Deformations of Associative Algebras

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Let k be an algebraically closed field of characteristic zero and A a vector space over k equipped with an associative multiplication $m : A \otimes_k A \to A$. For simplicity, $\otimes_k$ and $\text{Hom}_k(-,-)$ are written as $\otimes$ and Hom(-,-), respectively, and $A^{\otimes n}$ means $\underbrace{A \otimes \cdots \otimes A}_{\text{n copies}}$.

1 First-order deformations

Definition 1.1. *A first-order deformation of (A, m) is an associative and $k[t]/t^2$-bilinear multiplication*

$$F : A \otimes k[t]/t^2 \times A \otimes k[t]/t^2 \to A \otimes k[t]/t^2$$

of the form $F = m + tf$, $f \in Hom(A^{\otimes 2}, A)$.

The multiplication F is associative if, for any $a, b, c \in A$, the equation

$$F(F(a, b), c) = F(a, F(b, c))$$

holds. Expanding both sides and collecting the coefficient of t yields the equation

$$f(ab, c) + f(a, b)c = f(a, bc) + af(b, c) \quad (1)$$

where $ab := m(a, b)$.

Definition 1.2. *Let F_1, F_2 be first-order deformations of m. For any $a \in A$, $g \in Hom(A, A)$, we define an automorphism T of $A \otimes k[t]/t^2$ as $T(a) = a + tg(a)$. First-order deformations*

F_1 and F_2 are called equivalent if there exists $g \in Hom(A, A)$ such that, for any $a, b \in A$, the equation

$$F_1(a, b) = T(F_2(T^{-1}(a), T^{-1}(b))) \tag{2}$$

holds.

From 2, if $F_1 = m + tf_1$ is equivalent to $F_2 = m + tf_2$, then there exists $g \in \mathrm{Hom}(A, A)$ such that, for any $a, b \in A$, the equation

$$f_1(a, b) - f_2(a, b) = ag(b) - g(ab) + g(a)b \tag{3}$$

holds.

Observe that equivalence classes of first-order deformations of (A, m) can be expressed as the set of morphisms $f \in \mathrm{Hom}(A^{\otimes 2}, A)$ satisfying (1) modulo the equivalence relation (3). Furthermore, we can describe first-order deformations in terms of the second Hochschild cohomology.

Definition 1.3. *Hochschild cohomology $HH^*(A)$ of a k-algebra A is defined as the cohomology of the cochain complex $C^*(A, A)$*

$$0 \longrightarrow A \xrightarrow{d} Hom(A, A) \xrightarrow{d} Hom(A \otimes A, A) \xrightarrow{d} \cdots$$

with $C^n(A) := Hom(A^{\otimes n}, A)$ for $n \geq 0$ and differential $d : Hom(A^{\otimes n}, A) \to Hom(A^{\otimes n+1}, A)$, $d = \sum_{i=0}^{n+1} (-1)^i \partial^i$ where

$$\partial^i f(a_0, \ldots, a_n) = \begin{cases} a_0 f(a_1, \ldots, a_n) & i = 0 \\ f(a_0, \ldots, a_{i-1}a_i, \ldots, a_n) & 0 < i \leq n \\ f(a_0, \ldots, a_{n-1})a_n & i = n + 1 \end{cases}$$

Remark. $C^0(A, A) = \mathrm{Hom}_k(k, A) \cong A$

Example 1.4. Compute $HH^0(A)$, $HH^1(A)$ and $HH^2(A)$.

For the differential $d : C^0(A, A) \to C^1(A, A)$, an element $b \in A$ is a 0-cocycle if, for any $a \in A$, $db(a) = ab - ba = 0$. Hence

$$H^0(A) = \{b \in A |\ ab = ba\ \forall a \in A\} = Z(A).$$

For a cocycle $g \in C^1(A, A)$, $dg = 0$ is equivalent to $g(ab) = ag(b) + g(a)b$ for any $a, b \in A$. Such a function is called a *k-derivation* and the collection of all k-derivations is denoted by $Der(A)$. If $h \in C^1(A, A)$ is a coboundary, then there exists $b \in A$ such that, for any $a \in A$, we have $h(a) = ab - ba$. This type of functions are called *inner derivations*, denote by $InnDer(A)$. Therefore,

$$HH^1(A) = Der(A)/InnDer(A).$$

Let $f \in C^2(A, A)$ be a cocycle. Then, for any $a, b, c \in A$, we have $df(a, b, c) = af(b, c) - f(ab, c) + f(a, bc) + f(a, b)c = 0$, which is equivalent to (1). Let $f' \in C^2(A, A)$ be a coboundary, then there exists $g \in C^1(A, A)$ such that, for any $a, b \in A$, we have $f'(a, b) = dg(a, b) = ag(b) - g(a)b$. This relation is the same as (3). Therefore, we have the following theorem.

Theorem 1.5. *There is a bijection*

$$\left\{\text{first-order deformations of } (A, m)\right\} /\sim \quad \simeq HH^2(A, A)$$

We observe that first-order deformations can be classified by $HH^2(A, A)$, and our next goal is to understand the relation between higher order deformations and the Hochschild cochain complex.

2 Higher-order deformations

Definition 2.1. *A second-order deformation of (A, m) is an associative and $k[t]/t^3$-bilinear multiplication*

$$F : A \otimes k[t]/t^3 \times A \otimes k[t]/t^3 \to A \otimes k[t]/t^3$$

of the form $F = m + tf_1 + t^2 f_2$, with $f_1, f_2 \in Hom(A^{\otimes 2}, A)$.

We now discuss the question of whether a first-order deformation can be extended to a second-order deformation.

2.1 Extension to second-order

Let $f_1 \in C^2(A, A)$ be a cocyle, so that $F = m + tf_1$ is a first-order deformation. We want to find a $f_2 \in \mathrm{Hom}(A^{\otimes 2}, A)$ to extend F to $\widetilde{F} = m + tf_1 + t^2 f_2$ such that $\widetilde{F}$ is associative.

The associativity condition yields two equations corresponding to the coefficients of t and t^2, respectively: for any $a, b, c \in A$,

$$f_1(ab, c) + f_1(a, b)c = f_1(a, bc) + af_1(b, c), \tag{4}$$

$$f_1(a, f_1(b, c)) - f_1(f_1(a, b), c) = f_2(ab, c) + f_2(a, b)c - f_2(a, bc) - af_2(b, c). \tag{5}$$

Observe that (4) automatically holds since f_1 is a 2-cocycle, and the right-hand side of (5) is a 3-coboundary in $\mathrm{Hom}(A^{\otimes 3}, A)$. In order to describe the left-hand side of (5), we introduce Gerstenhaber bracket which was first defined by Murray Gerstenhaber [Ger63]. As we will now explain, this bracket defines a differential graded Lie algebra structure on the Hochschild cochain complex.

2.2 DGLA and Gerstenhaber bracket

Definition 2.2. *A differential graded Lie algebra (DGLA) (L, [,], d) is the data of a $\mathbb{Z}$-graded vector space $L = \oplus_{i\in\mathbb{Z}} L^i$ together with a bilinear bracket $[,] : L \times L \to L$ and a linear map $d : L \to L$ satisfying the following conditions:*

(1) $[,]$ is a homogeneous skew-symmetric, i.e., $[L^i, L^j] \subset L^{i+j}$ and $[a, b] + (-1)^{|a||b|}[b, a] = 0$ where $|a|$ is the degree of a.

(2) Every a, b, c homogeneous satisfy the Jacobi identity

$$[a, [b, c]] = [[a, b], c] + (-1)^{|a||b|}[b, [a, c]]$$

(3) $d(L^i) \subset L^{i+1}, d^2 = 0$ and $d[a, b] = [da, b] + (-1)^{|a|}[a, db]$ the map d is called the differential of L.

We define $\Gamma := \oplus_{i\in\mathbb{Z}}\Gamma^i$ to be the graded vector space with $\Gamma^n := \mathrm{Hom}(A^{\otimes n+1}, A)$, for $n \geq -1$, and $\Gamma^n = 0$ for $n < -1$.

Definition 2.3. *(i) For $f \in \Gamma^m(A, A)$, $g \in \Gamma^n$, $m \geq 0$, let us denote the element $f \circ_i g \in \Gamma^{m+n}$, for $i = 1, \ldots, m+1$, defined by*

$$\begin{aligned} &f \circ_i g(a_1 \otimes \cdots \otimes a_{m+n+1-1}) \\ &\qquad = f(a_0 \otimes \cdots \otimes a_{i-1} \otimes g(a_i \otimes \cdots \otimes a_{i+n}) \otimes a_{i+n+1} \otimes \cdots \otimes a_{m+n+1}) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1, 2, \ldots, m+1 \end{aligned}$$

(ii) For every m, n, we define a homomorphism $\circ$ called circle product sending $\Gamma^m(A, A) \otimes \Gamma^n(A, A)$ into Γ^{m+n} by setting for $f \in \Gamma^m(A, A)$, $g \in \Gamma^n(A, A)$, with $m \geq 0$,

$$f \circ g = \sum_{i=1}^{m+1} (-1)^{(i+1)n} f \circ_i g \tag{6}$$

and $f \circ g = 0$, for $m < 0$

Note that for $m = n = 0$, the operation is in fact the commutator. Explicitly,

$$f \circ g = \begin{cases} f \circ_1 g + f \circ_2 g + \cdots + f \circ_m g & \text{if n even} \\ f \circ_1 g - f \circ_2 g + \cdots + (-1)^{m+1} f \circ_m g & \text{if n odd} \end{cases}$$

Remark. $df = -[f, m]$, for $f \in \Gamma^m$.

Proposition 2.4. *For elements $f \in \Gamma^m$ and $\Gamma \in A^n$, we define the Gerstenhaber bracket*

$$[f, g] = f \circ g - (-1)^{mn} g \circ f \tag{7}$$

then $(\Gamma, [,], \delta)$ where $\delta := -d$ for is a differential graded Lie algebra.

Proof. The skew-symmetry follows directly from equation (6). For the verification of the Jacobi identity, we refer the reader to original paper [Ger63, Theorem 1&2]. The compatibility of differential follows from the fact that the circle product we defined satisfies the graded Leibnitz rule. □

Corolary 2.5. *If f an n-cocycle in Γ^n, then $[f, f]$ is a $2n$-cocycle.*

Let $m = n = 1$, for any $f \in \Gamma^1$, $a, b, c \in A$, we have

$$[f, f](a, b, c) = 2f \circ f(a, b, c) = 2(f(f(a, b), c) - f(a, f(b, c))). \tag{8}$$

Combing it with equation (4), we get the formula

$$\frac{1}{2}[f_1, f_1] = df_2. \tag{9}$$

From Corollary 2.5, we see that the left-hand side of this equation is a cocycle in Γ^2, but the right-hand side is a coboundary in Γ^2. Equation (9) says that to find f_2, making $m + tf_1 + t^2 f_2$ a second-order deformation, the left hand side has to be zero in $HH^3(A, A)$, and therefore we obtain the following theorem.

Theorem 2.6. *Let $F = m + tf_1$ be a first-order deformation. The class in $HH^3(A,A)$ defined by $[f_1, f_1]$ is the obstruction to extend the first-order deformation $F = m + tf_1$ to a second-order deformation.*

In particular, if $HH^3(A,A) = 0$, then any first-order deformation can be extended. However, the vanishing of $HH^3(A,A)$ is not a necessary condition for unobstructedness.

Example 2.7. Let $A = k[x,y,z]/(xy - z, x^2, y^2, z^2)$. Then A is a 4-dimensional algebra and we choose a basis $(1, x, y, z)$. Consider two cocycles $f, g \in \Gamma^1$. Let $f(y,x) = z$ and zero for other basis vector, and $g(x,x) = y$ and zero for the other basis vector. So, $[f,f] = 0$ and $[g,g] = 0$, and we obtain two unobstructed first-order deformations $m + tf$, $m + tg$. But $f + g$ is also a cocyle in Γ^1, and defines a obstructed first-order deformation $m + t(f+g)$. $[f+g, f+g](x,x,x) = 2f \circ g(x,x,x) = z$, and for any $h \in \Gamma^1$, $dh(x,x,x) = xh(x,x) - h(x^2,x) + h(x,x^2) - h(x,x)x = 0$ since A is a commutative algebra. Hence $[f+g, f+g] \neq 0$ in $HH^3(A,A)$.

In general, given a deformation over $k[t]/t^{n+1}$, we want to extent it to be a deformation over $k[t]/t^{n+2}$. By expanding and collecting coefficient of $t^k, k = 0, 1, \ldots, n+1$, the associativity condition implies that, for any $a, b, c \in A$, the equation

$$\sum_{i+j=0}^{k} f_j(a, f_i(b,c)) = \sum_{i+j=0}^{k} f_j(f_i(a,b), c) \tag{10}$$

holds, where $f_0 = m$ is the multiplication of A. For $k = n+1$, equation (10) can be rewritten as

$$\frac{1}{2} \sum_{\substack{i+j=1 \\ i,j\neq 0}}^{n+1} [f_j, f_i](a,b,c) = -\delta f_{n+1}(a,b,c) \tag{11}$$

by dg-Lie structure. If the left hand side of (11) is a cocycle, then it is the obstruction class in $HH^3(A,A)$ to extend the nth-order deformation $m + tf_1 + \cdots + t^n f_n$ to a (n+1)th-order deformation.

To prove the claim, we need to use equation (10) for $k = 1, \ldots, n$ which yields

$$\frac{1}{2} \sum_{\substack{i+j=k \\ i,j\neq k}} [f_j, f_i] = -\delta f_k, \quad k = 1, \ldots, n$$

Thus, by applying the graded Lebnitz rule,

$$\sum_{j=1}^{n} \delta[f_j, f_{n+1-j}] = \sum_{j=1}^{n}[\delta f_j, f_{n+1-k}] - [f_j, \delta f_{n+1-j}] = \sum_{\substack{i+j+k=n+1 \\ i,j,k\neq 0}} [f_i, [f_k, f_j]] = 0,$$

and hence a cocycle.

Definition 2.8. *$HH^3(A, A)$ is called the obstruction space of A.*

We conclude by noticing that any finite-order deformation problem of an associative algebra can be interpreted by associate DGLA. Our next goal is to find the connection between formal deformations of (A, m) and the DGLA $(\Gamma, [,], \delta)$.

3 Formal deformations

Define $\widetilde{A} := A \otimes k[[t]]$

Definition 3.1. *A one-parameter formal deformation of (A, m) is an associative, $k\,[[t]]$-bilinear multiplication*

$$F : \widetilde{A} \times \widetilde{A} \to \widetilde{A}$$

of the form $F = m + tf_1 + t^2 f_2 + \cdots$ with $f_i \in Hom(A^{\otimes 2}, A)$.

Denote $f := tf_1 + t^2 f_2 + \cdots$.

Note that $\widetilde{\Gamma} := \Gamma \otimes t \cdot k[[t]]$ is a DGLA, with elements of degree n expressed as $t\psi_1 + t^2\psi_2 + \ldots,\ \psi_i \in \Gamma^n$. The dg-Lie structure of $\widetilde{\Gamma}$ is obtained from Γ by extending $[,]$ and δ $k[[t]]$-linearly. Hence, $\widetilde{\Gamma}$ inherits the relation between $[,]$ and δ from Γ such that

$$\delta = [-, m].$$

Proposition 3.2. *Let $F = m + f,\ f \in \widetilde{\Gamma}^1$. Then F is a formal deformation of m if and only if f satisfies the Maurer-Cartan equation*

$$\delta f + \frac{1}{2}[f, f] = 0 \tag{12}$$

Proof. $[m,m]=0$ since m is associative. The associativity condition of F yields

$$\begin{aligned}
&F(a,F(b,c)) = F(F(a,b),c)\\
\iff& \sum_{i+j=n} f_j(a,f_i(b,c)) = \sum_{i+j=n} f_j(f_i(a,b),c), n\geq 1\\
\iff& \sum_{\substack{i+j=n\\ i,j\neq 0}} f_j(a,f_i(b,c)) - f_j(f_i(a,b),c) = f_n(a,b)c + f_n(ab,c) - af_n(b,c) - f_n(a,bc), n\geq 1\\
\iff& \frac{1}{2}\sum_{\substack{i+j=n\\ i,j\neq 0}} [f_j\circ f_i] + \delta f_n = 0, n\geq 1\\
\iff& \frac{1}{2}[f,f] + \delta f = 0
\end{aligned}$$

the last equivalence is obtained by combing all the equations for $n=1,2,\ldots,$ and bringing t back, the result follows. □

Denote by $MC(\widetilde{\Gamma})$ solutions of Maurer-Cartan equation of the DGLA $(\widetilde{\Gamma},[,],\delta)$, i.e.,

$$MC(\widetilde{\Gamma}) := \left\{ f\in\widetilde{\Gamma}^1 \, |\delta f + \frac{1}{2}[f,f] = 0 \right\}.$$

Definition 3.3. *For every $x\in\widetilde{\Gamma}^1$, define the automorphism of $\widetilde{A}$*

$$\psi = \exp(x) = \exp(x) = 1 + x + \frac{x^2}{2!} + \cdots .$$

Formal deformations $F=m+f$ and $F'=m+f'$ are called equivalent if there exists a $x\in\widetilde{\Gamma}^0$ such that, for any $a,b\in A$, the equation

$$(m+f')(a,b) = \psi((m+f)(\psi^{-1}a,\psi^{-1}b)) \tag{13}$$

holds.

Note that $\Gamma^0\otimes k[[t]]$ is a group with normal multiplication. If we equip $\widetilde{\Gamma}^0$ with the multiplication law given by the Baker-Campbell-Hausdorff formula,

$$x\cdot y = x + y + \frac{1}{2}[x,y] + \frac{1}{12}([x,[x,y]] + [y,[y,x]]) + \ldots,$$

then the exp map is a group homomorphism sending $\widetilde{\Gamma}^0$ to $\Gamma^0\otimes k[[t]]$ such that $\exp(x\cdot y) = \exp(x)\exp(y)$. The inverse of the map $\exp:\widetilde{\Gamma}^0\to\exp(\widetilde{\Gamma}^0)$ is the homomorphism

$$\log:\exp(\widetilde{\Gamma}^0)\to\widetilde{\Gamma}^0,$$

defined as

$$1 + x \mapsto x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots,$$

and hence $(\widetilde{\Gamma}^0, \cdot) \cong \exp(\widetilde{\Gamma}^0)$.

Definition 3.4. *The group* $\exp(\widetilde{\Gamma}^0)$ *is called gauge group and denoted by* $G(\widetilde{\Gamma})$. *The action of the group* $G(\widetilde{\Gamma})$ *on* $\widetilde{\Gamma}^1$ *is defined by the formula, for any* $x \in \widetilde{\Gamma}^0$, $f \in \widetilde{\Gamma}^1$,

$$x \star f = f + [x, m + f] + \frac{1}{2!}[x, [x, m + f]] + \frac{1}{3!}[x, [x, [x, m + f]]] + \dots, \tag{14}$$

and this action is called gauge action. Elements f, f' *in* $\widetilde{\Gamma}^1$ *are called gauge equivalent if there exists* $x \in \widetilde{\Gamma}^0$ *such that* $f' = x \star f$.

Remark. Equation (14) can be rewritten as a form without m,

$$x \star f = f + \delta x + [x + f] + \frac{1}{2!}\{[x, \delta x] + [x, [x, f]]\} + \frac{1}{3!}\{[x, [x, \delta x]] + [x, [x, [x, f]]]\} + \cdots, \tag{15}$$

by $\delta(x) = [m, x]$, so that this gauge action only depends on dg-Lie structure of $\widetilde{\Gamma}$.

Given equivalent formal deformations $F = m + f$ and $F' = m + f'$, there exists $x \in \widetilde{\Gamma}^0$ such that $F' = \psi F(\psi^{-1}, \psi^{-1})$, where $\psi = \exp(x)$. By expanding the equation, we obtain

$$\begin{aligned} m + f' &= \exp(x)(m + f)(\exp(-x), \exp(-x)) \\ &= (1 + x + \frac{x^2}{2!} + \dots)(m + f)(1 - x + \frac{x^2}{2!} - \dots, 1 - x + \frac{x^2}{2!} - \dots) \\ &= m + f + \delta x + [x, f] + \frac{1}{2!}\{[x, \delta x] + [x, [x, f]]\} + \cdots = m + x \star f. \end{aligned}$$

So the gauge action can be obtained by expressing the right hand side of

$$x \star f = \exp(x)(m + f)(\exp(-x), \exp(-x)) - m$$

in terms of Lie bracket. Therefore, formal deformations $F = m+f, F' = m+f'$ are equivalent if and only if f, f' are gauge equivalent. Hence, we have the following theorem.

Theorem 3.5. *Let A be an associative algebra. There is a bijection*

$$\Big\{\textit{formal deformations of (A,m)}\Big\} /_{\sim} \; \simeq MC(\widetilde{\Gamma})/G(\widetilde{\Gamma}).$$

Remark. Solutions of Maurer-Cartan equation only depends on associate DGLA, and the action of gauge group $G(\widetilde{\Gamma})$ on $MC(\widetilde{\Gamma})$ only depends on dg-Lie structure of $\widetilde{\Gamma}$.

Every deformation problem of an associative algebra can be interpreted by constructing DGLA depending on given data. We only discussed DGLA whose differential has the form $\delta = [-, m]$, more generally, one may assume a nilpotent DGLA. For further study about DGLA and deformation theory, the reader can refer to [Man05] and [DMZ07].

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