Homological Algebra - Problem Set 3

Problem 1. Let $\mathcal{A}$ be an abelian category.

(1) Show that, for every object A in $\mathcal{A}$, the functor

$$\mathrm{Hom}_{\mathcal{A}}(A, -) : \mathcal{A} \to \mathbf{Ab}$$

is left exact. Show that it is exact if and only if A is projective.

(2) Show that, for every object B in $\mathcal{A}$, the functor

$$\mathrm{Hom}_{\mathcal{A}}(-, B) : \mathcal{A}^{\mathrm{op}} \to \mathbf{Ab}$$

is left exact. Show that it is exact if and only if B is injective.

Problem 2. Let $\mathcal{A}$ be an abelian category. Let $C_\bullet$ be a chain complex in $\mathcal{A}$ and let $P_\bullet$ be a chain complex of projectives with $P_n = 0$ for $n < 0$. Show that any quasi-isomorphism $f : C_\bullet \to P_\bullet$ has a quasi-inverse, i.e., a morphism $g : P_\bullet \to C_\bullet$ which induces an inverse of f on all homology groups. *Hint:* Show that any morphism from $P_\bullet$ into an exact complex is homotopic to the zero morphism. Apply this to $P_\bullet \to \mathrm{cone}(f)$.

Problem 3. Let $\mathcal{C}$, $\mathcal{D}$ be categories. An *adjunction*

$$F : \mathcal{C} \longleftrightarrow \mathcal{D} : G$$

is a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ equipped with, for every pair of objects A of $\mathcal{C}$ and B of $\mathcal{D}$, a bijection

$$\rho_{A,B} : \mathrm{Hom}_{\mathcal{D}}(F(A), B) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(A, G(B))$$

such that, for every pair of morphisms $f : A' \to A$ and $g : B \to B'$, the diagram

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathcal{D}}(F(A), B) & \xrightarrow{-\circ F(f)} & \mathrm{Hom}_{\mathcal{D}}(F(A'), B) & \xrightarrow{g\circ -} & \mathrm{Hom}_{\mathcal{C}}(F(A'), B') \\
\downarrow{\scriptstyle \rho_{A,B}} & & \downarrow{\scriptstyle \rho_{A',B}} & & \downarrow{\scriptstyle \rho_{A',B'}} \\
\mathrm{Hom}_{\mathcal{D}}(A, G(B)) & \xrightarrow{-\circ f} & \mathrm{Hom}_{\mathcal{D}}(A', G(B)) & \xrightarrow{G(g)\circ -} & \mathrm{Hom}_{\mathcal{C}}(A', G(B'))
\end{array}$$

commutes. We say F is *left adjoint* to G and, vice versa, G is *right adjoint* to F.

(1) Show (or rather convince yourself) that an adjunction can equivalently be described as a pair of functors F,G as above together with a natural isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(F(-), -) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(-, G(-))$$

of functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathbf{Set}$.

(2) Given an adjunction

$$F : \mathcal{A} \longleftrightarrow \mathcal{B} : G$$

between additive categories $\mathcal{A}$ and $\mathcal{B}$ with F and G additive functors, show that all bijections $\rho_{A,B}$ are group isomorphisms. We call such adjunctions *additive*.

(3) Given an additive adjunction

$$F : \mathcal{A} \longleftrightarrow \mathcal{B} : G$$

between abelian categories $\mathcal{A}$ and $\mathcal{B}$ show that F is right exact and G is left exact.

(4) Let

$$F : \mathcal{A} \longleftrightarrow \mathcal{B} : G$$

be an additive adjunction between abelian categories $\mathcal{A}$ and $\mathcal{B}$ with G exact. Show that F preserves projectives. Formulate and prove the dual statement.

(5) Let R be a ring and let N be a left R-module. Show that there is an additive adjunction

$$- \otimes_R N : \mathbf{mod}\text{–}\, R \longleftrightarrow \mathbf{Ab} : \mathrm{Hom}_{\mathbf{Ab}}(N, -).$$

Here, $\mathrm{Hom}_{\mathbf{Ab}}(N, A)$ denotes the right R-module of homomorphisms between the underlying abelian group of N and the abelian group A. The R-linear structure is given by the formula $f.r(n) := f(r.n)$ where $f : N \to A$, $r \in R$ and $n \in N$. Deduce that $- \otimes_R N$ is right exact and, for every injective abelian group I, the right R-module $\mathrm{Hom}_{\mathbf{Ab}}(R, I)$ is injective.

Problem 4. Let k be a field. A *Frobenius algebra* over k is a finite dimensional k-algebra A equipped with a nondegenerate bilinear form $\sigma : A \times A \to k$ satisfying $\sigma(ab, c) = \sigma(a, bc)$.

(1) For $n \geq 1$ let $M_n(k)$ denote the k-algebra of n by n matrices with entries in k. Show that the bilinear form $\sigma(X, Y) = \mathrm{tr}(XY)$ makes $M_n(k)$ a Frobenius algebra.

(2) Let G be a finite group. We denote by kG the group algebra of G, i.e., the vector space

$$kG = \bigoplus_{g \in G} ke_g$$

spanned by the formal symbols $\{e_g\}$ with multiplication given by extending the formula $e_g e_h = e_{gh}$ bilinearly. Given elements x,y of kG we define $\sigma(x, y)$ to be the coefficient of e_{id} in the product xy. Show that kG equipped with this form is a Frobenius algebra.

(3) Let A be a Frobenius algebra. Show that A is injective both as left and right A-module. Show that a finitely generated left (resp. right) A-module is projective if and only if it is injective. (*Hint:* Construct an isomorphism $A \cong \mathrm{Hom}_k(A, k)$ and use ideas from the previous problem.)