Homological Algebra - Problem Set 1

Problem 1. Find products and coproducts in the category **Ab** of abelian groups and the category **Group** of all groups.

Problem 2. Let G be a group, and let BG denote the corresponding category with one object $*$ and $\mathrm{Hom}_{BG}(*,*) := G$. Let $I : BG \to BG$ be the identity functor. Show that the natural transformations from I to I form a group with respect to composition. Describe this group in terms of G.

Problem 3. Let $\mathcal{C}$ be a small category and let $\mathbf{Set}_{\mathcal{C}}$ denote the category of functors from $\mathcal{C}^{\mathrm{op}}$ to the category of sets. Show that the association $X \mapsto h_X = \mathrm{Hom}_{\mathcal{C}}(-, X)$ naturally extends to a functor

$$h : \mathcal{C} \longrightarrow \mathbf{Set}_{\mathcal{C}}.$$

Show that, for every pair of objects X, Y of $\mathcal{C}$, the corresponding map on morphisms

$$h : \mathrm{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathbf{Set}_{\mathcal{C}}}(h_X, h_Y)$$

is bijective (*Hint*: Consider $\mathrm{id}_X \in h_X(X)$.)

Problem 4. Let $\mathcal{A}$ be an additive category and let X,Y be objects of $\mathcal{A}$.

(1) Denote by e the neutral element of the abelian group $\mathrm{Hom}_{\mathcal{A}}(X, Y)$. Denote by e' the composite of the unique maps $X \to 0$ and $0 \to Y$, where 0 denotes a zero object of $\mathcal{A}$. Show that $e' = e$. In what follows we will refer to e as 0.

(2) We denote by $X \oplus Y$ the sum of X and Y, equipped with inclusions $i_1 : X \to X \oplus Y$, $i_2 : Y \to X \oplus Y$ and projections $p_1 : X \oplus Y \to X$, $p_2 : X \oplus Y \to Y$ which characterize $X \oplus Y$ as both a product and a coproduct, respectively. We define the diagonal map

$$\delta_X : X \to X \oplus X$$

to be the map uniquely determined by the properties $p_1 \circ \delta_X = \mathrm{id}_X$ and $p_2 \circ \delta_X = \mathrm{id}_X$. Dually, we define the codiagonal map

$$\sigma_Y : Y \oplus Y \to Y$$

determined by $\sigma_Y \circ i_1 = \mathrm{id}_Y$ and $\sigma_Y \circ i_2 = \mathrm{id}_Y$. Given morphisms $f : X \to Y$, $g : X' \to Y'$, we define the map

$$f \oplus g : X \oplus X' \to Y \oplus Y'$$

determined by $p_1 \circ (f \oplus g) \circ i_1 = f$, $p_2 \circ (f \oplus g) \circ i_2 = g$, and $p_j \circ (f \oplus g) \circ i_k = 0$ for $j \neq k$. Show that, for $f, g : X \to Y$, the sum $f + g$, computed with respect to the group structure on $\mathrm{Hom}_{\mathcal{A}}(X, Y)$, equals the composition

$$X \xrightarrow{\delta_X} X \oplus X \xrightarrow{f \oplus g} Y \oplus Y \xrightarrow{\sigma_Y} Y.$$

(3) Show that the abelian group structure on the Hom-sets of $\mathcal{A}$ is uniquely determined by the ordinary category underlying $\mathcal{A}$. Conclude that an additive category is a category which satisfies certain properties as opposed to a category with additional structure. Formulate those properties, giving an alternative definition to the one in class. Extend this statement to abelian categories.

Problem 5. Let $\mathcal{A}$ be an abelian category. Show that the category $\mathbf{Ch}(\mathcal{A})$ of chain complexes in $\mathcal{A}$ is an abelian category.