Lecture notes on homological algebra

Hamburg SS 2019

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June 17, 2019

Contents

1 Chain complexes

Let R be a ring. A *chain complex* $(C_\bullet, d)$ of (left) R-modules consists of a family $\{C_n | n \in \mathbb{Z}\}$ of R-modules equipped with R-linear maps $d_n : C_n \to C_{n-1}$ satisfying, for every $n \in \mathbb{Z}$, the condition $d_n \circ d_{n+1} = 0$. The maps $\{d_n\}$ are called *differentials* and, for $n \in \mathbb{Z}$, the module C_n is called the *module of n-chains*.

Remark 1.1. To keep the notation light we usually abbreviate $C_\bullet = (C_\bullet, d)$ and simply write d for d_n.

Example 1.2 (Syzygies)**.** Let $R = \mathbb{C}[x_1, \dots, x_n]$ denote the polynomial ring with complex coefficients, and let M be a finitely generated R-module.

(i) Suppose the set $\{a_1, a_2, \dots, a_k\} \subset M$ generates M as an R-module, then we obtain a sequence of R-linear maps

$$\ker(\varphi) \hookrightarrow R^k \overset{\varphi}{\twoheadrightarrow} M \tag{1.3}$$

where φ is defined by sending the basis element e_i of R^k to a_i. We set $\mathrm{Syz}^1(M) := \ker(\varphi)$ and, for now, ignore the fact that this R-module may depend on the chosen generators of M. Following D. Hilbert, we call $\mathrm{Syz}^1(M)$ the *first syzygy module* of M. In light of (1.3), every element of $\mathrm{Syz}^1(M)$, also called *syzygy*, can be interpreted as a relation among the chosen generators of M as follows: every element of $r \in \mathrm{Syz}^1(M)$ can be expressed as an R-linear combination $\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_k e_k$ of the basis elements of R^k. The fact that, by definition, $\varphi(r) = 0$, now reads as the equation

$$\lambda_1 m_1 + \lambda_2 m_2 + \cdots + \lambda_k m_k = 0$$

which is a relation between the generators $\{m_i\}$ in M.

(ii) By Hilbert's basis theorem, the module $\mathrm{Syz}^1(M)$ is again finitely generated, so we may choose a set $\{b_1, b_2, \ldots, b_l\}$ of generators. Repeating (1), we obtain a sequence of R-linear maps

$$\ker(\psi) \hookrightarrow R^l \overset{\psi}{\twoheadrightarrow} M \tag{1.4}$$

where ψ is defined by sending the basis element e_i of R^l to b_i. We set $\mathrm{Syz}^2(M) := \ker(\psi)$, called the *second syzygy module* of M. In light of (1.3) and (1.4), every element of $\mathrm{Syz}^2(M)$, also called *second syzygy*, can be interpreted as a relation among the relations among generators of M.

(iii) We can reiterate this procedure ad infinitum. Piecing together the sequences (1.3), (1.4), ..., we obtain a sequence of R-linear maps

$$\begin{tikzcd}
 & & \mathrm{Syz}^2(M) \arrow[dr, hook] & & & \\
\cdots \arrow[r, "d"] & R^m \arrow[rr, "d"] \arrow[ur, two heads] & & R^l \arrow[rr, "d"] \arrow[dr, two heads] & & R^m \\
 & & & & \mathrm{Syz}^1(M) \arrow[ur, hook] &
\end{tikzcd} \tag{1.5}$$

which, by construction, satisfies $d^2 = 0$.

The chain complex $C = (C_\bullet, d)$ of free R-modules constructed in (1.5) is called a *free resolution* of M. Note that, by construction, C comes equipped with a surjective map $C_0 \to M$ called *augmentation map*. Hilbert's phisolophy behind the construction of the sequence of higher syzygies is that, while the original module M may be very complicated, the modules $\mathrm{Syz}^k(M)$ become easier to understand as k increases. This statement is made precise in his famous *syzygy theorem*[1]: Every finitely generated R-module M admits a free resolution of length $\leq n$. We will give a proof a local variant of this theorem in the course. Hilbert's original proof was very explicit, our proof will be a corollary of a fairly abstract modern theory.

[1]Hilbert stated and proved this theorem under the additional assumption that M is a *graded* R-module. The statement is true in general but relies on a deep theorem of Quillen and Suslin which says that every finitely generated projective module over a polynomial ring is free.

Example 1.6 (Simplicial complexes)**.** Let N be a natural number. An *(abstract) simplicial complex* K on $[N] := \{0, 1, \ldots, N\}$ is a collection of subsets of $[N]$ such that

- if $\sigma \in K$ and $\tau \subset \sigma$, then $\tau \in K$.

One thinks of the collection of all subsets of $[N]$ as corresponding to all subsimplices of a geometric N-dimensional simplex Δ^N. A simplicial complex K on $[N]$ then corresponds to a union of subsimplices in Δ^N. For every $n \geq 0$, we introduce the set

$$K_n := \{\sigma \in K \big| \, |\sigma| = n + 1\}$$

of *n-simplices* in K. Any $\sigma \in K_n$ can be uniquely expressed as

$$\sigma = \{x_0, x_1, \ldots, x_n\}$$

with $x_0 < x_1 < \cdots < x_n$. We then define, for every $0 \leq i \leq n$, the face map

$$\partial_i : K_n \longrightarrow K_{n-1}, \ \sigma \mapsto \{x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\}.$$

Let R be a ring and define

$$C_n(K, R) := \bigoplus_{\sigma \in K_n} R e_\sigma$$

the free R-module on the set K_n whose basis elements we label by $\{e_\sigma\}$. We define differentials

$$d_n : C_n(K, R) \longrightarrow C_{n-1}(K, R), \ e_\sigma \mapsto \sum_{i=0}^{n} (-1)^i e_{\partial_i \sigma}$$

where, as $C_n(K, R)$ is a free R-module, it suffices to specify the map d_n on a the basis $\{e_\sigma\}$. By explicit verification we have $d^2 = 0$ and hence we obtain a chain complex $(C_\bullet(K, R), d)$ called the simplicial chain complex of K with coefficients in R. As an explicit example, consider the simplicial complex on $[2]$ given by the collection of subsets

$$\{\{0, 1\}, \{1, 2\}, \{0, 2\}, \{0\}, \{1\}, \{2\}, \emptyset\}.$$

The corresponding simplicial chain complex with coefficients in the ring $\mathbb{Z}$ has two non-zero components given by $C_1(K, \mathbb{Z}) = \mathbb{Z}e_{\{0,1\}} \oplus \mathbb{Z}e_{\{1,2\}} \oplus \mathbb{Z}e_{\{0,2\}}$ and $C_0(K, \mathbb{Z}) = \mathbb{Z}e_{\{0\}} \oplus \mathbb{Z}e_{\{1\}} \oplus \mathbb{Z}e_{\{2\}}$ while the differential $d : C_1 \to C_0$ is given, with respect to the indicated basis, by the matrix

$$\begin{pmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Note that, by convention, we omit the trivial modules of n-chains corresponding to the zero R-module.

Example 1.7 (Cyclic bar complex)**.** Let k be a field, and let A be an associative k-algebra. We define, for every $n \geq 0$, the k-vector space

$$C_n(A) = \underbrace{A \otimes_k A \otimes_k \cdots \otimes_k A}_{n+1 \text{ copies}}$$

given by iterating the above tensor product construction. This construction only depends on the k-vector space structure underlying A. Further, we define maps $d : C_n(A) \to C_{n-1}(A)$ by k-linearly extending the formula

$$\begin{aligned} d(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = {} & (a_0 a_1) \otimes a_2 \otimes \cdots \otimes a_n \\ & + \sum_{i=1}^{n-1} (-1)^i a_0 \otimes \cdots \otimes (a_i a_{i+1}) \otimes \cdots \otimes a_n \\ & + (-1)^n (a_n a_0) \otimes a_1 \otimes \cdots \otimes a_{n-1}. \end{aligned}$$

An explicit computation shows $d^2 = 0$. We obtain a chain complex $(C_\bullet(A), d)$ of k-vector spaces called the *cyclic bar complex* of A.

We introduce some terminology: Given a chain complex $C_\bullet$ of R-modules, we call $Z_n := \ker(d_n)$ the *module of n-cycles*, and $B_n := \operatorname{im}(d_{n+1})$ the *module of n-boundaries*. Note that $B_n \subset Z_n \subset C_n$ so that we can further define the quotient module

$$H_n(C_\bullet) := Z_n / B_n,$$

called the *nth homology module* of $C_\bullet$. The chain complex $C_\bullet$ is called *exact* at $n \in \mathbb{Z}$, if $H_n(C_\bullet) \cong 0$.

Example 1.8. (1) Let $R = \mathbb{C}[x_1, \ldots, x_n]$ be the polynomial ring, M a finitely generated R-module and $C_\bullet$ the free resolution of M as defined above. Then we have

$$H_i(C_\bullet) \cong \begin{cases} M & \text{for } i = 0, \\ 0 & \text{for } i \neq 0. \end{cases}$$

(2) Let K be a simplicial complex with corresponding simplicial chain complex $C_\bullet(K, R)$ with coefficients in a ring R. The homology modules $H_i(C_\bullet(K, R))$ are called *simplicial homology* of K with coefficients in R. These modules, and variants, are of central importance in algebraic topology. In the explicit example above, given by the boundary of a 2-simplex, we have

$$H_i(C_\bullet(K, \mathbb{Z})) \cong \begin{cases} \mathbb{Z} & \text{for } i = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

(3) Let A be an associative k-algebra. The homology modules of the cyclic bar complex $C_\bullet(A)$ are called *Hochschild homology*. Their calculation is in general very involved, but, for example if A is a finitely generated, commutative k-algebra satisfying a certain smoothness condition, then the so-called *Hochschild-Kostant-Rosenberg* theorem states that, for all $i \geq 0$, we have

$$H_i(C_\bullet(A)) \cong \Omega^i_{A/k}$$

where $\Omega^i_{A/k}$ denotes the module of Kähler differentials of A over k.

Given a chain complex $C_\bullet$ of R-modules, we introduced, for every n, modules of cycles $Z_n = \ker(d_n)$, boundaries $B_n = \mathrm{im}(d_{n+1})$, and the homology module given by the quotient $H_n(C_\bullet) = Z_n/B_n$. We say that $C_\bullet$ is *exact* at $n \in \mathbb{Z}$, if $H_n(C_\bullet) \cong 0$.

Convention 1.9. We say that a sequence

$$C_k \xrightarrow{d} C_{k-1} \xrightarrow{d} \dots \xrightarrow{d} C_{l-1} \xrightarrow{d} C_l$$

of R-modules with $d^2 = 0$ is *exact* if $H_n(C_\bullet) = 0$ for all $l < n < k$. Note the strict inequality so that no exactness condition is required at the end terms l and k.

Example 1.10. An exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is called *short exact sequence*. The exactness conditions translate into the requirements that f be injective, g be surjective, and $\ker(g) = \mathrm{im}(f)$. The sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

of abelian groups is a short exact sequence. The sequence

$$\mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

is exact but can not be extended to form a short exact sequence, since the map given by multiplication by 2 is not injective.

A key insight of modern mathematics is that, to understand any class of objects of a certain kind, one ought to understand what kind of maps to allow between these objects. This leads to various concepts of *morphisms*: a morphism between groups is a group homomorphism, a morphism between R-modules is an R-linear map, a morphism between topological spaces is a continuous map, etc. A collection of objects and their morphisms forms a category.

Definition 1.11. A *category* $\mathcal{C}$ consists of

- a class $\mathrm{ob}(\mathcal{C})$ of *objects*,
- for every pair A, B of objects, a set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ of *morphisms*,
- for every object A, an *identity* morphism id_A,

equipped with, for every triple of objects A, B, C, a map

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A, C),\ (g, f) \mapsto f \circ g$$

called *composition* such that

(1) for morphisms $h \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, and $f \in \mathrm{Hom}_{\mathcal{C}}(C, D)$, we have

$$(f \circ g) \circ h = f \circ (g \circ h),$$

(2) for every morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, we have

$$f = f \circ \mathrm{id}_A = \mathrm{id}_B \circ f.$$

Example 1.12. (1) The category **Set** of sets with all maps as morphisms, the category **Top** of topological spaces with continuous maps as morphisms, the category **Ab** of abelian groups with group homomorphisms as morphisms, the category $R-$**mod** of left R-modules with R-linear maps as morphisms.

(2) Let G be a group. We define a category BG with one object, denoted $*$, and morphism set $\mathrm{Hom}_{BG}(*, *) = G$ where composition is given by the group law.

Definition 1.13. A morphism $f : C_\bullet \to D_\bullet$ between chain complexes of R-modules is a family $\{f_n : C_n \to D_n\}$ of R-linear maps satisfying, for every n, the condition $d_n \circ f_n = f_{n-1} \circ d_n$. We obtain a category of chain complexes of R-modules which we denote by $\mathbf{Ch}(R-\mathbf{mod})$.

Any morphism of chain complexes preserves, in each degree, the modules of cycles and boundaries, and therefore induces a map on homology modules. A morphism $f : C_\bullet \to D_\bullet$ of chain complexes is called a *quasi-isomorphism* if, for every n, the induced map $H_n(f) : H_n(C) \to H_n(D)$ is an isomorphism. Since the homology modules of a chain complex are the invariants we are typically interested in, we are lead to the natural problem of studying complexes "up to quasi-isomorphism". The main conceptual difficulty in this problem lies in the following fact:

Problem 1.14. Given a morphism $f : C_\bullet \to D_\bullet$ of chain complexes, a *quasi-inverse* of f is a morphism $g : D_\bullet \to C_\bullet$ of chain complexes such that, for every n, the maps $H_n(f)$ and $H_n(g)$ are inverses of one another. Show that quasi-isomorphisms do, in general, *not* have quasi-inverses.

2 Abelian categories

Above, we introduced chain complexes in the category $R-$**mod**. It turns out that all of the above makes sense more generally for chain complexes in any *abelian category*. Examples of abelian categories include, besides $R-$**mod**, the categories of

- graded modules over a graded ring
- sheaves of abelian groups
- complexes of R-modules
- ...

Our next goal is to introduce the notion of an abelian category. We start with some preliminaries in general category theory. Various "element-theoretic" concepts can be defined in an "arrow-theoretic" way, using *universal mapping properties*.

element-theoretic	arrow-theoretic
The singleton $\{*\}$ has one element.	For every set A, there exists a unique map $A \to \{*\}$.
The empty set $\emptyset$ has no elements.	For every set A, there exists a unique map $\emptyset \to A$.
The Cartesian product $X \times Y$ of sets X, Y has elements given by pairs (x, y) where $x \in X$ and $y \in Y$.	The Cartesian product $X \times Y$ comes equipped with two projection maps $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ such that, for any set A, equipped with maps $f : A \to X$ and $g : Z \to Y$, there exists a unique map $\varphi : A \to X \times Y$ such that the diagram $$\begin{tikzcd} & A \arrow[d, dashed, "\varphi"] \arrow[ddl, "f"'] \arrow[ddr, "g"] & \\ & X \times Y \arrow[dl, "\pi_X"] \arrow[dr, "\pi_Y"'] & \\ X & & Y \end{tikzcd}$$ commutes.
$\vdots$	$\vdots$

The advantage of the arrow-theoretic formulations is that they make sense in any category therefore leading to abstract versions of "singleton", "empty set", "Cartesian product", etc, in any category $\mathcal{C}$:

Definition 2.1. Let $\mathcal{C}$ be a category.

(1) An object X in $\mathcal{C}$ is called a *final (or terminal) object* if, for every object A in $\mathcal{C}$, there exists a unique morphism $A \to X$.

(2) An object X in $\mathcal{C}$ is called an *initial object* if, for every object A in $\mathcal{C}$, there exists a unique morphism $X \to A$.

(3) An object X in $\mathcal{C}$ is called a *zero object* if it is both initial and final.

(4) Given objects X, Y in $\mathcal{C}$, an object Z, equipped with morphisms $p_X : Z \to X$ and $p_Y : Z \to Y$, is called a *product* of X and Y if, for any object A, equipped with morphisms $f : A \to X$ and $g : A \to Y$, there exists a unique morphism $\varphi : A \to Z$ such that the diagram

$$\begin{tikzcd} & A \arrow[d, dashed, "\exists! \varphi"'] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "g"] & \\ & Z \arrow[dl, "\pi_X"] \arrow[dr, "\pi_Y"'] & \\ X & & Y \end{tikzcd}$$

commutes.

(5) Given objects X, Y in $\mathcal{C}$, an object Z, equipped with morphisms $i_X : X \to Z$ and $i_Y : Y \to Z$, is called a *coproduct* of X and Y if, for any object A, equipped with morphisms $f : X \to A$ and $g : Y \to A$, there exists a unique morphism $\varphi : Z \to A$ such that the diagram

$$\begin{tikzcd} & A & \\ & Z \arrow[u, dashed, "\exists! \varphi"] & \\ X \arrow[ur, "i_X"'] \arrow[uur, bend left, "f"] & & Y \arrow[ul, "i_Y"] \arrow[uul, bend right, "g"'] \end{tikzcd}$$

commutes.

Remark 2.2. Note that we obtain the definitions of initial object and coproduct from the definitions of final object and product, respectively, by simply reversing all arrows. This observation can be formulated more precisely as follows. Given a category $\mathcal{C}$, we can introduce the *opposite category* $\mathcal{C}^{\mathrm{op}}$ which has the same objects as $\mathcal{C}$, but

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(A, B) := \mathrm{Hom}_{\mathcal{C}}(B, A).$$

An object A is a final object of $\mathcal{C}$ if and only if it is an initial object of $\mathcal{C}^{\mathrm{op}}$. An object Z of $\mathcal{C}$, equipped with morphisms $Z \to X$ and $Z \to Y$ is a product in $\mathcal{C}$ if and only if it is a coproduct in $\mathcal{C}^{\mathrm{op}}$.

Remark 2.3. Note that the existence of an object characterized by a universal mapping property is not automatic and needs to be verified. Once existence is established, the object is uniquely determined up to unique isomorphism. We hence slightly abuse language and denote *a* product of two objects X and Y, if it exists, by $X \times Y$ referring to it as *the* product. Similarly, we denote a coproduct of X and Y, if it exists, by $X \amalg Y$ referring to it as the coproduct.

Example 2.4. (1) In the category **Set** of sets, initial objects, final objects, products, and coproducts, correspond to the empty set, singletons, Cartesian product, and disjoint union, respectively. There does not exist a zero object.

(2) In the category **Grp** of groups, a group with one element is initial and final, and hence a zero object. The product of two groups G and H, is given by the product group $G \times H$ whose underlying set is the Cartesian product with componentwise multiplication. The coproduct of G and H is the free product $G * H$. For example, $\mathbb{Z} * \mathbb{Z}$ is the free group on two generators.

(3) In the category **Ab** of abelian groups, a group with one element is initial and final, and hence a zero object. The product of two abelian groups A and B, is given by the product group $A \times B$ equipped with the two projection homomorphisms to A and B. The coproduct of A and B is also given by the product group $A \times B$, but equipped with the inclusion homomorphisms $A \to A \times B, a \mapsto (a, 0)$ and $B \to A \times B, a \mapsto (0, b)$.

We continue with introducing various general concepts from category theory which will be used throughout the course.

Definition 2.5. A *functor* $F : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ is a rule that associates

- to every object A of $\mathcal{C}$ an object $F(A)$ of $\mathcal{D}$,
- to every morphism $f : A \to B$ in $\mathcal{C}$ a morphism $F(f) : F(A) \to F(B)$ in $\mathcal{D}$,

such that,

(1) for every object A of $\mathcal{C}$, we have $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$,

(2) for every pair of composable morphisms $f : A \to B$ and $g : B \to C$, we have $F(g \circ f) = F(g) \circ F(f)$.

Example 2.6. (1) There is a functor $F : \mathbf{Set} \longrightarrow \mathbf{Top}$ which assigns to a set X the topological space $F(X)$ equipped with the discrete topology (every subset is open). For a map of sets $f : X \to Y$, we set $F(f)$ to equal f, noting that any map of sets is continuous with respect to the discrete topology.

(2) There is a forgetful functor $F : \mathbf{Top} \longrightarrow \mathbf{Set}$ such that, for a topological space X, the set $F(X)$ is the set underlying X, forgetting the topology. Similarly, for a continuous map $f : X \to Y$, the map $F(f)$ equals f, but we forget that f is continuous.

(3) Similarly, there is a forgetful functor $F : \mathbf{Ab} \to \mathbf{Set}$ which forgets the abelian group structure and only remembers the underlying set.

(4) There is a functor $F : \mathbf{Set} \to \mathbf{Ab}$, which assigns to a set X the free abelian group $F(X)$ on the set X, i.e.,
$$F(X) := \bigoplus_{x \in X} \mathbb{Z}.$$
Since the set X can be interpreted as a basis of $F(X)$, given a map $f : X \to Y$, we obtain a homomorphism $F(f) : F(X) \to F(Y)$ of abelian groups by extending f linearly.

(5) For any left R-module N, the association $M \mapsto M \otimes_R N$ extends to a functor $R{-}\mathbf{mod} \to \mathbf{Ab}$.

(6) Given a category $\mathcal{C}$ and an object $X \in \mathcal{C}$, we obtain a functor
$$\mathcal{C} \longrightarrow \mathbf{Set},\ A \mapsto \mathrm{Hom}_{\mathcal{C}}(X, A)$$
which we denote by h^X. Similarly, we obtain a functor
$$\mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Set},\ A \mapsto \mathrm{Hom}_{\mathcal{C}}(A, X)$$
denoted by h_X.

Example 2.7. A category is called *small* if its objects form a set. Small categories form a category $\mathbf{Cat}$ with functors as morphisms.

Definition 2.8. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* $\eta : F \Rightarrow G$ consists of, for every object A of $\mathcal{C}$, a morphism $\eta_A : F(A) \to G(A)$ in $\mathcal{D}$, such that, for every morphism $f : A \to B$ in $\mathcal{C}$, the diagram

$$\begin{tikzcd} F(A) \arrow[r, "\eta_A"'] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\eta_B"'] & G(B) \end{tikzcd}$$

commutes.

Example 2.9. Let I be the category with two objects 0 and 1 and, besides the identity morphisms of 0 and 1, one morphism $0 \to 1$. Let $\mathcal{C}$ be a category. A functor $F : I \to \mathcal{C}$ corresponds to a morphism $f : A_0 \to A_1$ in $\mathcal{C}$. Let $G : I \to \mathcal{C}$ be another functor corresponding to a morphism $g : B_0 \to B_1$. A natural transformation $\eta : F \Rightarrow G$ consists of a commutative square

$$\begin{tikzcd} A_0 \arrow[r, "f"] \arrow[d, "\eta_0"'] & A_1 \arrow[d, "\eta_1"] \\ B_0 \arrow[r, "g"'] & B_1. \end{tikzcd}$$

Example 2.10. Let $\mathbf{Vect}_k$ denote the category of vector spaces over a field k. Consider the identify functor $\mathrm{id} : \mathbf{Vect}_k \to \mathbf{Vect}_k$ and the functor $D : \mathbf{Vect}_k \to \mathbf{Vect}_k, V \mapsto (V^*)^*$ given by forming the double dual, where $V^* := \mathrm{Hom}_k(V, k)$. Then the family of linear maps

$$V \longrightarrow (V^*)^*, v \mapsto (f \mapsto f(v))$$

defines a natural transformation $\mathrm{id} \Rightarrow D$.

Example 2.11. Let $\mathcal{C}, \mathcal{D}$ be categories and assume that $\mathcal{C}$ is small. Then the collection of functors $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ forms a category with natural transformations as morphisms.

Definition 2.12. An **Ab**-*category* is a category $\mathcal{A}$ together with, for every pair A, B of objects, an abelian group structure on the set $\mathrm{Hom}_{\mathcal{A}}(A, B)$ such that all composition maps

$$\mathrm{Hom}_{\mathcal{A}}(A, B) \times \mathrm{Hom}_{\mathcal{A}}(B, C) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(A, C)$$

are bilinear. An *additive category* is an **Ab**-category which has a zero object and admits products $A \times B$ for objects A, B.

Proposition 2.13. Let $\mathcal{A}$ be an additive category, and let A, B objects of $\mathcal{A}$. Then there exist canonical maps $i_A : A \to A \times B$ and $i_B : B \to A \times B$ which exhibit $A \times B$ as a *coproduct* of A and B.

Proof. The product $A \times B$ comes equipped with projection maps $p_A : A \times B \to A$ and $p_B : A \times B \to B$. By the universal property of the product, there exists a unique map $i_A : A \to A \times B$ such that $p_A \circ i_A = \mathrm{id}_A$ and $p_B \circ i_A = 0$. Similarly, there exists a unique map $i_B : B \to A \times B$ such that $p_A \circ i_B = 0$ and $p_B \circ i_B = \mathrm{id}_B$. We claim that $A \times B$ equipped with the maps i_A and i_B is a coproduct. To show this, we have to construct, given morphisms $f : A \to Z$ and $g : B \to Z$, a unique morphism $\varphi : A \times B \to Z$ such that $\varphi \circ i_A = f$ and $\varphi \circ i_B = g$. We set $\varphi := f \circ p_A + g \circ p_B$. To show that φ is unique, we first claim that $i_A \circ p_A + i_B \circ p_B = \mathrm{id}_{A\times B}$. By the universal property of the product it suffices to check that this identity holds after applying $p_A \circ -$ and $p_B \circ -$ on both sides where we obtain $p_A = p_A$ and $p_B = p_B$ showing the claim. Now, given any φ satisfying $\varphi \circ i_A = f$ and $\varphi \circ i_B = g$, we have

$$\begin{aligned}\varphi &= \varphi \circ \mathrm{id}_{A\times B}\\ &= \varphi \circ (i_A \circ p_A + i_B \circ p_B)\\ &= f \circ p_A + g \circ p_B\end{aligned}$$

showing uniqueness. $\square$

Convention 2.14. In other words, in any additive category, coproducts and products coincide. We use the notation

$$A \oplus B := A \times B \cong A \amalg B$$

to emphasize this fact.

We need to introduce a few more basic definitions. A morphism $f : X \to Y$ in a category $\mathcal{C}$ is called

- *monic* if, for every $g_1 : A \to X$, $g_2 : A \to X$, we have the left cancellation property

$$f \circ g_1 = f \circ g_2 \quad \Rightarrow \quad g_1 = g_2.$$

- *epic* if, for every $g_1 : Y \to A$, $g_2 : Y \to A$, we have the right cancellation property

$$g_1 \circ f = g_2 \circ f \quad \Rightarrow \quad g_1 = g_2.$$

Example 2.15. In the category **Set** of sets the monic morphisms are precisely the injective morphisms while the epic morphisms are precisely the surjective morphisms. However, in the category of rings, the embedding $\mathbb{Z} \to \mathbb{Q}$ is epic even though not surjective.

Remark 2.16. Note that a morphism f in $\mathcal{C}$ is epic if and only if the corresponding morphism in $\mathcal{C}^{\mathrm{op}}$ is monic.

Let $\mathcal{A}$ be an additive category and let $f : X \to Y$ be a morphism in $\mathcal{A}$.

- A *kernel of* f is a morphism $i : K \to X$ which is universal with the property $f \circ i = 0$ in the following sense: for every $j : A \to X$ such that $f \circ j = 0$, there exists a unique morphism $\varphi : A \to K$ such that $j = i \circ \varphi$. We may depict this in a diagram as

$$\begin{tikzcd} & A \arrow[dl, dashed, "\exists!\varphi"'] \arrow[d, "j"] \arrow[dr, "0"] & \\ K \arrow[r, "i"] & X \arrow[r, "f"] & Y. \end{tikzcd}$$

- A *cokernel of f* is a morphism $p : Y \to C$ which is universal with the property $p \circ f = 0$ in the following sense: for every $q : Y \to A$ such that $q \circ f = 0$, there exists a unique morphism $\varphi : C \to A$ such that $q = \varphi \circ p$. We depict this in a diagram as

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "0"'] & Y \arrow[r, "p"] \arrow[d, "q"] & C \arrow[dl, dashed, "\exists!\varphi"] \\ & A & \end{tikzcd}$$

Remark 2.17. Given an additive category $\mathcal{C}$, the opposite category $\mathcal{C}^{\mathrm{op}}$ is again additive. In this context, a morphism is a kernel in $\mathcal{C}$ if and only if it is a cokernel in $\mathcal{C}^{\mathrm{op}}$.

Remark 2.18. We will typically simply call the object K the kernel of f, denoted by $\ker(f)$, leaving the morphism $i : K \to X$ implicit. Similarly, we will sometimes refer to the object C as the cokernel of f, denoted by $\mathrm{coker}(f)$, leaving the morphism $p : Y \to C$ implicit.

Proposition 2.19. Kernels are monic and cokernels are epic.

Proof. Let $i : K \to X$ be a kernel of $f : X \to Y$. Let $g_1 : A \to X$, $g_2 : A \to X$, such that $i \circ g_1 = i \circ g_2$. We have to show that $g_1 = g_2$. Equivalently, for $g = g_1 - g_2$, we have $i \circ g = 0$, and have to show $g = 0$. The diagram

$$\begin{tikzcd} & A \arrow[dl, "g"'] \arrow[d, "0"] \arrow[dr, "0"] & \\ K \arrow[r, "i"] & X \arrow[r, "f"] & Y. \end{tikzcd}$$

is commutative and, by the universal property of the kernel, g is the unique morphism from $A \to K$ which makes the diagram commutative. But the diagram also commutes if we replace g by 0, and hence $g = 0$. This argument also implies the statement for cokernels which are kernels in the opposite category. □

Definition 2.20. An additive category $\mathcal{A}$ is called *abelian* if

(1) Every morphism in $\mathcal{A}$ has a kernel and a cokernel.

(2) Every monic morphism is the kernel of its cokernel.

(3) Every epic morphism is the cokernel of its kernel.

Example 2.21. (1) Let R be a ring. Then the categories $R-\mathbf{mod}$ and $\mathbf{mod}-R$ are abelian.

(2) Let $\mathcal{A}$ be an abelian category. Then the category $\mathbf{Ch}(\mathcal{A})$ of chain complexes in $\mathcal{A}$ is abelian.

(3) Let I be a small category, and let $\mathcal{A}$ be an abelian category. Then the category $\mathrm{Fun}(I, \mathcal{A})$ of functors from I to $\mathcal{A}$ is abelian.

(4) Let G be a group. A functor from $BG \to \mathbf{Ab}$ corresponds to a *G-module*, i.e., an abelian group A equipped with an action of G via group homomorphisms. The category $\mathrm{Fun}(BG, \mathbf{Ab})$ of all G-modules is abelian by (3) and we denote it by $G{-}\mathbf{mod}$. Alternatively, for the abelian category $\mathbf{Vect}_k = k{-}\mathbf{mod}$ of k-vector spaces over a field k, we obtain the abelian category $\mathrm{Fun}(BG, \mathbf{Vect}_k)$ of *k-linear representations of G*.

(5) Let X be a topological space. We define a small category $\mathrm{Open}(X)$ with objects given by open subsets $U \subset X$ and morphisms given by inclusions of open subsets. A *presheaf of abelian groups on* X is a functor $F : \mathrm{Open}(X)^{\mathrm{op}} \to \mathbf{Ab}$. The category $\mathrm{Fun}(\mathrm{Open}(X)^{\mathrm{op}}, \mathbf{Ab})$ of all such presheaves is abelian by (3) and we denote it by $\mathbf{Psh}(X, \mathbf{Ab})$. An example of a presheaf of abelian groups on X is given by assigning to an open subset $U \subset X$ the abelian group of continuous functions $U \to \mathbb{R}$.

Remark 2.22. One of the original motivations for introducing the concept of an abelian category was to give a unified treatment of homological algebra for G-modules and presheaves (or, rather, *sheaves*, a concept introduced later in the course).

Example 2.23. We mention a few examples of additive categories which are *not* abelian.

(1) Consider the category $R{-}\mathbf{mod}^{\mathrm{fg}}$ of finitely generated modules over the polynomial ring $R = \mathbb{C}[x_1, x_2, x_3, \dots]$ with variables indexed by the natural numbers. Then the morphism $R \to \mathbb{C}$ given by sending all variables x_i to 0 does not have a kernel. The kernel of this morphism in the category of all R-modules is given by the maximal ideal $\mathfrak{m} = (x_1, x_2, \dots)$ but, since this R-module is not finitely generated, it does not provide a kernel in $R{-}\mathbf{mod}^{\mathrm{fg}}$. Note that this does *not* immediately imply that a kernel does not exist in $R{-}\mathbf{mod}^{\mathrm{fg}}$, since the defining universal property involves less objects. Nevertheless, a little more work shows that the above morphism does indeed not have a kernel. Note that a similar example can be given for any ring which is non-Noetherian.

(2) Consider the category $\mathbf{Ab}^{\mathrm{ff}}$ of free abelian groups of finite rank. Let $f : X \to Y$ be a morphism in $\mathbf{Ab}^{\mathrm{ff}}$. The kernel $\ker(f)$ taken in $\mathbf{Ab}$ is a subgroup of X, hence free of finite rank, and so provides a kernel in $\mathbf{Ab}^{\mathrm{ff}}$. The cokernel C of f in $\mathbf{Ab}$, however, is not necessarily free. But it is immediate to verify that the quotient of C modulo its torsion subgroup is a cokernel of f in $\mathbf{Ab}^{\mathrm{ff}}$. Therefore, the category $\mathbf{Ab}^{\mathrm{ff}}$ is additive and admits kernels and cokernels. However, the morphism $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ is an example of a monic morphism which is not the kernel of its cokernel (its cokernel is the 0 group).

The morphism $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ in the category of free finite rank abelian groups from the above example is monic and epic without having an inverse. This phenomenon does not occur in abelian categories:

Proposition 2.24. Let $\mathcal{A}$ be an abelian category. A morphism in $\mathcal{A}$ has an inverse if and only if it is monic and epic.

Proof. In any category, a morphism with an inverse satisfies both left- and right cancellation and is therefore both monic and epic. Let $f : X \to Y$ be a morphism in $\mathcal{A}$ which is both monic and epic. Since $\mathcal{A}$ is abelian, f is the kernel of its cokernel. We claim that this implies that the kernel of f is 0. Let $g : A \to X$ be a morphism such that $f \circ g = 0$, then we have a commutative diagram

$$\begin{tikzcd}
 & A \arrow[d, "g"'] \arrow[dr, "0"] \arrow[drr, bend left, "0"] & & \\
0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r] & \operatorname{coker}(f)
\end{tikzcd}$$

where, due to the fact that $X \to Y$ is a kernel of $Y \to \operatorname{coker}(f)$, the map g is the unique map making the diagram commute. But, since replacing g by 0 also makes the diagram commute, we conclude $g = 0$. Further, the map $0 : A \to X$ coincides with the composite of $A \to 0$ and $0 \to X$ and hence factors uniquely over $0 \to X$. By definition, this means that $0 \to X$ is the kernel of f. We now apply the universal property of f as a cokernel of $0 \to X$ to obtain a unique map $\varphi : Y \to X$ such that

$$\begin{tikzcd}
0 \arrow[r] \arrow[dr, "0"'] & X \arrow[r, "f"] \arrow[d, "\mathrm{id}"'] & Y \arrow[dl, dashed, "\exists!\varphi"] \\
 & X &
\end{tikzcd}$$

commutes, in other words, φ is a left inverse of f. A similar argument, using that f is the kernel of $Y \to 0$, we obtain a right inverse of f which must therefore coincide with φ. $\square$

Let $f : X \to Y$ be a morphism in an abelian category $\mathcal{A}$. We define the *image of f* to be

$$\operatorname{im}(f) := \ker(\operatorname{coker}(f))$$

which is an object of $\mathcal{A}$ equipped with a monic map $\operatorname{im}(f) \to Y$. Given a sequence of morphisms

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{2.25}$$

in $\mathcal{A}$ such that $g \circ f = 0$, we obtain a commutative diagram

$$\begin{tikzcd}
 & \operatorname{im}(f) \arrow[d, "i"] \arrow[dr, "h"] & \\
A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[d, "p"] & C \\
 & \operatorname{coker}(f) \arrow[ur] &
\end{tikzcd}$$

which implies that $h = 0$, since $p \circ i = 0$. Hence, we obtain a unique map $\varphi : \operatorname{im}(f) \to \ker(g)$. We say that the sequence (2.25) is *exact* if φ is an isomorphism. More generally, we form the *homology* which is the object in $\mathcal{A}$ given by $\operatorname{coker}(\varphi)$. Using this terminology, all concepts which we introduced for chain complexes in R−**mod** can be introduced for chain complexes in any abelian category $\mathcal{A}$.

Definition 2.26. Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories. A functor $F : \mathcal{A} \to \mathcal{B}$ is called *additive* if, for every pair of objects X,Y of $\mathcal{A}$, the corresponding map

$$\operatorname{Hom}_{\mathcal{A}}(X, Y) \to \operatorname{Hom}_{\mathcal{B}}(F(X), F(Y))$$

are homomorphisms of abelian groups. An additive functor F is called *exact* if it preserves exact sequences.

We finally mention the following famous theorem which will allow us to apply element-theoretic arguments to finite diagrams in a general abelian category $\mathcal{A}$.

Theorem 2.27 (Freyd-Mitchell)**.** Let $\mathcal{A}$ be a small abelian category. Then there exists a ring R and an exact functor

$$F : \mathcal{A} \longrightarrow R\mathbf{-mod}$$

which is *fully faithful*, i.e., for every pair of objects X, Y of $\mathcal{A}$, the map

$$\operatorname{Hom}_{\mathcal{A}}(X, Y) \to \operatorname{Hom}_{R\mathbf{-mod}}(F(X), F(Y))$$

induced by F is a bijection.

3 Derived functors

We have introduced abelian categories and exact functors as natural way to relate two abelian categories. In practice, however, it turns out that additive functors between abelian categories are rarely exact. The theory of *derived functors*, which we will establish in the next few lectures, provides us with a tool to understand this situation quantitatively.

Example 3.1. Let R be a ring, and let N be a left R-module. Then, in general, the additive functor

$$- \otimes_R N : \mathbf{mod}{-}R \longrightarrow \mathbf{Ab},\ M \mapsto M \otimes_R N$$

is not exact. For example, for $R = \mathbb{Z}$, the functor $- \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ takes the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

in **Ab**, to the sequence

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\cong} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

which is not exact. Note, however, that exactness only fails at the leftmost position.

We capture the phenomenon observed in Example 3.1 in a definition: An additive functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ of abelian categories is called *right exact* if, for every short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in $\mathcal{A}$, the sequence

$$F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow 0$$

is exact. Dually, F is called *left exact* if, for every short exact sequence in $\mathcal{A}$ as above, the sequence

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C).$$

While exact functors are rare, it turns out that most functors of interest are either left or right exact. Nevertheless, there is a certain class of short exact sequences which every additive functor preserves. A sequence in $\mathcal{A}$ of the form

$$0 \longrightarrow A \xrightarrow{i_A} A \oplus B \xrightarrow{\pi_B} B \longrightarrow 0$$

is called *split exact*. Note that every split exact sequence is indeed exact.

Lemma 3.2. Let A, B be objects in an additive category $\mathcal{A}$. Let X be an object in $\mathcal{A}$ equipped with morphisms

$$\begin{tikzcd} A \arrow[dr, "i_A"] & & A \\ & X \arrow[ur, "\pi_A"] \arrow[dr, "\pi_B"] & \\ B \arrow[ur, "i_B"] & & B \end{tikzcd}$$

such that the following equations hold

$$i_A \circ \pi_A + i_B \circ \pi_B = \mathrm{id}_X \tag{3.3}$$
$$\pi_A \circ i_B = 0 = \pi_B \circ i_A \tag{3.4}$$
$$\pi_A \circ i_A = \mathrm{id}_A \tag{3.5}$$
$$\pi_B \circ i_B = \mathrm{id}_B\,. \tag{3.6}$$

Then the morphisms i_A, i_B, π_A, π_B exhibit $X \cong A \oplus B$ as a sum of A and B. Conversely, for any sum $A \oplus B$, we can choose i_A, i_B, π_A, π_B such that the equations (3.3)-(3.6) hold.

Proof. The argument is similar to our proof that products and coproducts in additive categories coincide. We leave it to the reader. □

Note that, by definition, any additive functor preserves the equations (3.3)-(3.6). In particular, additive functors preserve sums so that we have the following corollary.

Corollary 3.7. Let $F : \mathcal{A} \to B$ be an additive functor. Then F preserves split exact sequences.

Let $\mathcal{A}$ be an abelian category. An object P of $\mathcal{A}$ is called *projective* if, for every exact sequence $B \to C \to 0$, and every morphism $p : P \to C$ in $\mathcal{A}$, there exists a morphism $\widetilde{p} : P \to B$, such that the diagram

$$\begin{tikzcd} & P \arrow[dl, dashed, "\widetilde{p}"'] \arrow[d, "p"] & \\ B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

commutes.

Example 3.8. Every free R-module is a projective object of $R-\mathbf{mod}$.

Proposition 3.9. Let $\mathcal{A}$ be an abelian category. Every short exact sequence of projective objects

$$0 \longrightarrow P' \xrightarrow{f} P \xrightarrow{g} P'' \longrightarrow 0$$

in $\mathcal{A}$ is split exact. In fact, it suffices to assume that P'' is projective.

Proof. We construct morphisms $i_{P'}$, $i_{P''}$, $\pi_{P'}$, and $\pi_{P''}$, satisfying equations (3.3)-(3.6). We set $i_{P'} = f$ and $\pi_{P''} = g$. Since P'' is projective, we obtain a morphism $\widetilde{\mathrm{id}} : P'' \to P$ making the diagram

$$\begin{tikzcd} & P'' \arrow[dl, dashed, "\widetilde{\mathrm{id}}"'] \arrow[d, "\mathrm{id}"] & \\ P \arrow[r, "g"] & P'' \arrow[r] & 0 \end{tikzcd}$$

commute. We set $i_{P''} = \widetilde{\mathrm{id}}$. We set $r = \mathrm{id}_P - i_{P''} \circ \pi_{P''}$ and note that $g \circ r = 0$ such that, by the universal property of f being the kernel of g, there exists a morphism $p' : P \to P'$ making the diagram

$$\begin{tikzcd} & P \arrow[dl, dashed, "p'"'] \arrow[d, "r"] & \\ P' \arrow[r, "f"] & P \arrow[r, "g"] & P'' \end{tikzcd}$$

commute. We set $\pi_{P'} = p'$. Equations (3.3)-(3.6) are easily verified and hence, by Lemma 3.2, the sequence is split exact. $\square$

Remark 3.10. In particular, we deduce that, given an additive functor $F : \mathcal{A} \to \mathcal{B}$ of abelian categories, and a short exact sequence

$$0 \longrightarrow P' \longrightarrow P \longrightarrow P'' \longrightarrow 0$$

in $\mathcal{A}$, then the resulting sequence

$$0 \longrightarrow F(P') \longrightarrow F(P) \longrightarrow F(P'') \longrightarrow 0$$

is an exact sequence (of not necessarily projective objects) in $\mathcal{B}$. In other words, additive functor behave in a very controlled way if restricted to the class of projective objects.

In light of this observation, one way to prevent the uncontrolled behaviour of a non-exact additive functor is to simply refrain from applying it to objects which are not projective. This agreement seems unreasonable for obvious pragmatic reasons (after all we *are* interested in objects which are not necessarily projective). Nevertheless, this strategy turns out to work well for the following reason: in many abelian categories, we can express every object in terms of projectives. We study a particular case: Consider the functor

$$- \otimes_R N : \mathbf{mod}{-}R \longrightarrow \mathbf{Ab},\ M \mapsto M \otimes_R N$$

where N is a left R-module. Given a module M, we are not supposed to apply $- \otimes_R N$ to M itself. However, we may choose a free resolution

$$P_\bullet = \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0$$

of M and then apply $- \otimes_R N$ to $P_\bullet$.

Definition 3.11. Let N a left R-module. Let M be a right R-module with free resolution $P_\bullet$. We introduce the complex of abelian groups

$$M \otimes_R^L N := P_\bullet \otimes_R N$$

called the *left derived tensor product* of M and N. Further, we define, for every $i \geq 0$, the abelian group

$$\mathrm{Tor}_i^R(M, N) := H_i(P_\bullet \otimes_R N)$$

called the *ith Tor group* of M and N.

Remark 3.12. Note that both the left derived tensor product and the ith Tor group depend, a priori, on the choice of a free resolution of M. We will clarify this ambiguity later.

Therefore, instead of obtaining one module $M \otimes_R N$, we obtain a sequence of modules $\{\mathrm{Tor}_i^R(M, N)\}$. We can easily compute $\mathrm{Tor}_0^R(M, N)$: Using the fact that $- \otimes_R N$ is right exact, we obtain an exact sequence

$$P_1 \otimes_R N \xrightarrow{f} P_0 \otimes_R N \longrightarrow M \otimes_R N \longrightarrow 0$$

which exhibits $M \otimes_R N$ as the cokernel of f. But $\mathrm{coker}(f) = H_0(P_\bullet \otimes_R N)$ and hence we obtain a canonical isomorphism

$$\mathrm{Tor}_0^R(M, N) \cong M \otimes_R N.$$

We will see how the higher Tor groups naturally provide a quantitative measure for the nonexactness of the functor $- \otimes_R N$. Our theme for the next few lectures will be to develop an analogous theory for a general right exact (or left exact) functor.

Example 3.13. Let $R = \mathbb{C}[x, y]$ and $\mathbb{C} = \mathbb{C}[x, y]/(x, y)$. We compute $\mathrm{Tor}_*^R(\mathbb{C}, \mathbb{C})$. A free resultion of $\mathbb{C}$ is given by

$$\cdots \longrightarrow 0 \longrightarrow R \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} R.$$

Applying $- \otimes_R \mathbb{C}$, we obtain the complex

$$0 \longrightarrow \mathbb{C} \xrightarrow{0} \mathbb{C}^2 \xrightarrow{0} \mathbb{C} \longrightarrow 0$$

and hence

$$\mathrm{Tor}_i^R(\mathbb{C}, \mathbb{C}) \cong \begin{cases} \mathbb{C} & i = 0, \\ \mathbb{C}^2 & i = 1, \\ \mathbb{C} & i = 2, \\ 0 & i > 2. \end{cases}$$

Example 3.14. Let R be a ring, let $r \in R$ and assume that left multiplication by r is injective on R. Under this assumption, the complex

$$\cdots \longrightarrow 0 \longrightarrow R \xrightarrow{r\cdot -} R \longrightarrow 0 \longrightarrow \cdots,$$

concentrated in degrees 0 and 1, is a free resolution of the right R-module R/rR. Then, for every left R-module N, we have

$$\mathrm{Tor}_*^R(R/rR, N) \cong \begin{cases} N/rN & i = 0, \\ {}_rN & i = 1, \\ 0 & i > 1. \end{cases}$$

Here ${}_rN$ denotes the r-torsion subgroup of N consisting of those $n \in N$ such that $rn = 0$.

The interpretation of $\mathrm{Tor}_1^R(R/rR, N)$ in terms of torsion explains the choice of terminology "Tor". The following proposition is the historical origin of Tor groups:

Proposition 3.15 (Universal Coefficient Theorem)**.** Let $C_\bullet$ be a chain complex of free abelian groups, and let A be any abelian group. Then, for every $n \in \mathbb{Z}$, there is a short exact sequence of abelian groups

$$0 \longrightarrow A \otimes_{\mathbb{Z}} H_n(C_\bullet) \longrightarrow H_n(A \otimes_{\mathbb{Z}} C_\bullet) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(A, H_{n-1}(C_\bullet)) \longrightarrow 0$$

This context appears naturally in algebraic topology when comparing the homology of a simplicial complex K (cf. Lecture 1) with coefficients in $\mathbb{Z}$ and A, respectively. For $A = \mathbb{Z}/r\mathbb{Z}$, we can further interpret the right hand term of the short exact sequence as the r-torsion subgroup of $H_{n-1}(C_\bullet)$.

Proof. Exercise. □

We will now give a conceptual meaning to the higher Tor groups. Let N be a left R-module, and suppose

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0 \tag{3.16}$$

is a short exact sequence of right R-modules. Following our principle we should *not* apply the functor $- \otimes_R N$ directly to this sequence, since its terms may not be projective, but rather find projective resolutions of its terms and *then* apply $- \otimes_R N$. The following lemma ensures that we can find projective resolutions of M', M, and M'' which are well-adapted to (3.16).

Lemma 3.17 (Horseshoe Lemma)**.** Let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be a short exact sequence in an abelian category $\mathcal{A}$. Let $P_\bullet \to M'$, $P''_\bullet \to M''$ be projective resolutions. Then there exists a short exact sequence

$$0 \longrightarrow P'_\bullet \longrightarrow P_\bullet \longrightarrow P''_\bullet \longrightarrow 0$$

in $\mathbf{Ch}(\mathcal{A})$ where $P_\bullet \to M$ is a projective resolution and the diagram

$$\begin{tikzcd}
0 \arrow[r] & P_0' \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & P_0'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0
\end{tikzcd}$$

commutes.

Proof. We construct the resolution $P_\bullet \to M$ inductively using universal mapping properties. First, we set $P_0 = P_0' \oplus P_0''$, and obtain

$$\begin{tikzcd}
0 \arrow[r] & P_0' \arrow[r] \arrow[d, "p'"'] \arrow[dr, "f \circ p'"'] & P_0' \oplus P_0'' \arrow[r] \arrow[d, dashed, "\exists p"] & P_0'' \arrow[r] \arrow[d, "p''"] \arrow[dl, dashed, "\exists q"] & 0 \\
0 \arrow[r] & M' \arrow[r, "f"'] & M \arrow[r] & M'' \arrow[r] & 0
\end{tikzcd}$$

where q can be found using the projectivity of P_0'' and the existence of p follows from the universal mapping property of $P_0' \oplus P_0''$ as a coproduct, applied to $f \circ p'$ and q. The commutativity of the left square follows by construction, the commutativity of the right square follows, again, from the universal property of $P_0' \oplus P_0''$ as a coproduct. The snake lemma implies $\operatorname{coker}(p) = 0$ hence p is an epic. We obtain a diagram

$$\begin{tikzcd}
0 \arrow[r] & P_1' \arrow[d, "p_1'"] & & P_1'' \arrow[r] \arrow[d, "p_1''"] & 0 \\
0 \arrow[r] & \ker(p') \arrow[r] \arrow[d] & \ker(p) \arrow[r] \arrow[d] & \ker(p'') \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0' \arrow[r] & P_0' \oplus P_0'' \arrow[r] & P_0'' \arrow[r] & 0
\end{tikzcd}$$

where the sequence of kernels is exact by the snake lemma, and the maps p_1' and p_1'' are epic due to the assumption that P' and P'' are resolutions. We are hence precisely in the same situation as in the first step so that we may proceed inductively, setting $P_1 = P_1' \oplus P_1''$. $\square$

By the horseshoe lemma, we can find a short exact sequence

$$0 \longrightarrow P_\bullet' \longrightarrow P_\bullet \longrightarrow P_\bullet'' \longrightarrow 0$$

of projective resolutions of the modules M', M, and M'', respectively. In fact, it is immediate from the proof that we can choose the resolutions to be free. Applying the functor $- \otimes_R N$ (in accordance with our principle), we obtain a short exact (!) sequence

$$0 \longrightarrow P_\bullet' \otimes_R N \longrightarrow P_\bullet \otimes_R N \longrightarrow P_\bullet'' \otimes_R N \longrightarrow 0 \tag{3.18}$$

in $\mathbf{Ch}(\mathbf{Ab})$. The result can now be analyzed using the following general proposition.

Proposition 3.19. Let $\mathcal{A}$ be an abelian category, and let

$$0 \longrightarrow C'_\bullet \longrightarrow C_\bullet \longrightarrow C''_\bullet \longrightarrow 0$$

be a short exact sequence in $\mathbf{Ch}(\mathcal{A})$. Then there exists a long exact sequence in homology

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{n+1}(C'') \arrow[dll, out=-10, in=170] \\
H_n(C') \arrow[r] & H_n(C) \arrow[r] & H_n(C'') \arrow[dll, out=-10, in=170] \\
H_{n-1}(C') \arrow[r] & H_{n-1}(C) \arrow[r] & H_{n-1}(C'') \arrow[dll, out=-10, in=170] \\
H_{n-2}(C') \arrow[r] & \cdots &
\end{tikzcd}$$

Proof. For every n, we may apply the snake lemma to the diagram

$$\begin{tikzcd}
0 \arrow[r] & C'_{n+1} \arrow[r] \arrow[d] & C_{n+1} \arrow[r] \arrow[d] & C''_{n+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C'_n \arrow[r] & C_n \arrow[r] & C''_n \arrow[r] & 0
\end{tikzcd}$$

to obtain, for every n, the exact sequences

$$0 \longrightarrow Z'_{n+1} \longrightarrow Z_{n+1} \longrightarrow Z''_{n+1}$$

and

$$C'_n/B'_n \longrightarrow C_n/B_n \longrightarrow C''_n/B''_n \longrightarrow 0.$$

Applying the snake lemma once again, for every n, to the diagram

$$\begin{tikzcd}
 & C'_n/B'_n \arrow[r] \arrow[d] & C_n/B_n \arrow[r] \arrow[d] & C''_n/B''_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z'_{n-1} \arrow[r] & Z_{n-1} \arrow[r] & Z''_n &
\end{tikzcd}$$

we obtain the exact sequence

$$\begin{tikzcd}
H'_n \arrow[r] & H_n \arrow[r] & H''_n \arrow[dll, out=-10, in=170] \\
H'_{n-1} \arrow[r] & H_{n-1} \arrow[r] & H''_{n-1}.
\end{tikzcd}$$

Piecing together those sequences for the various $n \in \mathbb{Z}$ gives the claimed long exact sequence. $\square$

Finally, we apply this Proposition to (3.18) to obtain the long exact sequence in Tor

$$\begin{tikzcd}
 & \cdots \arrow[r] & \mathrm{Tor}_3^R(M'', N) \arrow[dll, out=-10, in=170] \\
\mathrm{Tor}_2^R(M', N) \arrow[r] & \mathrm{Tor}_2^R(M, N) \arrow[r] & \mathrm{Tor}_2^R(M'', N) \arrow[dll, out=-10, in=170] \\
\mathrm{Tor}_1^R(M', N) \arrow[r] & \mathrm{Tor}_1^R(M, N) \arrow[r] & \mathrm{Tor}_1^R(M'', N) \arrow[dll, out=-10, in=170] \\
M' \otimes_R N \arrow[r] & M \otimes_R N \arrow[r] & M'' \otimes_R N \arrow[r] & 0.
\end{tikzcd}$$

We conclude by observing that this long exact sequence gives an interpretation of the higher Tor groups as successive corrections to the failure of the functor $- \otimes_R N$ to be left exact. In the next lecture, we will clarify the dependence of the invariant $M \otimes_R^L N$ and the system of invariants $\{\mathrm{Tor}_*^R(M, N)\}$ on the chosen free resolution of M. We will then move on to develop a general formalism of derived functors using the theory we have developed of $- \otimes_R N$ as a guiding principle.

Our next goal will be to analyze the dependency of the left derived tensor product $M \otimes_R^L N$ and the Tor groups $\mathrm{Tor}_*^R(M, N)$ on the choice of a free resolution of M. We need to introduce a few concepts.

Let $f, f' : C_\bullet \to D_\bullet$ be morphisms of chain complexes in an abelian category $\mathcal{A}$. A *homotopy* h between f and f' is a family $\{h_n : C_n \to D_{n+1} \mid n \in \mathbb{Z}\}$ of morphisms in $\mathcal{A}$ such that, for every n, we have

$$f'_n - f_n = d_{n+1} \circ h_n + h_{n-1} \circ d_n.$$

When no confusion can arise, we drop the indices and simply write $f' - f = dh + hd$. The morphisms f and f' are called *homotopic*, written $f \sim f'$, if there exists a homotopy between f and f'. It is immediate to verify that the relation $\sim$ defines an equivalence relation on each morphism set in $\mathbf{Ch}(\mathcal{A})$. The equivalence classes of this relation are called *homotopy classes* of morphisms. The morphism f is called a *homotopy equivalence* if there exists a morphism $g : D_\bullet \to C_\bullet$ such that $f \circ g \sim \mathrm{id}_D$ and $g \circ f \sim \mathrm{id}_C$. The morphism g is called a *homotopy inverse* of f.

Remark 3.20. It is immediate from the definition that homotopic morphisms induce the same morphism on homology. In particular, homotopy equivalences induce isomorphism on homology and are hence quasi-isomorphisms.

Given an object M in an abelian category $\mathcal{A}$, a chain complex $Q_\bullet$ in $\mathcal{A}$, equipped with a morphism $\epsilon : Q_0 \to M$, is called a *resolution* of M if

- $Q_n = 0$ for $n < 0$,
- $\epsilon \circ d = 0$,

- the resulting augmented complex

$$\cdots \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow Q_0 \xrightarrow{\epsilon} M \longrightarrow 0$$

  is exact.

Note that a resolution $Q_\bullet$ of M can be interpreted as a morphism of complexes $Q_\bullet \to M$ where M denotes the complex whose only nonzero term is M in degree 0. A resolution $Q_\bullet \to M$ is called *projective* if all terms of $Q_\bullet$ are projective.

Proposition 3.21. Let $\mathcal{A}$ be an abelian category, let M, M' be objects of $\mathcal{A}$. Let $P_\bullet \to M$ be a projective resolution, and let $Q_\bullet \to M'$ be any resolution. Then, for every morphism $f : M \to M'$, there exists a morphism $\widetilde{f} : P_\bullet \to Q_\bullet$ of complexes, called a *lift* of f, such that the diagram

$$\begin{tikzcd}
P_0 \arrow[r] \arrow[d, "\widetilde{f}_0"'] & M \arrow[d, "f"] \\
Q_0 \arrow[r] & M'
\end{tikzcd}$$

commutes. Further, this property uniquely determines the morphism $\widetilde{f}$ up to homotopy.

Proof. Let $f : M \to M'$ be a morphism. We first show the existence of a lift. Consider the diagram

$$\begin{tikzcd}
P_0 \arrow[r] \arrow[d, dashed, "\exists \widetilde{f}_0"'] \arrow[dr] & M \arrow[d, "f"] \\
Q_0 \arrow[r] & M'
\end{tikzcd}$$

where the dotted arrow can be filled due to the projectivity of P_0. Next, consider the diagram

$$\begin{tikzcd}
P_1 \arrow[rr, "d"] \arrow[dd, dashed, "\exists \widetilde{f}_1"'] \arrow[dr, dashed, "\exists r"'] \arrow[ddrr, bend left, "\widetilde{f}_0 \circ d"] & & P_0 \arrow[r] \arrow[dd, "\widetilde{f}_0"] & M \arrow[dd, "f"] \\
& Z_0 \arrow[dr, hook] & & \\
Q_1 \arrow[ur, "q"'] \arrow[rr] & & Q_0 \arrow[r, "\pi'"] & M'
\end{tikzcd}$$

where $Z_0 \to Q_0$ is the kernel of π', the morphism r is obtained using the universal property of Z_0 as a kernel, and the morphism $\widetilde{f}_1$ is obtained using the lifting property of the projective object P_1. Note that the morphism q is epic, since $Q_\bullet$ is a resolution and hence has zero homology in positive degrees. We continue inductively, showing the existence of a lift.

To show uniqueness, it suffices to show that any lift of the zero morphism $M \to M'$ is homotopic to the zero morphism $P_\bullet \to Q_\bullet$. Let $r : P_\bullet \to Q_\bullet$ be such a lift. We fill in the diagram

$$\begin{tikzcd}
 & & P_0 \arrow[r] \arrow[ddll, dashed, bend right, "\exists h_0"'] \arrow[dl, dashed, "\exists \widetilde{r}_0"] \arrow[dd, "r_0"] \arrow[ddr, "0"] & M \arrow[dd, "0"] \\
 & Z_0 \arrow[dr, hook] & & \\
Q_1 \arrow[ur, two heads] \arrow[rr, "d"] & & Q_0 \arrow[r] & M'
\end{tikzcd}$$

obtaining h_0 such that $d \circ h_0 = r_0$. Similarly, we can fill in the diagram

$$\begin{tikzcd}
 & & P_1 \arrow[ddll, dashed, bend right, "\exists h_1"'] \arrow[dl, dashed, "\exists \widetilde{r'}"] \arrow[dd, "r'"] \arrow[ddr, "0"] & \\
 & Z_1 \arrow[dr, hook] & & \\
Q_2 \arrow[ur, two heads] \arrow[rr, "d"] & & Q_1 \arrow[r, "d"] & Q_0
\end{tikzcd}$$

where $r' = r_1 - h_0 \circ d$, noting that $d \circ r' = 0$. Therefore, we obtain the equation $d \circ h_1 = r' = r_1 - h_0 \circ d$ so that $r_1 = d \circ h_1 + h_0 \circ d$ as needed. We continue inductively to construct a sequence $\{h_n\}$ which is a homotopy between 0 and r. □

Corollary 3.22. Let N be a left R-module. Let M be a right R-module, and let $P_\bullet \to M$, $P'_\bullet \to M$ be projective resolutions of M. Then, for every n, there is a canonical isomorphism

$$H_n(P_\bullet \otimes_R N) \xrightarrow{\cong} H_n(P'_\bullet \otimes_R N).$$

Proof. Let $f : P_\bullet \to P'_\bullet$ be a lift of the identity morphism $\mathrm{id} : M \to M$. Let further $g : P'_\bullet \to P_\bullet$ be a lift of the identity morphism $\mathrm{id} : M \to M$. Since $f \circ g$ and id'_P are both lifts of $\mathrm{id} : M \to M$ of the identity, they must be homotopic. The same argument for $g \circ f$ implies that f and g are homotopy inverses. Since any additive functor preserves the relation of being homotopic, it follows that $f \otimes_R N$ and $g \otimes_R N$ are homotopy inverses. Therefore, the morphism $f \otimes_R N$ induces, for every i, an isomorphism

$$H_n(P_\bullet \otimes_R N) \xrightarrow{\cong} H_n(P'_\bullet \otimes_R N). \tag{3.23}$$

Further, since the lift f is unique up to homotopy, the morphism $f \otimes_R N$ is unique up to homotopy and hence defines a unique map on homology. Therefore, the isomorphisms in (3.23) are canonical, i.e., do not depend on any choices. □

Remark 3.24. The fact that the isomorphisms in Corollary 3.22 are *canonical* justifies the notation $\mathrm{Tor}_n^R(M, N)$ without reference to a particular chosen projective resolution of M. A little more subtly, the notation $M \otimes_R^L N$ for the left derived tensor product is justified since for any two chosen projective resolutions, there exists a canonical homotopy class of homotopy equivalences between $P_\bullet \otimes_R N$ and $P'_\bullet \otimes_R N$.

Corollary 3.25. Let N be a right R-module. Then, for every n, there exists an additive functor

$$\mathrm{Tor}_n^R(-, N) : \textbf{mod}\text{–}R \longrightarrow \textbf{Ab},\ M \mapsto \mathrm{Tor}_n^R(M, N).$$

Proof. To define the functor, we will have to choose, for every R-module M, a projective resolution $P_\bullet^M$ of M. If the collection of objects of $\mathbf{mod}{-}R$ formed a set, then the axiom of choice would ensure the existence of our needed collection of resolutions. However, the objects in $\mathbf{mod}{-}R$ form a *class*. There are various ways to deal with this issue (e.g., Grothendieck's theory of universes). In this course, we do not have the time to indulge in this any further and will simply assume the existence of our needed collection of projective resolutions.

On objects, we then define the functor by the assignment

$$M \mapsto \operatorname{Tor}_n^R(M, N) := H_n(P_\bullet^M \otimes_R N).$$

Given a morphism $f : M \to M'$ of right R-modules, we choose, using Proposition 3.21, a lift $\widetilde{f} : P_\bullet^M \to P_\bullet^{M'}$ of f. Since $\widetilde{f}$ is unique up to homotopy, and hence $\widetilde{f} \otimes_R N$ is unique up to homotopy, it induces a *unique* morphism

$$\operatorname{Tor}_n^R(f, N) : H_n(P_\bullet^M \otimes_R N) \longrightarrow H_n(P_\bullet^{M'} \otimes_R N)$$

of abelian groups. To see that the assigment $f \mapsto \operatorname{Tor}_n^R(f, N)$ respects the composition law, we argue as follows. For a morphism $g : M' \to M''$, both morphisms $\widetilde{g \circ f}$ and $\widetilde{g} \circ \widetilde{f}$ are lifts of $g \circ f$ and therefore homotopic. But this implies the equality

$$\operatorname{Tor}_n^R(g \circ f, N) = \operatorname{Tor}_n^R(g, N) \circ \operatorname{Tor}_n^R(f, N).$$

Similarly, $\operatorname{Tor}_n^R(\mathrm{id}_M, N) = \mathrm{id}_{\operatorname{Tor}_n^R(M,N)}$ since any lift of id_M is homotopic to id_{P^M}. □

Remark 3.26. Analyzing the proof of Corollary 3.25, it is clear that the functor $\operatorname{Tor}_n^R(-, N)$ is not unique but depends on the simultaneous choice of projective resolutions of all right R-modules. Choosing another collection of projective resolutions leads to a different functor $\operatorname{Tor}_n^R(-, N)'$. However, the argument of Corollary 3.22 in fact shows the stronger statement that there exists a canonical natural transformation of functors

$$\operatorname{Tor}_n^R(-, N) \longrightarrow \operatorname{Tor}_n^R(-, N)'$$

which induces, for every R-module M, the canonical isomorphism

$$\operatorname{Tor}_n^R(M, N) \longrightarrow \operatorname{Tor}_n^R(M, N)'$$

from Corollary 3.22. Such a natural transformation is called a *natural isomorphism.* We can therefore summarize the above by saying the functor $\operatorname{Tor}_n^R(-, N)$ is uniquely determined up to canonical natural isomorphism.

The formal properties of the system of functors $\{\operatorname{Tor}_*^R(-, N)\}$ are captured in the following definition.

Definition 3.27. A *homological δ-functor* between abelian categories $\mathcal{A}$ and $\mathcal{B}$ consists of

- a collection $\{T_n : \mathcal{A} \to \mathcal{B}\}$ of additive functors indexed by $n \in \mathbb{Z}$,

- for every short exact sequence

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

in $\mathcal{A}$, there is, for every n, a morphism

$$\delta_n : T_n(A'') \longrightarrow T_{n-1}(A')$$

such that the following conditions hold:

(1) For $n < 0$, we have $T_n = 0$.

(2) For every short exact sequence as above, there is a long exact sequence

$$\begin{tikzcd}
 & \cdots \arrow[r] & T_{n+1}(A'') \arrow[dll, "\delta_{n+1}"'] \\
T_n(A') \arrow[r] & T_n(A) \arrow[r] & T_n(A'') \arrow[dll, "\delta_n"'] \\
T_{n-1}(A') \arrow[r] & \cdots &
\end{tikzcd}$$

(3) For every morphism of short exact sequences, i.e., every commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] \arrow[d, "f'"] & A \arrow[r] \arrow[d, "f"] & A'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0
\end{tikzcd}$$

in $\mathcal{A}$ with exact rows, and for every n, the diagram

$$\begin{tikzcd}
T_n(A'') \arrow[r, "\delta_n"] \arrow[d, "T_n(f'')"'] & T_{n-1}(A') \arrow[d, "T_{n-1}(f')"] \\
T_n(B'') \arrow[r, "\delta_n"] & T_{n-1}(B')
\end{tikzcd}$$

commutes.

Theorem 3.28. Let N be a left R-module. Then the collection of functors $\{\mathrm{Tor}_*^R(-, N)\}$ forms a homological δ-functor.

Proof. Given a short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0 \tag{3.29}$$

we have to construct a long exact sequence for $\mathrm{Tor}_*^R(-, N)$. We abbreviate $T_n := \mathrm{Tor}_n^R(-, N)$. Note that, for the definition of the functor T_n, we have chosen a collection of projective resolutions for all objects in $\mathbf{mod}{-}R$, in particular, we have chosen projective resolutions $P_\bullet^{M'} \to M'$,$P_\bullet^M \to M$, and $P_\bullet^{M''} \to M''$. It is *not* a priori guaranteed, that these choices fit together to form a short exact sequence of complexes which is compatible with (3.29)! However, we may discard $P_\bullet^M$ and use the horseshoe lemma to construct an alternative projective resolution $\widetilde{P_\bullet^M} \to M$ such that we do have a short exact sequence

$$0 \longrightarrow P_\bullet^{M'} \longrightarrow \widetilde{P^M}_\bullet \longrightarrow P_\bullet^{M''} \longrightarrow 0 \tag{3.30}$$

of complexes compatible with (3.29). However, this different choice of projective resolution will lead to a different functor $\widetilde{T}_n$. To the short exact sequence of complexes obtained by applying $- \otimes_R N$ to (3.30), we have an associated long exact sequence

$$\cdots \longrightarrow \widetilde{T}_n(M) \longrightarrow \widetilde{T}_n(M'') \xrightarrow{\delta} \widetilde{T}_{n-1}(M') \longrightarrow \widetilde{T}_{n-1}(M) \longrightarrow \cdots \tag{3.31}$$

involving the modified functors $\widetilde{T}_*$. By Lecture 7, there is, for every n, a canonical natural isomorphism $T_n \to \widetilde{T}_n$. Applying this isomorphism to (3.31), we obtain a diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \widetilde{T}_n(M) \arrow[r] & \widetilde{T}_n(M'') \arrow[r, "\delta"] & \widetilde{T}_{n-1}(M') \arrow[r] & \widetilde{T}_{n-1}(M) \arrow[r] & \cdots \\
\cdots \arrow[r] & T_n(M) \arrow[r] \arrow[u, "\cong"] & T_n(M'') \arrow[r, dashed, "\exists!\delta"] \arrow[u, "\cong"] & T_{n-1}(M') \arrow[r] \arrow[u, "\cong"] & T_{n-1}(M) \arrow[r] \arrow[u, "\cong"] & \cdots
\end{tikzcd}$$

where there is a unique filling of the dashed arrow which makes the diagram commute. The bottom long exact sequence is the desired long exact sequence for the original, unmodified functors T_*.

It remains to demonstrate property (3). Given a commutative diagram of right R-modules

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0
\end{tikzcd} \tag{3.32}$$

with exact rows, we can, via the technique we used to establish the long exact sequence for $\{T_*\}$, without loss of generality assume that the chosen projective resolutions of all terms fit into a diagram

$$\begin{tikzcd}
0 \arrow[r] & P^{A'} \arrow[r] \arrow[d, dashed] & P^{A} \arrow[r] \arrow[d, dashed] & P^{A''} \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & P^{B'} \arrow[r] & P^{B} \arrow[r] & P^{B''} \arrow[r] & 0
\end{tikzcd} \tag{3.33}$$

with exact rows. However, it is not immediate that we are able to fill in the vertical dashed arrows by lifts of the respective vertical morphisms in (3.32) in such a way that the resulting diagram commutes. The proof that we can find those compatible lifts is somewhat technical and given in Lemma 3.34 below. Provided the existence of those lifts, we can then conclude our argument by applying Lemma 3.38 below to the diagram obtained from (3.33) by application of $- \otimes_R N$. □

The following is an improvement of the horseshoe lemma.

Lemma 3.34 (Horseshoe lemma for morphisms)**.** Let $\mathcal{A}$ be an abelian category, and consider a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d, "f'"] & A \arrow[r] \arrow[d, "f"] & A'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0 \end{tikzcd} \tag{3.35}$$

in $\mathcal{A}$ with exact rows. Suppose we are given projective resolutions $P^{A'}$, $P^{A''}$, $P^{B'}$, and $P^{B''}$ of the indicated objects together with lifts $\widetilde{f'} : P^{A'} \to P^{B'}$ and $\widetilde{f''} : P^{A''} \to P^{B''}$ of f' and f''. Then there exist projective resolutions P^A and P^B equipped with a lift $\widetilde{f}$ of f such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & P^{A'} \arrow[r] \arrow[d, "\widetilde{f'}"] & P^{A} \arrow[r] \arrow[d, "\widetilde{f}"] & P^{A''} \arrow[r] \arrow[d, "\widetilde{f''}"] & 0 \\ 0 \arrow[r] & P^{B'} \arrow[r] & P^{B} \arrow[r] & P^{B''} \arrow[r] & 0 \end{tikzcd} \tag{3.36}$$

in $\mathbf{Ch}(\mathcal{A})$ commutes and lifts (3.35).

Proof. Using the horseshoe lemma, we construct $P^A \to A$ and $P^B \to B$ to obtain a diagram

$$\begin{tikzcd} 0 \arrow[r] & P^{A'} \arrow[r] \arrow[d, "\widetilde{f'}"] & P^{A} \arrow[r] \arrow[d, dotted, "\widetilde{f}"] & P^{A''} \arrow[r] \arrow[d, "\widetilde{f''}"] & 0 \\ 0 \arrow[r] & P^{B'} \arrow[r] & P^{B} \arrow[r] & P^{B''} \arrow[r] & 0 \end{tikzcd}$$

where the only missing piece is the dotted arrow $\widetilde{f'}$. For $n \in \mathbb{Z}$, we have $P^A_n = P^{A'}_n \oplus P^{A''}_n$ and $P^B_n = P^{B'}_n \oplus P^{B''}_n$. Any morphism $\widetilde{f}$ which makes the above diagram commute must, for every n, necessarily be given by the formula

$$\widetilde{f}_n = \begin{pmatrix} \widetilde{f'}_n & r_n \\ 0 & \widetilde{f''}_n \end{pmatrix} : P^{A'}_n \oplus P^{A''}_n \longrightarrow P^{B'}_n \oplus P^{B''}_n . \tag{3.37}$$

for some morphism $r_n : P^{A''}_n \to P^{B'}_n$. Our task is therefore to find a sequence $\{r_n\}$ such that Formula (3.37) determines a morphism of chain complexes which lifts f. For the first step, we denote by $\gamma_{A''}$ and $\gamma_{B''}$ the restriction of the augmentation morphisms $P^A_0 \to A$ and $P^B_0 \to B$ to the summands $P^{A''}_0$ and $P^{B''}_0$, respectively. We can form the diagram

$$\begin{tikzcd} P^{A''}_0 \arrow[r, "\widetilde{f''}_0"] \arrow[d, "\gamma_{A''}"'] \arrow[dr, phantom, "\boxtimes"] & P^{B''}_0 \arrow[d, "\gamma_{B''}"] \\ A \arrow[r, "f"] \arrow[d] & B \arrow[d, "d"] \\ A'' \arrow[r, "f''"] & B'' \end{tikzcd}$$

where the lower square, which is the respective square in (3.35), and the exterior rectangle commute, but, and this is the key difficulty, the upper square does *not* necessarily commute. We denote $s = f \circ \gamma_{A''} - \gamma_{B''} \circ \widetilde{f''}_0$. Note, that the commutativity of the exterior square implies $d \circ s = 0$. By the exactness of the lower row of (3.35), the morphism s therefore factors as

$$\begin{tikzcd} & P_0^{A''} \arrow[dl, dashed, "\exists s'"'] \arrow[d, "s"] \arrow[dr, "0"] & \\ B' \arrow[r] & B \arrow[r, "d"] & B''. \end{tikzcd}$$

Due to the projectivity of $P_0^{A''}$, we can lift s' to a morphism $r_0 : P_0^{A''} \longrightarrow P_0^{B'}$ which will serve as the first member of our desired sequence $\{r_*\}$. Namely, setting

$$\widetilde{f}_0 := \begin{pmatrix} \widetilde{f'}_0 & r_0 \\ 0 & \widetilde{f''}_0 \end{pmatrix},$$

we claim that the diagram

$$\begin{tikzcd} P_0^A \arrow[r, "\widetilde{f}_0"] \arrow[d, "\varepsilon^A"'] & P_0^B \arrow[d, "\varepsilon^B"] \\ A \arrow[r, "f"] & B, \end{tikzcd}$$

where ε^A and ε^B denote the augmentation morphisms, commutes by construction. Using the universal property of P_0^A as a sum, it suffices to show that the equation $f \circ \varepsilon^A = \varepsilon^B \circ \widetilde{f}_0$ holds after restriction to both $P_0^{A'}$ and $P_0^{A''}$. This leads to the equations

$$f \circ \varepsilon^{A'} = \varepsilon^{B'} \circ \widetilde{f'}_0$$

and

$$f \circ \gamma_{A''} = \underbrace{f \circ \gamma_{A''} - \gamma_{B''} \circ \widetilde{f''}_0}_{s} + \gamma_{B''} \circ \widetilde{f''}_0,$$

respectively, which both hold. We obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & P_0^{A'} \arrow[r] \arrow[d, "\widetilde{f'}_0"] & P_0^A \arrow[r] \arrow[d, "\widetilde{f}_0"] & P_0^{A''} \arrow[r] \arrow[d, "\widetilde{f''}_0"] & 0 \\ 0 \arrow[r] & P_0^{B'} \arrow[r] & P_0^B \arrow[r] & P_0^{B''} \arrow[r] & 0 \end{tikzcd}$$

with exact rows which lifts the original diagram (3.35). Passing to the kernels of all augmentation maps, we further obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & Z_0^{A'} \arrow[r] \arrow[d] & Z_0^A \arrow[r] \arrow[d] & Z_0^{A''} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & Z_0^{B'} \arrow[r] & Z_0^B \arrow[r] & Z_0^{B''} \arrow[r] & 0 \end{tikzcd}$$

where the rows are exact by the snake lemma applied in the category $\mathrm{Mor}(\mathcal{A})$. We can now repeat the above construction with this diagram in place of the original diagram (3.35) to obtain the morphism r_1 and hence $\widetilde{f}_1$. Proceeding inductively leads to the desired lift $\widetilde{f}$. □

Lemma 3.38. Let $\mathcal{A}$ be an abelian category and let

$$\begin{tikzcd} 0 \arrow[r] & A'_\bullet \arrow[r] \arrow[d] & A_\bullet \arrow[r] \arrow[d] & A''_\bullet \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B'_\bullet \arrow[r] & B_\bullet \arrow[r] & B''_\bullet \arrow[r] & 0 \end{tikzcd} \tag{3.39}$$

be a commutative diagram in $\mathbf{Ch}(\mathcal{A})$ with exact rows. Then the long exact sequences associated to the top and bottom row, fit into a commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & H_n(A) \arrow[r] \arrow[d] & H_n(A'') \arrow[r, "\delta"] \arrow[d] & H_{n-1}(A') \arrow[r] \arrow[d] & H_{n-1}(A) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & H_n(B) \arrow[r] & H_n(B'') \arrow[r, "\delta"] & H_{n-1}(B') \arrow[r] & H_{n-1}(B) \arrow[r] & \cdots \end{tikzcd} \tag{3.40}$$

where δ denotes the connecting morphisms.

Proof. Let I denote the category

$$0 \xrightarrow{f} 1$$

with two objects, denoted 0 and 1, and, besides the identity morphisms, one morphism $f : 0 \to 1$. A functor from $F : I \to \mathcal{A}$ corresponds to a morphism $F(f) : F(0) \to F(1)$ in $\mathcal{A}$. A natural transformation between functors F, G from I to $\mathcal{A}$ corresponds to a commutative diagram

$$\begin{tikzcd} F(0) \arrow[r] \arrow[d, "F(f)"'] & G(0) \arrow[d, "G(f)"] \\ F(1) \arrow[r] & G(1). \end{tikzcd}$$

We have seen that functors from I to $\mathcal{A}$ form an abelian category $\mathrm{Fun}(I, \mathcal{A})$ with natural transformations as morphisms. We denote this category by $\mathrm{Mor}(\mathcal{A})$ and call it the *category of morphisms* in $\mathcal{A}$. Diagram (3.39) can be interpreted as a short exact sequence in $\mathbf{Ch}(\mathrm{Mor}(\mathcal{A}))$. Since $\mathrm{Mor}(\mathcal{A})$ is an abelian category, there exists an associated long exact sequence in $\mathrm{Mor}(\mathcal{A})$. Writing out this long exact sequence explicitly, we obtain precisely the diagram (3.40). $\square$

The homological δ-functor $\{\mathrm{Tor}^R_*(-, N)\}$ extends the functor $\mathrm{Tor}^R_0(-, N) \cong - \otimes_R N$ which we were originally interested in. However, as the following example shows, there may be other δ-functors with this property.

Example 3.41. Let R be a ring and let $r \in R$, and consider the functor $- \otimes_R R/Rr$. Given a short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of right R-modules, we consider the associated commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] \arrow[d, "r"] & M \arrow[r] \arrow[d, "r"] & M'' \arrow[r] \arrow[d, "r"] & 0 \\ 0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0 \end{tikzcd}$$

where the vertical morphisms are given by right multiplication by r. The snake lemma provides the long exact sequence

$$
\begin{tikzcd}
0 \arrow[r] & M'_r \arrow[r] & M_r \arrow[r] & M''_r \arrow[dll, out=-10, in=170, "\delta"'] \\
& M'/M'r \arrow[r] & M/Mr \arrow[r] & M''/M''r \arrow[r] & 0
\end{tikzcd}
$$

where $(-)_r$ denotes the abelian group of right r-torsion elements. This implies that the collection of functors $T_0 = - \otimes_R R/Rr$, $T_1 = (-)_r$, $T_n = 0$ for $n > 1$, forms a δ-functor extending $- \otimes_R R/Rr$. However, in general, this δ-functor does not coincide with $\{\mathrm{Tor}_*^R(-, N)\}$: For $R = \mathbb{C}[x]/(x^2)$ and $r = x$, we have $\mathrm{Tor}_n^R(R/xR, R/Rx) \cong \mathbb{C}$ for all $n \geq 0$, while $T_n(R/xR) = 0$ for $n > 1$.

To characterize the δ-functor $\{\mathrm{Tor}_*^R(-, N)\}$ we introduce the following terminology. A *morphism* of homological δ-functors $\{S_*\}$ and $\{T_*\}$ between abelian categories $\mathcal{A}$ and $\mathcal{B}$ is a family

$$\{\eta_n : S_n \Rightarrow T_n \mid n \geq 0\}$$

of natural transformations commuting with δ. A homological δ-functor $\{T_*\}$ is called *universal* if, given a homological δ-functor $\{S_*\}$ and a natural transformation $\eta_0 : S_0 \Rightarrow T_0$, there exists a unique morphism $\eta : \{S_*\} \Rightarrow \{T_*\}$ extending η_0.

Theorem 3.42. The δ-functor $\{\mathrm{Tor}_*^R(-, N)\}$ is universal.

Proof. Exercise. $\square$

Remark 3.43. Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor. Note that it is immediate from the definition that, provided it exists, a universal δ-functor $\{T_*\}$ with $T_0 \cong F$ is unique up to canonical isomorphism of δ-functors.

The theory for $- \otimes_R N$ and $\{\mathrm{Tor}_*^R(-, N)\}$ developed above generalizes as follows. We say an abelian category $\mathcal{A}$ *has enough projectives* if, for every object A, there exists a projective object P and an epic morphism

$$P \twoheadrightarrow A.$$

Note that, if $\mathcal{A}$ has enough projectives, then every object admits a projective resolution.

Theorem 3.44. Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor of abelian categories and assume that $\mathcal{A}$ has enough projectives. Then there exists a universal homological δ-functor $\{L_*F\}$ with $L_0F \cong F$. For every n, the functor L_nF is given explicitly by

$$L_nF(A) = H_n(F(P_\bullet^A))$$

where $P_\bullet^A \to A$ is a projective resolution. The functor L_nF is called the *nth left derived functor* of F.

Proof. We have formulated all key arguments needed to construct $\{\mathrm{Tor}_*^R(-, N)\}$ and characterize it as a universal δ-functor in terms of general abelian categories. The only needed conditions are the right exactness of $- \otimes_R N$ and the existence of projective resolutions. Therefore, all constructions and proofs generalize verbatim. $\square$

There is a dual theory for left exact functors. An object I of an abelian category $\mathcal{A}$ is called *injective* if I is projective when considered as an object of the abelian category $\mathcal{A}^{\mathrm{op}}$. A chain complex

$$I_\bullet = I_0 \longrightarrow I_{-1} \longrightarrow I_{-2} \longrightarrow \cdots$$

in $\mathcal{A}$, equipped with a morphism $A \to I_0$, is called an injective resolution of A if $A \to I_\bullet$ is a projective resolution when considered a chain complex in $\mathcal{A}^{\mathrm{op}}$. To avoid the appearance of negative indices, it is common practice to introduce and upper index notation as follows.

Definition 3.45. A *cochain complex* $C^\bullet$ in an abelian category $\mathcal{A}$ consists of a collection $\{C^n \mid n \in \mathbb{Z}\}$ of objects of $\mathcal{A}$ equipped with differentials $d : C^n \to C^{n+1}$ such that $d^2 = 0$. Further, we use the expressions *cochains*, *cocycles*, *coboundaries*, and *cohomology* to refer to the cochain analogues of the respective notions for chain complexes.

Note that this terminology is of purely linguistic nature. One can translate between chain complexes and cochain complexes via $I^n := I_{-n}$ and vice versa. Using this language, we can express an injective resolution as a cochain complex of the form

$$I^\bullet = I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

An abelian category $\mathcal{A}$ is said to have enough injectives if $\mathcal{A}^{\mathrm{op}}$ has enough projectives. Note that if $\mathcal{A}$ has enough injectives then every object admits an injective resolution. The concept of a (universal) cohomological δ-functor is dual to the homological variant. To avoid confusion, we explicitly spell out the definition.

Definition 3.46. A *cohomological δ-functor* between abelian categories $\mathcal{A}$ and $\mathcal{B}$ consists of

- a collection $\{T^n : \mathcal{A} \to \mathcal{B}\}$ of additive functors indexed by $n \in \mathbb{Z}$,
- for every short exact sequence

  $$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

  in $\mathcal{A}$, there is, for every n, a morphism

  $$\delta_n : T^n(A'') \longrightarrow T^{n+1}(A')$$

such that the following conditions hold:

(1) For $n < 0$, we have $T^n = 0$.

(2) For every short exact sequence as above, there is a long exact sequence

$$\cdots \longrightarrow T^{n-1}(A'') \xrightarrow{\delta_{n-1}} T^n(A') \longrightarrow T^n(A) \longrightarrow T^n(A'') \xrightarrow{\delta_n} T^{n+1}(A') \longrightarrow \cdots$$

(3) For every morphism of short exact sequences, i.e., every commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d, "f'"] & A \arrow[r] \arrow[d, "f"] & A'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0 \end{tikzcd}$$

in $\mathcal{A}$ with exact rows, and for every n, the diagram

$$\begin{tikzcd} T^n(A'') \arrow[r, "\delta_n"] \arrow[d, "T^n(f'')"] & T^{n+1}(A') \arrow[d, "T^{n+1}(f')"] \\ T^n(B'') \arrow[r, "\delta_n"] & T^{n-1}(B') \end{tikzcd}$$

commutes.

A cohomological δ-functor $\{T^*\}$ is called *universal* if, given a cohomological δ-functor $\{S^*\}$ and a natural transformation $\eta^0 : T^0 \Rightarrow S^0$, there exists a unique morphism $\eta : \{T^*\} \Rightarrow \{S^*\}$ extending η^0.

Theorem 3.47. Let $G : \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories. Assume that $\mathcal{A}$ has enough injectives. Then there exists a universal cohomological δ-functor $\{R^*G\}$ with $R^0G \cong G$. For every n, the functor R_nG is given explicitly by

$$R^nG(A) = H^n(G(I_A^\bullet))$$

where $A \to I_A^\bullet$ is an injective resolution. The functor R^nG is called the *nth right derived functor* of G.

Proof. By passing to opposite categories the statement is equivalent to Theorem 3.44. □

4 Application: Syzygy Theorem

Let R be a commutative ring. A sequence $x_1, \ldots, x_n$ of elements in R is called *regular* if, for every $1 \leq i \leq n$, the element x_i is a nonzerodivisor in $R/(x_1, \ldots, x_{i-1})$. On Problem Set 3, we have introduced the *Koszul complex*

$$K(x_1, x_2, \ldots, x_n) = K(x_1) \otimes_R K(x_2) \otimes_R \cdots \otimes_R K(x_n).$$

and shown that, for a regular sequence $x_1, x_2, \ldots, x_n$ in R, the Koszul complex is a free resolution of $R/(x_1, x_2, \ldots, x_n)$.

A Noetherian local ring R is called *regular* if the maximal ideal $\mathfrak{m}$ can be generated by a regular sequence $x_1, x_2, \ldots, x_n \in \mathfrak{m}$. The number n is independent of the chosen sequence and is called the dimension of R. For example, a power series ring $k[[x_1, x_2, \ldots, x_n]]$ with coefficients in any field k is regular local.

Theorem 4.1. Let R be a regular local ring of dimension n, and let M be a finitely generated R-module. Then M has a free resolution of length $\leq n$.

Proof. We claim that M admits a *minimal* free resolution

$$\cdots \xrightarrow{d} F_2 \xrightarrow{d} F_1 \xrightarrow{d} F_0 \xrightarrow{\varepsilon} M$$

by finite rank free R-modules such that, for every $i > 0$, we have $d(F^i) \subset \mathfrak{m}F^{i-1}$. Note that, after choosing bases of the free modules, this simply means that the matrices corresponding to d have entries in the maximal ideal $\mathfrak{m}$. We will prove the claim in Lemma 4.2 below.

Let $k = R/\mathfrak{m}$ denote the residue field. We compute the Tor groups $\mathrm{Tor}_i(M,k)$ in terms of a minimal free resolution of M to obtain

$$\mathrm{Tor}_i^R(M,k) \cong H_i(F_\bullet \otimes_R k) \cong k^{b_i}$$

where b_i denotes the rank of F_i. On the other hand, since Tor is balanced, we may compute the same Tor groups in terms of the Koszul resolution

$$K(x_1, x_2, \dots, x_n) \to k$$

of the residue field k. We have

$$\mathrm{Tor}_i^R(M,k) \cong H_i(M \otimes_R K(x_1, x_2, \dots, x_n)).$$

But, since the Koszul complex has length n, this implies that $k^{b_i} \cong \mathrm{Tor}_i^R(M,k) \cong 0$, hence $b_i = 0$, for $i > n$. Thus, we have shown that any minimal free resolution of M must have length $\leq n$. $\square$

Lemma 4.2. Let R be a Noetherian local ring, and let M a finitely generated R-module. Then there exists a *minimal* free resolution of M as defined above.

Proof. We choose a free resolution $F_\bullet$ of M such that, for every i, the rank of F_i is minimal. In other words, the rank of F^i equals the minimal number of generators of $\ker(d_{i-1})$. We claim that the resulting resolution is minimal. Assume it is not, then there exists i such that $d(F_{i+1}) \not\subset \mathfrak{m}F_i$. Choosing a basis $e_1, \dots, e_r$ of F_i, this means that there exists $f \in F_{i+1}$ such that

$$d(f) = \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_r e_r$$

where at least one coefficient λ_l does not lie in $\mathfrak{m}$. After permuting the basis we may assume $\lambda_1 \neq \mathfrak{m}$. Since R is local, this implies that λ_1 is a unit in R so that we can divide the above equation to obtain

$$d(\frac{1}{\lambda_1} f) = e_1 + \frac{\lambda_2}{\lambda_1} e_2 + \cdots + \frac{\lambda_r}{\lambda_1} e_r.$$

Applying d to the equation, we obtain the relation

$$d(-e_1) = \frac{\lambda_2}{\lambda_1} d(e_2) + \cdots + \frac{\lambda_r}{\lambda_1} d(e_r)$$

in $\ker(d_{i-1})$. But this means that the basis element e_1 of F_i is superfluous, contradicting the premise that we have chosen a minimal number of generators of $\ker(d_{i-1})$. $\square$

Remark 4.3. Let R be a regular local ring. The proof of Theorem 4.1 implies the formula

$$\dim(R) = \sup\{d \mid \operatorname{Tor}_d^R(M,N) \neq 0 \text{ for some f.g. } R\text{-modules } M,N\}.$$

This connection between homological algebra and dimension theory has led to various remarkable applications. For example, the statement that every localization of a regular local ring is again regular local has a very elegant proof using homological algebra (see Weibel 4.4.18).

We conclude by noting that the arguments given above can be used to prove the following statement which is Hilbert's original theorem.

Theorem 4.4 (Hilbert's Syzygy Theorem)**.** Let $R = k[x_1, x_2, \dots, x_n]$ be the polynomial ring over a field k considered as a graded ring with $|x_i| = 1$. Then every finitely generated graded R-module M admits a free resolution of length $\leq n$.

5 Ext and extensions

Let R be a ring. An *extension* of R-modules A by B is a short exact sequence

$$\xi: \qquad 0 \longrightarrow B \longrightarrow X \longrightarrow A \longrightarrow 0.$$

Extensions ξ and ξ' are called *equivalent* if there exists a commutative diagram

$$\begin{array}{rccccccccc} \xi: & 0 & \longrightarrow & B & \longrightarrow & X & \longrightarrow & A & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle \mathrm{id}_B} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mathrm{id}_A} & & \\ \xi': & 0 & \longrightarrow & B & \longrightarrow & X' & \longrightarrow & A & \longrightarrow & 0. \end{array}$$

Remark 5.1. The five lemma implies that any morphism φ making the above diagram commutative is automatically an isomorphism. This implies that the relation for extensions to be equivalent is in fact an equivalence relation.

An extension is called *split* if it is equivalent to

$$0 \longrightarrow B \longrightarrow B \oplus A \longrightarrow A \longrightarrow 0.$$

Given an extension ξ as above, we consider the coboundary map $\delta : \operatorname{Hom}(B,B) \longrightarrow \operatorname{Ext}^1(A,B)$ from the long exact sequence associated to $\operatorname{Ext}^*(-,B)$ and define the element

$$\Theta(\xi) = \delta(\mathrm{id}_B) \in \operatorname{Ext}^1(A,B)$$

called the *obstruction class* of ξ.

Proposition 5.2. An extension ξ is split if and only if $\Theta(\xi) = 0$.

Proof. Given

$$\xi: \qquad 0 \longrightarrow B \xrightarrow{f} X \longrightarrow A \longrightarrow 0.$$

Suppose $\Theta(\xi) = 0$. Considering the part

$$\mathrm{Hom}(X, B) \longrightarrow \mathrm{Hom}(B, B) \xrightarrow{\delta} \mathrm{Ext}^1(A, B)$$

of the long exact sequence associated to $\mathrm{Ext}^*(-, B)$, we obtain that there exists $\pi \in \mathrm{Hom}(X, B)$ such that $\pi \circ f = \mathrm{id}_B$ which implies that ξ is split. Conversely, given a split extension

$$\begin{tikzcd} \xi: & 0 \arrow[r] & B \arrow[r] & B \oplus A \arrow[r, "g"] & A \arrow[r] \arrow[l, bend left, "q"] & 0, \end{tikzcd}$$

the map q induces a splitting

$$\begin{tikzcd} \mathrm{Ext}^1(A, B) \arrow[r, "\mathrm{Ext}^1(g,B)"] & \mathrm{Ext}^1(B \oplus A, B) \arrow[l, bend left, "\mathrm{Ext}^1(q,B)"] . \end{tikzcd}$$

In particular, the map $\mathrm{Ext}^1(f, B)$ is injective so that the exact sequence

$$\mathrm{Hom}(B, B) \xrightarrow{\delta} \mathrm{Ext}^1(A, B) \xrightarrow{\mathrm{Ext}^1(f,B)} \mathrm{Ext}^1(B \oplus A, B) .$$

forces δ to be 0. $\square$

Remark 5.3. Proposition 5.2 explains our terminology: the class $\Theta(\xi)$ is the obstruction for the extension ξ to split.

Theorem 5.4. There is a natural bijection

$$\left\{ \begin{array}{c} \text{equivalence classes of} \\ \text{extensions of } A \text{ by } B \end{array} \right\} \xrightarrow{\cong} \mathrm{Ext}^1(A, B)$$

given by the association $\xi \mapsto \Theta(\xi)$.

Proof. To show that Θ is well-defined, assume that ξ and ξ' are equivalent extensions. Then, by naturality of the coboundary morphism δ, there exists a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}(X, B) \arrow[r] & \mathrm{Hom}(B, B) \arrow[r, "\delta"] & \mathrm{Ext}^1(A, B) \\ \mathrm{Hom}(X', B) \arrow[r] \arrow[u] & \mathrm{Hom}(B, B) \arrow[r, "\delta"] \arrow[u, "\mathrm{id}"] & \mathrm{Ext}^1(A, B). \arrow[u, "\mathrm{id}"] \end{tikzcd}$$

Since $\Theta(\xi)$ is defined to be the image of id_B under the upper coboundary map and $\Theta(\xi')$ is defined to be the image of id_B under the lower coboundary map, we have $\Theta(\xi) = \Theta(\xi')$. This shows that Θ yields a well-defined map as claimed above.

To show that Θ is a bijection, we construct an explicit inverse Ψ. First, we fix the choice of a short exact sequence

$$0 \longrightarrow N \xrightarrow{\psi} P \longrightarrow A \longrightarrow 0$$

with P projective and consider the long exact sequence in $\mathrm{Ext}^*(-, B)$ to obtain the exact sequence

$$\mathrm{Hom}(N, B) \xrightarrow{\delta} \mathrm{Ext}^1(A, B) \longrightarrow \mathrm{Ext}^1(P, B)$$

with $\mathrm{Ext}^1(P,B)=0$. Now suppose we are given an element $e \in \mathrm{Ext}^1(A,B)$. We can find $\varphi \in \mathrm{Hom}(N,B)$ with $\delta(\varphi)=e$. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "\psi"] \arrow[d, "\varphi"'] & P \arrow[r] \arrow[d] & A \arrow[r] & 0 \\
& B \arrow[r] \arrow[rr, bend right, "0"'] & B \amalg_N P \arrow[ur, dashed] & &
\end{tikzcd}$$

where the existence of the dashed arrow follows from the universal property of the pushout $B \amalg_N P$ from Lemma 5.5 below. It is immediate to verify that the resulting sequence

$$0 \longrightarrow B \longrightarrow B \amalg_N P \longrightarrow A \longrightarrow 0$$

is exact and we define $\Psi(e)$ to be the extension given by this exact sequence.

To verify that Ψ is well-defined, we have to show that it is independent of the choice of φ. Let $\varphi' : N \to B$ be another morphism such that $\delta(\varphi')=e$. Then we have $\delta(\varphi-\varphi')=0$ and hence, by the long exact sequence for $\mathrm{Ext}^*(-,B)$, there exists a morphism $\rho \in \mathrm{Hom}(P,B)$ such that $\rho \circ \psi = \varphi - \varphi'$. We obtain a morphism

$$\alpha : B \amalg_N^{\varphi} P \longrightarrow B \amalg_N^{\varphi'} P,\ (b,p) \mapsto (b+\rho(p),p)$$

which makes the diagram

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, "\mathrm{id}_B"] & B \amalg_N^{\varphi} P \arrow[r] \arrow[d, "\alpha"] & A \arrow[r] \arrow[d, "\mathrm{id}_A"] & 0 \\
0 \arrow[r] & B \arrow[r] & B \amalg_N^{\varphi'} P \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

commute so that Ψ does not depend on the choice of φ. It remains to show that $\Theta \circ \Psi = \mathrm{id}$ and $\Psi \circ \Theta = \mathrm{id}$. To show $\Theta \circ \Psi = \mathrm{id}$, let $e \in \mathrm{Ext}^1(A,B)$. By construction of $\Psi(e)$, we have a morphism of short exact sequences

$$\begin{tikzcd}
& 0 \arrow[r] & N \arrow[r, "\psi"] \arrow[d, "\varphi"] & P \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\mathrm{id}_A"] & 0 \\
\Psi(e): & 0 \arrow[r] & B \arrow[r] & B \amalg_N P \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

which induces, by naturality of δ, a commutative square

$$\begin{tikzcd}
\mathrm{Hom}(N,B) \arrow[r, "\delta"] & \mathrm{Ext}^1(A,B)\ . \\
\mathrm{Hom}(B,B) \arrow[r, "\delta"] \arrow[u, "-\circ\varphi"] & \mathrm{Ext}^1(A,B) \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

We obtain $\Theta(\Psi(e)) = \delta(\mathrm{id}_B) = \delta(\varphi) = e$. To show $\Psi \circ \Theta = \mathrm{id}$, let

$$\xi: \qquad 0 \longrightarrow B \longrightarrow X \longrightarrow A \longrightarrow 0$$

be an extension with obstruction class $e = \Theta(\xi)$. Since P is projective, we may fill in the dashed morphism in

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "\psi"] \arrow[d, dotted, "\exists\varphi"'] & P \arrow[r] \arrow[d, dashed, "\exists"] & A \arrow[r] \arrow[d, "\mathrm{id}_A"] & 0 \\ 0 \arrow[r] & B \arrow[r] & X \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

and further obtain φ as the induced map on kernels. By naturality of δ, we have $\delta(\varphi) = e$. By the universal property of the pushout, we obtain a morphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, "\mathrm{id}_B"'] & B \amalg_N P \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\mathrm{id}_A"] & 0 \\ 0 \arrow[r] & B \arrow[r] & X \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

and hence $\xi = \Theta(\Psi(\xi))$. $\square$

Lemma 5.5. For morphisms of R-modules $\varphi : N \to B$ and $\psi : N \to P$, the pushout

$$B \amalg_N P := B \oplus P / \{(\varphi(n), 0) - (0, \psi(n))\}$$

has the following universal property: Given morphisms $r : B \to M$ and $s : P \to M$ such that $r \circ \varphi = s \circ \psi$, there exists a unique morphism $t : B \amalg_N P \to M$ making the diagram

$$\begin{tikzcd} N \arrow[r, "\psi"] \arrow[d, "\varphi"'] & P \arrow[d] \arrow[ddr, bend left, "s"] & \\ B \arrow[r] \arrow[drr, bend right, "r"'] & B \amalg_N P \arrow[dr, dashed, "t"] & \\ & & M \end{tikzcd}$$

commute.

we constructed a natural bijection given by the association $\xi \mapsto \Theta(\xi)$. Here, $\Theta(\xi)$ denotes the obstruction class of ξ which was defined as $\delta(\mathrm{id}_B)$ where

$$\delta : \mathrm{Hom}_R(B, B) \to \mathrm{Ext}^1_R(A, B)$$

denotes the coboundary morphism associated to the long exact sequence for $\mathrm{Ext}^*_R(-, B)$. We compute an explicit example.

Example 5.6. Let $\mathbb{Z}[x]$ denote the ring of polynomials with integral coefficients. Consider the $\mathbb{Z}[x]$-module $\mathbb{Z} = \mathbb{Z}[x]/(x)$. The short exact sequence

$$\xi : \qquad 0 \longrightarrow \mathbb{Z} \xrightarrow{x} \mathbb{Z}[x]/(x^2) \longrightarrow \mathbb{Z} \longrightarrow 0$$

exhibits $\mathbb{Z}[x]/(x^2)$ as an extension of $\mathbb{Z}$ by $\mathbb{Z}$. We will compute the obstruction class $\Theta(\xi) \in \mathrm{Ext}^1_{\mathbb{Z}[x]}(\mathbb{Z}, \mathbb{Z})$. To compute the latter group, we choose a free resolution

$$0 \longrightarrow \mathbb{Z}[x] \xrightarrow{x} \mathbb{Z}[x] \xrightarrow{\varepsilon} \mathbb{Z}$$

of $\mathbb{Z}$ and obtain the formula

$$\operatorname{Ext}^1_{\mathbb{Z}[x]}(\mathbb{Z},\mathbb{Z}) = H^1(\mathbb{Z} \xrightarrow{0} \mathbb{Z}) \cong \mathbb{Z}.$$

To compute the coboundary map $\delta : \operatorname{Hom}_{\mathbb{Z}[x]}(\mathbb{Z},\mathbb{Z}) \to \operatorname{Ext}^1_{\mathbb{Z}[x]}(\mathbb{Z},\mathbb{Z})$ we choose a horseshoe free resolution of the short exact sequence given by ξ:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}[x] \arrow[r, "\binom{1}{0}"] \arrow[d, "x"'] & \mathbb{Z}[x]^2 \arrow[r, "(0\ 1)"] \arrow[d, "\begin{pmatrix} x & -1 \\ 0 & x \end{pmatrix}"] & \mathbb{Z}[x] \arrow[r] \arrow[d, "x"] & 0 \\
0 \arrow[r] & \mathbb{Z}[x] \arrow[r, "\binom{1}{0}"'] \arrow[d] & \mathbb{Z}[x]^2 \arrow[r, "(0\ 1)"'] \arrow[d, "(x\ 1)"] & \mathbb{Z}[x] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "x"] & \mathbb{Z}[x]/(x^2) \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

Applying the functor $\operatorname{Hom}_{\mathbb{Z}[x]}(-,\mathbb{Z})$ to this resolution, we obtain the diagram

$$\begin{tikzcd}
0 & \mathbb{Z} \arrow[l] & \mathbb{Z}^2 \arrow[l, "(1\ 0)"'] & \mathbb{Z} \arrow[l, "\binom{0}{1}"'] & 0 \arrow[l] \\
0 & \mathbb{Z} \arrow[l] \arrow[u, "0"'] & \mathbb{Z}^2 \arrow[l, "(1\ 0)"] \arrow[u, "\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}"'] & \mathbb{Z} \arrow[l, "\binom{0}{1}"] \arrow[u, "0"'] & 0 \arrow[l]
\end{tikzcd}$$

The element $\mathrm{id} \in \operatorname{Hom}_{\mathbb{Z}[x]}(\mathbb{Z},\mathbb{Z})$ is represented by $1 \in \mathbb{Z}$ in the lower left corner of the diagram. To compute the coboundary map $\delta(\mathrm{id})$, we choose the preimage $\binom{1}{0} \in \mathbb{Z}^2$ which gets mapped to $\binom{0}{-1} \in \mathbb{Z}^2$ under $\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$. Finally, $\binom{0}{-1} \in \mathbb{Z}^2$ is the image of $-1 \in \mathbb{Z} = \operatorname{Ext}^1_{\mathbb{Z}[x]}(\mathbb{Z},\mathbb{Z})$ in the upper right corner of the diagram. We have calculated $\delta(\mathrm{id}) = -1 = \Theta(\xi)$.

Observe that

$$\left\{ \begin{array}{c} \text{equivalence classes of} \\ \text{extensions of } A \text{ by } B \end{array} \right\} \xrightarrow{\cong} \operatorname{Ext}^1_R(A,B) \tag{5.7}$$

is a natural bijection of sets where the right hand side carries an abelian group structure. This implies that there exists a natural abelian group structure on the set of extension classes of A by B. Our next goal is to give an explicit description of this addition law.

Let

$$\xi : \qquad 0 \longrightarrow B \xrightarrow{f} X \xrightarrow{g} A \longrightarrow 0$$

and

$$\xi' : \qquad 0 \longrightarrow B \xrightarrow{f'} X' \xrightarrow{g'} A \longrightarrow 0$$

be extensions of A by B. We form the pushout $X \amalg_B X'$ and define

$$X'' := \ker(X \amalg_B X' \overset{(g,0)-(0,g')}{\longrightarrow} A). \tag{5.8}$$

We obtain a sequence

$$0 \longrightarrow B \xrightarrow{f''} X'' \xrightarrow{g''} A \longrightarrow 0$$

where the map f'' is given by $b \mapsto (f(b), 0)$ (which coincides with $b \mapsto (0, f'(b))$ and the map g'' is given by $(x, x') \mapsto g(x)$ (which coincides, by construction, with $(x, x') \mapsto g(x')$). An explicit verification shows that the sequence is exact. We call the resulting extension the *Baer sum* of ξ and ξ', denoted $\xi + \xi'$. It is immediate that the Baer sum is compatible with equivalences of extensions and hence descends to give an addition law on extension classes.

Example 5.9. Let

$$\xi : \qquad 0 \longrightarrow B \xrightarrow{f} X \xrightarrow{g} A \longrightarrow 0$$

be an extension of A by B. Consider the split extension

$$\eta : \qquad 0 \longrightarrow B \longrightarrow B \oplus A \longrightarrow A \longrightarrow 0$$

To compute $\xi + \eta$, note that we have $X \amalg_B (B \oplus A) \cong X \oplus A$, and the map $X \to X \oplus A$ given by $x \mapsto (x, f(x))$ identifies X as the kernel from (5.8). Thus, we have $\xi + \eta \sim \xi$ and conclude that the split extension class is a neutral element for the addition law on extension classes given by the Baer sum.

Example 5.10. Let

$$\xi : \qquad 0 \longrightarrow B \xrightarrow{f} X \xrightarrow{g} A \longrightarrow 0$$

be an extension of A by B. Consider the extension

$$\xi' : \qquad 0 \longrightarrow B \xrightarrow{-f} X \xrightarrow{g} A \longrightarrow 0.$$

We have

$$X'' = \{(x, y) \in X \amalg_B X | g(x) = g(y)\}.$$

Thus, we have a morphism $\pi : X'' \to B$ given by $(x, y) \mapsto x - y$ where $x - y \in B$ since $g(x - y) = 0$ and B is the kernel of g. Note further, that this map is in fact well-defined since an element $(f(b), -(-f(b)))$ in the ideal of $X \oplus X$ defining $X \amalg_B X$ is mapped to $f(b) - f(b) = 0$. The map π provides a splitting of the Baer sum

$$\begin{tikzcd} \xi + \xi' : & 0 \arrow[r] & B \arrow[r] & X'' \arrow[r] \arrow[l, bend left, "\pi"] & A \arrow[r] & 0 \end{tikzcd}$$

so that the extension class of $\xi + \xi'$ is the split class. We conclude that $\xi' = -\xi$ is the inverse of ξ with respect to the addition law given by the Baer sum.

We claim that the addition law on extension classes given by the Baer sum is in fact the one induced from Ext^1 via the bijection (5.7).

Theorem 5.11. For extension classes ξ, ξ' of A by B, we have $\Theta(\xi + \xi') = \Theta(\xi) + \Theta(\xi')$.

Proof. We may equivalently show that, for $e, e' \in \mathrm{Ext}^1_R(A, B)$, we have $\Psi(e + e') = \Psi(e) + \Psi(e')$ where Ψ is the inverse to Θ constructed in Lecture 11. To construct Ψ, we chose a short exact sequence

$$0 \longrightarrow N \xrightarrow{\psi} P \xrightarrow{\alpha} A \longrightarrow 0$$

with P projective. Given $e \in \mathrm{Ext}^1(A, B)$, we consider the exact sequence

$$\mathrm{Hom}(N, B) \longrightarrow^{\delta} \mathrm{Ext}^1(A, B) \longrightarrow 0$$

and choose $\varphi \in \mathrm{Hom}(N, B)$ such that $\delta(\varphi) = e$. Then we defined $\Psi(e)$ to be given by the extension

$$\Psi(e): \qquad 0 \longrightarrow B \longrightarrow P \amalg^{\varphi}_N B \longrightarrow A \longrightarrow 0$$

Here the superscript φ indicates that the map $\varphi : N \to B$ is used to form the pushout while the map $N \to P$ will be given by ψ throughout the proof. Choosing φ' such that $\delta(\varphi') = e'$ we have $\delta(\varphi + \varphi') = e + e'$ and hence, we are reduced to showing that the extension given by $P \amalg^{\varphi+\varphi'}_N B$ is equivalent to the Baer sum of the extensions given by $P \amalg^{\varphi}_N B$ and $P \amalg^{\varphi'}_N B$. This latter Baer sum is given by the kernel X'' of the difference of the two maps from

$$(P \amalg^{\varphi}_N B) \amalg_B (P \amalg^{\varphi'}_N B) \cong P \amalg^{\varphi}_N B \amalg^{\varphi'}_N P$$

to A given by $(p, b, p') \mapsto \alpha(p)$ and $(p, b, p') \mapsto \alpha(p')$, respectively. The morphism

$$P \amalg^{\varphi+\varphi'}_N B \longrightarrow P \amalg^{\varphi}_N B \amalg^{\varphi'}_N P$$

given by $(p, b) \mapsto (p, b, p)$ factors through an isomorphism

$$P \amalg^{\varphi+\varphi'}_N B \xrightarrow{\cong} X''$$

proving our claim. $\square$

Corollary 5.12. The Baer sum equips the set of extension classes of A by B with an abelian group structure such that the map Φ is a group isomorphism.

6 Quiver representations

Our next goal is to compute Ext groups in a certain class of abelian categories given by representations of quivers. A *quiver* Q consists of

- a set Q_0 of *vertices*,
- a set Q_1 of *arrows*,
- a pair of maps $s : Q_1 \to Q_0$ and $t : Q_1 \to Q_0$ called *source* and *target* maps.

Example 6.1. (1) $Q = \bullet_1$ where $Q_0 = \{1\}$ and $Q_1 = \emptyset$.

(2) $Q = \bullet_1 \longrightarrow \bullet_2$ where $Q_0 = \{1, 2\}$ and $Q_1 = \{\rho\}$ and $s(\rho) = 0$, $t(\rho) = 1$.

(3) $Q = \overset{\circlearrowright}{\bullet_1}$ where $Q_0 = \{1\}$ and $Q_1 = \{\rho\}$ and $s(\rho) = t(\rho) = 1$.

Let k be a field. A k-linear *representation* of a quiver Q consists of

- for every vertex $x \in Q_0$, a vector space V_x,
- for every arrow $\rho \in Q_1$, a k-linear map $V_{s(\rho)} \to V_{t(\rho)}$.

Given representations V and W of a quiver Q, a morphism $f : V \to W$ consists of, for every vertex $x \in Q_0$, a k-linear map $f_x : V_x \to W_x$ such that, for every arrow $\rho \in Q_1$, the diagram

$$\begin{tikzcd} V_{s(\rho)} \arrow[r] \arrow[d, "f_{s(\rho)}"'] & V_{t(\rho)} \arrow[d, "f_{t(\rho)}"] \\ W_{s(\rho)} \arrow[r] & W_{t(\rho)} \end{tikzcd}$$

commutes.

We assume that Q has finitely many vertices and arrows and fix a bijection $Q_0 = \{1, 2, \ldots, n\}$. A *nontrivial path* in Q is a finite sequence

$$\rho_m \cdots \rho_2 \rho_1, \ m \geq 1,$$

of arrows in Q such that, for all $1 \leq i < m$, we have $s(\rho_{i+1}) = t(\rho_i)$. For every vertex i, we introduce a *trivial path* e_i with $s(e_i) = t(e_i) = i$. We define the *path algebra* kQ of Q to be the k-vector space generated by all paths. Multiplication of nontrivial paths x, y is given by

$$x \cdot y = \begin{cases} xy & \text{if } s(x) = t(y), \\ 0 & \text{otherwise.} \end{cases}$$

For any path x, we further define

$$e_i \cdot x = \begin{cases} x & \text{if } t(x) = i, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$x \cdot e_i = \begin{cases} x & \text{if } s(x) = i, \\ 0 & \text{otherwise.} \end{cases}$$

The product in kQ is now given by k-linearly extending the above multiplication laws. Note that the element $e_1 + e_2 + \cdots + e_n$ given by the sum over all trivial paths is a unit in kQ so that the path algebra forms an associative unital k-algebra.

Example 6.2. (1) $Q = \overset{\overset{\rho}{\circlearrowright}}{\bullet_1}$. Then, we have

$$kQ \cong ke_1 \oplus k\rho \oplus k\rho\rho \oplus k\rho\rho\rho \oplus \cdots$$

The association $e_1 \mapsto 1$, $\rho \mapsto x$ provides an isomorphism $kQ \cong k[x]$ with the polynomial ring over k in one variable.

(2) $Q = \bullet_1 \xrightarrow{\rho_1} \bullet_2 \xrightarrow{\rho_2} \bullet_3$. Then we have

$$kQ \cong ke_1 \oplus ke_2 \oplus ke_3 \oplus k\rho_1 \oplus k\rho_2 \oplus k\rho_2\rho_1$$

with $e_i \cdot e_i = e_i$, $e_i \cdot e_j = 0$ for $i \neq j$, $\rho_1 \cdot e_1 = \rho_1 = e_2 \cdot \rho_1$, $\rho_2 \cdot \rho_1 = \rho_2\rho_1$, $\rho_1 \cdot \rho_2 = 0$, and various other identities involving the trivial paths e_i.

Example 6.3. Let Q be any quiver and let $A = kQ$ be its path algebra. For a vertex i, the left ideal $Ae_i \subset A$ is the k-vector space generated by those paths which leave i. Note, that we have $A \cong Ae_i \oplus A(1 - e_i)$. In particular, the left A-module Ae_i is a summand of a free module and hence projective. Similarly, the right ideal $e_iA \subset A$ is the k-vector space generated by all paths which end in i. By the same argument, e_iA is a projective right A-module.

Proposition 6.4. Let Q be quiver and kQ its path algebra. There is a canonical equivalence of categories

$$\mathrm{Rep}_k(Q) \xrightarrow{\simeq} kQ\text{--}\mathbf{mod}$$

between the category of k-linear representations of Q and the category of left kQ-modules.

Proof. We define the functor by associating to a representation V of Q the k-vector space $\bigoplus_{i \in Q_0} V_i$ with the following kQ-module structure:

(1) For a trivial path e_i corresponding to a vertex i, we define

$$e_i.(v_1, v_2, \ldots, v_n) = (0, \cdots, 0, v_i, 0, \cdots, 0)$$

the vector with only one nonzero entry v_i in position i.

(2) For an arrow ρ with $s(\rho) = i$ and $t(\rho) = j$, we define

$$\rho.(v_1, v_2, \ldots, v_n) = (0, \cdots, 0, f(v_i), 0, \cdots, 0)$$

the vector with only one nonzero entry $f(v_i)$ in position j where $f : V_{s(\rho)} \to V_{t(\rho)}$ is the k-linear map which the representation V associated to the arrow ρ.

Since the path algebra is generated as a k-algebra by trivial paths and arrows, formulas (1) and (2) uniquely determine the desired left kQ-module structure.

We construct an inverse functor. Given a left kQ-module M, we define a representation of Q by associating to the vertex i the vector space $e_iM \subset M$. To an arrow $\rho : i \to j$, we associate the k-linear map $f : e_iM \longrightarrow e_jM$ given by $m \mapsto \rho.m$. It is straightforward to verify that the functors defined above are inverse to one another establishing the desired equivalence (in fact isomorphism) of categories. $\square$

The following result will be the key statement allowing us to compute Ext groups.

Lemma 6.5. Let Q be a quiver and let A be its path algebra. Then there is an exact sequence of A-bimodules

$$0 \longrightarrow \bigoplus_{\rho \in Q_1} Ae_{t(\rho)} \otimes_k e_{s(\rho)}A \xrightarrow{f} \bigoplus_{i \in Q_0} Ae_i \otimes_k e_iA \xrightarrow{g} A \longrightarrow 0. \tag{6.6}$$

Proof. We set $S = \bigoplus_{\rho\in Q_1} Ae_{t(\rho)} \otimes_k e_{s(\rho)}A$ and $T = \bigoplus_{i\in Q_0} Ae_i \otimes_k e_iA \xrightarrow{g} A$. To simplify notation, we refer to an element $(v_1, v_2, \ldots, v_n) \in T$ as $\sum_i v_i$ and to an element $(v_\rho)_{\rho\in Q_1} \in S$ as $\sum_\rho v_\rho$. The map $g : T \to A$ is defined by the formula $g(a \otimes b) = ab \in A$ where $a \otimes b \in T_i$, $i \in Q_0$. The map $f : S \to T$ is defined as follows. Given $a \otimes b \in S_\rho$, $\rho \in Q_1$, we define $f(a \otimes b) := a\rho \otimes b - a \otimes \rho b$ having components $a\rho \otimes b \in T_{s(\rho)}$ and $-a \otimes \rho \in T_{t(\rho)}$ and all other components 0.

To show that (6.6) is exact, we will show that the identity morphism is homotopic to 0. We define the homotopy h as follows. For $a \in A$, we define $h(a) \sum_i e_i \otimes e_ia \in T_i$. For $p \otimes b \in T_i$, where p is a path starting at i and ending at j, we define $h(p \otimes b) = \sum_{p=p'\rho p''} p' \otimes p''b$ where the sum ranges over all factorizations of p into a path p', an arrow ρ, and a path p''. We extend h k-linearly to obtain a map $h : T \to S$. An explicit computation shows that $dh + hd = \mathrm{id}$ verifying our claim. □

Corollary 6.7. Let M be a left A-module. Then the sequence of left A-modules

$$0 \longrightarrow \bigoplus_{\rho\in Q_1} Ae_{t(\rho)} \otimes_k e_{s(\rho)}M \longrightarrow \bigoplus_{i\in Q_0} Ae_i \otimes_k e_iM \longrightarrow M \longrightarrow 0,$$

obtained by applying $-\otimes_A M$ to (6.6), is exact and exhibits a resolution of M by projective left A-modules.

Proof. The homotopy constructed in the proof of Lemma 6.5 is in fact right A-linear. Therefore, it remains a k-linear homotopy between id and 0 after applying $-\otimes_A M$. This implies that the complex is exact. We have an isomorphism of left A-modules

$$Ae_i \otimes_k e_iM \cong \coprod_{b\in B} Ae_i$$

where B is a basis of the k-vector space e_iM. This shows that $Ae_i \otimes_k e_iM$ is a coproduct of projectives and hence projective. □

Corollary 6.8. Let M, N be left A-modules. Then we have $\mathrm{Ext}^i_A(M, N) = 0$ for $i \geq 2$.

Let M,N be finitely generated A-modules. Since A is a k-algebra, the abelian group $\mathrm{Hom}_A(M, N)$ is naturally a k-vector space. Therefore, we obtain a natural k-vector space structure on all Ext groups $\mathrm{Ext}^*_A(M, N)$ by deriving the functor

$$\mathrm{Hom}_A(-, N) : (kQ\mathbf{-mod})^{\mathrm{op}} \longrightarrow \mathbf{Vect}_k.$$

We apply $\mathrm{Hom}_A(-, N)$ to the above projective resolution of M and augment on both sides to obtain the exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(M, N) \longrightarrow X \longrightarrow Y \longrightarrow \mathrm{Ext}^1_A(M, N) \longrightarrow 0 \qquad (6.9)$$

of k-vector spaces with

$$X = \bigoplus_{i\in Q_0} \mathrm{Hom}_A(Ae_i \otimes_k e_iM, N)$$
$$Y = \bigoplus_{\rho\in Q_1} \mathrm{Hom}_A(Ae_{t(\rho)} \otimes_k e_{s(\rho)}M, N).$$

We compute

$$\begin{aligned}\mathrm{Hom}_A(Ae_i \otimes_k e_jM, N) &\cong \mathrm{Hom}_A(Ae_i, N) \otimes_k e_jM \\ &\cong e_iN \otimes_k e_jM\end{aligned}$$

Consequently, we obtain the following main result.

Theorem 6.10. Let Q be a quiver with path algebra A. For finitely generated left A-modules M and N, the number

$$\chi(M, N) := \dim_k \mathrm{Hom}_A(M, N) - \dim_k \mathrm{Ext}^1_A(M, N)$$

is given by the formula

$$\chi(M, N) = \sum_{i \in Q_0} \dim_k(e_iM) \dim_k(e_iN) - \sum_{\rho \in Q_1} \dim_k(e_{s(\rho)}M) \dim_k(e_{t(\rho)}N).$$

Proof. For an exact bounded complex of finite dimensional k-vector spaces, the Euler characteristic, i.e., the alternating sum over the dimensions of all terms, vanishes. Applying this to (6.9) yields the result. $\square$

Remark 6.11. Note that, interpreting a left kQ-module M as a k-linear representation of Q, the number $\dim_k(e_iM)$ is simply the dimension of the vector space associated to the vertex i. We can collect all dimensions at the various vertices of Q to form a vector $\underline{\dim}(M) \in \mathbb{N}^n$ called the *dimension vector of* M. The remarkable feature of the formula in Theorem 6.10 is that the number $\chi(M, N)$ only depends on the dimension vectors of M and N.

7 Group homology

Let G be a group. A *G-module* is an abelian group A equipped with an additive left action of G. A *trivial G-module* is an abelian group with G-action given by $g.a = a$ where $g \in G$ and $a \in A$. A *G-equivariant homomorphism* between G-modules A and B is a homomorphism $f : A \to B$ of underlying abelian groups such that, for every $g \in G$ and $a \in A$, we have $f(g.a) = g.f(a)$. The collection of G-modules with G-equivariant homomorphisms form a category denoted $G{-}\mathbf{mod}$. Denoting by BG the category with one object and automorphism group G, we have an isomorphism of categories

$$G{-}\mathbf{mod} \cong \mathrm{Fun}(BG, \mathbf{Ab})$$

which implies that $G{-}\mathbf{mod}$ is abelian since any functor category from a small categrory into an abelian category is abelian.

Remark 7.1. We can more generally study G-actions on objects in any abelian category $\mathcal{A}$ by forming the functor category $\mathrm{Fun}(BG, \mathcal{A})$. For example, setting $\mathcal{A} = \mathbf{Vect}_k$ to be the category of vector spaces over a field, we obtain the abelian category of *k-linear representations of G*.

Given a G-module A, we are interested in the abelian groups of

(1) *invariants*: $A^G = \{a \in A \mid g.a = a\}$,

(2) and *coinvariants*: $A_G = A/\langle g.a - a \mid g \in G,\ a \in A\rangle$ where the brackets denote the subgroup generated by the indicated set.

Proposition 7.2. There are adjunctions

$$\mathrm{triv} : \mathbf{Ab} \longleftrightarrow G\mathbf{-mod} : (-)^G$$
$$(-)_G : G\mathbf{-mod} \longleftrightarrow \mathbf{Ab} : \mathrm{triv}\,.$$

In particular, the functor $(-)^G$ of invariants is left exact and the functor $(-)_G$ of coinvariants is right exact.

Proof. For an abelian group A and a G-module B, we have

$$\mathrm{Hom}_{G-\mathbf{mod}}(\mathrm{triv}(A), B) \cong \mathrm{Hom}_{\mathbf{Ab}}(A, B^G)$$

since a G-equivariant homomorphism from a trivial G-module can be identified with a homomorphism into the invariants. Similarly, for a G-module A and an abelian group B, we have

$$\mathrm{Hom}_{\mathbf{Ab}}(A_G, B) \cong \mathrm{Hom}_{G-\mathbf{mod}}(A, \mathrm{triv}(B))$$

since any G-equivariant homomorphism into a trivial G-module factors uniquely over a homomorphism from the coinvariants. $\square$

We will see below that the abelian category $G\mathbf{-mod}$ has enough projectives and injectives so that we can make the following definition.

Definition 7.3. Let G be a group.

(1) The right derived functors of the functor of invariants $(-)^G$ are called *group cohomology* denoted by $\{H^*(G,-)\}$.

(2) The left derived functors of the functor of coinvariants $(-)_G$ are called *group homology* denoted by $\{H_*(G,-)\}$.

Given a group G, we introduce the *integral group ring* $\mathbb{Z}G$ to be the free $\mathbb{Z}$-module on the set G

$$\mathbb{Z}G \cong \bigoplus_{g \in G} \mathbb{Z}g$$

with product given by additively extending $g \cdot h = gh$. As for quivers, there is a more or less tautological identification of categories

$$G\mathbf{-mod} \cong \mathbb{Z}G\mathbf{-mod}$$

which is the identity functor on the underlying abelian groups. We typically leave this identification implicit. As an immediate consequence, we obtain that the abelian category $G\mathbf{-mod}$ has enough injectives and projectives.

Proposition 7.4. Let G be a group. We have isomorphisms of δ-functors

(1) $H^*(G,-) \cong \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z},-)$

(2) $H_*(G,-) \cong \mathrm{Tor}^{\mathbb{Z}G}_*(\mathbb{Z},-)$

where $\mathbb{Z}$ denotes the trivial left (resp. right) $\mathbb{Z}G$-module.

Proof. (1) By the universality of δ-functors given by derived functors, it suffices to construct isomorphisms

$$A^G \cong \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, A),$$

functorial in A, which, after rewriting $A^G \cong \mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}, A^G)$ we obtain from the adjunction in Proposition 7.2.
(2) From Set 4, Problem 3(5), we have an adjunction

$$\mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z} \otimes_{\mathbb{Z}G} A, B) \cong \mathrm{Hom}_{\mathbb{Z}G}(A, \mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}, B)),$$

where the left $\mathbb{Z}G$-module structure on $\mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}, B)$ is given by $r.f(x) = f(x.r)$. Since $\mathbb{Z}$ is the trivial right $\mathbb{Z}G$-module, $\mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}, B)$ is the trivial $\mathbb{Z}G$-module $\mathrm{triv}(B)$. Therefore $\mathbb{Z} \otimes_{\mathbb{Z}G} -$ is left adjoint to triv. By the uniqueness of left adjoints, we have

$$\mathbb{Z} \otimes_{\mathbb{Z}G} - \cong (-)_G.$$

□

Proposition 7.4 is of central importance: The balance of Tor and Ext allows us to compute group homology and cohomology by using projective resolutions of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$.

Example 7.5. Let $T \cong \mathbb{Z}$ be an infinite cyclic group with generator t. Then $\mathbb{Z}T \cong \mathbb{Z}[t, t^{-1}]$. We have an exact sequence

$$0 \longrightarrow \mathbb{Z}T \xrightarrow{t-1} \mathbb{Z}T \longrightarrow \mathbb{Z} \longrightarrow 0$$

of left (or right) $\mathbb{Z}T$-modules which exhibits a free resolution of $\mathbb{Z}$. We obtain for a G-module A

$$H^n(T, A) \cong \begin{cases} A^G & \text{for } n = 0, \\ A_G & \text{for } n = 1, \\ 0 & \text{for } n > 1, \end{cases}$$

and

$$H_n(T, A) \cong \begin{cases} A_G & \text{for } n = 0, \\ A^G & \text{for } n = 1, \\ 0 & \text{for } n > 1. \end{cases}$$

8 The bar construction and group cohomology in low degrees

For every $n \geq 1$, we introduce the free left $\mathbb{Z}G$-module on the set $(G \setminus \{1\})^n$

$$B_n = \bigoplus_{(g_i) \in (G \setminus \{1\})^n} \mathbb{Z}G[g_1|g_2|\ldots|g_n]$$

where the formal bracket symbol $[g_1|g_2|\dots|g_n]$ denotes a basis element corresponding to the n-tupel $(g_1,\dots,g_n)$. By convention, we set $B_0 = \mathbb{Z}G[\,]$ to be the cyclic $\mathbb{Z}G$-module with generator denoted by the empty bracket $[\,]$. We define a differential $d : B_n \to B_{n-1}$ given by the formula

$$\begin{aligned} d([g_1|g_2|\dots|g_n]) = g_1[g_2|g_3|\dots|g_n] + \sum_{i=1}^{n-1}(-1)^i[g_1|\dots|g_ig_{i+1}|\dots|g_n] \\ + (-1)^n[g_1|\dots|g_{n-1}], \end{aligned}$$

extended $\mathbb{Z}G$-linearly to B_n, where, on the right-hand side, we declare any symbol $[h_1|h_2|\dots|h_{n-1}]$ with $h_i = 1$ for some i to be 0. We obtain a complex $(B_\bullet, d)$ called the *bar complex of* G.

In low degree terms, we have the explicit formulas

(1) For $[g] \in B_1$, we have $d[g] = g[\,] - [\,] = (g-1)[\,] \in B_0$.

(2) For $[g_1|g_2] \in B_2$, we have $d([g_1|g_2]) = g_1[g_2] - [g_1g_2] + [g_2] \in B_1$

(3) For $[g_1|g_2|g_3] \in B_3$, we have $d([g_1|g_2|g_3]) = g_1[g_2|g_3] - [g_1g_2|g_3] + [g_1|g_2g_3] - [g_1|g_2] \in B_2$.

Note that, from the formula for $d : B_1 \to B_0$, we deduce that the augmentation map $\varepsilon : \mathbb{Z}G \to \mathbb{Z}, \sum_g n_g g \mapsto \sum_g n_g$ defines an augmentation $\varepsilon : B_\bullet \to \mathbb{Z}$ of the bar complex of G.

Theorem 8.1. The bar complex $B_\bullet$ equipped with the augmentation ε is a free resolution of the trivial left $\mathbb{Z}G$-module $\mathbb{Z}$.

Proof. We have to show that the augmented complex

$$\cdots \longrightarrow B_2 \longrightarrow B_1 \longrightarrow B_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

is exact. We will show this by constructing a $\mathbb{Z}$-linear homotopy h between $\mathrm{id}_{B_\bullet}$ and 0. To this end, we set $h_{-1} : \mathbb{Z} \to B_0, n \mapsto n[\,]$ and, for $n \geq 0$,

$$h_n(g[g_1|g_2|\dots|g_n]) := [g|g_1|\dots|g_n]$$

extended $\mathbb{Z}$-linearly to B_n. We have $\varepsilon \circ h_{-1} = \mathrm{id}_{\mathbb{Z}}$ and, an explicit calculation shows that, for every $n \geq 0$, we have $d_{n+1} \circ h_n + h_{n-1} \circ d_n = \mathrm{id}_{B_n}$. □

Remark 8.2. In complete analogy, we obtain a free resolution of the *right* $\mathbb{Z}G$-module $\mathbb{Z}$, by mirroring the above constructions and define

$$B'_n = \bigoplus_{(g_i)\in(G\setminus\{1\})^n} [g_n|g_{n-1}|\dots|g_1]\mathbb{Z}G$$

with differentials given mutatis mutandis by the formula

$$\begin{aligned} d([g_n|g_{n-1}|\dots|g_1]) = [g_n|g_{n-1}|\dots|g_2]g_1 + \sum_{i=1}^{n-1}(-1)^i[g_n|\dots|g_{i+1}g_i|\dots|g_1] \\ + (-1)^n[g_{n-1}|\dots|g_1]. \end{aligned}$$

Given a left $\mathbb{Z}G$-module A, we obtain, as an immediate consequence of Theorem 8.1, explicit complexes

$$C^\bullet(G,A) := \mathrm{Hom}_{\mathbb{Z}G}(B_\bullet, A) \qquad\qquad C_\bullet(G,A) := B'_\bullet \otimes_{\mathbb{Z}G} A$$

with $H^n(C^\bullet(G,A)) \cong H^n(G,A)$ and $H_n(C_\bullet(G,A)) \cong H_n(G,A)$.

Remark 8.3. We have introduced the bar complex via explicit ad hoc formulas without any reference to its origin. In fact, the complex arises very naturally in algebraic topology. As will be discussed later in more detail, there is a CW-complex $K(G,1)$ which is uniquely determined up to homotopy equivalence by the requirements

- $K(G,1)$ is connected.
- The fundamental group $\pi_1(K(G,1),*)$ of $K(G,1)$ is isomorphic to the group G.
- For every $n \geq 2$, the homotopy group $\pi_n(K(G,1),*)$ is trivial.

In this context, the bar construction arises as the cellular complex of a specifically chosen CW-model for the universal cover $\widetilde{K(G,1)}$ of $K(G,1)$. The fact that $\widetilde{K(G,1)}$ is contractible explains why the bar complex is exact. Further, the cellular chain complex of $K(G,1)$ itself with coefficients in $\mathbb{Z}$ can be identified with the complex $C_\bullet(G,\mathbb{Z})$. This implies an isomorphism

$$H_n(G,\mathbb{Z}) \cong H_n(K(G,1),\mathbb{Z})$$

so that group homology computes singular homology of the topological space $K(G,1)$. Similarly, group cohomology computes singular cohomology of $K(G,1)$.

Example 8.4. Let G be a group. We use the bar complex to compute $H_1(G,\mathbb{Z})$. Explicitly, we have $C_0(G,\mathbb{Z}) = [\ ]\mathbb{Z}$, $C_1(G,\mathbb{Z}) = \bigoplus [g]\mathbb{Z}$, and $C_2(G,\mathbb{Z}) = \bigoplus [g|h]\mathbb{Z}$. The differential $d : C_1 \to C_0$ is given by

$$d([g]) = [\ ].g - [\ ] = [\ ] - [\ ]$$

so that all elements of C_1 are cycles. The differential $d : C_2 \to C_1$ is given by

$$d([g|h]) = [g]h - [gh] + [h] = [g] - [gh] + [h].$$

Therefore, the group $H_1(G,\mathbb{Z})$ is the free abelian group generated by the symbols $[g]$, $g \neq 1$, and the relations $[g] + [h] = [gh]$. Thus, we have

$$H_1(G,\mathbb{Z}) \cong G/[G,G]$$

given by the abelianization of G.

Let G be a group and A a left $\mathbb{Z}G$-module. Explicitly, an element $\varphi \in C^n(G,A)$ can be represented by a map of sets

$$\varphi : G^n \longrightarrow A, (g_1,\dots,g_n) \mapsto \varphi(g_1,\dots,g_n)$$

which vanishes if $g_i = 1$ for some $1 \leq i \leq n$. Then $d\varphi \in C^{n+1}(G,A)$ is given by

$$\begin{aligned} d\varphi(g_1,\dots,g_{n+1}) = g_1\varphi(g_2,\dots,g_{n+1}) + \sum_{i=1}^{n-1}(-1)^i\varphi(g_1,\dots,g_ig_{i+1},\dots,g_n) \\ + (-1)^n\varphi(g_1,\dots,g_{n-1}). \end{aligned}$$

Example 8.5. We explicitly compute $H^1(G, A)$. An element $\varphi \in C^0$ can be identified with the element $a = \varphi([\,]) \in A$ and the differential $d : C^0 \to C^1$ is given by

$$d\varphi(g) = g.a - a,$$

for $d : C^1 \to C^2$, we have the formula

$$d\varphi(g, h) = g.\varphi(h) - \varphi(gh) + \varphi(g).$$

We define: Given a group G, and a G-module A, a map of sets $\varphi : G \to A$ is called a *crossed homomorphism* if, for every $g, h \in G$, we have

$$\varphi(gh) = \varphi(g) + g.\varphi(h).$$

Every element $a \in A$ defines a crossed homomorphism

$$\varphi_a : G \to A,\ g \mapsto g.a - a.$$

Crossed homomorphisms of the form φ_a are called *principal crossed homomorphisms.* With this terminology, we have

$$Z^1(G, A) \cong \{\text{crossed homomorphisms from } G \text{ to } A\},$$

and

$$B^1(G, A) \cong \{\text{principal crossed homomorphisms from } G \text{ to } A\},$$

so that $H^1(G, A)$ can be described as the quotient of all crossed homomorphisms by the subgroup of principal ones. As a special case, suppose that A is a trivial G-module. Then any crossed homomorphism $G \to A$ is a homomorphism of groups and every principal crossed homomorphism is trivial. Therefore, in this case, we have

$$H^1(G, A) \cong \mathrm{Hom}_{\mathbf{Grp}}(G, A) \cong \mathrm{Hom}_{\mathbf{Ab}}(G/[G, G], A).$$

We develop the context from which the terminology "crossed homomorphism" historically stems. Given a group G and a G-module A, we define a group called the *semidirect product* $A \rtimes G$ as follows: The underlying set is given by the Cartesian product of sets $A \times G$ equipped with the multiplication law

$$(a, g) \cdot (a', g') = (a + g.a', gg').$$

Note that we have a short exact sequence of groups

$$0 \longrightarrow A \longrightarrow A \rtimes G \xrightarrow{\pi} G \longrightarrow 1.$$

To emphasize the fact that G and $A \rtimes G$ may be nonabelian, we write the composition law multiplicatively while we write the composition in A additively. We say that an automorphism σ of the group $A \rtimes G$ *stabilizes* A *and* G if the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\mathrm{id}_A"] & A \rtimes G \arrow[r] \arrow[d, "\sigma"] & G \arrow[r] \arrow[d, "\mathrm{id}_G"] & 0 \\
0 \arrow[r] & A \arrow[r] & A \rtimes G \arrow[r] & G \arrow[r] & 0
\end{tikzcd}$$

commutes. Given a crossed homomorphism $\varphi \in Z^1(G, A)$, we obtain an automorphism of $A \rtimes G$, stabilizing A and G, defined by the formula

$$\sigma_\varphi : (a, g) \mapsto (a + \varphi(g), g).$$

Proposition 8.6. The homomorphism

$$\psi : Z^1(G,A) \longrightarrow \mathrm{Aut}(A \rtimes G), \varphi \mapsto \sigma_\varphi$$

is an isomorphism onto the subgroup of automorphisms of $A \rtimes G$ which stabilize A and G. A principal crossed homomorphism $\varphi_a \in B^1(G,A)$ is mapped to the automorphism of $A \rtimes G$ given by conjugation with $a \in A$.

Proof. The inverse of ψ is given by associating to an automorphism σ of $A \rtimes G$ which stabilizes A and G, the crossed homomorphism $\varphi(g) := \sigma(g) - g \in A$. The remaining statements are obtained by unravelling the defining formulas. □

Example 8.7. Consider the cyclic group $C_2 \cong \{1, \tau\}$ of order 2 acting on the cyclic group $\mathbb{Z}/m\mathbb{Z}$ via $\tau.1 = -1$. The corresponding semidirect product $\mathbb{Z}/m\mathbb{Z} \rtimes C_2$ is isomorphic to the dihedral group D_m. The complex $C^\bullet(C_2, \mathbb{Z}/m\mathbb{Z})$ is explicitly given by

$$\mathbb{Z}/m\mathbb{Z} \xrightarrow{-2} \mathbb{Z}/m\mathbb{Z} \xrightarrow{0} \mathbb{Z}/m\mathbb{Z} \xrightarrow{-2} \mathbb{Z}/m\mathbb{Z} \longrightarrow \dots$$

so that we have

$$H^1(C_2, \mathbb{Z}/m\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{for } m \text{ even,} \\ 0 & \text{for } m \text{ odd.} \end{cases}$$

Therefore, for n odd, every automorphism of D_m which stabilizes C_2 and $\mathbb{Z}/m\mathbb{Z}$ is given by conjugation with an element of $\mathbb{Z}/m\mathbb{Z}$. For m even, there is one automorphism which stabilizes C_2 and $\mathbb{Z}/m\mathbb{Z}$ and is not given by conjugation with an element of $\mathbb{Z}/m\mathbb{Z}$. It is explicitly given by $(a,1) \mapsto (a,1)$, $(a,\tau) \mapsto (a+1,\tau)$ where $a \in \mathbb{Z}/m\mathbb{Z}$.

Next, we will given an interpretation of $H^2(G,A)$. Let G be a group and A an abelian group. An *extension of G by A* is a short exact sequence of groups

$$0 \longrightarrow A \xrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 1.$$

We observe that the action of E on A by conjugation factors over π to give an action of G on A. Two extensions E,E' of G by A are called *equivalent* if there is a homomorphism $E \to E'$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "\mathrm{id}_A"'] & E \arrow[r, "\pi"] \arrow[d] & G \arrow[r] \arrow[d, "\mathrm{id}_G"] & 1 \\ 0 \arrow[r] & A \arrow[r, "i'"] & E' \arrow[r, "\pi'"] & G \arrow[r] & 1 \end{tikzcd}$$

commutes. Note that any such a homomorphism is necessarily an isomorphism. An extension of G by A is called *split* if there exists a group homomorphism $s : G \to E$ such that $\pi s = \mathrm{id}_G$. Such a homomorphism s is called a *section* of π.

Proposition 8.8. An extension is split if and only if it is equivalent to the extension

$$0 \longrightarrow A \longrightarrow A \rtimes G \longrightarrow G \longrightarrow 1$$

where $A \rtimes G$ denotes the semidirect product with respect to the G-action on A determined by the given extension.

Proof. The semidirect product extension admits the section $s(g) = (0, g)$. Given a split extension E of G by A, the map

$$E \longrightarrow A \rtimes G,\ e \mapsto (e(s\pi(e)^{-1}), \pi(e))$$

defines the desired homomorphism. □

Consider an extension

$$0 \longrightarrow A \xrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 1.$$

of G by A. While the extension may not be split, we may always find a *set-theoretic* section: a map of sets $s : G \to E$ such that $\pi s = \mathrm{id}_G$ and $s(1) = 1$. Given such a splitting, we define, for every $g, h \in G$, there exists $f(g, h) \in A$, such that

$$i(f(g, h)) = s(g)s(h)s(gh)^{-1}.$$

The resulting function

$$f : G \times G \longrightarrow A$$

is called the *factor set* associated to the extension and s. It measures the deviation of the set-theoretic splitting s from being a group homomorphism. We claim that we can completely recover the group law on E from the action of G on A and the function f. Namely, for every $a, b \in A$ and $g, h \in G$, we calculate

$$\begin{aligned} i(a)s(g)i(b)s(h) &= i(a)s(g)i(b)s(g^{-1})s(g)s(h) \\ &= i(a)i(g.b)s(g)s(h) \\ &= i(a + g.b)i(f(g, h))s(gh) \\ &= i(a + g.b + f(g, h))s(gh) \end{aligned}$$

so that we can describe the group law on $E \cong A \times G$ via

$$(a, g)(b, h) = (a + g.b + f(g, h), gh).$$

In other words, the function f describes the group law on E as a perturbation of the group law on $A \rtimes G$. Note that, since $s(1) = 1$, we have, for every $g \in G$,

$$f(g, 1) = f(1, g) = 0.$$

Further, the associativity of the group law on E means the equality of

$$((a, g)(b, h))(c, k) = (a + g.b + f(g, h) + (gh).c + f(gh, k), ghk)$$

and

$$(a, g)((b, h)(c, k)) = (a + g.b + (gh).c + g.f(h, k) + f(g, hk), ghk)$$

which translates into

$$g.f(h, k) - f(gh, k) + f(g, hk) - f(g, h) = 0$$

so that f is in fact a 2-cocycle in $C^\bullet(G, A)$. The construction of f depends on the choice of a set-theoretic section s. Let s' be another section leading to another factor set f'. For every $g \in G$, there exists $\varphi(g) \in A$ such that $i(\varphi(g)) = s'(g)s(g)^{-1}$ so that we obtain a function

$$\varphi : G \to A.$$

We compute

$$\begin{aligned} s'(g)s'(h) &= i(\varphi(g))i(g.\varphi(h))s(g)s(h) \\ &= i(\varphi(g) + g.\varphi(h) + f(g,h))i(\varphi(gh))^{-1}s'(gh) \\ &= i(\varphi(g) + g.\varphi(h) + f(g,h) - \varphi(gh))s'(gh) \end{aligned}$$

so that we have

$$f_{s'}(g,h) - f_s(g,h) = g.\varphi(h) - \varphi(gh) + \varphi(g).$$

We can summarize the above in the following result.

Theorem 8.9. Let G be a group and A a G-module. Then there is a natural bijection between $H^2(G,A)$ and the set of extension classes of G by the abelian group A such that the induced G-action on A coincides the specified G-module structure.

Example 8.10. Consider $G = C_2$ and $A = \mathbb{Z}/4\mathbb{Z}$ equipped with the trivial C_2-module structure. The complex $C^\bullet(C_2, \mathbb{Z}/4\mathbb{Z})$ is given by

$$\mathbb{Z}/4\mathbb{Z} \xrightarrow{0} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{0} \dots$$

so that $H^2(C_2, \mathbb{Z}/4\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. The two corresponding extension classes are represented by the split extension

$$0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow \mathbb{Z}/4\mathbb{Z} \times C_2 \longrightarrow C_2 \longrightarrow 1$$

and the nonsplit extension

$$0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow C_8 \longrightarrow C_2 \longrightarrow 1$$

given by the cyclic group C_8 of order 8.

Example 8.11. Consider $C_2 = \{1, \tau\}$ and $A = \mathbb{Z}/4\mathbb{Z}$ equipped with the C_2-action given by $\tau.1 = -1$. The complex $C^\bullet(C_2, \mathbb{Z}/4\mathbb{Z})$ is given by

$$\mathbb{Z}/4\mathbb{Z} \xrightarrow{-2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{0} \mathbb{Z}/4\mathbb{Z} \xrightarrow{-2} \dots$$

so that $H^2(C_2, \mathbb{Z}/4\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. The two corresponding extension classes are represented by the split extension

$$0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow D_4 \longrightarrow C_2 \longrightarrow 1$$

given by the dihedral group $D_4 = \mathbb{Z}/4\mathbb{Z} \rtimes C_2$ and the nonsplit extension

$$0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow Q_8 \longrightarrow C_2 \longrightarrow 1$$

where $Q_8 = \langle -1, i, j, k \mid (-1)^2 = 1,\ i^2 = j^2 = k^2 = ijk = -1 \rangle$ denotes the quaternion group and $\mathbb{Z}/4\mathbb{Z} \subset Q_8$ is the subgroup generated by i.

9 Periodicity in group homology

Our next goal is to prove a combinatorial version of the following remarkable result:

Theorem 9.1. Let G be a group. Suppose G acts freely on a sphere of dimension $2k-1$. Then G has $2k$-periodic group homology.

Let I be a finite set. A *simplicial complex K on I* is a collection of subsets of I such that, for every $\sigma \in K$ and $\tau \subset \sigma$, we have $\tau \in K$. We denote by K_n the collection of subsets in K of cardinality $n+1$ which we refer to as *n-simplices of K*. To define the simplicial chain complex of a simplicial complex K, we first define, for every $n \geq 0$, the free abelian group

$$\widetilde{C}_n(K) := \bigoplus_{(x_0,x_1,\ldots,x_n)\in K} \mathbb{Z}e_{(x_0,x_1,\ldots,x_n)}$$

on the set of all ordered $n+1$-tupels $(x_0, x_1, \ldots, x_n)$ such that the set $\{x_0, x_1, \ldots, x_n\}$ is in K_n.

Example 9.2. Let K be the simplicial complex on $\{0,1\}$ consisting of all subsets. We have $\widetilde{C}_1(K) \cong \mathbb{Z}e_{(0,1)} \oplus \mathbb{Z}e_{(1,0)}$.

We define a differential $d : \widetilde{C}_n(K) \longrightarrow \widetilde{C}_{n-1}(K)$ by linearly extending the formula

$$d(e_{(x_0,x_1,\ldots,x_n)}) = \sum_{i=0}^{n}(-1)^i e_{(x_0,x_1,\ldots,\widehat{x_i},\ldots,x_n)} \tag{9.3}$$

where, as usual, the hat indicates that the respective element is missing. We finally define the quotient $C_n(K) := \widetilde{C}_n(K)/R_n$ where R_n is the subgroup generated by all elements of the form

$$e_{(x_0,x_1,\ldots,x_n)} - \operatorname{sign}(\rho)e_{(x_{\rho(0)},x_{\rho(1)},\ldots,x_{\rho(n)})}$$

where $\{x_0, x_1, \ldots, x_n\} \in K_n$, $\rho \in S_{n+1}$ is a permutation, and $\operatorname{sign}(\rho)$ denotes the sign of ρ.

Example 9.4. Let K be the simplicial complex on $\{0,1\}$ consisting of all subsets. In $C_1(K)$, we have $e_{(0,1)} = -e_{(1,0)}$.

Proposition 9.5. Let K be a simplicial complex on I. Show that formula (9.3) yields a well-defined differential

$$d : C_n(K) \to C_{n-1}(K).$$

Further, after choosing a bijection $I \cong \{0, 1, \ldots, N\}$, there exists a canonical isomorphism between the complex $C_\bullet(K)$ and the simplicial chain complex of K as defined previously. In other words, the complex we have defined is a "coordinate-free" version of the one from class.

Given a simplicial complex K and an abelian group A, we define the simplicial chain complex of K with coefficients in A as $C_\bullet \otimes_{\mathbb{Z}} A$ and denote its homology groups by

$$H_i(K, A) := H_i(C_\bullet \otimes_{\mathbb{Z}} A),$$

called *simplicial homology groups of K with coefficients in A*. We define an *automorphism* f of a simplicial complex K on I to be a bijection $f : I \to I$ which preserves K so that, for every $\sigma \in K$, we have $f(\sigma) \in K$.

Proposition 9.6. Let K be a simplicial complex, and let f be an automorphism of K. Then f induces an automorphism of chain complexes

$$f : C_\bullet(K) \to C_\bullet(K)$$

by defining, on generators, $f_n(e_{(v_0,v_1,\dots,v_n)}) = e_{(f(v_0),f(v_1),\dots,f(v_n))}$.

Given a simplicial complex K with automorphism f, we introduce the *Lefschetz number*

$$\Lambda_f := \sum_{i\geq 0}(-1)^i \operatorname{tr}(f|H_i(K,\mathbb{Q}))$$

where $\operatorname{tr}(f|H_i(K,\mathbb{Q}))$ denotes the trace of the endomorphism induced by f on the $\mathbb{Q}$-vector space $H_i(K,\mathbb{Q})$. We say f *has no fixed points* if, for every $\sigma \in K$, we have $f(\sigma) \neq \sigma$. The following statement is a combinatorial version of the *Lefschetz fixed point theorem.*

Proposition 9.7. Let K be a simplicial complex, and let f be an automorphism of K. Assume f has no fixed points. Then $\Lambda_f = 0$.

Proof. The number Λ_f can be computed on the chain level before passing to homology (similarly to the Euler characteristic). But the diagonals of the matrices corresponding to the maps induced by f on simplicial chains have zeros on their diagonals so that the result follows. □

We call a simplicial complex K a *simplicial n-dimensional homology sphere*, if $K_i = \emptyset$ for $i > n$, and

$$H_i(K) \cong \begin{cases} \mathbb{Z} & \text{for } i \in \{0, n\}, \\ 0 & \text{else.} \end{cases}$$

Proposition 9.8. Let K be a simplicial $(2k-1)$-dimensional homology sphere. Assume that f is an automorphism of K which has no fixed points. Then f induces the identity map on $H_*(K)$.

Proof. This is an immediate consequence of Proposition 9.7. □

Let K be a simplicial complex and let G be a group. An *action* of G on K is a group homomorphism from G into the group of automorphisms of K. We say that G *acts freely* on K if, for every $g \in G$, the induced automorphism of K has no fixed points.

Proposition 9.9. An action of G on K makes the complex $C_\bullet(K)$ canonically a complex of $\mathbb{Z}G$-modules. If G acts freely then $C_\bullet(K)$ is a complex of free $\mathbb{Z}G$-modules.

Theorem 9.10. Let K be a simplicial $(2k-1)$-dimensional homology sphere and let G be a group which acts freely on K. Then the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ has a $2k$-periodic resolution by free $\mathbb{Z}G$-modules. Deduce that, for every $\mathbb{Z}G$-module A, the associated group homology $\{H_*(G,A)\}$ is $2k$-periodic so that, for every n, we have $H_n(G,A) \cong H_{n+2k}(G,A)$.

Proof. It follows from the above results that there is an exact complex of $\mathbb{Z}G$-modules

$$0 \longrightarrow \mathbb{Z} \longrightarrow C_{2k-1}(K) \longrightarrow C_{2k-2}(K) \longrightarrow \cdots \longrightarrow C_0(K) \longrightarrow \mathbb{Z} \longrightarrow 0$$

where $\mathbb{Z}$ denotes the trivial $\mathbb{Z}G$-module. We may then splice copies of this exact complex to produce the desired resolution. □