{aligned} P_U^{(n)}(x) = \frac{1}{3}\Bigg(&\sum_{\substack{k=0,\\ k \text{ even}}}^{\infty} \binom{k+n-1}{k} x^k + \sum_{\substack{k=0,\\ 3|k}}^{\infty} \binom{\frac{k}{3} + \left[\frac{n}{3}\right] - 1}{\frac{k}{3}} x^k \cdot \frac{1}{(1-x)^{n-3\left[\frac{n}{3}\right]}} \\ &+ \sum_{\substack{k=0,\\ 3|k}}^{\infty} (-1)^k \binom{\frac{k}{3} + \left[\frac{n}{3}\right] - 1}{\frac{k}{3}} x^k \cdot \frac{1}{(1+x)^{n-3\left[\frac{n}{3}\right]}}\Bigg). \end{aligned} \tag{5.19}$$

Note that $n - 3\left[\frac{n}{3}\right] = \{m \in \{-1, 0, 1\} \mid m \equiv n \mod 3\}$. So if we further have $n \equiv 0 \mod 6$ then (5.19) yields

$$P_U^{(n)}(x) = \frac{1}{3}\left(\sum_{\substack{k=0,\\ k \text{ even}}}^{\infty} \binom{k+n-1}{k} x^k + 2\sum_{\substack{k=0,\\ 6|k}}^{\infty} \binom{\frac{k}{3} + \frac{n}{3} - 1}{\frac{k}{3}} x^k\right). \tag{5.20}$$

Now assume $n \equiv 4 \mod 6$. Then

$$\frac{1}{(1-x)^{n-3\left[\frac{n}{3}\right]}} = \sum_{k=0}^{\infty} x^k, \qquad \frac{1}{(1+x)^{n-3\left[\frac{n}{3}\right]}} = \sum_{k=0}^{\infty} (-1)^k x^k$$

and hence (5.19) yields

$$\begin{aligned} P_U^{(n)}(x) &= \frac{1}{3}\Bigg(\sum_{\substack{k=0,\\ k \text{ even}}}^{\infty} \binom{k+n-1}{k} x^k + \sum_{k=0}^{\infty}\Bigg(\sum_{\substack{i=0,\\ 3|i}}^{k} \binom{\frac{i}{3} + \left[\frac{n}{3}\right] - 1}{\frac{i}{3}}\Bigg) x^k \\ &\quad + \sum_{k=0}^{\infty} (-1)^k \Bigg(\sum_{\substack{i=0,\\ 3|i}}^{k} \binom{\frac{i}{3} + \left[\frac{n}{3}\right] - 1}{\frac{i}{3}}\Bigg) x^k\Bigg) \\ &\overset{(5.16)}{=} \frac{1}{3}\sum_{\substack{k=0,\\ k \text{ even}}}^{\infty}\left(\binom{k+n-1}{k} + 2\binom{\left\lfloor\frac{k}{3}\right\rfloor + \left[\frac{n}{3}\right]}{\left\lfloor\frac{k}{3}\right\rfloor}\right) x^k. \end{aligned} \tag{5.21}$$

Now let $n \equiv 2 \mod 6$. Then (5.19) yields

$$P_U^{(n)}(x) = \frac{1}{3}\left(\sum_{\substack{k=0,\\ k \text{ even}}}^{\infty} \binom{k+n-1}{k} x^k + 2\left(\sum_{\substack{k=0,\\ 6|k}}^{\infty} \binom{\frac{k}{3}+\left[\frac{n}{3}\right]-1}{\frac{k}{3}} x^k - \binom{\frac{k}{3}+\left[\frac{n}{3}\right]}{\frac{k}{3}+1} x^{k+4}\right)\right). \tag{5.22}$$

Now assume n is odd. Then appendix B.2 yields that

$$P_U^{(n)}(x) = \frac{1}{3}\left(\sum_{k=0}^{\infty} \binom{k+n-1}{k} x^k + \sum_{\substack{k=0,\\ 3|k}}^{\infty} \binom{\frac{k}{3}+\left[\frac{n}{3}\right]-1}{\frac{k}{3}} x^k \cdot \frac{2}{(1-x)^{n-3\left[\frac{n}{3}\right]}}\right). \tag{5.23}$$

We consider the remaining residue classes of n modulo 6. First assume $n \equiv 3 \mod 6$. Then (5.23) yields

$$P_U^{(n)}(x) = \frac{1}{3}\left(\sum_{k=0}^{\infty} \binom{k+n-1}{k} x^k + 2\sum_{\substack{k=0,\\ 3|k}}^{\infty} \binom{\frac{k}{3}+\frac{n}{3}-1}{\frac{k}{3}} x^k\right). \tag{5.24}$$

Now assume $n \equiv 1 \mod 6$. Then (5.23) yields that

$$\begin{aligned} P_U^{(n)}(x) &= \frac{1}{3}\left(\sum_{k=0}^{\infty} \binom{k+n-1}{k} x^k + \sum_{\substack{k=0,\\ 3|k}}^{\infty} \binom{\frac{k}{3}+\left[\frac{n}{3}\right]-1}{\frac{k}{3}} x^k \cdot \frac{2}{(1-x)}\right) \\ &= \frac{1}{3}\left(\sum_{k=0}^{\infty} \binom{k+n-1}{k} x^k + 2\sum_{k=0}^{\infty}\left(\sum_{\substack{i=0,\\ 3|i}}^{k} \binom{\left\lfloor\frac{i}{3}\right\rfloor+\left[\frac{n}{3}\right]-1}{\left\lfloor\frac{i}{3}\right\rfloor}\right) x^k\right) \\ &\overset{(5.16)}{=} \frac{1}{3}\left(\sum_{k=0}^{\infty} \binom{k+n-1}{k} x^k + 2\sum_{k=0}^{\infty} \binom{\left\lfloor\frac{k}{3}\right\rfloor+\left[\frac{n}{3}\right]}{\left\lfloor\frac{k}{3}\right\rfloor} x^k\right). \end{aligned} \tag{5.25}$$

Lastly, assume $n \equiv 5 \mod 6$. Then (5.23) yields

$$\begin{aligned} P_U^{(n)}(x) &= \frac{1}{3}\left(\sum_{k=0}^{\infty} \binom{k+n-1}{k} x^k + 2\sum_{\substack{k=0,\\ 3|k}}^{\infty} \binom{\frac{k}{3}+\left[\frac{n}{3}\right]-1}{\frac{k}{3}} x^k \cdot (1-x)\right) \\ &= \frac{1}{3}\left(\sum_{k=0}^{\infty} \binom{k+n-1}{k} x^k + 2\sum_{\substack{k=0,\\ 3|k}}^{\infty} \binom{\frac{k}{3}+\left[\frac{n}{3}\right]-1}{\frac{k}{3}} (x^k - x^{k+1})\right). \end{aligned} \tag{5.26}$$

To conclude the proof, note that the spaces $A_k^{(n)}$ and $C_k^{(n)}$ contain polynomials of degree $k-n$. Hence i) follows by considering the coefficient of x^{k-n} in (5.15), (5.17) and (5.18) for $\dim A_k^{(n)}$ as well as in (5.20), (5.21), (5.22), (5.24), (5.25) and (5.26) for $\dim C_k^{(n)}$. $\square$

In order to use (5.14) to compute the dimension of $W_k^{(n)}$ we still need the dimension of $E_k^{(n)} = A_k^{(n)} \cap C_k^{(n)}$. Recall the analogue spaces from sections 2.4 and 3.4. In the case of period polynomials, this space is trivial. In the case of bi-period polynomials, the key argument was a special case of Weitzenböck's Theorem 3.21 which showed that the algebra of $|T$-invariant polynomials is finitely generated. In the proof, we considered the action of T as a differential operator. This derivation was given on $\mathbb{Q}[x_1, x_2, y_1, y_2]$ as

$$\mathrm{d}_{\mathsf{bi}} \coloneqq x_2 \cdot \frac{\partial}{\partial x_1} - y_1 \cdot \frac{\partial}{\partial y_2}.$$

The algebra of invariants is then equal to the kernel of this derivation. By considering the restriction to $\mathcal{V}_3$, we obtain an endomorphism with transformation matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The Jordan form of this is given by

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This is interesting in light of the following theorem by Tyc. For a proof of this theorem we refer to the original paper.

Theorem 5.33 ([Tyc98])**.** *Let $A = \mathbb{C}[x_1, \ldots, x_n]$ and let $\mathrm{d}\colon A \to A$ be a non-zero locally nilpotent derivation such that $\mathrm{d}(W) \subseteq W$ with $W = \mathbb{C}x_1 + \cdots + \mathbb{C}x_n$. Then* $\ker(\mathrm{d})$ *is*

1. *a Gorenstein ring and*
2. *a polynomial algebra if and only if $W = W_0 \oplus W'$ for some subspaces $W_0, W' \subset W$ such that $\mathrm{d}(W_0) = 0$, $\mathrm{d}(W') \subset W'$ and the Jordan matrix of the endomorphism $\mathrm{d}|_{W'}\colon W' \to W'$ is one of the following*

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that Jordan form corresponding to the derivation d_{bi} coincides with the last matrix from Theorem 5.33. However, this is not the case in this more general setting.

Corollary 5.34. The ring of invariants $\mathbb{C}[x_1, \ldots, x_n]^{\tilde{T}}$ is a polynomial algebra if and only if $n \leq 3$.

Proof. For $n \geq 2$ and $\lambda \in \mathbb{R}$ we consider the matrix

$$\left(\tilde{T}^{(n)}\right)^{\lambda} = \left(\binom{n-i}{j-i}\lambda^{j-i}\right)_{i,j=1,\dots,n}.$$

Now let $f \in V_k^{(n)}$ for some $k \geq n$. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(f\left|\tilde{T}^{\lambda}\right.\right)\Big|_{\lambda=0} &= \frac{\mathrm{d}}{\mathrm{d}\lambda} f\left(\sum_{j=1}^{n}\binom{n-1}{j-1}\lambda^{j-1}x_j,\dots,x_n\right)\Big|_{\lambda=0} \\ &= \sum_{i=1}^{n-1}(n-i)\cdot x_{i+1}\cdot\frac{\partial f}{\partial x_i}\end{aligned}$$

where the last equality follows from the product rule and the fact that after evaluating the expression at $\lambda = 0$ only one summand remains in each entry.

Now, if f is invariant under T, we obtain that

$$\mathrm{d}_n := \sum_{i=1}^{n-1}(n-i)\cdot x_{i+1}\cdot\frac{\partial f}{\partial x_i} = 0$$

so we can identify $\mathbb{C}[x_1,\dots,x_n]^{\tilde{T}}$ with $\ker(\mathrm{d}_n)$. However, the transformation matrix of d_n is given by

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ n-1 & 0 & \dots & 0 & 0 \\ 0 & n-2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

and the entries of the Jordan form are 0 everywhere except for the superdiagonal, i. e.

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Now let $W = \mathbb{C}x_1 + \dots + \mathbb{C}x_n$ and consider a decomposition $W = W_0 \oplus W'$ with $\mathrm{d}_n(W_0) = 0$ and $\mathrm{d}(W') \subset W'$. The kernel of $\mathrm{d}_n|_W$ is one-dimensional with $\mathrm{d}_n(x_n) = 0$. And since $\mathrm{d}_n(x_i)$ is a multiple of x_{i+1} for all $i \in \{1,\dots,n-1\}$, the condition $\mathrm{d}(W') \subset W'$ implies that $W' = W$ and W_0 is trivial. The proof now follows from the second part of Theorem 5.33. □

By Corollary 5.34 it seems to be more complicated to find the generating series for $E_k^{(n)}$ if $n \geq 4$ compared to the space $\mathcal{E}_k$ from section 3.2. Unfortunately, it was not possible to compute the dimensions of $E_k^{(n)}$ within the scope of this thesis.

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A. Representation theory of the dihedral group of order 6

As an example for a discussion on representation theory and isotypical decompositions we consider the dihedral group D_6. Geometrically, this group describes the isometries of a regular triangle. Algebraically, the group is given by the presentation

$$D_6 = \left\langle r, s \;\middle|\; r^3, s^2, (sr)^2 \right\rangle \tag{A.1}$$

where r and s correspond to a rotation of 120° and a reflection, respectively. We have by (A.1) that $|D_6| = 6$.

For the following discussion, we will first introduce some fundamental notions and statements. The statements, however, will not be proven for the sake of simplicity. For the proofs as well as more details on the subject see, e. g., [FH13], [Ser+77], [Pan06] and [Tel05].

Throughout this discussion, let $\mathbb{F}$ be an algebraically closed field of characteristic 0, V a n-dimensional $\mathbb{F}$-vector space and $G \subset \mathrm{GL}(V)$ be a finite group.

Definition A.1. We define a *representation of* G, *invariant subspaces* of a representation and *irreducible* representations of G analogously to Definitions 4.26 and 4.29, respectively.

Definition A.2. Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation of G. The *character* of ρ is given by $\chi_\rho\colon G \to \mathbb{F}$, $g \mapsto \mathrm{trace}(\rho(g))$. The character χ_ρ is *irreducible* if ρ is irreducible. We denote the set of irreducible characters of G by $X(G)$. If G is clear from the context we write X instead.

Proposition A.3. *The number of conjugacy classes in G equals $|X|$.*

Lemma A.4. *Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation of G and $W \subset V$ an invariant subspace. Then there is an invariant subspace $W' \subset V$ such that*

$$V = W \oplus W'.$$

Definition A.5. Let $\rho\colon G \to \mathrm{GL}(V)$ and $\eta\colon G \to \mathrm{GL}(W)$ be representations of G. A linear map $\phi\colon V \to W$ such that

$$\phi\bigl(\rho(g)(v)\bigr) = \eta(g)(\phi(v))$$

for all $g \in G$, $v \in V$ is called an *intertwining map.*

Lemma A.6 (Schur's lemma)**.** *Let $\rho\colon G \to \mathrm{GL}(V)$ and $\eta\colon G \to \mathrm{GL}(W)$ be irreducible representations of G and let $\phi\colon V \to W$ be an intertwining map. Then*

1. *ϕ is either an isomorphism or $\phi \equiv 0$ and*
2. *if $V \cong W$, then ϕ is a homothety.*

By Lemmas A.4 and A.6 we obtain the following corollary.

Corollary A.7. Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation of G. Then there exists a unique decomposition

$$V = \bigoplus_{\chi \in X} V_\chi \tag{A.2}$$

where, for each $\chi \in X$, $V_\chi = T_\chi^{\oplus k_\chi}$ such that the representation $\rho_\chi\colon G \to \mathrm{GL}(T_\chi)$ is irreducible and $k_\chi \in \mathbb{N}_0$.

Definition A.8. The decomposition (A.2) is called *isotypical decomposition* and the spaces V_χ are *isotypical components*. To emphasize the group G, we also write V_χ^G instead of V_χ.

We now set $R := \mathbb{F}[x_1, \ldots, x_n]$ and denote the subalgebra of G-invariant polynomials by

$$R^G = \{f \in R \mid \text{for all } M \in G : M(f) = f\}$$

where the G-action on R is induced by the action on V, i. e.

$$M(f) := f\left(M \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right)^{\mathsf{t}}.$$

Proposition A.9 ([Hil90])**.** *The algebra R^G is finitely generated. The generators have bounded degree of $\leq |G|$.*

Since R is infinite-dimensional, we consider the grading

$$R = \bigoplus_{k=0}^{\infty} R_k$$

where $R_k = \{f \in R \mid f \text{ homogeneous}, \deg(f) = k\}$. Note that the spaces R_k, called *homogeneous components*, are finite-dimensional. Our previous discussion thus applies to the homogeneous components of R.

Remark A.10. Since the G-action is degree preserving, we obtain a grading on R^G where the homogeneous components are given by $\left(R^G\right)_k := R^G \cap R_k$ for all $k \in \mathbb{N}_0$. For $\chi \in X$ we further set $\left(R_\chi^G\right)_k := R_\chi^G \cap R_k$ for all $k \in \mathbb{N}_0$. We thus obtain that R_χ^G is a graded R^G-module since $R^G \cdot R_\chi^G \subseteq R_\chi^G$.

Definition A.11. For an irreducible character $\chi \in X$ we call

$$F_{G,\chi}(x) := \chi(1)^{-1} \sum_{k=0}^{\infty} \dim_{\mathbb{F}}\left((R_\chi^G)_k\right)\, x^k$$

the *Molien series of* (G, χ).

Theorem A.12 (Molien)**.** *Let $\chi \in X$ be an irreducible character, then*

$$F_{G,\chi}(x) = \frac{1}{|G|} \sum_{M \in G} \frac{\overline{\chi(M)}}{\det(1 - xM)}.$$

In order to compute isotypical decompositions for D_6, we first specify the irreducible representations of D_6. Since D_6 contains 3 conjugacy classes, we have $|X| = 3$. In fact, the 1-dimensional irreducible representations of D_6 are given by

$$\rho_1(g) = 1 \qquad \text{for all } g \in D_6$$
$$\rho_2(g) = (-1)^{\mathrm{ord}(g)+1} \qquad \text{for all } g \in D_6.$$

The 2-dimensional irreducible representation $\rho_3\colon D_6 \to \mathrm{GL}(\mathbb{C}^2)$ can be described on the generators r and s by

$$\rho_3(r) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix} \qquad \text{and} \qquad \rho_3(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $\omega = e^{\frac{2\pi i}{3}}$ is a third root of unity. Note that the eigenvalues are $\{\omega, \omega^2\}$ and $\{\pm 1\}$, respectively.

Computing the corresponding characters yields $\chi_i(g) = \rho_i(g)$ for all $g \in D_6$ and $i \in \{1, 2\}$ as well as

$$\chi_3(g) = \begin{cases} 2, & g = 1 \\ -1, & g \in \{r, r^2\} \\ 0, & g \in \{s, rs, r^2 s\} \end{cases}$$

Now, let $\pi\colon D_6 \to \mathrm{GL}(V)$ be an arbitrary representation of D_6. Since $\pi(r)$ is diagonalizable we obtain the decomposition

$$V = V_1 \oplus V_\omega \oplus V_{\omega^2}$$

where V_λ is the eigenspace of $\pi(r)$ to the eigenvalue $\lambda \in \{1, \omega, \omega^2\}$. The transformation $\pi(s)$ preserves V_1 since $srs^{-1} = r^{-1}$. The space V_1 therefore decomposes into a direct sum of copies of the irreducible representations ρ_1 and ρ_2 from above. For the remaining spaces V_ω and V_{ω^2} we observe that $\pi(s)$ is a self-inverse isomorphism $V_\omega \to V_{\omega^2}$. By choosing a basis $e_1, \dots, e_n$ for V_ω, we obtain a basis $e'_1, \dots, e'_n$ for V_{ω^2} where $e'_i = \pi(s)(e_i)$. Hence $\pi(s)$ acts on $\mathbb{C}e_i \oplus \mathbb{C}e'_i$ as $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\pi(r)$ acts as $\left(\begin{smallmatrix} \omega & 0 \\ 0 & \omega^2 \end{smallmatrix}\right)$. This yields a decomposition of $V_\omega \oplus V_{\omega^2}$ in n copies of the 2-dimensional representation ρ_3.

As an explicit example, let $R = \mathbb{C}[X, Y]$. The isotypical decomposition

$$R_k = \left(R^{D_6}_{\chi_1}\right)_k \oplus \left(R^{D_6}_{\chi_2}\right)_k \oplus \left(R^{D_6}_{\chi_3}\right)_k$$

for $k \in \mathbb{N}$ is then given by

$$\left(R^{D_6}_{\chi_1}\right)_k = \langle f \in R_k \mid M(f) = f \text{ for all } M \in D_6 \rangle_{\mathbb{C}}$$
$$\left(R^{D_6}_{\chi_2}\right)_k = \langle f \in R_k \mid r(f) = f, s(f) = -f \rangle_{\mathbb{C}}$$
$$\left(R^{D_6}_{\chi_3}\right)_k = \left\langle (f, g) \in R_k^2 \;\middle|\; r(f) = \omega f, r(g) = \omega^2 g, s(f) = g, s(g) = f \right\rangle_{\mathbb{C}}.$$

By Proposition A.9 R^G is a finitely generated $\mathbb{C}$-algebra with generators of degree ≤ 6. Hence it suffices to compute the isotypical components $\left(R^{D_6}_{\chi_1}\right)_k$ for $k \in \{0, \dots, 6\}$. This yields

$$R^{D_6}_{\chi_1} = \mathbb{C}[XY, X^3 + Y^3].$$

We further have by Remark A.10 that $R^{D_6}_{\chi_2}$ and $R^{D_6}_{\chi_3}$ are finitely generated R^{D_6}-modules. Explicitly, we have

$$\begin{aligned} R^{D_6}_{\chi_2} &= R^{D_6} \cdot \left(X^3 - Y^3\right) \\ R^{D_6}_{\chi_3} &= R^{D_6} X \oplus R^{D_6} Y \oplus R^{D_6} X^2 \oplus R^{D_6} Y^2. \end{aligned}$$

The respective Molien series can now be computed with Molien's theorem A.12, hence

$$\begin{aligned} F_{D_6,\chi_1}(x) &\coloneqq \sum_{k=0}^{\infty} \dim_{\mathbb{C}} \left(R^{D_6}_{\chi_1}\right)_k x^k = \frac{1}{x^5 - x^3 - x^2 + 1} \\ F_{D_6,\chi_2}(x) &\coloneqq \sum_{k=0}^{\infty} \dim_{\mathbb{C}} \left(R^{D_6}_{\chi_2}\right)_k x^k = x^3 \cdot F_{D_6,\chi_1}(x) \\ F_{D_6,\chi_3}(x) &\coloneqq \sum_{k=0}^{\infty} \dim_{\mathbb{C}} \left(R^{D_6}_{\chi_3}\right)_k x^k = (x + x^2) \cdot F_{D_6,\chi_1}(x). \end{aligned}$$

This concludes our discussion.

B. Hilbert-Poincaré series of the spaces $A^{(n)}_k$ and $C^{(n)}_k$

We denote the Hilbert-Poincaré series by $P^{(n)}_S(x)$ and $P^{(n)}_U(x)$ respectively.

B.1. The space $A^{(n)}_k$

For even n, the group generated by $\tilde{S}^{(n)}$ is given by $\{\mathrm{id}, -\mathrm{id}, \tilde{S}, -\tilde{S}\}$. Hence Molien's theorem A.12 yields that

$$\begin{aligned} P^{(n)}_S(x) &= \frac{1}{4}\left(\frac{1}{\det(\mathrm{id} - x\,\mathrm{id})} + \frac{1}{\det(\mathrm{id} + x\,\mathrm{id})} + \frac{1}{\det(\mathrm{id} - x\tilde{S})} + \frac{1}{\det(\mathrm{id} + x\tilde{S})}\right) \\ &= \frac{1}{4}\left(\frac{1}{(1-x)^n} + \frac{1}{(1+x)^n} + \frac{2}{(1+x^2)^{\frac{n}{2}}}\right) \\ &= \frac{1}{2}\sum_{\substack{k=0,\\ k \text{ even}}}^{\infty} \left(\binom{k+n-1}{k} + (-1)^{\frac{k}{2}}\binom{\frac{k}{2} + \frac{n}{2} - 1}{\frac{k}{2}}\right) x^k. \end{aligned}$$

For odd n, the group generated by $\tilde{S}^{(n)}$ is given by $\{\mathrm{id}, \tilde{S}\}$, as $\tilde{S}^2 = \mathrm{id}$ in this case. Hence Molien's theorem A.12 yields that

$$\begin{aligned}
P_S^{(n)}(x) &= \frac{1}{2}\left(\frac{1}{\det(\mathrm{id} - x\,\mathrm{id})} + \frac{1}{\det(\mathrm{id} - x\tilde{S})}\right) \\
&= \frac{1}{2}\left(\frac{1}{(1-x)^n} + \frac{1}{(1+x)^{2\left[\frac{n}{4}\right]}(1-x)^{n-2\left[\frac{n}{4}\right]}}\right) \\
&= \frac{1}{2}\left(\frac{1}{(1-x)^n} + \frac{1}{(1-x^2)^{2\left[\frac{n}{4}\right]}} \cdot \frac{1}{(1-x)^{n-4\left[\frac{n}{4}\right]}}\right) \\
&= \frac{1}{2}\left(\sum_{k=0}^{\infty} \binom{k+n-1}{k} x^k + \sum_{\substack{k=0,\\ k \text{ even}}}^{\infty} \binom{\frac{k}{2} + 2\left[\frac{n}{4}\right] - 1}{\frac{k}{2}} x^k \cdot \frac{1}{(1-x)^{n-4\left[\frac{n}{4}\right]}}\right).
\end{aligned}$$

B.2. The space $C_k^{(n)}$

For even n, the group generated by $\tilde{U}^{(n)}$ is given by $\{\mathrm{id}, -\mathrm{id}, \tilde{U}, -\tilde{U}, \tilde{U}^2, -\tilde{U}^2\}$. Hence Molien's theorem A.12 yields that

$$\begin{aligned}
P_U^{(n)}(x) &= \frac{1}{6}\Bigg(\frac{1}{\det(\mathrm{id} - x\,\mathrm{id})} + \frac{1}{\det(\mathrm{id} + x\,\mathrm{id})} + \frac{1}{\det(\mathrm{id} - x\tilde{U})} \\
&\quad + \frac{1}{\det(\mathrm{id} + x\tilde{U})} + \frac{1}{\det(\mathrm{id} - x\tilde{U}^2)} + \frac{1}{\det(\mathrm{id} + x\tilde{U}^2)}\Bigg) \\
&= \frac{1}{6}\left(\frac{1}{(1-x)^n} + \frac{2}{(1+x+x^2)^{\left[\frac{n}{3}\right]}(1-x)^{n-2\left[\frac{n}{3}\right]}} + \frac{1}{(1+x)^n} + \frac{2}{(1-x+x^2)^{\left[\frac{n}{3}\right]}(1+x)^{n-2\left[\frac{n}{3}\right]}}\right) \\
&= \frac{1}{6}\left(\frac{1}{(1-x)^n} + \frac{1}{(1+x)^n} + \frac{2}{(1-x^3)^{\left[\frac{n}{3}\right]}(1-x)^{n-3\left[\frac{n}{3}\right]}} + \frac{2}{(1+x^3)^{\left[\frac{n}{3}\right]}(1+x)^{n-3\left[\frac{n}{3}\right]}}\right) \\
&= \frac{1}{6}\Bigg(\sum_{k=0}^{\infty} \binom{k+n-1}{k} x^k + \sum_{k=0}^{\infty} (-1)^k \binom{k+n-1}{k} x^k \\
&\quad + \sum_{\substack{k=0,\\ 3|k}}^{\infty} \binom{\frac{k}{3} + \left[\frac{n}{3}\right] - 1}{\frac{k}{3}} x^k \cdot \frac{2}{(1-x)^{n-3\left[\frac{n}{3}\right]}} + \sum_{\substack{k=0,\\ 3|k}}^{\infty} (-1)^k \binom{\frac{k}{3} + \left[\frac{n}{3}\right] - 1}{\frac{k}{3}} x^k \cdot \frac{2}{(1+x)^{n-3\left[\frac{n}{3}\right]}}\Bigg) \\
&= \frac{1}{3}\Bigg(\sum_{\substack{k=0,\\ k \text{ even}}}^{\infty} \binom{k+n-1}{k} x^k + \sum_{\substack{k=0,\\ 3|k}}^{\infty} \binom{\frac{k}{3} + \left[\frac{n}{3}\right] - 1}{\frac{k}{3}} x^k \cdot \frac{1}{(1-x)^{n-3\left[\frac{n}{3}\right]}} \\
&\quad + \sum_{\substack{k=0,\\ 3|k}}^{\infty} (-1)^k \binom{\frac{k}{3} + \left[\frac{n}{3}\right] - 1}{\frac{k}{3}} x^k \cdot \frac{1}{(1+x)^{n-3\left[\frac{n}{3}\right]}}\Bigg).
\end{aligned}$$

For odd n, the group generated by $\tilde{U}^{(n)}$ is given by $\{\mathrm{id}, \tilde{U}, \tilde{U}^2\}$, as $\tilde{U}^3 = \mathrm{id}$ in this case. Hence Molien's theorem A.12 yields that

$$\begin{aligned} P_U^{(n)}(x) &= \frac{1}{3}\left(\frac{1}{\det(\mathrm{id} - x\,\mathrm{id})} + \frac{1}{\det(\mathrm{id} - x\tilde{U})} + \frac{1}{\det(\mathrm{id} - x\tilde{U}^2)}\right) \\ &= \frac{1}{3}\left(\frac{1}{(1-x)^n} + \frac{2}{(1+x+x^2)^{\left[\frac{n}{3}\right]}(1-x)^{n-2\left[\frac{n}{3}\right]}}\right) \\ &= \frac{1}{3}\left(\frac{1}{(1-x)^n} + \frac{1}{(1-x^3)^{\left[\frac{n}{3}\right]}} \cdot \frac{2}{(1-x)^{n-3\left[\frac{n}{3}\right]}}\right) \\ &= \frac{1}{3}\left(\sum_{k=0}^{\infty}\binom{k+n-1}{k}x^k + \sum_{\substack{k=0,\\ 3|k}}^{\infty}\binom{\frac{k}{3}+\left[\frac{n}{3}\right]-1}{\frac{k}{3}}x^k \cdot \frac{2}{(1-x)^{n-3\left[\frac{n}{3}\right]}}\right). \end{aligned}$$

C. Source code

The computer based calculations in this thesis were done using the free and open-source computer algebra system SageMath [The22]. In order to compute period polynomials we first need to define the respective matrices. Since they are needed in most computations below, we will state the corresponding code here.

```
S = matrix([[0,-1],[1,0]])
T = matrix([[1,1],[0,1]])
U = matrix([[1,-1],[1,0]])
eps = matrix([[0,1],[1,0]])
```

C.1. Dimension of period polynomials

The following code was used to compute the dimensions of W_k, $W_k^{\pm}$ and L_k in table 1.

```
k = 8
P = PolynomialRing(SR, 'x,y')
poly = 0
for i in range(k-1):
    j=k-2-i
    tmp = var("a%d"%(i))
    poly += tmp*P.gens()[0]^j*P.gens()[1]^i
relS = poly + S.act_on_polynomial(poly)
relU = poly + U.act_on_polynomial(poly) + (U^2).act_on_polynomial(poly)
invEps = poly - eps.act_on_polynomial(poly)
relEps = poly + eps.act_on_polynomial(poly)
relLewis = poly - T.act_on_polynomial(poly) -
    (eps*T*eps).act_on_polynomial(poly)
A = matrix(ZZ,[[rel.coefficient(coeff) for coeff in poly.coefficients()]
    for rel in relS.coefficients() + relU.coefficients()])
```

```
B = matrix(ZZ,[[rel.coefficient(coeff) for coeff in poly.coefficients()] for rel in relS.coefficients() + relU.coefficients() + invEps.coefficients()])
C = matrix(ZZ,[[rel.coefficient(coeff) for coeff in poly.coefficients()] for rel in relS.coefficients() + relU.coefficients() + relEps.coefficients()])
D = matrix(ZZ,[[rel.coefficient(coeff) for coeff in poly.coefficients()] for rel in relLewis.coefficients()])
print(A.right_nullity(),B.right_nullity(),C.right_nullity(),D.right_nullity())
```

C.2. Dimension of bi-period polynomials

The following code was used to compute the dimensions of $\mathcal{W}_k$, $\mathcal{W}_k^{\pm}$ and $\mathcal{L}_k$ in table 2.

```
k = 8
P = PolynomialRing(SR, 'x1,x2,y1,y2')
poly = 0
for i,y in enumerate(sorted(list(set([prod(x) for x in cartesian_product_iterator([P.gens() for _ in range(k-2)])])))[::-1]):
    tmp_var = var("a%d"%(i))
    poly += tmp_var*y
relS = poly + bi_slash_mat(S).act_on_polynomial(poly)
relU = poly + bi_slash_mat(U).act_on_polynomial(poly) + bi_slash_mat(U^2).act_on_polynomial(poly)
invEps = poly - bi_slash_mat(eps).act_on_polynomial(poly)
relEps = poly + bi_slash_mat(eps).act_on_polynomial(poly)
relLewis = poly - bi_slash_mat(T).act_on_polynomial(poly) - bi_slash_mat(eps*T*eps).act_on_polynomial(poly)
A = matrix(ZZ,[[rel.coefficient(coeff) for coeff in poly.coefficients()] for rel in relS.coefficients() + relU.coefficients()])
B = matrix(ZZ,[[rel.coefficient(coeff) for coeff in poly.coefficients()] for rel in relS.coefficients() + relU.coefficients() + invEps.coefficients()])
C = matrix(ZZ,[[rel.coefficient(coeff) for coeff in poly.coefficients()] for rel in relS.coefficients() + relU.coefficients() + relEps.coefficients()])
D = matrix(ZZ,[[rel.coefficient(coeff) for coeff in poly.coefficients()] for rel in relLewis.coefficients()])
print(A.right_nullity(),B.right_nullity(),C.right_nullity(),D.right_nullity())
```

C.3. Transformation matrices under irreducible representations

The transformation matrices that were introduced in section 5.1 were computed using SageMath [The22]. As this computation was frequently needed in the code of C.4, we defined a function to return said matrices. The source code is given below.

```
def trafo_mat(n,mat):
    R.<var1,var2> = PolynomialRing(QQ)
    lst = []
    for i in range(n):
```

```
        tmp = var1^(n-1-i)*var2^i
        poly = mat.act_on_polynomial(tmp)
        tmp_lst = [poly[n-1-i, i] for i in range(n)]
        lst.append(tmp_lst)
    return matrix(QQ, lst)
```

C.4. Dimension of generalized period polynomials

The dimensions in table 3 have been computed using SageMath [The22]. The source code for this computation is given below. Note that the occurring function `trafo_mat` in the code is defined in appendix C.3.

```
n = 5
k = 8
P = PolynomialRing(SR, 'x', n)
poly = 0
for i,y in enumerate(sorted(list(set([prod(x) for x in
    cartesian_product_iterator([P.gens() for _ in range(k-n)])])))[::-1]):
    tmp_var = var("a%d"%(i))
    poly += tmp_var*y
relS = poly + trafo_mat(n,S).act_on_polynomial(poly)
relU = poly + trafo_mat(n,U).act_on_polynomial(poly) +
    trafo_mat(n,U^2).act_on_polynomial(poly)
relLewis = poly - trafo_mat(n,T).act_on_polynomial(poly) -
    trafo_mat(n,eps*T*eps).act_on_polynomial(poly)
A = matrix([[rel.coefficient(coeff) for coeff in poly.coefficients()] for
    rel in relS.coefficients()+relU.coefficients()])
B = matrix([[rel.coefficient(coeff) for coeff in poly.coefficients()] for
    rel in relLewis.coefficients()])
print(A.right_nullity(),B.right_nullity())
```