Exercise Sheet 2

Bonus-Problem 1 (8 points)**.** If you are unfamiliar with the concept of a module over a ring, then read Chapter 12 in Artin's algebra book. Solve 4 problems of your choice.

Problem 2 (4 points)**.** Let F be a ∂-field of characteristic 0, and let $F \subset E$ be an extension of ∂-fields.

1. The set of constants $K_F \subset F$ forms a subfield.
2. Suppose that $x \in E$ is algebraic over K_F. Show that $x \in K_E$.
3. Suppose that $x \in K_E$ is algebraic over F. Show that x algebraic over K_F.

Problem 3 (4 points)**.** Let F be a ∂-field.

1. Let $l \in F[\partial]$ be a ∂-operator, and let $M_l = F[\partial]/F[\partial]l$ be the corresponding $F[\partial]$-module. Determine the system of 1st order differential equations that corresponds to M_l upon choosing an F-basis (there is a "canonical" choice of basis in this case).
2. Let M be an $F[\partial]$-module with $\dim_F(M) = n$, let $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ be F-bases of M. Suppose that
$$\partial(x_1, \dots, x_n)^{\mathrm{tr}} = A(x_1, \dots, x_n)^{\mathrm{tr}}$$
and
$$\partial(y_1, \dots, y_n)^{\mathrm{tr}} = B(y_1, \dots, y_n)^{\mathrm{tr}}$$
with $A, B \in F^{n \times n}$. Show that there exists $C \in \mathrm{GL}(n, F)$ such that
$$A = C^{-1}BC - C^{-1}\partial(C).$$

Problem 4 (4 points)**.** Let F be a ∂-field.

1. Let $A \in F^{n \times n}$ be a matrix and let $Y \in \mathrm{GL}_n(F)$ with $\partial(Y) = AY$. Show that $\partial(\det(Y)) = \mathrm{tr}(A)\det(Y)$.
2. Let
$$l = \partial^n + f_{n-1}\partial^{n-1} + \cdots + f_0 \in F[\partial]$$
be a ∂-operator and suppose that $y_1, \dots, y_n$ is a basis of $\mathrm{Sol}_F(l)$. Show that
$$\partial(\mathrm{wr}(y_1, \dots, y_n)) = -f_{n-1}\,\mathrm{wr}(y_1, \dots, y_n).$$

Problem 5 (*, 4 points)**.** Let F be a ∂-field of characteristic 0, and let M be an $F[\partial]$-module with F-basis $e_0, \dots, e_{n-1}$. Assume that there exists an element $t \in F$ with $\partial(t) = 1$. Show that, for all except finitely many constants $\lambda \in K_F$, the element
$$v = \sum_{j=0}^{n-1} \frac{(t-\lambda)^j}{j!} \sum_{k=0}^{j} (-1)^k \binom{j}{k} \partial^k(e_{j-k})$$
generates M as an $F[\partial]$-module, i.e., the elements $v, \partial(v), \dots, \partial^{n-1}(v)$ form an F-basis of M. Deduce that there exists a differential operator $l \in F[\partial]$ such that M is isomorphic to M_l as an $F[\partial]$-module.

*. Hints will be posted in a separate file. □