# Exercise Sheet 3

**Problem 1.** Read the proof of Hilbert's Nullstellensatz (Chapter 10, 7.6) in Artin's algebra book and explain why the proof works, more generally, for any uncountable algebraically closed field $K$ instead of $\mathbb{C}$.

**Problem 2.** Let $F$ be a $\partial$-field of characteristic 0 with algebraically closed field of constants $K_F$. Let $f \in F$ such that there does not exist $y \in F$ with $\partial(y) = f$. Consider the matrix

$$A = \begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix} \in F^{2\times 2}.$$

Show that the $\partial$-ring $F[T]$ with $\partial(T) = f$ is a Picard-Vessiot ring for $A$.

**Problem 3.** Let $F$ be a $\partial$-field of characteristic 0 with algebraically closed field of constants $K_F$ and let $a \in F^*$. Consider the matrix

$$A = (a) \in F^{1\times 1}.$$

and let

$$R = F[T, T^{-1}]$$

with $\partial(T) = aT$.

1. Suppose that, for every $n \in \mathbb{N} \setminus \{0\}$, there does not exist $y \in F$ with $\partial(y) = nay$. Show that $R$ is a Picard-Vessiot ring for $A$.
2. Suppose that there exists $n \in \mathbb{N} \setminus \{0\}$ and $0 \neq y \in F$ with $\partial(y) = nay$. We assume $n > 0$ to be minimal. Show that

   $$\overline{R} = R/(T^n - y)$$

   is a Picard-Vessiot ring for $A$.
3. Find explicit examples for $F$ and $a$ which realize the above two cases.

**Problem 4.** Let $F$ be a $\partial$-field of characteristic 0 with algebraically closed field of constants $K_F$. Let $E/F$ be a finite Galois extension of $F$ with Galois group $G$. Prove that $E/F$ is a Picard-Vessiot extension as follows:

1. Show that the derivation $\partial$ on $F$ admits a unique extension to a derivation of $E$ such that $F \subset E$ is an embedding of $\partial$-fields.
2. Show that, for every $\sigma \in G$ and every $x \in E$, we have $\sigma(\partial(x)) = \partial(\sigma(x))$. Hint: Consider the map $x \mapsto \sigma^{-1}(\partial(\sigma(x)))$.
3. Argue that we may find $\alpha_1, \ldots, \alpha_r$ such that $E = F(\alpha_1, \ldots, \alpha_r)$ and $G$ permutes the set $\{\alpha_i\}$. Deduce that the $K_F$-linear span $V$ of the set $\{\alpha_i\}$ is $G$-invariant and choose a $K_F$-basis $v_1, \ldots, v_n$ of $V$. Let $W = \mathrm{Wr}(v_1, \ldots, v_n)$ be the Wronskian matrix of $v_1, \ldots, v_n$.
   (a) Show that, for every $\sigma \in G$, there exists $C_\sigma \in \mathrm{GL}(n, K_F)$ such that $\sigma(W) = WC_\sigma$.
   (b) Show that $\mathrm{wr}(v_1, \ldots, v_n) \neq 0$ so that $W$ is invertible.
   (c) Show that the entries of $A = \partial(W)W^{-1}$ are invariant under $G$ and that $W$ is a fundamental solution matrix for $A \in F^{n\times n}$.