Homological Algebra - Problem Set 2 (due Thursday, April 25)

Problem 1. Let $\mathcal{A}$ be an abelian category. Suppose we are given a commutative diagram

$$\begin{array}{ccccccc} & & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & & \end{array}$$

in $\mathcal{A}$ such that the rows are exact sequences. Show that there is an exact sequence

$$\ker(f) \longrightarrow \ker(g) \longrightarrow \ker(h) \longrightarrow \operatorname{coker}(f) \longrightarrow \operatorname{coker}(g) \longrightarrow \operatorname{coker}(h)$$

relating the kernels and cokernels of the maps f, g, h.

Problem 2. Let R be a ring.

(1) Show that an R-module M is projective if and only if there exists a module M' and a free module F such that $F \cong M \oplus M'$ (we say M is a *summand* of F).

(2) Show that the $\mathbb{Z}/6\mathbb{Z}$-modules $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ are examples of projective modules which are not free.

Problem 3. Let N be a left R-module. Show that the functor

$$- \otimes_R N : \mathbf{mod}\text{--}\, R \longrightarrow \mathbf{Ab}, M \mapsto M \otimes_R N$$

from the category of right R-modules to the category of abelian group is right exact.

Problem 4. A left R-module N is called *flat* if the functor $- \otimes_R N$ is exact.

(1) Let N be a left R-module. Show that the following are equivalent:

(a) N is flat.

(b) $\operatorname{Tor}_i^R(M, N) \cong 0$ for every right R-module M and every $i > 0$.

(c) $\operatorname{Tor}_1^R(M, N) \cong 0$ for every right R-module M.

(2) Let

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of left R-modules and assume that B and C are flat. Show that A is flat.

(3) Let k be a field and let $R = k[x, y]$ denote the polynomial ring. Show that the ideal $I = (x, y)$ is an example of a torsionfree R-module which is not flat.

Problem 5. Let R be a commutative ring. Note that, in this case, there is no difference between left and right R-modules. Further, given two R-modules M and N, we can define a natural R-module structure on $M \otimes_R N$ by letting $r.(m \otimes n) = (rm) \otimes n$. We define the tensor product $X \otimes_R Y$ of chain complexes of R-modules X and Y to be given by

$$(X \otimes_R Y)_n = \bigoplus_{i+j=n} X_i \otimes_R Y_j$$

and differential obtained by linearly extending the formula $d(x_i \otimes y_j) = d_X(x_i) \otimes y_j + (-1)^i x_i \otimes d_Y(y_j)$.

(1) For $x \in R$, we define the complex

$$K(x) = \overset{1}{R} \xrightarrow{x} \overset{0}{R},$$

where 0 and 1 indicate the degrees of the terms. Let X be a complex of R-modules and assume that, for every n, multiplication by x is injective on the homology module $H_n(X)$. Show that, for every n, we have

$$H_n(X \otimes_R K(x)) \cong H_n(X)/xH_n(X).$$

(2) Let R be the polynomial ring $k[x_1, x_2, \ldots, x_n]$ over a field k. Show that the complex

$$K(x_1) \otimes_R K(x_2) \otimes_R \cdots \otimes_R K(x_n)$$

is a free resolution of the residue field $k \cong R/(x_1, x_2, \ldots, x_n)$. Compute the Tor groups $\mathrm{Tor}_*^R(k, k)$.

Problem 6. Let $\mathcal{A}$ be an abelian category, and let $f : X \to Y$ be a map of chain complexes in $\mathcal{A}$. We define the *mapping cone* of f to be the chain complex given by

$$\mathrm{cone}(f)_n = X_{n-1} \oplus Y_n$$

with differential $d = (-d_X, d_Y - f)$. Show that we have natural maps of complexes $Y \to \mathrm{cone}(f)$ and $\mathrm{cone}(f) \to X[-1]$. Here we define $X[k]$, $k \in \mathbb{Z}$, to be chain complex given by

$$X[k]_n = X_{k+n}$$

and differential $(-1)^k d_X$. Show that the resulting diagram

$$\cdots \to X \to Y \to \mathrm{cone}(f) \to X[-1] \to Y[-1] \to \ldots$$

induces a long exact sequence

$$\cdots \to H_n(X) \to H_n(Y) \to H_n(\mathrm{cone}(f)) \to H_{n-1}(X) \to H_{n-1}(Y) \to \ldots$$

in $\mathcal{A}$. Deduce that f is a quasi-isomorphism if and only if $\mathrm{cone}(f)$ is exact.