Classical operads

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1. Operads

Operads, and algebras over operads, are good for many things. One thing they are particularly good for is talking about how a binary operation can fail to be associative and/or commutative.

Suppose we have some sort of binary operation μ on some space M:

$$\mu\colon M \times M \to M.$$

We can form a ternary operation in two different ways:

$$\mu(\mu(-,-),-) \qquad \text{and} \qquad \mu(-,\mu(-,-)).$$

If our operation is not associative, then these two ternary operations will not be equal. We may want to study the ways in which such operations can fail to be associative. One way of formalizing this is to define a space (say, a topological space) $\mathcal{P}(3)$ of ternary operations; the operations $\mu(\mu(-,-),-)$ and $\mu(-,\mu(-,-))$ then correspond to points in this space. If these two points are the same, then the operation μ is associative. If they are not, then μ fails to be associative, but may be considered to be weakly associative if, for example, they are connected by a path.

One may also want to study the extent to which an operation is associative at higher arity; one would then define a space of operations $\mathcal{P}(n)$.

The (hand-wavy) idea behind the theory of operads is that they define 'stock' spaces of operations $\mathcal{P}$, and for an object X we can try to interpret $\mathcal{P}(n)$ as a space of n-ary operations on X.

Warning 1. What I'm going to say works in basically any symmmetric monoidal category. However, to save time I'll occasionally state parts of definitions using elements. All statements about elements can be replaced with element-free statements.

Notation 2. For the remainder of this document, $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ will denote a closed symmetric monoidal category with internal hom $\underline{\mathrm{Hom}}(-,-)$.

1.1. Planar operads

Planar operads are good for talking about spaces which are associative.

Definition 3 (planar operad)**.** Let $(\mathcal{C}, \otimes, I)$ be a symmetric monoidal category. An planar operad in $\mathcal{C}$ consists of, for each $n \in \mathbb{N}$, an object $\mathcal{P}(n)$ in $\mathcal{C}$, together with a family of morphisms

$$\gamma\colon \mathcal{P}(n) \otimes \mathcal{P}(k_1) \otimes \cdots \otimes \mathcal{P}(k_n) \longrightarrow \mathcal{P}(k_1 + \cdots + k_n)$$

and a morphism $I \longrightarrow P(1)$, satisfying the following conditions.

- **Associativity:** For any choices of integers

  $$n, \quad j_1, \ldots, j_n, \quad k_1^1, k_1^2, \ldots, k_1^{j_1}, k_2^1, \ldots, k_n^{j_n},$$

  (which correspond respectively to the choice of an n-ary operation, an operation for each input, and operations for each of *their* inputs) with

  $$j = \sum_{r=1}^{n} j_r, \qquad k_r = \sum_{s=1}^{j_r} k_r^s, \qquad \text{and} \qquad k = \sum_{r=1}^{n} \sum_{s=1}^{j_r} k_r^s,$$

  the diagram

  $$\begin{array}{ccc} \mathcal{P}(n) \otimes \left(\bigotimes_r \mathcal{P}(j_r)\right) \otimes \left(\bigotimes_{s,t} \mathcal{P}(k_s^t)\right) & \xrightarrow{\text{shuffle}} & \mathcal{P}(n) \otimes \left(\bigotimes_r \mathcal{P}(j_r) \otimes \left(\bigotimes_s \mathcal{P}(k_r^s)\right)\right) \\ & & \downarrow{\scriptstyle \mathrm{id}\otimes\gamma} \\ \downarrow{\scriptstyle \gamma\otimes\mathrm{id}} & & \mathcal{P}(n) \otimes \left(\bigotimes_r \mathcal{P}(k_r)\right) \\ & & \downarrow{\scriptstyle \gamma} \\ \mathcal{P}(j) \otimes \left(\bigotimes_{s,t} \mathcal{P}(k_s^t)\right) & \xrightarrow{\gamma} & \mathcal{P}(k) \end{array} \tag{1}$$

  commutes.

- **Identity:** The diagrams

$$\begin{array}{ccc} \mathcal{P}(n) \otimes I^n & \xrightarrow{\mathrm{id}\otimes e^n} & \mathcal{P}(n) \otimes \mathcal{P}(1)^{\otimes n} \\ & \searrow^{\rho^{\otimes n}} & \downarrow{\scriptstyle \gamma} \\ & & \mathcal{P}(n) \end{array} \qquad \begin{array}{ccc} I \otimes \mathcal{P}(n) & \xrightarrow{e\otimes\mathrm{id}} & \mathcal{P}(1) \otimes \mathcal{P}(n) \\ & \searrow^{\lambda} & \downarrow{\scriptstyle \gamma} \\ & & \mathcal{P}(n) \end{array} \tag{2}$$

  commute.

For any closed symmetric monoidal category $\mathcal{C}$, the planar operads in $\mathcal{C}$, live in a category $\mathbf{PlanarOp}_{\mathcal{C}}$, whose object are planar operads in $\mathcal{C}$, and whose morphisms are morphisms of operads.

Definition 4 (morphism of planar operads)**.** Let $\mathcal{P}$, $\mathcal{P}'$ be planar operads in a symmetric monoidal category $\mathcal{C}$. A morphism $\alpha\colon \mathcal{P} \longrightarrow \mathcal{P}'$ consists of, for each n, a map $\alpha_n\colon \mathcal{P}(n) \longrightarrow \mathcal{P}'(n)$ such that the following diagrams commute.

- **Compatibility with composition:**

$$\begin{array}{ccc} \mathcal{P}(n) \otimes \mathcal{P}(k_1) \otimes \cdots \otimes \mathcal{P}(k_n) & \xrightarrow{\alpha_n \otimes \alpha_{k_1} \otimes \cdots \otimes \alpha_{k_n}} & \mathcal{P}'(n) \otimes \mathcal{P}'(k_1) \otimes \cdots \otimes \mathcal{P}'(k_n) \\ {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle \gamma'} \\ \mathcal{P}(k) & \xrightarrow{\qquad \alpha_k \qquad} & \mathcal{P}'(k) \end{array}$$

- **Unitality:**

$$\begin{array}{ccc} & I & \\ {\scriptstyle e_{\mathcal{P}}}\swarrow & & \searrow{\scriptstyle e_{\mathcal{P}'}} \\ \mathcal{P}(1) & \xrightarrow[\alpha_1]{} & \mathcal{P}'(1) \end{array}$$

Example 5. The *associative operad* over $\mathcal{C}$, denoted $\mathcal{A}ss_{\mathcal{C}}$, is the operad $\mathcal{A}ss_{\mathcal{C}}(n) = I$ for all n.

In $\mathcal{C} = \mathbf{Set}$, $I = \{*\}$, so the associative operad has exactly one n-ary operation for each arity n. In particular, any operation of arity n can be built by composing operations of arity 2. This is the operation-space for a strictly associative operation.

Example 6 (little intervals operad)**.** The *little intervals operad* is the planar operad $\mathcal{O}$ in **Top** defined by

$$\mathcal{O}(n) = \mathrm{Emb}^+(I_1 \amalg I_2 \amalg \cdots \amalg I_n, I),$$

i.e. the set of orientation-preserving embeddings of n non-overlapping intervals into the interval such that the image of I_1 precedes the image of I_2, etc. This is given the obvious topology; the configuration space of n intervals $2n$ numbers, corresponding to n locations and n lengths; thus, $\mathcal{O}(n)$ carries the subspace topology as a subset of on $\mathbb{R}^n$.

Composition is given by inserting big intervals into smaller intervals.

Given any object X in a closed monoidal category, there is a very natural candidate for the 'space of n-ary operations on X': the space $\underline{\mathrm{Hom}}(X^{\otimes n}, X)$. This means that any object carries a canonical operad.

Definition 7 (endomorphism operad)**.** Let $X \in \mathcal{C}$. The endomorphism operad over X, denoted $\mathcal{E}\mathrm{nd}_X$, is defined by

$$\mathcal{E}\mathrm{nd}_X(n) = \underline{\mathrm{Hom}}_{\mathcal{C}}(X^{\otimes n}, X).$$

The unit is given by the adjunct to the identity under the adjunction

$$\mathrm{Hom}(X, X) \simeq \mathrm{Hom}(I, \underline{\mathrm{Hom}}(X, X)).$$

The composition morphisms

$$\underline{\mathrm{Hom}}(X^{\otimes n}, X) \otimes \bigotimes_s \underline{\mathrm{Hom}}(X^{\otimes k_s}, X) \to \underline{\mathrm{Hom}}(X^{\otimes k}, X)$$

are given by the composition

$$\underline{\mathrm{Hom}}(X^{\otimes n}, X) \otimes \textstyle\bigotimes_s \underline{\mathrm{Hom}}(X^{\otimes k_s}, X) \xrightarrow{\mathrm{id} \otimes \mathrm{tensor}} \underline{\mathrm{Hom}}(X^{\otimes n}, X) \otimes \underline{\mathrm{Hom}}(X^{\otimes k}, X^{\otimes n}) \xrightarrow{\mathrm{compo}\circ\gamma} \underline{\mathrm{Hom}}(X^{\otimes k}, X) \ .$$

Definition 8 (algebra over a planar operad)**.** Let $\mathcal{P}$ be a planar operad in a closed symmetric monoidal category $\mathcal{C}$, and let X in $\mathcal{C}$. A $\mathcal{P}$-algebra over X is any of the following equivalent things.

- An operad morphism $\mathcal{P} \to \mathcal{E}\mathrm{nd}_X$.
- For each n, a morphism $\mathcal{P}(n) \to \underline{\mathrm{Hom}}(X^{\otimes n}, X)$, satisfying compatibility conditions.
- For each n, a morphism $\mathcal{P}(n) \otimes X^{\otimes n} \to X$ satisfying compatibility conditions.

The last item follows from the adjunction defining the internal hom.

Example 9. Let $\mathcal{C} = \mathbf{Vect}_k$, and let V be a k-vector space. The endomorphism operad $\mathcal{E}\mathrm{nd}_V$ consists of, for each $n \geq 0$, the vector space of linear functions from n copies of V to V:

$$\mathcal{E}\mathrm{nd}_V(n) = \underline{\mathrm{Hom}}(V^{\otimes n}, V).$$

Now consider an $\mathcal{A}$ss-algebra over V. This consists of, for each n, a morphism

$$\{*\} \to \mathbf{Vect}_k(V^{\otimes n}, V),$$

i.e. a map $V^{\otimes n} \to V$, related by composition. In particular, there is a map $\mu\colon V \otimes V \to V$, which generates the other maps by

$$\mu(\mu(\mu(\cdots), -), -).$$

This satisfies the relation

$$\mu(\mu(-, -), -) = \mu(-, \mu(-, -)).$$

We also get a distinguished element of $\underline{\mathrm{Hom}}(*, V)$ which functions as the identity.

That is, an $\mathcal{A}$ss-algebra in $\mathbf{Vect}_k$ is simply an associative algebra.

Pulling exactly the same trick in **Set** gives a monoid.

Example 10. Let X be a pointed topological space, and ΩX its loop space. The loop space is naturally a little intervals algebra in the following way.

The elements of ΩX are loops, i.e. maps $I \to X$ starting and ending at the basepoint. For an embedding $\phi\colon I_1 \amalg \cdots \amalg I_n \to I$ as described above, we interpret this as a loop by assigning loops to each of the I_i, and interpreting the whole interval as a big loop.

1.2. Symmetric operads

Just as planar operads are good for talking about how an operation fails to be associative, symmetric operads are good for talking about how an operation fails to commute. However, as we will see, symmetric operads are more general than planar operads in the sense that the theory of symmetric operads contains the theory of planar operads. For this reason, one simply calls symmetric operads 'operads.'

Definition 11 (operad)**.** Let $(\mathcal{C}, \otimes, I)$ be a symmetric monoidal category. An operad $\mathcal{O}$ in $\mathcal{C}$ is a collection $\mathcal{O}(n)$ of objects of $\mathcal{C}$, together with morphisms γ and I as in Definition 3, and an action of S_n on $\mathcal{O}(n)$ which satisfy the associativity (Diagram 1) and identity (Diagram 2) conditions, together with the the following **symmetry** condition:

- For any $\sigma \in S_n$, let $j = \sum_s j_s$ and denote by

$$\sigma(j_1, \ldots, j_n)$$

the element of S_j which permutes n blocks of letters as σ permutes n letters. Then the following diagram commutes

$$\begin{array}{ccc} \mathcal{P}(n) \otimes \left(\bigotimes_s \mathcal{P}(j_s) \right) & \xrightarrow{\sigma \otimes \text{permute}} & \mathcal{P}(n) \otimes \left(\bigotimes_s \mathcal{P}(j_{\sigma(s)}) \right) \\ \downarrow \gamma & & \downarrow \gamma \\ \mathcal{P}(n) & \xrightarrow{\sigma(j_1, \ldots, j_n)} & \mathcal{P}(n) \end{array}$$

Furthermore, with $\tau_s \in S_{j_s}$ for $s = 1, \ldots, n$, and denoting by $\tau_1 \oplus \cdots \oplus \tau_n$ the block sum, the following diagram commutes.

$$\begin{array}{ccc} \mathcal{P}(n) \otimes \left(\bigotimes_s \mathcal{P}(j_s) \right) & \xrightarrow{\text{id} \otimes \tau_1 \otimes \cdots \otimes \tau_n} & \mathcal{P}(n) \otimes \left(\bigotimes_s \mathcal{P}(j_s) \right) \\ \downarrow \gamma & & \downarrow \gamma \\ \mathcal{P}(n) & \xrightarrow{\tau_1 \oplus \cdots \oplus \tau_n} & \mathcal{P}(n) \end{array}$$

Definition 12 (morphism of operad)**.** A morphism of operads is a morphism of the underlying planar operads such that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{O}(n) & \xrightarrow{\alpha} & \mathcal{O}'(n) \\ \downarrow \sigma & & \downarrow \sigma \\ \mathcal{O}(n) & \xrightarrow{\alpha} & \mathcal{O}'(n) \end{array}$$

The definition of the endomorphism operad $\mathcal{E}\text{nd}_X$ survives unchanged to the symmetric case. The structure morphisms permute the factors $X^{\otimes n}$. An algebra over an operad is still a morphism of operads $\mathcal{O} \to \mathcal{E}\text{nd}_X$.

Example 13. Consider the operad $\mathcal{O}$ with $\mathcal{O}(n) = I$ for all n. This has one operation of each arity, but now with the added condition that it does not matter which order we plug in our operations. We call this operad $\mathcal{C}\text{omm}$.

Consider an $\mathcal{O}$-operad over some object X. This will give us, for each n, a map $I \otimes X^{\otimes n} \to X$, but now the condition that the diagram

$$\begin{array}{ccc} I \otimes X^{\otimes n} & \longrightarrow & X \\ {\scriptstyle \sigma_{\text{triv}} \otimes \sigma} \downarrow & & \downarrow {\scriptstyle \text{id}} \\ I \otimes X^{\otimes n} & \longrightarrow & X \end{array}$$

commutes tells us that we get the same result no matter how we permute the arguments of our operation. That is, the algebras over $\mathcal{C}\text{omm}$ are commutative in the standard sense.

One tends not to see planar algebras used very often, except as an introductory tool. The reason for this is that one can create from a planar operad an ordinary operad which behaves equivalently by adding an S_n's worth of extraneous copies of each n-ary operation, and letting the symmetric group act by shuffling them around. That is, we allow an action that takes $\mu(a, b) \mapsto \mu(b, a)$, but view them as different operations.

Example 14. The operad $\mathcal{A}\text{ss}$ is the defined by

$$\mathcal{A}\text{ss}(n) = S_n,$$

with the standard right action.

Because $\{*\}$ is terminal in **Set**, there is a canonical operad morphism $\mathcal{A}\text{ss} \to \mathcal{C}\text{omm}$. This is a reflection of the fact that any commutative monoid, i.e. morphism

$$\mathcal{C}\text{omm} \to \mathcal{E}\text{nd}_X,$$

is also an associative monoid via the pullback

$$\mathcal{C}\text{omm} \to \mathcal{E}\text{nd}_X \quad \mapsto \quad \mathcal{A}\text{ss} \to \mathcal{C}\text{omm} \to \mathcal{E}\text{nd}.$$

The fact that there is no operad morphism $\mathcal{A}\text{ss} \to \mathcal{C}\text{omm}$ is a reflection of the fact that not every associative monoid is commutative.

2. The A_∞ operad

Recall from Toby's lecture that we were interested in configurations of particles on some space, say a manifold M. Let us take the simplest case, in which $M = \mathbb{R}$.

The idea is to construct a topological operad whose n-ary operations correspond, roughly, to the configuration space of n distinct points in $\mathbb{R}$, modulo symmetries (i.e. translation, positive re-scaling), which are allowed to collide.

Unfortunately, it is not immediately clear how to define any sort of operadic composition.

We will get the configurations corresponding to the collisions as limit points of configurations with no collisions. Therefore, we for the moment confine our attention to non-collided states.

The action of S_n will permute the points around. We will factor this out now by choosing to look at embeddings

$$f \in \mathrm{Emb}(\{1, 2, \ldots, n\}, \mathbb{R})$$

such that $f(1) < f(2) < \ldots < f(n)$. This means we are now looking for a planar operad.

We will denote the space of embeddings $\mathrm{Emb}(\{1, 2, \ldots, n\}, \mathbb{R})$ by $C_n(\mathbb{R})$. We will denote the space $C_n(\mathbb{R})$ modulo translation and rescaling by $\widetilde{C_n(\mathbb{R})}$.

We can coordinatize $\widetilde{C_n(\mathbb{R})}$ as follows. By translating and re-scaling, fix the first and last point to lie at 0 and 1 respectively. The other points are free to move without overlapping; this amounts to an embedding of $\widetilde{C_n(\mathbb{R})}$ into the interior of the 2-simplex. Taking the closure in this situation gives the closed simplex whose boundary gives configurations of collisions. The vertices of the simplex give the points at which all three points have collided.

We would like to define an operad structure similar to that on the little-intervals operad. However, we don't have a way of plugging in a configuration of n particles into a single particle; we need to dig deeper.

This is not the only way we can take limiting points, and finding an ambient topology in a smarter way will allow us to hold on to more information. We can define functions

$$r_{ijk} \colon C_n(\mathbb{R}) \longrightarrow [0, \infty), \qquad i < j < k,$$

which takes a configuration with the particles at $(x_1, \ldots, x_n)$ to the relative distance

$$\frac{|x_2 - x_1|}{|x_3 - x_1|}.$$

These maps respect translation and scaling, so they descend to $\widetilde{C_n(\mathbb{R})}$.

These maps provide coordinates on $\widetilde{C_n(\mathbb{R})}$. They are redundant because $\binom{n}{3}$ is greater than $n - 2$, but in the limit they allow us to keep track of more information.

For example, consider $n = 4$. Using our freedom to fix x_1 at 0 and x_4 at 1, we see that r_{124} and r_{134} reproduce the coordinates of x_2 and x_3 respectively. The functions r_{123} tells us about how far along particle 2 is between particles 1 and 3, and r_{234} tells us how far 3 is between 2 and 4.

When we take limit points, this extraneous data gives us new information. Consider a situation in which particle 3 is twice as far from particle 1 as particle 2, and in which particles 2 and 3 are moving towards particle 1, with particle 3 moving twice as fast as particle 1. In this situation, we have that $r_{123} = 2$. If instead particle 3 is three times away and moving three times as fast, then $r_{123} = 3$. That is, in situations in which three particles collide, this extra data keeps track of the relative velocities.

This means in taking the closure, these collisions are not represented by a point, but instead by moduli spaces.

We can draw trees to keep track of which part of the closed moduli space we are in. When we draw things in this form, the operadic composition law is clear: we plug into each particle the moduli space for another system of particles.

We then define the A_∞ operad (in **Top**) as follows:

$$A_\infty(n) = \{\text{closure of image of } \widetilde{C_n(\mathbb{R})}\}.$$

A. Symmetric monoidal categories

A.1. The definition of a symmetric monoidal category

Definition 15 (monoid)**.** A monoid is a triple $(M, \cdot, e)$, where M is a set, $e \in M$ and '$\cdot$' is a function $\cdot\colon M \times M \to M$, which satisfies the following conditions for all $m, n, p \in M$.

- $e \cdot m = e$
- $m \cdot e = e$
- $(m \cdot n) \cdot p = m \cdot (n \cdot p)$

A monoid M is called commutative if it satisfies $m \cdot n = n \cdot m$ for all $m, n \in M$.

(Symmetric) monoidal categories are a categorification of (commutative) monoids. Roughly speaking, we have make the following replacements.

- Sets $\to$ categories
- Functions $\to$ functors
- Elements $\to$ objects
- Equality $\to$ isomorphism

The last of these is actually fairly subtle.

Definition 16 (symmetric monoidal category)**.** A monoidal category is a tuple $(\mathcal{C}, \otimes, I, \lambda, \rho, \alpha)$, where

- $\mathcal{C}$ is a category
- $\otimes$ is a functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$
- I is an object of $\mathcal{C}$
- λ is a natural isomorphism with components

  $$\lambda_A\colon I \otimes A \to A,$$

  called the *left unitor*
- ρ is a natural isomorphism with components

  $$\rho_A\colon A \otimes I \to A,$$

  called the *right unitor*
- α is a natural isomorphism with components

  $$\alpha_{A,B,C}\colon (A \otimes B) \otimes C \to A \otimes (B \otimes C),$$

  called the *associator.*

The natural isomorphisms λ, ρ, and α satisfy coherence conditions, which we'll talk about later.

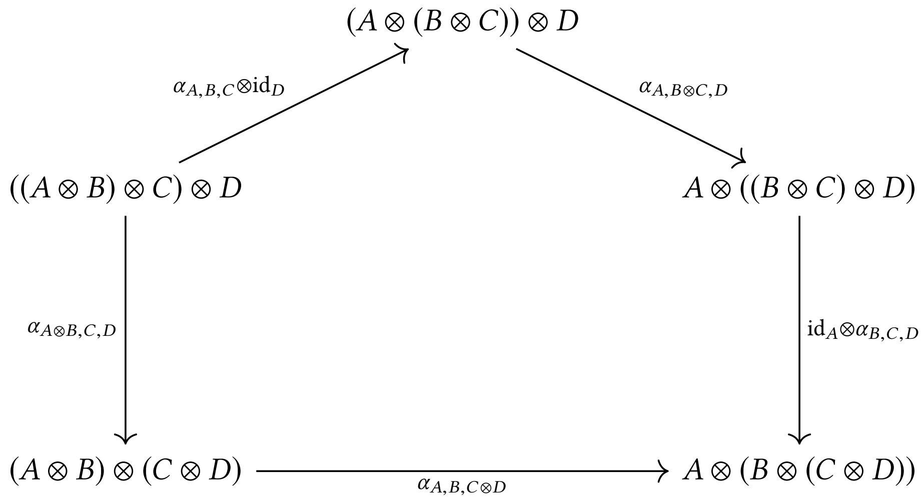
A monoidal category is called *symmetric* if it is equipped with another natural isomorphism τ, called the *braiding,* with components

$$\tau_{A,B}\colon A \otimes B \to B \otimes A,$$

such that $\tau_{B,A} \circ \tau_{A,B} = \mathrm{id}$.

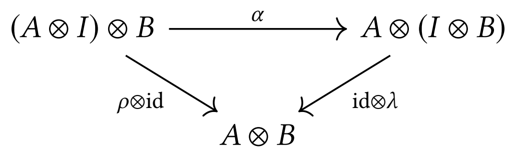
This also has to satisfy coherence conditions.

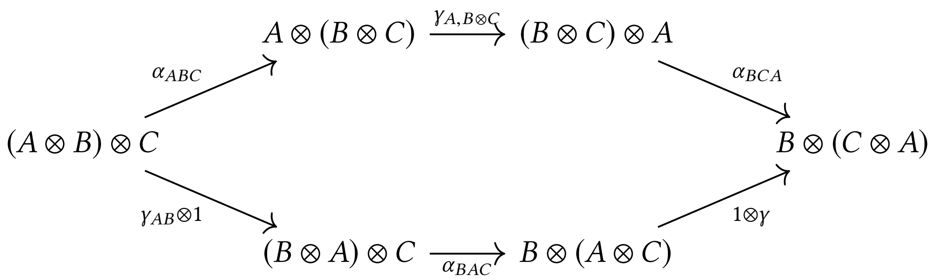
What are these coherence conditions? Consider the following diagram.

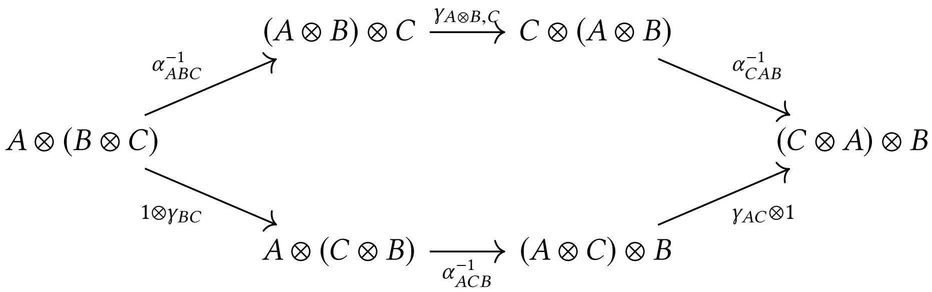


There is no guarantee that these two ways of going from top left to bottom right agree; we have to make this an axiom.

Similarly, we need to assert 'by hand' that the following diagrams commute.







It turns out that these axioms are enough to guarantee that, given two bracketings of an n-fold tensor product, any two ways of formally[1] composing associators, left and right unitors, and the braiding to go between them agree. This is known as *Mac Lane's coherence theorem.*

[1] The word *formally* means that there may be diagrams which fail to commute because some formally different vertices accidently coincide, causing a diagram to fail to commute. The diagrams which have to commute are the 'formal' ones.

Example 17. The following are all examples of symmetric monoidal categories

- **Set**, with $\otimes = \times$, or $\otimes = \amalg$
- **Top** with $\times$, $\amalg$
- $\mathbf{Top}_*$, the category of pointed topological spaces, with $\wedge$
- R-**Mod** with $\otimes$
- $\mathrm{Ch}(R)$ with $\otimes$

And many others.

A.2. Closed monoidal categories

In many monoidal categories, it is possible to add extra structure to hom-sets to interpret them as categorical objects rather than sets. For example, in $\mathbf{Vect}_k$, the hom-set $\mathrm{Hom}(V, W)$ has the structure of a vector space.

More generally, we may want to formally replace hom-sets $\mathrm{Hom}_{\mathcal{C}}(A, B)$ with appropriate objects from the category $\mathcal{C}$.

Definition 18 (internal hom)**.** Let $\mathcal{C}$ be a symmetric monoidal category. An internal hom for $\mathcal{C}$ is a functor $\underline{\mathrm{Hom}} \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ such that for every $c \in \mathcal{C}$, there is an adjunction

$$\mathcal{C} : - \otimes c \leftrightarrow \underline{\mathrm{Hom}}(c, -),$$

called the *tensor-hom adjunction.*

The above condition equivalently says that there is a natural bijection

$$\mathrm{Hom}(b \otimes c, d) \simeq \mathrm{Hom}(b, \underline{\mathrm{Hom}}(c, d)).$$

Definition 19 (closed monoidal category)**.** A monoidal category which admits an internal hom functor is known as a closed monoidal category.

Example 20. The following are closed monoidal categories.

- **Set**, with $\underline{\mathrm{Hom}}(A, B) = \mathrm{Hom}(A, B)$.
- $\mathbf{Vect}_k$, with $\underline{\mathrm{Hom}}(V, W) = \mathrm{Hom}(V, W)$ carrying the natural vector space structure.
- **CGHaus**, with $\underline{\mathrm{Hom}}(X, Y) = \mathrm{Hom}(X, Y)$ carrying the compact-open topology.
- $\mathbf{Set}_\Delta$, with $\underline{\mathrm{Hom}}(X, K) = \mathrm{Maps}(X, K)$.

Many of the familiar properties of the hom functor translate to the internal hom.

Definition 21 (evaluation map)**.** For objects A, B, the evaluation map

$$\mathrm{ev}_{A,B} \colon A \times \underline{\mathrm{Hom}}(A, B) \to B$$

is given by the adjunct to the identity in the natural bijection

$$\mathrm{Hom}(\underline{\mathrm{Hom}}(A, B), \underline{\mathrm{Hom}}(A, B)) \dashv \mathrm{Hom}(A \times \underline{\mathrm{Hom}}(A, B), B).$$

Definition 22 (composition map)**.** The composition map

$$\underline{\mathrm{Hom}}(A,B)\otimes\underline{\mathrm{Hom}}(B,C)\to\underline{\mathrm{Hom}}(A,C)$$

is the tensor-hom adjunct to the composition

$$A\otimes\underline{\mathrm{Hom}}(A,B)\otimes\underline{\mathrm{Hom}}(B,C)\xrightarrow{\mathrm{ev}} B\otimes\underline{\mathrm{Hom}}(B,C)\xrightarrow{\mathrm{ev}} C\ .$$

Definition 23 (tensor map)**.** The tensor map

$$\underline{\mathrm{Hom}}(A,B)\otimes\underline{\mathrm{Hom}}(C,D)\to\underline{\mathrm{Hom}}(A\otimes C,B\otimes D)$$

is the adjunct to composition

$$A\otimes C\otimes\underline{\mathrm{Hom}}(A,B)\otimes\underline{\mathrm{Hom}}(C,D)\xrightarrow{\gamma} A\otimes\underline{\mathrm{Hom}}(A,B)\otimes C\otimes\underline{\mathrm{Hom}}(C,D)\xrightarrow{\mathrm{ev}\otimes\mathrm{ev}} B\otimes C\ .$$

A.3. Colored operads

There is a straightforward generalization, the definition of which is unfortunately combinatorially tedious. We do not write out the commutative diagrams here explicitly.

Definition 24 (colored operad)**.** Let $(\mathcal{C},\otimes,I)$ be a monoidal category. A colored operad in $\mathcal{C}$ consists of a set of colors $\mathcal{O}=\{X,Y,Z,\ldots\}$, together with,

- For each finite set I and set of I-indexed colors $\{X_i\}_{i\in I}$, an object $\mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i\in I})\in\mathcal{C}$, which we interpret as an 'object of operations' from the X_i to Y
- For each color X a morphism $\mathrm{id}_X\colon I\to\mathrm{Mul}_{\mathcal{O}}(\{X\},X)$
- For each $(n+1)$-tuple $(Z,Y_1,\ldots,Y_n)$ and n other tuples $(X_1^1,\ldots,X_1^{k_1}),\ldots,(X_n^1,\ldots,X_n^{k_n})$, a morphism

$$\gamma\colon\ \mathrm{Mul}_{\mathcal{O}}(\{Y_i\}_{i=1}^n,Z)\otimes\bigotimes_{i=1}^n\mathrm{Mul}_{\mathcal{O}}(\{X_i^j\}_{j=1}^{k_i},Y_i)\to\mathrm{Mul}_{\mathcal{O}}(\{X_i^j\},Z)$$

- For each n-tuple (X_i), and each Y, a right action of S_n

$$\mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i=1}^n,Y)\to\mathrm{Mul}_{\mathcal{O}}(\{X_{\sigma(i)}\}_{i=1}^n,Y)$$

This data is required to be compatible in the sense that composition should be associative, unital, and S_n-equivariant.

Example 25. A colored operad with a single color is a (symmetric) operad.

Example 26. Let $\mathcal{C}$ be any category. We can define a corresponding colored **Set**-operad via

$$\mathrm{Mul}_{\mathcal{C}}(\{c\},d)=\mathcal{C}(c,d),\qquad c,d\in\mathcal{C},$$

and $\mathrm{Mul}_{\mathcal{C}}(\{c_i\}_{i\in I},d)=\emptyset$ for $|I|\neq 1$. This is known as the *free operad over* $\mathcal{C}$.

Conversely, given any colored operad, we can define a category by forgetting operations of all arities except 1. Both of these operations extend to functors, giving us an adjunction

$$\mathrm{Free}:\mathbf{Cat}\leftrightarrow\mathbf{ColoredOperad}_{\mathbf{Set}}:\mathrm{Forget}.$$

Example 27. Let A be an algebra, and M an A-module. We can define a two-colored operad whose colors are A and M, and whose operations of arity n are operations

$$\overbrace{A \times A \times \cdots \times A}^{n-1 \text{ times}} \times M \to M.$$

Example 28. Let $\mathcal{C}$ be a symmetric monoidal category. We can define from this a colored operad, whose colors are the objects of $\mathcal{C}$ and whose 'multilinear' maps are

$$\mathrm{Mul}_{\mathcal{C}}(\{c_i\}_{i \in I}, d) = \mathcal{C}(c_1 \otimes \cdots \otimes c_n, d).$$

In fact, this process is lossless; from the data of the $\mathrm{Mul}_{\mathcal{C}}$, one can recover, up to natural isomorphism, the functor $\otimes$ via the Yoneda lemma.

More precisely, we have for each pair c, c' of objects, a set

$$\mathcal{C}(c \otimes c', d).$$

This extends to a functor $\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$. The object $c \otimes c'$ is the object representing this functor.

A.4. A more symmetric definition of colored operads

The definition of a colored operad is lopsided in the sense that we consider morphisms from many objects to one object. There is an alternative definition of colored operads which is in many ways cleaner.

Definition 29 (category of finite pointed sets)**.** Consider the category $\mathcal{F}\mathrm{in}_*$, whose objects are

$$\langle n \rangle = \{*, 1, \ldots, n\}, \qquad n \geq 0$$

and whose morphisms $\langle m \rangle \to \langle n \rangle$ are maps of the underlying sets sending $* \mapsto *$.

Let $\mathcal{O}$ be a colored operad. We can define a category $\mathcal{O}^{\otimes}$ as follows.

- The objects of $\mathcal{O}$ are finite sequences

$$X_1, \ldots, X_n, \qquad \text{written} \quad \{X_i\}_{i=1}^n.$$

- The morphisms $\{X_i\}_{i=1}^m \to \{Y_j\}_{j=1}^n$ consist of a map

$$f \colon \langle m \rangle \to \langle n \rangle$$

  in $\mathcal{F}\mathrm{in}_*$ and a collection

$$\{\phi_j \in \mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i \in f^{-1}(j)}, Y_j)\}_{1 \leq j \leq n}$$

B. Operads as presheaves on the category of permutations

There is another way of looking at operads, as a categorification of formal power series. In this section we investivate this point of view.

Definition 30 (permutation category)**.** Denote by $\mathbb{P}$ the category whose objects are finite cardinals

$$\mathrm{ob}(\mathbb{P}) = \{\emptyset, \{1\}, \{1,2\}, \ldots\},$$

and whose morphisms are bijections. Denote $\{1, \ldots, n\}$ by $\langle n \rangle$.

Obviously, we can also write

$$\mathbb{P} = \coprod_{n \geq 0} \mathbf{B}S_n,$$

where S_n is the nth symmetric group.

The category $\mathbb{P}$ has a strict symmetric monoidal structure with bifunctor given by the cardinal sum: $\langle m \rangle \oplus \langle n \rangle = \langle m + n \rangle$, and unit object given by the empty set.

Proposition 31. The category $\mathbb{P}$ is the free strict symmetric monoidal category on one generator. That is, for any other strict symmetric monoidal category $(\mathcal{C}, \otimes, I)$ and functor $\{*\} \to \mathcal{C}$, there is a unique strict braided monoidal functor $\mathbb{P} \to \mathcal{C}$ making the following diagram commute.

$$\begin{array}{ccc} \{*\} & \xrightarrow{\{1\}} & \mathbb{P} \\ & {\scriptstyle g} \searrow & \big\downarrow {\scriptstyle \exists ! \hat{g}} \\ & & \mathcal{C} \end{array}$$

Proof. This is easy to see; we have to send $\langle n \rangle \mapsto g^{\otimes n}$. □

Consider the category $\mathbf{Set}_{\mathbb{P}}$ of functors $\mathbb{P}^{\mathrm{op}} \to \mathbf{Set}$. For any two such functors X and Y, we can define a functor

$$X \bar{\otimes} Y \colon \mathbb{P}^{\mathrm{op}} \times \mathbb{P}^{\mathrm{op}} \to \mathbf{Set}; \qquad (\langle m \rangle, \langle n \rangle) \mapsto X(\langle m \rangle) \times Y(\langle n \rangle).$$

This, together with the ordinal sum $\oplus$ on $\mathbb{P}$, allows us to turn $\mathbf{Set}_{\mathbb{P}}$ into a symmetric monoidal category.

Definition 32 (Day convolution)**.** Let $X, Y \in \mathbf{Set}_{\mathbb{P}}$. The Day convolution of X and Y, denoted $X \otimes_{\mathrm{Day}} Y$, is the following left Kan extension.

$$\begin{array}{ccc} \mathbb{P}^{\mathrm{op}} \times \mathbb{P}^{\mathrm{op}} & \xrightarrow{\quad X \bar{\otimes} Y \quad} & \mathbf{Set} \\ & {\scriptstyle \oplus^{\mathrm{op}}} \searrow \quad \nearrow {\scriptstyle X \otimes_{\mathrm{Day}} Y} & \\ & \mathbb{P}^{\mathrm{op}} & \end{array}$$

By the colimit formula for left Kan extensions, we see that we can write

$$X \otimes_{\mathrm{Day}} Y = \coprod_{\langle n \rangle = \langle r \rangle \oplus \langle s \rangle} X(\langle r \rangle) \times Y(\langle s \rangle).$$

The Day convolution makes $\mathbf{Set}_{\mathbb{P}}$ into a symmetric, monoidal category. The unit object is $よ(1)$. The associators are inherited from those in set under the observation that

$$\begin{aligned} (X \otimes_{\mathrm{Day}} Y) \otimes_{\mathrm{Day}} Z &\simeq \coprod_{\langle n \rangle = \langle q \rangle \oplus \langle r \rangle} \left(\coprod_{\langle q \rangle = \langle s \rangle \oplus \langle t \rangle} X_s \times Y_t \right) \times Z_r \\ &\simeq \coprod_{\langle n \rangle = \langle r \rangle \oplus \langle s \rangle \oplus \langle t \rangle} X_r \times Y_s \times Z_t, \end{aligned}$$

and similarly for the other bracketing.

Note 33. One interprets elements of $\mathbf{Set}_{\mathbb{P}}$ as a categorifiation of formal power series. The Day convolution is then a clear categorifiation of the product of formal power series.

Lemma 34. We have the following bijection.

$$\mathbf{Set}_{\mathbb{P}}(X \otimes_{\mathrm{Day}} Y, Z) \simeq \mathbf{Set}_{\mathbb{P}\times\mathbb{P}}(X \bar{\otimes} Y, Z \circ \oplus)$$

We have the following theorem, which is exactly analogous to the creation of nerves from cosimplicial objects.

Theorem 35. For any symmetric monoidally cocomplete category[2] $\mathcal{D}$, and any symmetric monoidal functor $F \colon \mathbb{P} \longrightarrow \mathcal{D}$ there is, up to unique isomorphism, a symmetric monoidal cocontinuous functor $\hat{F} \colon \mathbf{Set}_{\mathbb{P}} \longrightarrow \mathcal{D}$.

Proof. The functor $\hat{F}$ is given by the Yoneda extension $\mathcal{Y}_! F$. This is left adjoint to the functor

$$\mathcal{D} \longrightarrow \mathbf{Set}_{\mathbb{P}}; \qquad d \mapsto \mathcal{D}(F(-), d).$$

□

Corollary 36. The category $\mathbf{Set}_{\mathbb{P}}$ is the free cocomplete symmetric monoidal category on a single generator.

Proof. We need to show that $\mathbf{Set}_{\mathbb{P}}$ satisfies a universal property given by the outer triangle below.

$$\begin{array}{ccccc} \{*\} & \xrightarrow{\{1\}} & \mathbb{P} & \xrightarrow{\mathcal{Y}} & \mathbf{Set}_{\mathbb{P}} \\ & \searrow^{d} & & \searrow^{\exists!\delta} & \downarrow{\scriptstyle \exists!\hat{\delta}} \\ & & & & \mathcal{D} \end{array}$$

□

Proposition 37. The object $\hat{\delta}(X)$ can be computed using the formula

$$\coprod_{k=0}^{\infty} X(k) \otimes_{S_k} d^{\otimes_{\mathrm{Day}} k},$$

where the notion $\otimes_{S_k}$ means that the two factors are compatible with the action of S_k.

Proof. Using the colimit formula for Kan extensions, we can compute $\hat{\delta}(X)$ for any presheaf X; we have

$$\hat{\delta}(X) = \operatorname{colim}\left[(\mathcal{Y} \downarrow X) \to \mathbb{P} \xrightarrow{\delta} \mathcal{D}\right].$$

In $\mathcal{D}$, we can express this in terms of a coequalizer:

$$\coprod_{k=0}^{\infty} \left(\coprod_{\substack{x \in X_k \\ \sigma \in S_k}} d^{\otimes_{\mathrm{Day}} k} \right) \underset{S_{k,x,\sigma}}{\overset{R_{k,x,\sigma}}{\rightrightarrows}} \coprod_{k=0}^{\infty} \coprod_{x \in X(k)} d^{\otimes_{\mathrm{Day}} k} \longrightarrow \hat{\delta}(x)$$

[2]I.e. a cocomplete symmetric monoidal category whose tensor product commutes with small colimits in both slots.

where

$$R_{x,k,\sigma} = \iota_{x,d^{\otimes_{\mathrm{Day}} k}}, \qquad S_{x,k,\sigma} = \iota_{\sigma(x),d^{\otimes_{\mathrm{Day}} k}} \circ F(\sigma).$$

That is, the map R does nothing and the map S acts on each factor of $d^{\otimes_{\mathrm{Day}} k}$ with $F(\sigma)$, and shuffles them around with $X(\sigma)$. The coequalizer says that these are the same; that is, we can write the coequalizer as

$$\coprod_{k=0}^{\infty} X(k) \otimes_{S_k} d^{\otimes_{\mathrm{Day}} k},$$

where the notation $\otimes_{S_k}$ means "mod out by the difference of the action of S_k on the first and the second factor in the product." □

Any functor $\mathbf{Set}_{\mathbb{P}} \to \mathcal{D}$ gives us an element of $\mathcal{D}$ by restricting to the tensor generator. Another way of looking at Corollary 36 is this: it tells us that this correspondence is actually a bijection; any $d \in \mathcal{D}$ gives us a functor $\hat{\delta} \colon \mathbf{Set}_{\mathbb{P}} \to \mathcal{D}$ by the formula

$$X \mapsto \coprod_{k=0}^{\infty} X(k) \otimes_{S_k} d^{\otimes_{\mathrm{Day}} k}.$$

In symbols, denoting by $\underline{\mathrm{Hom}}(\mathcal{C}, \mathcal{D})$ the category of symmetric monoidal cocontinuous functors $\mathcal{C} \to \mathcal{D}$, we have an equivalence of categories

$$\underline{\mathrm{Hom}}(\mathbf{Set}_{\mathbb{P}}, \mathcal{D}) \simeq \mathcal{D}.$$

In particular, this means that we have an equivalence of categories

$$\underline{\mathrm{Hom}}(\mathbf{Set}_{\mathbb{P}}, \mathbf{Set}_{\mathbb{P}}) \simeq \mathbf{Set}_{\mathbb{P}},$$

where to go from the right to the left uses the formula

$$K \mapsto \coprod_{k=0}^{\infty} X(k) \otimes_{S_k} K^{\otimes_{\mathrm{Day}} k}.$$

Definition 38 (substitution product)**.** Let $S, K \in \mathbf{Set}_{\mathbb{P}}$. Define the <u>substitution product</u> of S and K, denoted $S \circ K$ by carrying across composition of functors on $\underline{\mathrm{Hom}}(\mathbf{Set}_{\mathbb{P}}, \mathbf{Set}_{\mathbb{P}})$.

Note 39. We mentioned before that the Day convolution was to be thought of as a categorification of the product of formal power series. The substitution product, then, should be thought of as the composition of formal power series.

Theorem 40. The substitution product is given by the formula

$$S \circ K = \coprod_{k=0}^{\infty} S(k) \otimes_{S_k} K^{\otimes_{\mathrm{Day}} k}.$$

Theorem 41. The category $\mathbf{Set}_{\mathbb{P}}$ has a monoidal structure $(\circ, I, \alpha, \rho, \lambda)$, where

Definition 42 (Set-operad)**.** A <u>Set-operad</u> is a monoid internal to $(\mathbf{Set}_{\mathbb{P}}, \circ, I)$.

Let (M, γ, e) be a set-operad. Let us work out in some detail what data this gives us. From our multiplication map

$$\gamma \colon M \circ M \Rightarrow M,$$

we get the following string of equivalent things.

- A natural transformation $\gamma\colon M \circ M \Rightarrow M$
- For each $n \geq 0$, a morphism

$$\gamma_n\colon \coprod_{k=0}^{\infty} M(k) \otimes_{S_k} M^{\otimes_{\mathrm{Day}} k}(n) \to M(n),$$

  or expanding the Day convolution, a morphism

$$\gamma_n\colon \coprod_{k=0}^{\infty} M(k) \otimes_{S_k} \left(\coprod_{\langle n \rangle = \langle m_1 \rangle \oplus \cdots \oplus \langle m_k \rangle} M(m_1) \times \cdots \times M(m_k) \right) \to M(n)$$

- Pulling the coproduct out of the $\otimes_{S_k}$ using the fact that the Cartesian product in **Set** preserves coproducts in both slots, a map

$$\gamma_n\colon \coprod_{k=0}^{\infty} \coprod_{\text{partitions } (m_1, \ldots, m_k) \text{ of } n} M(k) \times M(m_1) \times \cdots \times M(m_k) \to M(n)$$

  compatible with the action of the symmetric group
- For each n, k, and each partition $n = \sum_{i=1}^{k} m_i$, a map

$$\gamma_{k, m_1, \ldots, m_k}\colon M(k) \times M(m_1) \times \cdots \times M(m_k) \to M(n).$$

  satisfying compatibility conditions with the action of the symmetric group. Specifically, we demand that for each $\sigma \in S_k$, the above function coequalizes the following maps.

$$F(k) \bigtimes_{i=1}^{k} M(m_i) \underset{\mathrm{id} \times \sigma}{\overset{\sigma \times \mathrm{id}}{\rightrightarrows}} F(k) \bigtimes_{i=1}^{k} M(m_i) \longrightarrow M(n)$$

Similarly, the unit $e\colon I \to M$ gives us a map $\{*\} \to F(1)$, i.e. an element of $F(1)$.