Exercise Sheet 5

Problem 1 (8 points)**.** Familiarize yourself with basic categorical concepts such as: categories, functors, natural transformations, fully faithful functors, equivalence of categories, products, coproducts, limits, colimits. To this end, choose a book of your liking (for example, Leinster: Basic category theory, `https://arxiv.org/abs/1612.09375`) and do 3 problems of your choice.

Problem 2. Let K be a field and let $\mathbf{Calg}_K$ denote the category of commutative K-algebras.

1. Show that, for objects A, B of $\mathbf{Calg}_K$, there exists a product $A \times B$ with underlying set given by the Cartesian product of the sets underlying A and B.

2. Show that, for objects A, B of $\mathbf{Calg}_K$, there exists a coproduct $A \amalg B$ with underlying set given by the tensor product $A \otimes_K B$.

Problem 3. Let K be and algebraically closed field.

1. Verify for each of the following examples, that the given subset X is an affine variety, determine generators for the vanishing ideal $I(X)$ and describe the coordinate ring $\mathcal{O}(X)$.

   (a) $X = \{p\} \subset K^n$ for $p \in K^n$.
   (b) $X = \{(t, t^2, t^3) | t \in K\} \subset K^3$.
   (c) $X = \{(t^2, t^3) | t \in K\} \subset K^2$.

2. Show that no two of the affine varieties of Part 1. are isomorphic.

Problem 4. Let $V \subset K^n$ and $W \subset K^m$ be affine K-varieties.

1. Show that the Cartesian product

   $$V \times W \subset K^n \times K^m = K^{n+m}$$

   is an affine K-variety.

2. Show that the Zariski topology on $V \times W$ is, in general, not the product topology of the Zariski topologies on V and W. Hint: Determine the Zariski closed subsets of the affine line $\mathbb{A}^1(K)$.

3. Show that the above product $V \times W$ is a product in the category $\mathbf{Var}_K$ of affine varieties and determine the coordinate ring of $V \times W$, in terms of the coordinate rings of V and W (cf. Problem 2.2).

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