Higher Topoi and Hypercoverings

Severin Bunk

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Abstract

The goal of this talk is twofold: first, we aim to give an introduction to ∞-topoi and some of their internal homotopy theory. There are several ways of approaching this topic, and in this talk we emphasise the sheaf-theoretic perspective on ∞-topoi. The second goal is to understand the notion of a hypercovering in an ∞-topos. We treat these as an alternative way of characterising hypercompleteness. We end by briefly considering applications to formal moduli problems. These extended notes contain a little more material than strictly necessary (and than covered in the talk), which hopefully makes them more easy to use.

Contents

1 Introduction

Topoi form an interesting class of categories, which arise (at least) in geometry, topology, and logic. Our first goal in this talk is to get an idea of what an (∞-)topos is. Topoi can be defined as follows:

Definition 1.1 *A* (Grothendieck) topos *is a category which is equivalent to the category of sheaves on a Grothendieck site.*

Roughly speaking, a Grothendieck site generalises the notion of open covering of a topological space to general categories. Definition 1.1 provides a very compact way of saying what a topos is; however, if we are given a category and have to use Definition 1.1, this would be an extremely hard task. Luckily, (though with a lot of work) topoi can be characterised in alternative ways (see e.g. [MLM94]):

Theorem 1.2 *Let $\mathcal{C}$ be a category. The following are equivalent:*

(1) $\mathcal{C}$ is a topos.

(2) $\mathcal{C}$ is a left-exact localisation of a presheaf category: there exists a small category $\mathcal{D}$ such that $\mathcal{C}$ is a left-exact localisation of $\mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$.

(3) $\mathcal{C}$ satisfies Giraud's axioms.

Whatever 'Giraud's axioms' are (we will see their ∞-categorical version below), they provide an intrinsic characterisation of topoi. In Sections 2, 3, and 4 we will understand the characterisations (2) and (3) from Theorem 1.2 in the ∞-categorical setting. Interestingly, the slogan that 'every statement from category theory carries over to ∞-categories' *fails* when we ask about characterisation (1): *not every ∞-topos is equivalent to an ∞-category of sheaves on a Grothendieck ∞-site.* Note that this is truly an ∞-categorical phenomenon, in that an analogue of Theorem 1.2 still holds true if one works with n-categories for any *finite* n [Lur09, Sec. 6.4].

Our second goal in this talk is to familiarise ourselves with the notion of a *hypercovering.* Roughly, this is a refined notion of covering in an ∞-topos, which subsumes that of Čech coverings. Hypercoverings will be introduced in Sections 5 and 6.

Finally, in Section 7 we make the connection to the main line of our seminar and consider formal moduli problems from the point of view of ∞-topoi. Since we have this application in mind, in our examples we will mainly focus on presheaf ∞-topoi.

For the reader's convenience, we have included precise references wherever possible.

2 Localisations

It is often interesting to consider properties of objects in an ∞-category $\mathcal{C}$ that are invariant under a class of morphisms in $\mathcal{C}$ which is larger than the class of equivalences. For example, one can ask when two chain complexes have isomorphic homology groups. In these situations, we would like to think of this larger class of morphisms in $\mathcal{C}$ as equivalences. This is made concrete by *localisation* constructions.

Definition 2.1 [Lur09, Def. 5.2.7.2] *A functor $L\colon \mathcal{C} \to \mathcal{D}$ between two ∞-categories is a* localisation functor *if it admits a fully faithful right adjoint.*

Proposition 2.2 [Lur09, Prop. 5.2.7.4] *Let $\mathcal{C}$ be an ∞-category, and let $L\colon \mathcal{C} \to \mathcal{C}$ be a functor with essential image $L\mathcal{C} \subseteq \mathcal{C}$. The following are equivalent:*

(1) There exists a localisation functor $F\colon \mathcal{C} \to \mathcal{D}$ with fully faithful right adjoint $G\colon \mathcal{D} \to \mathcal{C}$ and a natural equivalence $G \circ F \simeq L$.

(2) The functor $L\colon \mathcal{C} \to L\mathcal{C}$ is a left adjoint to the (fully faithful) inclusion $L\mathcal{C} \hookrightarrow \mathcal{C}$.

This is in the spirit of performing a Bousfield localisation of a simplicial model category and restricting to $(L_S\mathcal{C})^\circ$, observing that S-local weak equivalences between S-local objects are exactly the original weak equivalences. Hence, on the level of left Bousfield localisations, ∞-categorical localisation is merely a restriction to a full subcategory.

Caution: In contrast to Gabriel-Zisman localisation or left Bousfield localisation, we do not change any morphisms in $\mathcal{C}$ to being equivalences – we merely generate a consistent collection of morphisms from S (its saturation) and then restrict to the largest full subcategory of $\mathcal{C}$ where these morphisms are already(!) equivalences.

Definition 2.3 [Lur09, Def. 5.5.4.1] *Let $\mathcal{C}$ be an ∞-category, and let S be a collection of morphisms in $\mathcal{C}$.*

(1) An object $Z \in \mathcal{C}$ is called S-local *if for every morphism $s\colon X \to Y$ in S the induced morphism*

$$\mathrm{Map}_{\mathcal{C}}(Y, Z) \xrightarrow{s^*} \mathrm{Map}_{\mathcal{C}}(X, Z)$$

is an isomorphism in $\mathcal{H}$ (the homotopy category of spaces).

(2) *A morphism $f\colon X \to Y$ in $\mathcal{C}$ is called an S-*local equivalence *if for every S-local object $Z \in \mathcal{C}$ the induced morphism*

$$\mathrm{Map}_{\mathcal{C}}(Y,Z) \xrightarrow{f^*} \mathrm{Map}_{\mathcal{C}}(X,Z)$$

is an isomorphism in $\mathcal{H}$.

This looks a lot like left Bousfield localisation of model categories. In fact, this resemblance is made precise in [Lur09, Prop. A.3.7.8].

The following definition is included for completeness:

Definition 2.4 [Lur09, Prop. 5.4.2.2] *Let κ be a regular cardinal. An ∞-category is κ-*accessible *if it has the following properties:*

(1) $\mathcal{C}$ is locally small.

(2) $\mathcal{C}$ admits κ-filtered colimits.

(3) The subcategory $\mathcal{C}^\kappa$ of κ-compact objects is essentially small.

(4) $\mathcal{C}^\kappa$ generates $\mathcal{C}$ under small, κ-filtered colimits.

$\mathcal{C}$ is called accessible *if it is κ-accessible for some regular cardinal κ. A functor out of an accessible ∞-category $\mathcal{C}$ is* accessible *if it preserves κ-filtered limits for some regular cardinal κ.*

Definition 2.5 [Lur09, Def. 5.5.0.1] *An ∞-category $\mathcal{C}$ is* presentable *if it is accessible and small cocomplete.*

The following proposition gives a full understanding of accessible localisations of presentable ∞-categories:

Theorem 2.6 [Lur09, Prop. 5.5.4.15] *Let $\mathcal{C}$ be a presentable ∞-category, and let S be a small collection of morphisms in $\mathcal{C}$. We write $S^{-1}\mathcal{C} \subseteq \mathcal{C}$ for the full sub-∞-category on the S-local objects, and we write $\overline{S}$ for the strongly saturated class of morphisms in $\mathcal{C}$ generated by S (cf. [Lur09, Def. 5.5.4.5]). The following statements hold true:*

(1) For every $C \in \mathcal{C}$ there exists a morphism $s\colon C \to C'$ such that $C' \in S^{-1}\mathcal{C}$ and $s \in \overline{S}$.

(2) $S^{-1}\mathcal{C}$ is presentable.

(3) The inclusion $S^{-1}\mathcal{C} \hookrightarrow \mathcal{C}$ has a left adjoint $L\colon \mathcal{C} \to S^{-1}\mathcal{C}$.

(4) For every morphism f in $\mathcal{C}$, the following are equivalent:

(i) f is an S-local equivalence.

(ii) $f \in \overline{S}$.

(iii) Lf is an equivalence (in $\mathcal{C}$!).

Observe that, in particular, L establishes $S^{-1}\mathcal{C}$ as a localisation of $\mathcal{C}$, and part (4) identifies precisely the S-local equivalences in $\mathcal{C}$. Conversely, if $L\colon \mathcal{C} \to L\mathcal{C}$ is a localisation in the sense of Proposition 2.2, we set $S := \{f \mid Lf \text{ is equivalence}\}$ and obtain an equivalence $L\mathcal{C} \simeq S^{-1}\mathcal{C}$; that is, every localisation of a presentable ∞-category $\mathcal{C}$ is equivalent to one of the form $S^{-1}\mathcal{C}$ for some (small) class of morphisms S in $\mathcal{C}$. Finally, note that two classes S, T of morphisms in $\mathcal{C}$ yield the same localisation of $\mathcal{C}$ if and only if they generate the same strongly saturated class of morphisms, i.e. if and only if $\overline{S} = \overline{T}$.

Remark 2.7 From the fact that the inclusion $\iota\colon S^{-1}\mathcal{C} \hookrightarrow \mathcal{C}$ is fully faithful, we infer that the counit $L \circ \iota \to 1_{S^{-1}\mathcal{C}}$ is a natural equivalence. Then, part (4) of Theorem 2.6 implies that a morphism $f\colon X \to Y$ between S-local objects in $\mathcal{C}$ is an S-local equivalence (in $\mathcal{C}$) if and only if f is an (ordinary) equivalence in $\mathcal{C}$. ◁

The following theorem can be found in [Lur09, Prop. 5.5.1.1, Prop. A.3.7.6]. Parts (1), (2), and (3) follow ideas of C. Simpson, where part (3) is an ∞-categorical version of a famous theorem about combinatorial model categories by D. Dugger [Dug01].

Theorem 2.8 [Simpson] *Let $\mathcal{C}$ be an ∞-category. The following are equivalent:*

(1) $\mathcal{C}$ is presentable.

(2) There exists a small ∞-category $\mathcal{D}$ and an accessible localisation $\mathcal{P}(\mathcal{D}) \to \mathcal{C}$.

(3) There exists a combinatorial simplicial model category $\mathbf{A}$ and an equivalence $\mathrm{N}(\mathbf{A}^\circ) \simeq \mathcal{C}$.

Part (3) implies the following consequence:

Corollary 2.9 [Lur09, Cor. 5.5.2.4] *A presentable ∞-category $\mathcal{C}$ has all (small) limits and colimits.*

Definition 2.10 *Let $L\colon \mathcal{C} \to L\mathcal{C}$ be a localisation functor such that $\mathcal{C}$ admits finite limits (e.g. if $\mathcal{C}$ is presentable). Then L is called a* left exact *localisation if it preserves finite limits.*

3 Truncated objects and morphisms

Definition 3.1 [Lur09, Def. 5.5.6.1] *Let $\mathcal{C}$ be an ∞-category, $C \in \mathcal{C}$ an object, and let $k \geq -1$ be an integer. We say C is* (-2)-truncated *if it is a final object. We call C a* k-truncated *object if for every object $C' \in \mathcal{C}$ we have $\pi_n(\mathrm{Map}_{\mathcal{C}}(C', C)) = 0$ for all $n > k$ (for all choices of basepoints). A 0-truncated object will also be called* discrete*. We denote by $\tau_{\leq k}\mathcal{C}$ the full sub-∞-category of $\mathcal{C}$ on the k-truncated objects.*

Observe that a (-1)-truncated space is either empty, or it consists of precisely one point. By construction, the ∞-category $\tau_{\leq 0}\mathcal{C}$ is equivalent to the nerve of its own homotopy category; we denote this ordinary category by $\mathrm{Disc}(\mathcal{C})$.

Example 3.2 A space $K \in \mathcal{S}$ is k-truncated precisely if $\pi_n(K) = 0$ for all $n > k$. Thus, we obtain that, for a general ∞-category $\mathcal{C}$, an object $P \in \mathcal{P}(\mathcal{C})$ is k-truncated if and only if the functor $P \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ factors through $\tau_{\leq k}\mathcal{S}$. $\triangleleft$

Definition 3.3 [Lur09, Def. 5.5.6.8] *Let $k \geq -2$ be an integer. A map $f\colon K \to K'$ of spaces is* k-truncated *if all its (homotopy) fibres are k-truncated spaces. A morphism $f\colon C \to C'$ in a generic ∞-category $\mathcal{C}$ is* k-truncated *if the morphism*

$$\mathrm{Map}_{\mathcal{C}}(D, C) \xrightarrow{f_*} \mathrm{Map}_{\mathcal{C}}(D, C')$$

is a k-truncated morphism in $\mathcal{S}$ for every object $D \in \mathcal{C}$.

This is equivalent to $f \in \mathcal{C}_{/C'}$ being k-truncated. Further, A morphism is (-2)-truncated if and only if it is an equivalence.

Definition 3.4 *A morphism f in an ∞-category $\mathcal{C}$ is a* monomorphism *if it is (-1)-truncated.*

This is the ∞-categorical version of saying that a map $f\colon K \to K'$ of sets is injective if and only if the preimage $f^{-1}(\{x\})$ contains either zero or one elements for every $x \in K'$.

Theorem 3.5 [Lur09, Prop. 5.5.6.18] *Let $\mathcal{C}$ be a presentable ∞-category, and let $k \geq -2$. The inclusion $\tau_{\leq k}\mathcal{C} \hookrightarrow \mathcal{C}$ has an accessible left adjoint*

$$\tau_{\leq k}\colon \mathcal{C} \to \tau_{\leq k}\mathcal{C}\,.$$

This establishes $\tau_{\leq k}\mathcal{C}$ as an accessible localisation of $\mathcal{C}$.

Remark 3.6 If $\mathcal{C} = \mathcal{S}$ is the ∞-category of spaces and $X \in \mathcal{X}$, the sequence $\tau_{\leq k}X$ is a Postnikov tower for X. ◁

4 Definition of ∞-topoi

Let $n \in \mathbb{N}_0$, and let $\Delta_{\leq n} \subseteq \Delta$ denote the full subcategory of Δ on the objects $[0], \dots, [n]$. For $n = -1$ we set $\Delta_{\leq n} = \emptyset$. We denote the inclusion functor by ι_n. Further, let Δ_+ denote the simplex category with an initial object adjoined (denoted by $[-1]$), and let $\Delta_{+,\leq n} \subset \Delta_+$ denote the full subcategory on the objects $[-1], \dots, [n]$.

Definition 4.1 *Let $\mathcal{C}$ be an ∞-category that admits pullbacks, and let $f\colon Y \to X$ be a morphism in f. The* Čech nerve of f *is the simplicial object $\check{C}(f)\colon \mathrm{N}(\Delta^{\mathrm{op}}) \to \mathcal{C}_{/X}$ given as the right Kan extension*

$$\begin{tikzcd} \mathrm{N}\Delta^{\mathrm{op}}_{+,\leq 0} \arrow[r, "f"] \arrow[d, hook] & \mathcal{C} \\ \mathrm{N}\Delta^{\mathrm{op}}_{+} \arrow[ur, dashed, "\check{C}(f)"'] & \end{tikzcd}$$

In any Čech nerve there are canonical isomorphisms

$$\check{C}(f)_n \simeq Y \underset{X}{\times} \cdots \underset{X}{\times} Y\,.$$

However, in order to defined the Čech nerve as a coherent simplicial object in the ∞-category $\mathcal{C}$, it is not enough to use this as a definition; the questions of coherence are remedied by using the ∞-categorical Kan extension from Definition 4.1.

Definition 4.2 [NSS15, Def. 2.8] *Let $\mathcal{C}$ be an ∞-category with pullbacks and geometric realisations of simplicial objects (i.e. colimits of simplicial diagrams). A morphism $f\colon Y \to X$ in $\mathcal{C}$ is an* effective epimorphism *if the canonical morphism $|\check{C}(f)_\bullet| \to X$ is an equivalence.*

Example 4.3 The prototypical example of an effective epimorphism is the Čech nerve of an open covering of a topological space; following [DI04] we describe this in a model-categorical presentation. An open covering $\{V_a\}_{a\in A}$ of X gives rise to a map

$$U_0 \coloneqq \coprod_{a\in A} V_a \longrightarrow X\,.$$

We let U_n denote the n-th level of this Čech nerve and find that

$$U_n = \coprod_{a_0,\dots,a_n \in A} V_{a_0} \cap \cdots \cap V_{a_n} =: \coprod_{a_0,\dots,a_n \in A} V_{a_0\dots a_n}$$

is the disjoint union of all $(n{+}1)$-fold intersections of patches in the open cover. The result of [DI04] now states that the map

$$\underset{\Delta^{\mathrm{op}}}{\mathrm{hocolim}}^{\mathcal{T}\mathrm{op}}\, U_\bullet \longrightarrow X$$

is a weak equivalence of topological spaces. ◁

Definition 4.4 [Lur09, Def. 6.1.2.7] *Let $\mathcal{C}$ be an ∞-category. A* groupoid object *in $\mathcal{C}$ is a simplicial object $U_\bullet \colon \mathrm{N}\Delta^{\mathrm{op}} \to \mathcal{C}$ satisfying the following property: for every $n \in \mathbb{N}_0$ and for every partition $[n] = S \cup S'$ into subsets such that $S \cap S'$ consists of a single element s, the diagram*

$$\begin{tikzcd} U_n \arrow[r] \arrow[d] & U_S \arrow[d] \\ U_{S'} \arrow[r] & U_{\{s\}} \end{tikzcd}$$

is a pullback diagram in $\mathcal{C}$.

Observe that any subset $S \subset [n]$ can canonically be identified with an object $[k] \in \Delta$ together with an injective map $[k] \to [n]$. This is how the vertices and arrows arise in the above diagram. In particular, for any groupoid object $U_\bullet$ in $\mathcal{C}$ there are weak equivalences

$$U_n \simeq U_1 \times_{U_0} \cdots \times_{U_0} U_1 \,.$$

In other words, a groupoid object is, in particular, a Segal object.

Example 4.5 A groupoid object in the ordinary category $\mathcal{S}\mathrm{et}$ is, in particular, a simplicial object $U_\bullet \colon \Delta^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$. We understand U_0 as a set of objects and U_1 as a set of morphisms. The partition $[2] = (0,1) \sqcup_{(1)} (1,2)$ yields an isomorphism $U_1 \times_{U_0} U_1 \cong U_2$. The remaining face map $U_2 \to U_1$ (from the inclusion $(0,2) \hookrightarrow [2]$) is interpreted as a composition map. Associativity follows from the compositions $(0,1,2) \cup (2,3) = [3] = (0,1) \cup (1,2,3)$ (exercise: do this!). This is how a Segal object in $\mathcal{S}\mathrm{et}$ can be identified with the nerve of a small category C_U. For a *groupoid* object, however, we also have the partitions $(0,2) \cup (0,1) = [2] = (0,2) \cup (1,2)$, which implies that every morphism in C_U has an inverse. This justifies the term groupoid object: we have found a generalisation of a groupoid internal to a category to the ∞-world. $\triangleleft$

Definition 4.6 [NSS15, Def. 2.1] *An* ∞-topos *is an ∞-category $\mathcal{X}$ which satisfies the following properties, called the* ∞-Giraud's axioms*:*

(1) $\mathcal{X}$ is presentable.

(2) (Coproducts are disjoint) Any pushout diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & A \sqcup B \end{tikzcd}$$

is also a pullback diagram.

(3) (Pullbacks preserve colimits) For all morphisms $f \colon X \to B$ in $\mathcal{X}$ and for every small diagram $D \colon I \to \mathcal{X}_{/B}$, there exists an equivalence

$$\operatorname*{colim}_I\, (f^* D_i) \simeq f^* (\operatorname*{colim}_I\, D_i) \,.$$

(4) (Quotient maps are effective epimorphisms/groupoid objects are effective) Every groupoid object $A \colon \mathrm{N}\Delta^{\mathrm{op}} \to \mathcal{X}$ is the Čech nerve of its quotient projection: setting $A_{-1} \coloneqq |A_\bullet| = \mathrm{colim}(A_\bullet)$, there exists an equivalence of simplicial objects in $\mathcal{X}_{/X}$

$$A_\bullet \simeq \check{C}\big(A_0 \to A_{-1}\big)_\bullet \,.$$

In fact, property (3) implies that there is an equivalence between groupoid objects and Čech nerves of effective epimorphisms in any ∞-topos $\mathcal{X}$.

Theorem 4.7 [Lur09, Thm. 6.1.0.6] *Let $\mathcal{X}$ be an ∞-category. The following are equivalent:*

(1) $\mathcal{X}$ is an ∞-topos.

(2) There exists a small ∞-category $\mathcal{C}$ and an accessible, left exact localisation $\mathcal{P}(\mathcal{C}) \to \mathcal{X}$.

Observe that both an ∞-topos and a mere presentable ∞-category can be realised as localisations of presheaf ∞-categories. By Theorem 4.7 the difference between the two lies purely in the question whether the localisation is left exact.

Compare Theorem 4.7 to Theorem 1.2, and observe that we did not list an analogue of the characterisation of topoi in terms of sheaf categories. This is not an omission: while ∞-topoi are equivalent to left-exact localisations of presheaf ∞-categories, sheaf ∞-categories only correspond to a proper subclass of such localisations. These are called *topological localisations* in [Lur09], and one of their properties is that they are generated by a class of *mono*morphisms in the sense of Definition 3.4

Example 4.8 The ∞-category $\mathcal{S}$ of spaces is an ∞-topos. $\triangleleft$

Example 4.9 Let $\mathcal{C}$ be a small ∞-category. Then, $\mathcal{P}(\mathcal{C})$ is an ∞-topos. This follows from the facts that $\mathcal{S}$ is an ∞-topos and that limits and colimits in $\mathcal{P}(\mathcal{C})$ are computed objectwise. $\triangleleft$

Example 4.10 If $\mathcal{C}$ is a small ∞-category endowed with a Grothendieck topology (see [Lur09, Sec. 6.2.2]), then the ∞-category $\mathcal{S}\mathrm{h}(\mathcal{C})$ of sheaves on $\mathcal{C}$ forms an ∞-topos. However, the converse is not true, as outlined above. $\triangleleft$

We will need the following two facts [Lur09, Prop. 6.3.5.1, Lemma 6.5.1.2]:

Lemma 4.11 *Let $\mathcal{X}$ be an ∞-topos.*

(1) If $\mathcal{X}$ is an ∞-topos, then so is $\mathcal{X}_{/X}$, for any $X \in \mathcal{X}$.

(2) Further, the truncation functor $\tau_{\leq k} \colon \mathcal{X} \to \tau_{\leq k}\mathcal{X} \subseteq \mathcal{X}$ preserves finite products.

5 Hypercompleteness

Let $\mathcal{X}$ be an ∞-topos. In particular, it is complete and cocomplete, and hence it is cotensored over the ∞-category $\mathcal{S}$ of spaces. Let us briefly recall this cotensoring: given a space $K \in \mathcal{S}$ and an object $X \in \mathcal{X}$, let $cX \colon K \to \mathcal{X}$ denote the constant diagram with value X. Then,

$$X^K = \lim_K^{\mathcal{X}}(cX) \quad \in \mathcal{X}\,.$$

We have canonical isomorphisms in $\mathcal{H}$:

$$\begin{aligned} \mathrm{Map}_{\mathcal{X}}(Y, X^K) &\cong \lim_K^{\mathcal{S}}\big(c\,\mathrm{Map}_{\mathcal{X}}(Y,X)\big) \\ &\cong \mathrm{Map}_K\big(K, K \times \mathrm{Map}_{\mathcal{X}}(Y,X)\big) \\ &\cong \mathrm{Map}_{\mathcal{S}}\big(K, \mathrm{Map}_{\mathcal{X}}(Y,X)\big)\,. \end{aligned}$$

Let $\mathbb{S}^n := \partial\Delta^{n+1} \in \mathcal{S}$ be the simplicial n-sphere, and fix a base point $x_0 \colon * \to \mathbb{S}^n$ in $\mathbb{S}^n$. This induces a morphism

$$\mathrm{ev}_{x_0} \colon X^{\mathbb{S}^n} \to X$$

in $\mathcal{X}$. We regard ev_{x_0} as an object in the slice $\mathcal{X}_{/X}$. Given any two pointed spaces $K, K' \in \mathcal{S}_*$, there is an equivalence $X^{K \vee K'} \simeq X^K \times_X K^{K'}$ in $\mathcal{X}$, so that we have $X^{K \vee K'} \simeq X^K \times K^{K'}$ in $\mathcal{X}_{/X}$. Combining this with Lemma 4.11, we deduce that $\tau_{\leq 0} X^{\mathbb{S}^n}$ is a group object in $\mathcal{X}_{/X}$ for $n \geq 1$, which is commutative for $n \geq 2$. We focus on zero-truncations here because we would like an analogy with homotopy groups, and because the fold map $\mathbb{S}^n \to \mathbb{S}^n \vee \mathbb{S}^n$ is only co-unital and co-associative up to homotopy. Further, note that we work *over* X to keep track of all possible choices of basepoints in $X^{\mathbb{S}^n}$ (cf. [Jar15, p. 64]). The basepoint of $X^{\mathbb{S}^n}$ in $\mathcal{X}_{/X}$ is induced by the collapse map $\mathbb{S}^n \to *$ in $\mathcal{S}$.

Definition 5.1 [Lur09, Def. 6.5.1.1] *Let $\mathcal{X}$ be an ∞-topos, and let $X \in \mathcal{X}$. We set*

$$\pi_n(X) \coloneqq \tau_{\leq 0}(X^{\mathbb{S}^n}) \quad \in \mathrm{Disc}(\mathcal{X}_{/X}).$$

These are often referred to as the sheaves of homotopy groups *of X.*

We will view $\pi_n(X)$ as a (group) object of $\mathcal{X}_{/X}$ via the inclusion $\tau_{\leq 0}(\mathcal{X}_{/X}) \subset \mathcal{X}_{/X}$, or as a (group) object of $\mathrm{Disc}(\mathcal{X}_{/X})$, the homotopy category of $\tau_{\leq 0}(\mathcal{X}_{/X})$ (recall that $\tau_{\leq 0}\mathcal{C}$ is equivalent to the nerve of its homotopy category). We point out that $\mathrm{Disc}(\mathcal{X}_{/X})$ is an ordinary (Grothendieck) topos.

Remark 5.2 There exists a model-categorical presentation of hypercompletions of sheaf ∞-topoi, in which $\pi_n(X)$ is really understood as the *sheafification* of presheaves of homotopy groups of X (for reference, see e.g. [Jar15]). $\triangleleft$

Example 5.3 In the case where $\mathcal{X} = \mathcal{P}(\mathcal{C})$ is a presheaf ∞-topos, we have $X^K(c) \simeq X(c)^K$, since limits are computed pointwise. Hence, $\pi_n(X)$ is the presheaf $c \mapsto \pi_n(X(c))$ of homotopy groups with all possible base points. $\triangleleft$

Definition 5.4 *For $f\colon X \to Y$ a morphism in an ∞-topos $\mathcal{X}$, we define $\pi_n(f) \in \mathrm{Disc}(\mathcal{X}_{/X})$ as follows: we view f as an object in the ∞-topos $\mathcal{X}_{/Y}$, where we can now form sheaves of homotopy groups $\pi'_n(f) \in \mathrm{Disc}((\mathcal{X}_{/Y})_{/f})$ by means of Definition 5.1. Using the equivalences $(\mathcal{X}_{/Y})_{/f} \simeq \mathcal{X}_{/f} \simeq \mathcal{X}_{/X}$, we can now identify $\pi'_n(f)$ with an object $\pi_n(f) \in \mathrm{Disc}(\mathcal{X}_{/X})$.*

The intuition is that $\pi_n(f)$ takes homotopy groups of the (homotopy) fibres of f that contain given points of X. This notion of homotopy group interacts well with truncation. For example, if $X \in \mathcal{X}$ is m-truncated for *some* $m \gg 0$, then it is k-truncated if $\pi_n(X) = 0$ for all $n > k$. However, in general the assumption that X is truncated at all is crucial here, in contrast to the classical setting of spaces (this is [NSS15, Rmk. 2.7]).

Definition 5.5 [Lur09, Def. 6.5.1.10] *Let $\mathcal{X}$ be an ∞-topos and $n \in \mathbb{N}_0 \cup \{\infty\}$. A morphism $f\colon X \to Y$ in $\mathcal{X}$ is called* n-connective *if it is an effective epimorphism and $\pi_k(f) = 0$ for all $n \leq k \leq n-1$. An object $X \in \mathcal{X}$ is called* n-connective *if the collapse morphism $X \to *_{\mathcal{X}}$ is n-connective. We use the convention that every morphism is (-1)-connective.*

It follows from the definitions that every equivalence in $\mathcal{X}$ is ∞-connective, i.e. it induces isomorphisms on all sheaves of homotopy groups. However, the converse is not true in general: there are more ∞-connective morphisms than there are equivalences. In particular, we cannot expect to be able to invert every ∞-connective morphism in $\mathcal{X}$. Let S_∞ be the class of ∞-connective morphisms in $\mathcal{X}$.

Definition 5.6 *Let $\mathcal{X}$ be an ∞-topos. An object $X \in \mathcal{X}$ is called* hypercomplete *if it is S_∞-local. We define the* hypercompletion *$\mathcal{X}^\wedge \coloneqq S_\infty^{-1}\mathcal{X}$ of $\mathcal{X}$ to be the left accessible localisation of $\mathcal{X}$ at the ∞-connective morphisms. This is itself an ∞-topos (requires some work). The ∞-topos $\mathcal{X}$ is called* hypercomplete *if $\mathcal{X}^\wedge = \mathcal{X}$.*

In other words, $\mathcal{X}$ is hypercomplete precisely if every ∞-connective morphism in $\mathcal{X}$ is already an equivalence in $\mathcal{X}$. In this sense, one says that 'the Whitehead Theorem holds true in $\mathcal{X}$ if and only if $\mathcal{X}$ is hypercomplete'.

Example 5.7 Presheaf ∞-topoi $\mathcal{P}(\mathcal{C})$ are hypercomplete. This follows from the observation that here $\pi_n(X)$ is just the presheaf of homotopy groups $c \mapsto \pi_n(X(c))$, seen as objects over $X(c)$ via the choice of base point. $\triangleleft$

6 Hypercoverings

In order to understand how hypercoverings come up in the theory of sheaves, we recommend the introduction to [Lur09, Sec. 6.5.3]. The comparison between two procedures of ∞-sheafification – either via infinitely iterating Grothendieck's $+$-construction, or by squeezing everything into a single step – is enlightening, but to keep this talk (somewhat) brief, we do not go through that story here.

If $\mathcal{C}$ is an ∞-category with all small limits and colimits, we have a diagram of adjunctions

$$\begin{tikzcd} \mathcal{C}^{(\Delta^{\mathrm{op}})} \arrow[rr, "\iota_n^*" description] & & \mathcal{C}^{(\Delta^{\mathrm{op}}_{\leq n})} \arrow[ll, bend right=40, "\mathrm{cosk}_{\leq n}"', "\perp"] \arrow[ll, bend left=40, "\mathrm{sk}_{\leq n}", "\perp"'] \end{tikzcd}$$

where $\mathrm{sk}_{\leq n}$ and $\mathrm{cosk}_{\leq n}$ are defined via left and right Kan extension along ι_n, respectively. We define

$$\mathrm{sk}_n \coloneqq \mathrm{sk}_{\leq n} \circ \iota_n^* , \quad \text{and} \quad \mathrm{cosk}_n \coloneqq \mathrm{cosk}_{\leq n} \circ \iota_n^* .$$

These functors form an adjoint pair $\mathrm{sk}_n \dashv \mathrm{cosk}_n$ of endofunctors of $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$.

Definition 6.1 *Let $\mathcal{X}$ be an ∞-topos.*

(1) A simplicial object $U_\bullet \colon \mathrm{N}\Delta^{\mathrm{op}} \to \mathcal{X}$ is a hypercovering of $\mathcal{X}$ *if for each $n \in \mathbb{N}_0$ the unit morphism*

$$U_n \to \big(\mathrm{cosk}_{n-1}(U_\bullet)\big)_n \tag{6.2}$$

is an effective epimorphism.

(2) We call a hypercovering $U_\bullet$ effective *if its colimit $|U_\bullet| = \mathrm{colim}^{\mathcal{X}}_{\mathrm{N}\Delta^{\mathrm{op}}}(U_\bullet)$ is a final object of $\mathcal{X}$.*

(3) For $X \in \mathcal{X}$, we define a hypercovering of X *to be a hypercovering of the slice ∞-topos $\mathcal{X}_{/X}$.*

A hypercovering of an object $X \in \mathcal{X}$ is effective if and only if the morphism $|U_\bullet| \to X$ is a final object of the slice ∞-topos $\mathcal{X}_{/X}$. Since the identity 1_X is a final object in $\mathcal{X}_{/X}$, that is the case if and only if the morphism $|U_\bullet| \to X$ is an equivalence in $\mathcal{X}$.

Example 6.3 Consider the condition (6.2) for low levels. We can write

$$(\mathrm{cosk}_n U_\bullet)_k = \lim{}^{\mathcal{X}} \big([k]/\Delta^{\mathrm{op}}_{\leq n} \longrightarrow \Delta^{\mathrm{op}} \xrightarrow{U_\bullet} \mathcal{X} \big) .$$

Observe that $[k]/\Delta^{\mathrm{op}}_{\leq n} \simeq (\Delta_{\leq n}/[k])^{\mathrm{op}}$. We obtain

$$(\mathrm{cosk}_{-1} U_\bullet)_0 = *_{\mathcal{X}} , \qquad (\mathrm{cosk}_0 U_\bullet)_1 = U_0 \times U_0 ,$$

$$
(\operatorname{cosk}_1 U_\bullet)_2 = \lim \left(\begin{tikzcd}[column sep=small] & & U_0 & & \\ & U_1 \arrow[ur] \arrow[dl] & & U_1 \arrow[ul] \arrow[dr] & \\ U_0 & & U_1 \arrow[ll] \arrow[rr] & & U_0 \end{tikzcd} \right)
$$

Observe that if the ambient ∞-topos is a slice ∞-topos $\mathcal{X}_{/X}$, then the limit is taken in $\mathcal{X}_{/X}$. ◁

Example 6.4 Recall that every morphism $f\colon Y \to X$ in an ∞-topos $\mathcal{X}$ gives rise to a simplicial object $\check{C}(f)\colon \mathrm{N}\Delta^{\mathrm{op}} \to \mathcal{X}_{/X}$ (see Definition 4.1). Using its explicit form we see from Example 6.3 that $\check{C}(f)$ satisfies the hypercover condition (6.2) for $n = 0, 1, 2$ (we find $\operatorname{cosk}_1 \check{C}(f)_\bullet \cong Y \times_X Y \times_X Y = \check{C}(f)_2$). In fact, *the Čech nerve of every effective epimorphism $f\colon Y \to X$ is a hypercovering of X.* ◁

Example 6.5 By Definition 4.2 and Definition 6.1 we deduce that a morphism $f\colon Y \to X$ in $\mathcal{X}$ is an effective epimorphism if and only if its Čech nerve is an *effective* hypercovering of X. ◁

Example 6.6 We consider $\mathcal{T}\mathrm{op}$ as a model for the ∞-topos of spaces to give a rough idea of how a hypercovering generalises a covering. Recall (Example 4.3) that open coverings of topological spaces give rise to effective epimorphisms. Let $X \in \mathcal{T}\mathrm{op}$ be a topological space, let $\{V^{(0)}_a\}_{a\in A}$ an open covering and set $U_0 \coloneqq \coprod_{a\in A} V^{(0)}_a$. For each $a, b \in A$, let $\{V^{(1)}_{ab,i}\}_{i\in I_{ab}}$ be an open covering of the topological space $V^{(0)}_{ab}$. Set

$$
U_1 \coloneqq \coprod_{a,b\in A,\, i\in I_{ab}} V^{(1)}_{ab,i} \,.
$$

This comes with two maps $d_j\colon U_1 \rightrightarrows U_0$ in $\mathcal{T}\mathrm{op}_{/X}$, acting as $V^{(1)}_{ab,i} \hookrightarrow V^{(0)}_b$ and as $V^{(1)}_{ab,i} \hookrightarrow V^{(0)}_a$, for $j = 0, 1$, respectively. By construction, the resulting map $U_1 \to U_0 \times_X U_0$ is an open covering, and hence an effective epimorphism. One can now either form a hypercovering of X by Kan extending $U_1 \rightrightarrows U_0$ to a simplicial object, or by adding a level U_2 as an open covering of

$$
\coprod_{a,b,c\in A,\, i\in I_{ab},\, j\in I_{bc},\, k\in I_{ac}} V^{(1)}_{ab,i} \cap V^{(1)}_{bc,j} \cap V^{(1)}_{ac,k} \,,
$$

and so on. ◁

Lemma 6.7 [Lur09, Lemma 6.5.3.11] *Let $\mathcal{X}$ be an ∞-topos, and let $U_\bullet\colon \mathrm{N}\Delta^{\mathrm{op}} \to \mathcal{X}$ be a hypercovering of $\mathcal{X}$. Then $|U_\bullet| \in \mathcal{X}$ is ∞-connective.*

It turns out that hypercoverings provide a way of characterising hypercomplete ∞-topoi:

Theorem 6.8 [Lur09, Thm. 6.5.3.12] *Let $\mathcal{X}$ be an ∞-topos. The following are equivalent:*

(1) For every object $X \in \mathcal{X}$, every hypercovering $U_\bullet$ of $\mathcal{X}_{/X}$ (i.e. 'of X') is effective.

(2) $\mathcal{X}$ is hypercomplete.

Proof. (1) ⇒ (2): Let $f\colon U \to X$ be an ∞-connective morphism in $\mathcal{X}$. We need to show that f is an equivalence. Consider the constant simplicial object $c_\bullet f$ in $\mathcal{X}_{/X}$ with value f. Since f is ∞-connective, this is a hypercovering of X [Lur09, Lemma 6.5.3.5]. Thus, (1) implies that that $|c_\bullet f| \to 1_X$ is an equivalence in $\mathcal{X}_{/X}$. In other words, f is an equivalence in $\mathcal{X}$.

(2) ⇒ (1): First, if $\mathcal{X}$ is hypercomplete, then so is $\mathcal{X}_{/X}$ for any $X \in \mathcal{X}$. By Lemma 6.7, $|U_\bullet|$ is an ∞-connective object in $\mathcal{X}_{/X}$. Hence, if $\mathcal{X}$ is hypercomplete, then $|U_\bullet|$ is a final object in $\mathcal{X}_{/X}$. That is, $U_\bullet$ is effective. □

In particular, this applies to any presheaf ∞-topos $\mathcal{P}(\mathcal{C})$.

The following statement is [Lur09, Cor. 6.5.3.13]; it is the ∞-categorical reformulation of results obtained before in [TV, DHI04].

Corollary 6.9 *Let $\mathcal{X}$ be an ∞-topos. Let S_{hc} denote the class of morphisms in $\mathcal{X}$ consisting of all morphisms $|U_\bullet| \to X$ for any object $X \in \mathcal{X}$ and any hypercovering $U_\bullet$ of X. Then, $\mathcal{X}^\wedge = S_{hc}^{-1}\mathcal{X}$.*

That is, the classes S_∞ and S_{hc} of morphisms in $\mathcal{X}$ generate the same strongly saturated classes, and hence the same localisations of $\mathcal{X}$. There also exists a model-categorical version of Corollary 6.9 (combine [DHI04, Thm. 6.2] and [Lur09, Prop. 6.5.2.14]).

Remark 6.10 In fact, every ∞-topos can be obtained by starting with a small ∞-category $\mathcal{C}$, choosing a collection of augmented simplicial objects $U_\bullet \colon \Delta_+^{\mathrm{op}} \to \mathcal{P}(\mathcal{C})$, and localising at (i.e. 'inverting') the morphisms $|U_\bullet| \to U_{-1}$ [Lur09, Rmk. 6.5.3.14]. Observe that this is different from giving a Grothendieck topology on $\mathcal{C}$ and localising at the Čech coverings. Therefore, in stark contrast to the case of classical topoi, *not every ∞-topos is a sheaf topos.* The difference stems from the possible presence of non-invertible ∞-connected morphisms, or equivalently from non-effective unbounded hypercoverings; these are hypercoverings that are not n-coskeletal for any $n \in \mathbb{N}_0$. None of these arise in the classical context. In particular, by [DHI04, App. A] (see [Lur09, Lemma 6.5.3.9] for the ∞-categorical version) taking into account only *bounded* hypercoverings is equivalent to localising at Čech coverings. Hence, in this case one ends up with a sheaf ∞-topos, and the difference really stems from unbounded hypercoverings. (Caution: the notion of hypercovering in [DHI04] is slightly different from that in [Lur09]; it intrinsically relies on the presence of a Grothendieck topology.) $\triangleleft$

Remark 6.11 Recall that every presheaf ∞-topos is hypercomplete. In contrast, there exist sheaf ∞-topoi which are not hypercomplete [Lur09, Counterexample 6.5.4.5]. $\triangleleft$

Remark 6.12 [Lur09, Rmk. 6.5.4.7] An ∞-topos is hypercomplete precisely if it *has enough points.* $\triangleleft$

7 A brief look at formal moduli problems

Let $(\mathcal{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context (in the conventions of [Lur18]). We set $\mathcal{X}^{\mathcal{A}} := \mathrm{Fun}(\mathcal{A}^{art}, \mathcal{S})$. Observe that this is the presheaf ∞-topos

$$\mathcal{X}^{\mathcal{A}} = \mathcal{P}\big((\mathcal{A}^{art})^{\mathrm{op}}\big)\,.$$

Recall that we defined the ∞-category $\mathrm{FMP}^{\mathcal{A}}$ as the full sub-∞-category of $\mathcal{P}(\mathcal{C})$ on the formal moduli problems. We first realise that $L : \mathcal{X}^{\mathcal{A}} \rightleftarrows \mathrm{FMP}^{\mathcal{A}} : \iota$ is an accessible localisation [Lur18, Rmk. 12.1.3.5] of $\mathcal{P}(\mathcal{C}) = \mathrm{Fun}(\mathcal{A}^{art}, \mathcal{S})$ at the collection S of morphisms in $\mathcal{X}^{\mathcal{A}}$ consisting of

(1) the morphism $\emptyset_{\mathcal{X}^{\mathcal{A}}} \to \mathcal{A}^{art}(*_{\mathcal{A}^{art}}, -)$ and

(2) the morphisms $\mathcal{A}^{art}(A,-) \sqcup_{\mathcal{A}^{art}(B,-)} \mathcal{A}^{art}(B',-) \longrightarrow \mathcal{A}^{art}(A \times_B B', -)$ induced by any cospan $A \to B \leftarrow B'$ in $\mathcal{A}^{art}$ where at least one of the morphisms is small.

The co-Yoneda Lemma then shows that the S-local objects in $\mathcal{X}^{\mathcal{A}}$ are precisely the formal moduli problems. In particular, $\mathrm{FMP}^{\mathcal{A}}$ is presentable by Theorem 2.6, and hence admits all small limits and colimits. Note that this does not necessarily mean that $\mathrm{FMP}^{\mathcal{A}}$ is an ∞-topos since the localisation is not necessarily left exact (cf. Theorem 4.7).

Proposition 7.1 *Let $(\mathcal{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context and $X \in \mathrm{FMP}^{\mathcal{A}}$.*

(1) If $U_\bullet \to X$ is a hypercovering of X in $\mathcal{X}^{\mathcal{A}}$, then the canonical morphism $|U_\bullet|_{\mathcal{X}^{\mathcal{A}}} \longrightarrow X$ (colimit taken in $\mathcal{X}^{\mathcal{A}}$) is an equivalence in $\mathcal{X}^{\mathcal{A}}$.

(2) If the diagram $U_\bullet$ factors through $\mathrm{FMP}^{\mathcal{A}} \subseteq \mathcal{X}^{\mathcal{A}}$, then the canonical morphism $u\colon |U_\bullet|_{\mathrm{FMP}^{\mathcal{A}}} \longrightarrow X$ is an equivalence in $\mathrm{FMP}^{\mathcal{A}}$.

Proof. Claim (1) follows from Theorem 6.8 and Example 5.7.

Statement (2) follows from the fact that since L is a localisation there is a canonical equivalence $L(|\iota U_\bullet|_{\mathcal{X}^{\mathcal{A}}}) \simeq |U_\bullet|_{\mathrm{FMP}^{\mathcal{A}}}$. Using that the right adjoint ι of L is fully faithful, we obtain a diagram

$$\begin{tikzcd}
{|U_\bullet|_{\mathrm{FMP}^{\mathcal{A}}}} \arrow[r, "(\sim)"] & X \\
{|L\iota U_\bullet|_{\mathrm{FMP}^{\mathcal{A}}}} \arrow[u, "\sim"', "\iota\text{ f.f.}"] & \\
{L\big(|\iota U_\bullet|_{\mathcal{X}^{\mathcal{A}}}\big)} \arrow[u, "\sim"', "L\text{ l.adj.}"] \arrow[r, "\sim\text{ by (1)}", "Lu"'] & L\iota X \arrow[uu, "\sim"', "\iota\text{ f.f.}"]
\end{tikzcd}$$

The claim now follows from the (2/3)-property. □

A key step in the proof of [Lur18, 12.3.3.5] is to decompose a general formal moduli problem into simpler ones, for which we can already prove the theorem. The general claim is then assembled from the simpler constituents of the problem. More precisely, we will see in [Lur18, Prop. 12.5.3.3] that any formal moduli problem admits a (particularly good) hypercovering by prorepresentable formal moduli problems.

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