Homological Algebra - Problem Set 4

Problem 1. Show directly, without using the bijection with Ext^1, that the extension classes of the abelian group $\mathbb{Z}/(p)$ by itself are represented by the split exact sequence and the short exact sequences of the form

$$0 \to \mathbb{Z}/(p) \xrightarrow{p} \mathbb{Z}/(p^2) \xrightarrow{i} \mathbb{Z}/(p) \to 0$$

where $1 \leq i \leq p-1$.

Problem 2. Let R be a ring and let A, B be left R-modules. An *n-extension* of A by B is an exact sequence

$$\xi : 0 \to B \to X_n \to \cdots \to X_1 \to A \to 0.$$

We say ξ is pre-equivalent to an n-extension

$$\xi' : 0 \to B \to Y_n \to \cdots \to Y_1 \to A \to 0,$$

of A by B, if there exists a commutative diagram

$$\begin{array}{cccccccccccccc} \xi : & 0 & \longrightarrow & B & \longrightarrow & X_n & \longrightarrow & \dots & \longrightarrow & X_1 & \longrightarrow & A & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle \mathrm{id}_B} & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}_A} & & \\ \xi' : & 0 & \longrightarrow & B & \longrightarrow & Y_n & \longrightarrow & \dots & \longrightarrow & Y_1 & \longrightarrow & A & \longrightarrow & 0. \end{array}$$

We define *Yoneda equivalence* to be the smallest equivalence relation on the set of n-extensions of A by B which contains the pre-equivalences and write $\xi \sim \xi'$ if ξ and ξ' are Yoneda equivalent. Show that there is a natural bijection

$$\Theta : \left\{ \begin{array}{c} n\text{-extensions of } A \text{ by } B \\ \text{up to Yoneda equivalence} \end{array} \right\} \xrightarrow{\cong} \mathrm{Ext}^n_R(A, B)$$

generalizing the statement for 1-extensions proved in class. The map Θ, called the Yoneda bijection, is constructed as follows. Given an n-extension ξ as above, we consider ξ as a resolution

$$X_\bullet := (\cdots \to 0 \to B \to X_n \to X_{n-1} \to \cdots \to X_1) \xrightarrow{\simeq} A$$

of A and choose a projective resolution $P_\bullet \xrightarrow{\simeq} A$. The identity map $A \to A$ extends to a map of complexes $f : P_\bullet \to X_\bullet$ which is unique up to homotopy. We apply the functor $\mathrm{Hom}(-, B)$ and pass to nth cohomology to obtain a canonical map $\mathrm{Hom}(B, B) \to \mathrm{Ext}^n_R(A, B)$. Define $\Theta(\xi)$ to be the image of id_B under this map. *Hint:* Immitate the construction given, in the case $n = 1$, in class of an explicit inverse of Θ using the composition "$g \circ \nu$" from Problem 4.

Problem 3. We extend the definition of the Baer sum from class: For $n > 1$, we define the Baer sum of n-extensions ξ and ξ' to be the n-extension

$$0 \to B \to X_n \textstyle\coprod_B Y_n \to X_{n-1} \oplus Y_{n-1} \to \cdots \to X_2 \oplus Y_2 \to X_1 \times_A Y_1 \to A \to 0.$$

where the leftmost and rightmost terms denote a fibered coproduct and fibered product, respectively. Show that this operation provides an abelian group structure on the set of n-extension classes such that the map Θ becomes a group isomorphism.

Problem 4. Let R be a ring. Let A, B and C be R-modules. Given a map $f : A \to B$ and an n-extension

$$\xi : C \to X_n \to \cdots \to X_1 \to B \to 0$$

of B by C, we define an n-extension

$$\xi \circ f : 0 \to C \to X_n \to \cdots \to X_2 \to X_1 \times_B A \to A \to 0.$$

Given a map $g : B \to C$ and an m-extension

$$\nu : B \to Y_m \to \cdots \to Y_1 \to A \to 0,$$

of A by B, we define an m-extension

$$g \circ \nu : 0 \to C \to Y_m \coprod_B C \to Y_{m-1} \cdots \to Y_1 \to A \to 0.$$

Finally, given an n-extension ξ of B by C and an m-extension ν of A by B as above, we define an $n+m$-extension

$$\xi \circ \nu : C \to X_n \to \cdots \to X_1 \to Y_m \to \cdots \to Y_1 \to A \to 0$$

by composing $X_1 \to B$ and $B \to Y_m$.

(1) Use the Yoneda interpretation of Ext^* to show that, for every $n, m \geq 0$, the above constructions provide a well-defined bilinear map

$$\mathrm{Ext}^n_R(A, B) \times \mathrm{Ext}^m_R(B, C) \to \mathrm{Ext}^{n+m}_R(A, C)$$

called *Yoneda composition.*

(2) Let A be an R-module. Show that Yoneda composition provides an associative multiplication law on

$$\mathrm{Ext}^*_R(A, A) := \bigoplus_{n \geq 0} \mathrm{Ext}^n_R(A, A).$$

so that $\mathrm{Ext}^*_R(A, A)$ is canonically a ring.

(3) Consider the $\mathbb{Z}[x]$-module $\mathbb{Z} = \mathbb{Z}[x]/(x)$. Compute the ring $\mathrm{Ext}^*_{\mathbb{Z}[x]}(\mathbb{Z}, \mathbb{Z})$.

(4) Consider the $\mathbb{Z}[x]/(x^2)$-module $\mathbb{Z} = \mathbb{Z}[x]/(x)$. Compute the ring $\mathrm{Ext}^*_{\mathbb{Z}[x]/(x^2)}(\mathbb{Z}, \mathbb{Z})$.