Exercise Sheet 1

Problem 1. Show that the definition of the Galois group of an irreducible polynomial $f \in \mathbb{Q}[X]$ presented in Lecture 1, agrees with the definition of the Galois group as the group of automorphisms of $L/\mathbb{Q}$ where L is a splitting field of f.

Problem 2. Provide a rigorous calculation of the Galois group of the polynomial

$$X^4 - 2 \in \mathbb{Q}[X]$$

defined as the group of automorphisms of a splitting field.

Problem 3. Let K be a field of characteristic 0, $K(t) := \mathrm{Quot}(K[t])$ the *field of rational functions*, and $K((t)) := \mathrm{Quot}(K[[t]])$ the *field of formal Laurent series* in t. Here $K[[t]]$ denotes the ring of formal power series in t. Let

$$\exp(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \in K[[t]] \subset K((t)).$$

Show that there does not exist a polynomial $f(X) \in K(t)[X]$ such that $f(\exp(t)) = 0$. In other words: The exponential function is transcendental over $K(t)$. **Hints:**

1. Show that $\exp(t)$ is a solution of the differential equation $y' = y$.
2. Show that the differential equation $y' = y$ has no nonzero solutions in $K(t)$.
3. Conclude the argument with a proof by contradiction, cancelling the leading term of a potential minimal polynomial of $\exp(t)$ over $K(t)$ with a suitable multiple of its derivative.

Problem 4. Let R be a ring, M a right R-module, N a left R-module, and A an abelian group. A map of sets $\varphi : M \times N \to A$ is called *bilinear* if the following hold

1. for $m_1, m_2 \in M$, $n \in N$, we have $\varphi(m_1 + m_2, n) = \varphi(m_1, n) + \varphi(m_2, n)$,
2. for $m \in M$, $n_1, n_2 \in N$, we have $\varphi(m, n_1 + n_2) = \varphi(m, n_1) + \varphi(m, n_2)$,
3. for $r \in R$, $m \in M$, $n \in N$, we have $\varphi(mr, n) = \varphi(m, rn)$.

(1) Explicitly construct an abelian group $M \otimes_R N$ equipped with a bilinear map $\pi : M \times N \to M \otimes_R N$ which is *universal* in the following sense: For every bilinear map $\varphi : M \times N \to A$ there exists a unique homomorphism $\overline{\varphi} : M \otimes_R N \to A$ of abelian groups such that the diagram

$$\begin{tikzcd} M \times N \arrow[d, "\pi"'] \arrow[dr, "\varphi"] & \\ M \otimes_R N \arrow[r, "\overline{\varphi}"'] & A \end{tikzcd}$$

commutes.

(2) Show that the universal property in (1) uniquely determines the abelian group $M \otimes_R N$ up to isomorphism. We call $M \otimes_R N$ the *tensor product of M and N over R.*

(3) Let m, n be integers. Determine the tensor product $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}$. Compute $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}$.