Notes for Talk I

Deformation functors and Schlessinger's theorem

1) Local functors on Artinian rings
2) tangent spaces of functors
3) (Pro-)-representability
4) Schlessinger's theorem

1) Def. Art_k is category of local, Artinian k-algebras (R, m) with residue field $k = R/m$

local : $\exists!$ max ideal m

Artinian : descending chain condition on ideals, $R > I_1 > I_2 > I_3 > \ldots$ becomes stationary.

Implies that R is fin dim over k and m is nilpotent

Examples : $k[\epsilon] := k[t]/k(\epsilon^2)$, $k[t]/(t^n)$

Def A functor $F : Art_k \to Set$ is local if $F(k) = \{*\}$ one element set.

Examples : 1) Given k-algebra A_k, get functor of deformations

$$Def(A_k, -) : Art_k \to Set$$

$$R \longmapsto \{\text{def. of } A_k \text{ parametrized by } R\}^{\text{iso classes of}}$$

$$= \{A_R \ R\text{-algebras}, \mid A_R \otimes_R k = A_k\}^{\text{iso classes of}}$$

$$Def(A_k, k) = \{[A_k]\}$$

$$Def(A_k, k[\epsilon]) = \{\text{1st order def's of } A_k\}$$

Given $R \xrightarrow{f} S$ in Art_k, get

$$Def(A_k, -)(f) : A_R \longmapsto A_R \otimes_R S$$

2) Deformations of point p of scheme $X = \operatorname{Spec} B$ over k

point corresponds to $\phi_k : B \to k$ $\qquad$ $\operatorname{Spec} B \to \operatorname{Spec} k$

$F : Art_k \to Set$ $\qquad$ $k \to B$

$$R \mapsto \{ \operatorname{Spec} R \to X \mid p \text{ image of } \operatorname{Spec} k \subset \operatorname{Spec} R \}$$

$$= \{ \phi_R : B \to R \mid \phi_R \otimes_R k = \phi_k \}$$

$k \sim \begin{array}{c} R \\ \downarrow \\ k = R/m \end{array}$

For $R \xrightarrow{f} S$ define

$F(f)(\phi_R) = f \circ \phi_R \qquad F(k) = \{\phi_k\}$

3) Picard functor: Given scheme X over k, $\operatorname{Pic}(X) = \{\text{iso classes of line bundles on } X\}$

$= \{\text{iso classes of loc. free, rk 1 sheaves on } X\}$

$= H^1(X, \mathcal{O}_X^*)$

Let $\eta \in \operatorname{Pic} X$

construct $\operatorname{Pic}_\eta : Art_k \to Set$

For $R \in Art_k$, denote $X_R = X \times_{\operatorname{Spec} k} \operatorname{Spec} R$

$$\operatorname{Pic}_\eta(R) = \{ L \in \operatorname{Pic}(X_R) \mid L \otimes_R k = \eta \}$$

Here for $R \xrightarrow{f} S$ in Art_k

get $\operatorname{Spec} S \to \operatorname{Spec} R$ and induced

$$X_S \to X_R$$

$L \otimes_R S$ is pullback line bundle

in particular for $R \to k = R/m$

$\operatorname{Pic}_\eta(f)(L) = L \otimes_R S$

$\operatorname{Pic}_\eta(k) = \{\eta\}$

II 2)

Def. Given a local functor $F: Art_k \to Set$.
the tangent space of F is $t_F := F(k[\varepsilon])$

Example: For $R \in Art_k$, $h^R = hom(R,-)$

$$t_{h^R} = hom(R, k[\varepsilon]) = t_R \quad \text{Zariski tangent space of ring } R.$$

Def. Let $\mathcal{C}$ be a category with terminal object p and (binary) products.
A k-vector space object in $\mathcal{C}$ is $V \in \mathcal{C}$
with $\Sigma: V \times V \to V$, $\underline{0}: p \to V$, $\lambda^c: V \to V$ for $c \in k$.
satisfying axioms of vector space.
$(\Sigma \circ (id, \underline{0}) \circ can = id$, $\Sigma \circ (\Sigma, id) = \Sigma \circ (id, \Sigma)$
$can: V \to V \times p$ etc. $)$

Lemma: $k[\varepsilon]$ is k-vector space object in Art_k

proof: k terminal object in Art_k, product is $\times_k$

$$\Sigma: k[\varepsilon] \times_k k[\varepsilon] \to k[\varepsilon]$$

$$(a+b\varepsilon, a+c\varepsilon) \longmapsto a + (b+c)\varepsilon$$

$$\underline{0}: k \to k[\varepsilon]$$

$$a \longmapsto a$$

$$\lambda^c: k[\varepsilon] \to k[\varepsilon]$$

$$a + b\varepsilon \mapsto a + bc\varepsilon$$

vector space axioms easy to verify. $\square$

Cor. If $F: Art_k \to Set$ is local functor satisfying that
$F(k[\varepsilon] \times_k k[\varepsilon]) \longrightarrow F(k[\varepsilon]) \times F(k[\varepsilon])$ is bijective,
then t_F is a k-vector space.

proof: Set addition on t_F to be $F(\Sigma) \circ (*)^{-1}$

zero $0 := F(\underline{0})(F(k))$ (local)

scalar mult. $F(\lambda^c)$

Axioms of vs follow from axioms of vs object. $\square$

3) (Pre-) representability

Def: $\hat{Art}_k$ is category of Local, complete, Noetherian k-algebras (R, m), s.t. $R/m^n \in Art_k \quad \forall n \in \mathbb{N}$

Noetherian: ascending chain condition on ideals
$0 < I_1 < I_2 < I_3 \dots$ becomes stationary

Completion of local ring (R, m) is $\hat{R} = \varprojlim (R/m^n)$
$= \{(a_n)_{n \in \mathbb{N}} \mid a_n \in R/m^n,\ pr\, a_{n+1} = a_n\}$

e.g. $R = k[x_1, \dots, x_n]$, $I = (x_1, x_2, \dots, x_n)$

$\hat{R}_I = k[[x_1, \dots, x_n]]$ (completion at I (not local ring))

can. morph $R \to \hat{R}$, R complete iff $R \to \hat{R}$ is iso.

Rmk: $Art_k \subset \hat{Art}_k$ full subcategory. $(R, m) \in Art_k$, then m nilp. and hence $\hat{R} = R$

Def: Given $F: Art_k \to Set$, can extend to
$\hat{F}: \hat{Art}_k \to Set$ by $\hat{F}(R) = \varprojlim F(R/m^n)$

Def: $F: \mathcal{C} \to Set$ representable if $\exists X \in \mathcal{C}$ s.t. $F \simeq h^X = hom(X, -)$

Yoneda Lemma: $F: \mathcal{C} \to Set$, $hom_{Functor}(h^X, F) \simeq F(X)$

$$\phi \longmapsto \phi_X(id_X)$$

$$\phi_Y(f) = F(f)(\mu) \longleftarrow\!\shortmid\ \mu$$

III : For $R \in \hat{Art}_k$ we ~~also~~ set $h^R_{Art} : Art_k \longrightarrow Set$

$h^R_{Art_k} = h^R_{\hat{Art}_k}\big|_{Art_k}$ $\quad h^R(R,S) = \hom_{\hat{Art}_k}(R,S)$

Since each $f \in \hom_{\hat{Art}_k}(R,S)$ factors through some $f_n : R/_{m^n} \to S$,

we get $\hom_{Functor}(h^R, F) \simeq \hat{F}(R)$

Def: Given $F : Art_k \to Set$ local,

a pro-couple for F is (R, ξ) where $R \in \hat{Art}_k$ and $\xi \in \hat{F}(R)$.

Def: A pro-couple (R,ξ) pro-represents F if $h^R \to F$ induced by ξ is an isomorphism.

Def: A morphism of functors $F \to G$ is smooth $F, G : Art_k \to Set$ if for any surjection $R \twoheadrightarrow S$ in Art_k the map

$F(R) \longrightarrow F(S) \times_{G(S)} G(R)$ is surjective.

Def: A pro-couple (R,ξ) is a hull for F if $h^R \to F$ induced by ξ is smooth and $t_R \longrightarrow t_F$ is a bijection.

Def: A small extension $R \to S$ in Art_k is

$0 \to k \to R \twoheadrightarrow S \to 0$, i.e. surjection where kernel is 1d vs, (principal ideal (t) s.t $m_R(t) = 0$

Theorem (Schlessinger): Given a local functor $F : Art_k \to Set$, for $R \xrightarrow{f} S$, $R' \xrightarrow{g} S$ in Art_k, consider

$$(*) : F(R \times_S R') \longrightarrow F(R) \times_{F(S)} F(R')$$

Denote by the following conditions on F.

H_1: If $R' \xrightarrow{g} S$ is small extension, then $(*)$ is surjective

H_2: If $R' = k[\varepsilon]$, $S = k$, then $(*)$ is bijection

H_3: $\dim_k (t_F) < \infty$

H_4: If $R' \xrightarrow{g} S = R \xrightarrow{f} S$ is small extension, then $(*)$ is bijection.

Then
- F has a hull iff it satisfies H_1, H_2 and H_3
- F is pro-representable iff it satisfies H_1 to H_4